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Midpoint Locally Uniform Rotundity of Musielak-Orlicz-Bochner Function Spaces endowed with the Luxemburg Norm*

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Criteria for midpoint locally uniform rotundity of Musielak-Orlicz-Bochner function spaces equipped with the Luxemburg norm are given. We also prove that, in Musielak-Orlicz-Bochner function spaces generated by midpoint locally uniformly rotund Banach space, midpoint locally uniform rotundity and rotundity are equivalent. The topic of this paper is related to the topic of [1–6] and [9–16].

Keywords: Midpoint locally uniform rotundity, Musielak-Orlicz-Bochner function spaces, Luxemburg norm

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1. Introduction

A lot of rotundity concepts of Banach spaces are known. Among them rotundity (R for short), midpoint locally uniform rotundity are important notions. One of the reasons is that (see [7]) a Banach X space is midpoint locally uniformly rotund if and only if every closed ball in X is an approximative compact Chebyshev set. The criteria for midpoint locally uniform rotundity in the classical Orlicz function spaces have been given in [3, 4] already. However, because of the complicated structure of Musielak-Orlicz-Bochner function spaces, at present the criteria for midpoint

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locally uniform rotundity have not been discussed yet. In the paper, we will discuss criteria for midpoint locally uniform rotundity in Musielak-Orlicz-Bochner function spaces endowed with the Luxemburg norm.

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and the unit ball, respectively. Let us recall some geometrical notions concerning rotundity. A Banach space X is be said to be rotund if for any $x, y \in S(X)$ and $\|x+y\| = 2$ we have $x = y$. A point $x \in S(X)$ is said to be a strongly extreme point if for any $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset X$ with $\|x_n\| \to 1, \|y_n\| \to 1$ and $x = \frac{1}{2}(x_n + y_n)$, there holds $\|x_n - y_n\| \to 0 (n \to \infty)$. If the set of all strongly extreme points of $B(X)$ is equal to $S(X)$, then X is said to be midpoint locally uniformly rotund.

Let (T, Σ, μ) be a nonatomic finite measurable space. Suppose that that a function $M : T \times [0, \infty) \to [0, \infty]$ statisfies the following conditions:

(1) for $\mu - a.e.$, $t \in T$, $M(t, 0) = 0$, $\lim_{u \to \infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$.
(2) for $\mu - a.e.$, $t \in T$, $M(t, u)$ is a convex function on $[0, \infty)$ with respect to u.
(3) for each $u \in [0, \infty)$, $M(t, u)$ is a μ-measurable function of t on T.

Moreover, given any Banach space $(X, \|\cdot\|)$, we denote by X_T the set of all strongly μ-measurable function from T to X, and for each $u \in X_T$, we define the modular of u by

$$\rho_M(u) = \int_T M(t, \|u(t)\|) dt.$$

Put

$$L_M(X) = \{u \in X_T : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\}.$$

Then the Musielak-Orlicz-Bochner space $L_M(X)$ equipped with Luxemburg norm

$$\|u\| = \inf\left\{\lambda > 0 : \int_T M\left(t, \frac{\|u(t)\|}{\lambda}\right) dt \le 1\right\} \quad (u \in L_M(X))$$

is a Banach space. Set

$$e(t) = \sup\{u > 0 : M(t, u) = 0\} \quad \text{and} \quad E(t) = \sup\{u > 0 : M(t, u) < \infty\}$$

Definition 1.1 (see [1]). We say that $M(t, u)$ satisfies condition $\triangle$ ($M \in \triangle$ for short) if there exist $K \ge 1$ and a measureable nonnegative function $\delta(t)$ on T such that $\int_T M(t, \delta(t)) dt < \infty$ and $M(t, 2u) \le K M(t, u)$ for almost all $t \in T$ and all $u \ge \delta(t)$.

For fixed $t \in T$ and $v > 0$, if there exists $\epsilon \in (0, 1)$ such that

$$M(t, v) = \frac{1}{2} M(t, v + \varepsilon) + \frac{1}{2} M(t, v - \varepsilon) < \infty$$

then we call v a nonstrictly convex point of M with respect to t. The set of all nonstrictly convex point of M with respect to t is denoted by K_t.

If $K_t = \emptyset$ for $\mu - a.e.$ $t \in T$, then we call that $M(t, u)$ is strictly convex with respect u.

First let us recall a result that will be used in the further part of the paper.

Lemma 1.2 (see [1]). *Suppose $M \in \triangle$ and $e(t) = 0$ μ-a.e. on T. Then*

$$\rho_M(u_n) \to 0 \Leftrightarrow \|u_n\| \to 0 \quad \text{and} \quad \rho_M(u_n) \to 1 \Leftrightarrow \|u_n\| \to 1 (n \to \infty).$$

It is easy to see that if $M(t, u)$ is strictly convex with respect u for almost all $t \in T$, then $e(t) = 0$ for almost all $t \in T$.

2. Main results

Theorem 2.1 (see [1]). *$L_M(X)$ is rotund if and only if*

(a) $M \in \triangle$;
(b) X *is rotund;*
(c) $M(t, u)$ *is strictly convex with respect u for almost all $t \in T$.*

Theorem 2.2. *$L_M(X)$ is midpoint locally uniformly rotund if and only if*

(a) $M \in \triangle$;
(b) X *is midpoint locally uniformly rotund;*
(c) $M(t, u)$ *is strictly convex with respect u for almost all $t \in T$.*

In order to prove the theorem, we first give a lemma.

Lemma 2.3. *Let $\frac{z}{\|z\|}$ be a strongly extreme point. Then for any $\varepsilon > 0$, we have*

(a) $$\delta(z) := \inf_{x \in X} \left\{ \max(\|x\| - \|z\|, \|y\| - \|z\|) : \|x - z\| \geq \varepsilon,\ z = \frac{1}{2}(x + y) \right\} > 0$$

(b) *if $z_n \to z$, then $\delta(z_n) \to \delta(z)$.*

Proof. (a) Suppose that $\delta(z) = 0$. Then there exists $\{x_n\}_{n=1}^{\infty} \subset X, \{y_n\}_{n=1}^{\infty} \subset X$ such that $\|x_n - z\| \geq \varepsilon$, $z = \frac{1}{2}x_n + \frac{1}{2}y_n$, but $\|x_n\| - \|z\| < \frac{1}{n}$, $\|y_n\| - \|z\| < \frac{1}{n}$, hence

$$\left\| \frac{x_n}{\|z\|} \right\| \leq 1 + \frac{1}{n\|z\|}, \qquad \left\| \frac{y_n}{\|z\|} \right\| \leq 1 + \frac{1}{n\|z\|}, \qquad \frac{x_n}{\|z\|} + \frac{y_n}{\|z\|} = 2\frac{z}{\|z\|}.$$

Since $\frac{z}{\|z\|}$ is a strongly extreme point, we have

$$\left\| \frac{x_n}{\|z\|} - \frac{y_n}{\|z\|} \right\| \to 0 \Rightarrow \|x_n - y_n\| \to 0 \Rightarrow \|x_n - z\| \to 0 (n \to \infty),$$

a contradiction.

(b) Suppose that there exist $a > 0$ and a sequence $\{z_n\}_{n=1}^{\infty}$ such that $z_n \to z$ and $\delta(z_n) - \delta(z) \geq a$. By the definition of $\delta(z)$, there exist x_0 and y_0 such that

$$\delta(z) + \frac{1}{8}a > \max(\|x_0\| - \|z\|, \|y_0\| - \|z\|), \quad \|x_0 - z\| \geq \varepsilon,\ z = \frac{1}{2}(x_0 + y_0). \quad (1)$$

By $z_n \to z$, there exists n_1 such that $\|z_{n_1} - z\| < \frac{1}{8}a$. It is obvious that the following

$$z_{n_1} = \frac{1}{2}[(x_0 - z + z_{n_1}) + (y_0 - z + z_{n_1})], \qquad \|x_0 - z + z_{n_1} - z_{n_1}\| \geq \varepsilon.$$

hold. By (1), we get the following inequality

$$\begin{aligned}\|x_0 - z + z_{n_1}\| - \|z_{n_1}\| &\le \|x_0\| + \frac{1}{8}a - \left(\|z\| - \frac{1}{8}a\right)\\ &= \|x_0\| - \|z\| + \frac{1}{4}a\\ &< \delta(z) + \frac{1}{8}a + \frac{1}{4}a.\end{aligned}$$

Similarly, we have $\|y_0 - z + z_{n_1}\| - \|z_{n_1}\| \le \delta(z) + \frac{1}{8}a + \frac{1}{4}a$. Hence $\delta(z_{n_1}) \le \delta(z) + \frac{1}{8}a + \frac{1}{4}a < \delta(z) + \frac{1}{2}a$, a contradiction. Similarly, $\delta(z) - \delta(z_n) \ge b > 0$ is impossible. Hence (b) is true. This completes the proof. □

Proof of Theorem 2.2. *Necessity.* By Theorem 2.1, the necessity of (a) and (c) is obvious. We next will prove that (b) is true. Pick $h(t) \in S(L_M(X))$, then there exists $d > 0$ such that $\mu E > 0$, where $E = \{t \in T : \|h(t)\| \ge d\}$. Put $h_1(t) = d \cdot x_0 \cdot \chi_E(t)$, where $x_0 \in S(X)$. It is easy to see that $h_1(t) \in L_M(X)$. Hence there exists $k > 0$ such that $k \cdot h_1(t) \in S(L_M(X))$. By Lemma 1.2, we have

$$1 = \int_T M(t, \|k \cdot h_1(t)\|)dt = \int_E M(t, \|k \cdot d \cdot x_0\|)dt.$$

Let $\alpha = k \cdot d$. Then $\int_E M(t,\alpha)dt = 1$. The necessity of (b) follows from the fact that X is isometrically embedded into $L_M(X)$. Namely, defining the operator $I : X \to L_M(X)$ by

$$I(x) = \alpha \cdot x \cdot \chi_E(t), \quad x \in X.$$

Hence, for any $x \in X\backslash\{0\}$, we have

$$\rho_M\left(\frac{I(x)}{\|x\|}\right) = \int_E M\left(t, \left\|\frac{I(x)}{\|x\|}\right\|\right)dt = \int_E M\left(t, \frac{\alpha\,\|x\|}{\|x\|}\right)dt = \int_E M(t,\alpha)dt = 1.$$

By Lemma 1.2, we have $\left\|\frac{I(x)}{\|x\|}\right\|_{L_M(X)} = 1$, whence $\|I(x)\|_{L_M(X)} = \|x\|$.

Sufficiency. Let $u \in S(L_M(X))$ and $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty} \subset L_M(X)$ with $\|u_n\| \to 1(n \to \infty), \|v_n\| \to 1(n \to \infty)$ and $u = \frac{1}{2}(u_n + v_n)$. By Lemma 1.2, we have $\rho_M(u_n) \to 1(n \to \infty)$ and $\rho_M(v_n) \to 1(n \to \infty)$. We next will prove that for any $\sigma > 0$ and $\varepsilon > 0$ there exists N such that

$$\mu\{t \in T : \|u_n(t) - u(t)\| \ge \sigma\} < \varepsilon$$

whenever $n > N$. Otherwise, without loss of generality, we may assume that there exists $\varepsilon_0 > 0$ and $\sigma_0 > 0$ such that for each $n \in N$ we can find the set $E'_n \subseteq T$ such that $\mu E'_n \ge \varepsilon_0$, where

$$E'_n = \{t \in T : \|u_n(t) - u(t)\| \ge \sigma_0\}.$$

We define the function

$$\eta(t) = \inf_{x \in X}\left\{\max(\|x\| - \|u(t)\|, \|y\| - \|u(t)\|) : \|x - u(t)\| \ge \sigma_0,\ u(t) = \frac{1}{2}(x+y)\right\}$$

for $t \in T$. By Lemma 2.3, we have $\eta(t) > 0$ for $t \in \{t \in T : u(t) \neq 0\}$. Moreover, it is easy to see that $\eta(t) = \sigma_0$ for $t \in \{t \in T : u(t) = 0\}$. Hence we have $\eta(t) > 0$ for $t \in T$. Let $h_n(t) \to u(t)$ μ-a.e. on T where h_n are simple functions. Hence

$$\begin{aligned}&\eta_n(t)\\ &= \inf_{x \in X} \left\{ \max(\|x\| - \|h_n(t)\|, \|y\| - \|h_n(t)\|) : \|x - h_n(t)\| \geq \sigma_0,\ h_n(t) = \frac{1}{2}(x+y) \right\}\end{aligned}$$

is μ-measurable. By Lemma 2.3, we have $\eta_n(t) \to \eta(t)$ μ-a.e. on $\{t \in T : u(t) \neq 0\}$. Moreover, we know that $\eta(t) = \sigma_0$ for $t \in \{t \in T : u(t) = 0\}$. Hence $\eta(t)$ is μ-measurable. Using

$$T \supset \bigcup_{n=1}^{\infty} E'_n \supset \bigcup_{i=1}^{\infty} \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i} \right\}$$

and

$$\left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i} \right\} \cap \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{j+1} < \eta(t) \leq \frac{1}{j} \right\} = \phi, \quad i \neq j,$$

we get

$$\begin{aligned}\infty > \mu T &\geq \mu \left(\bigcup_{i=1}^{\infty} \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i} \right\} \right)\\ &= \sum_{i=1}^{\infty} \mu \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i} \right\}.\end{aligned}$$

Hence there exists $i_0 \in N$ such that $\sum_{i=i_0}^{\infty} \mu\{t \in \cup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i}\} < \frac{1}{8}\varepsilon_0$. By $\cup_{i=i_0}^{\infty}\{t \in \cup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i}\} = \{t \in \cup_{n=1}^{\infty} E'_n : 0 < \eta(t) \leq \frac{1}{i_0}\}$, we have

$$\begin{aligned}\mu \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : 0 < \eta(t) \leq \frac{1}{i_0} \right\} &= \mu \left(\bigcup_{i=i_0}^{\infty} \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i} \right\} \right)\\ &= \sum_{i=i_0}^{\infty} \mu \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \frac{1}{i+1} < \eta(t) \leq \frac{1}{i} \right\} < \frac{1}{8}\varepsilon_0.\end{aligned}$$

Let $\eta_0 = \frac{1}{2}\min\{\frac{1}{i_0}, \sigma_0\}$. Then $\mu H < \frac{1}{8}\varepsilon_0$, where

$$H = \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : 0 < \eta(t) \leq \eta_0 \right\} = \left\{ t \in \bigcup_{n=1}^{\infty} E'_n : \eta(t) \leq \eta_0 \right\}.$$

Notice that $E'_n = \{t \in T : \|u_n(t) - u(t)\| \geq \sigma_0\} = \{t \in T : \|u_n(t) - v_n(t)\| \geq 2\sigma_0\}$ and decompose E'_n into E'_{n1}, E'_{n2} and E'_{n3}, where

$$E'_{n1} = \left\{ t \in E'_n : |\|u_n(t)\| - \|v_n(t)\|| \geq \frac{\eta_0}{4} \right\},$$

$$E'_{n2} = \left\{ t \in E'_n : |\|u_n(t)\| - \|v_n(t)\|| < \frac{\eta_0}{4},\ u(t) = 0 \right\},$$

$$E'_{n3} = \left\{ t \in E'_n : |\|u_n(t)\| - \|v_n(t)\|| < \frac{\eta_0}{4},\ u(t) \neq 0 \right\}.$$

If $t \in E'_{n3}$, we have $\|u_n(t) - u(t)\| \geq \sigma_0, |\|u_n(t)\| - \|v_n(t)\|| < \frac{\eta_0}{4}$ and $u(t) \neq 0$. Since X is midpoint locally uniformly rotund, by Lemma 2.3, without loss of generality, we may assume that $\|u_n(t)\| - \|u(t)\| \geq \eta(t)$. Let $E''_{n3} = E'_{n3} \backslash H$. Then we have $\|u_n(t)\| - \|u(t)\| \geq \eta_0$, whenever $t \in E''_{n3}$. By $|\|u_n(t)\| - \|v_n(t)\|| < \frac{\eta_0}{4}$, we have

$$\|v_n(t)\| \geq \|u_n(t)\| - \frac{\eta_0}{4} \geq \|u(t)\| + \frac{3}{4}\eta_0.$$

Hence we get the following inequality

$$\begin{aligned} &\frac{1}{2}\|u_n(t)\| + \frac{1}{2}\|v_n(t)\| - \|u(t)\| \\ \geq &\frac{1}{2}\|u(t)\| + \frac{1}{2}\eta_0 + \frac{1}{2}\|u(t)\| + \frac{3}{8}\eta_0 - \|u(t)\| = \frac{7}{8}\eta_0. \end{aligned}$$

Let $E_n = E'_{n1} \cup E'_{n2} \cup (E'_{n3} \backslash H)$. Then $\mu E_n \geq \frac{7}{8}\varepsilon_0$. We define the sets

$$A_n = \left\{ t \in T : M(t, \|u_n(t)\|) > \frac{16}{\varepsilon_0} \right\} \quad \text{and} \quad B_n = \left\{ t \in T : M(t, \|v_n(t)\|) > \frac{16}{\varepsilon_0} \right\}.$$

Then

$$1 \leftarrow \int_T M(t, \|u_n(t)\|)dt \geq \int_{A_n} M(t, \|u_n(t)\|)dt \geq \frac{16}{\varepsilon_0}\mu A_n.$$

whence for all $n \in N$, $\mu A_n \leq \frac{1}{8}\varepsilon_0$. Similarly, we have $\mu B_n \leq \frac{1}{8}\varepsilon_0$ for all $n \in N$. For μ-a.e. $t \in T$, we define a bound closed set

$$C_t = \left\{ (u, v) \in R^2 : M(t, u) \leq \frac{16}{\varepsilon_0},\ M(t, v) \leq \frac{16}{\varepsilon_0},\ \ |u - v| \geq \frac{1}{4}\eta_0 \right\}$$

in the space R^2. Since C_t is compact, then for μ-a.e. $t \in T$ there exists $(u_t, v_t) \in C_t$ such that

$$1 > \frac{2M(t, \frac{1}{2}(u_t + v_t))}{M(t, u_t) + M(t, v_t)} \geq \frac{2M(t, \frac{1}{2}(u + v))}{M(t, u) + M(t, v)} \tag{2}$$

for any $(u, v) \in C_t$. We define a function

$$1 - \delta(t) = \frac{2M(t, \frac{1}{2}(u_t + v_t))}{M(t, u_t) + M(t, v_t)} \tag{3}$$

which is μ-measurable. In fact, pick a dense set $\{r_i\}_{i=1}^{\infty}$ in $[0, \infty)$. We define a function

$$1 - \delta_{r_i, r_j}(t) = \begin{cases} \dfrac{2M(t, \frac{1}{2}(r_i + r_j))}{M(t, r_i) + M(t, r_j)}, & M(t, r_i) \leq \dfrac{16}{\varepsilon_0} \text{ and } M(t, r_j) \leq \dfrac{16}{\varepsilon_0} \\ 0, & M(t, r_i) > \dfrac{16}{\varepsilon_0} \text{ or } M(t, r_j) > \dfrac{16}{\varepsilon_0} \end{cases}.$$

By the definition of $M(t, u)$, it is easy to see that $1 - \delta_{r_i, r_j}(t)$ is μ-measurable and

$$1 - \delta(t) \geq \sup\left\{ 1 - \delta_{r_i, r_j}(t) : |r_i - r_j| \geq \frac{1}{4}\eta_0 \right\}.$$

On the other hand, since $\{r_i\}_{i=1}^{\infty}$ is dense in $[0,\infty)$ then $\{(r_i,r_j)\}_{i=1,j=1}^{\infty}$ is dense in $[0,\infty)\times[0,\infty)$. By the definition of function $1-\delta(t)$, for μ-a.e. $t\in T, \epsilon>0$, there exists $(r_i,r_j)\in\{(u,v)\in R^2 : M(t,u)\le \frac{16}{\varepsilon_0}, M(t,v)\le\frac{16}{\varepsilon_0}, |u-v|\ge\frac{1}{4}\eta_0\}$ such that

$$1-\delta(t)-\varepsilon < 1-\delta_{r_i,r_j}(t) \le \sup\left\{1-\delta_{r_i,r_j}(t) : |r_i-r_j|\ge\frac{1}{4}\eta_0\right\}$$

μ-a.e. on T. Since ϵ was arbitrary, we have

$$1-\delta(t)\le \sup\left\{1-\delta_{r_i,r_j}(t) : |r_i-r_j|\ge\frac{1}{4}\eta_0\right\}$$

μ-a.e. on T. Therefore $1-\delta(t)=\sup\{1-\delta_{r_i,r_j}(t) : |r_i-r_j|\ge\frac{1}{4}\eta_0\}$ μ-a.e. on T. This implies that $\delta(t)$ is μ-measurable. By (2) and (3), we have

$$M\left(t,\frac{1}{2}(u+v)\right)\le\frac{1}{2}(1-\delta(t))(M(t,u)+M(t,v)),\quad u,v\in C_t$$

for μ-a.e- $t\in T$. Hence, for μ-a.e. $t\in E'_{n1}\setminus(A_n\cup B_n)$, we have

$$M\left(t,\frac{1}{2}(\|u_n(t)\|+\|v_n(t)\|)\right)\le\frac{1}{2}(1-\delta(t))[M(t,\|u_n(t)\|)+M(t,\|v_n(t)\|)].$$

We know that

$$T\supset\bigcup_{i=1}^{\infty}\left\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\right\}$$

and

$$\left\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\right\}\cap\left\{t\in T:\frac{1}{j+1}<\delta(t)\le\frac{1}{j}\right\}=\phi,\quad i\ne j,$$

so we get

$$\begin{aligned}\infty>\mu T&\ge\mu\left(\bigcup_{i=1}^{\infty}\left\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\right\}\right)\\&=\sum_{i=1}^{\infty}\mu\left\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\right\}.\end{aligned}$$

Hence there exists $i_1\in N$ such that $\sum_{i=i_1}^{\infty}\mu\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\}<\frac{1}{8}\varepsilon_0$. So we get the following inequality

$$\begin{aligned}\mu\left\{t\in T:0<\delta(t)\le\frac{1}{i_0}\right\}&=\mu\left(\bigcup_{i=i_0}^{\infty}\left\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\right\}\right)\\&=\sum_{i=i_0}^{\infty}\mu\left\{t\in T:\frac{1}{i+1}<\delta(t)\le\frac{1}{i}\right\}<\frac{1}{8}\varepsilon_0.\end{aligned}$$

Let $\delta_0=\frac{1}{2i_1}$. Then $\mu G_1<\frac{1}{8}\varepsilon_0$, where $G_1=\{t\in T:0<\delta(t)\le\delta_0\}$. By $\delta(t)>0$ μ-a.e. $t\in T$, we have $\mu G=\mu G_1<\frac{1}{8}\varepsilon_0$, where

$$G=\{t\in T:\delta(t)\le\delta_0\}.$$

Moreover, we have

$$T \supset \bigcup_{i=1}^{\infty} \left\{ t \in T : \frac{1}{i+1} < M(t, K) \le \frac{1}{i} \right\}$$

and

$$\left\{ t \in T : \frac{1}{i+1} < M(t, K) \le \frac{1}{i} \right\} \cap \left\{ t \in T : \frac{1}{j+1} < M(t, K) \le \frac{1}{j} \right\} = \phi, \quad i \neq j,$$

where $K = \min\{\sigma_0, \frac{7}{8}\eta_0\}$. Similarly, there exists $a > 0$ such that $\mu C_1 < \frac{1}{8}\varepsilon_0$, where $C_1 = \{t \in T : 0 < M(t, K) \le a\}$. Since $M(t, u)$ is strictly convex with respect u for almost all $t \in T$, we have $\mu C = \mu C_1 < \frac{1}{8}\varepsilon_0$, where

$$C = \{t \in T : M(t, K) \le a\}.$$

Let $H_n = E_n \setminus (G \cup C \cup A_n \cup B_n)$, $H_{n1} = E'_{n1} \setminus (G \cup C \cup A_n \cup B_n)$, $H_{n2} = E'_{n2} \setminus (G \cup C \cup A_n \cup B_n)$, $H_{n3} = E''_{n3} \setminus (G \cup C \cup A_n \cup B_n)$. Then $\mu H_n \ge \frac{3}{8}\varepsilon_0$, and so

$$\begin{aligned}
&\frac{1}{2}\rho_M(u_n) + \frac{1}{2}\rho_M(v_n) - \rho_M\left(\frac{1}{2}(u_n + v_n)\right) \\
&\ge \frac{1}{2}\int_{H_n} M(t, \|u_n(t)\|)dt + \frac{1}{2}\int_{H_n} M(t, \|v_n(t)\|)dt - \int_{H_n} M\left(t, \frac{1}{2}\|u_n(t) + v_n(t)\|\right)dt \\
&\ge \frac{1}{2}\int_{H_{n1}} M(t, \|u_n(t)\|)dt + \frac{1}{2}\int_{H_{n1}} M(t, \|v_n(t)\|)dt \\
&\qquad - \int_{H_{n1}} M\left(t, \frac{1}{2}\|u_n(t) + v_n(t)\|\right)dt + \frac{1}{2}\int_{H_{n2}} M(t, \|u_n(t)\|)dt \\
&\qquad + \frac{1}{2}\int_{H_{n2}} M(t, \|v_n(t)\|)dt - \int_{H_{n2}} M\left(t, \frac{1}{2}\|u_n(t) + v_n(t)\|\right)dt \\
&\qquad + \frac{1}{2}\int_{H_{n3}} M(t, \|u_n(t)\|)dt + \frac{1}{2}\int_{H_{n3}} M(t, \|v_n(t)\|)dt \\
&\qquad - \int_{H_{n3}} M\left(t, \frac{1}{2}\|u_n(t) + v_n(t)\|\right)dt \\
&\ge \int_{H_{n1}} \left[\frac{1}{2}M(t, \|u_n(t)\|) + \frac{1}{2}M(t, \|v_n(t)\|) - M\left(t, \frac{1}{2}\|u_n(t) + v_n(t)\|\right)\right]dt \\
&\quad + \frac{1}{2}\int_{H_{n2}} M(t, \|u_n(t)\|)dt + \frac{1}{2}\int_{H_{n2}} M(t, \|v_n(t)\|)dt \\
&\quad + \int_{H_{n3}} \left[\frac{1}{2}M(t, \|u_n(t)\|) + \frac{1}{2}M(t, \|v_n(t)\|) - M\left(t, \frac{1}{2}\|u_n(t) + v_n(t)\|\right)\right]dt
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{H_{n1}} \delta_0 \left[\frac{1}{2}M(t,\|u_n(t)\|)+\frac{1}{2}M(t,\|v_n(t)\|)\right]dt \\
&\quad + \int_{H_{n2}} \left[\frac{1}{2}M(t,\|u_n(t)\|)+\frac{1}{2}M(t,\|v_n(t)\|)\right]dt \\
&\quad + \int_{H_{n3}} \left[M\left(t,\frac{1}{2}\|u_n(t)\|+\frac{1}{2}\|v_n(t)\|\right)-M\left(t,\frac{1}{2}\|u_n(t)+v_n(t)\|\right)\right]dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 \left[\frac{1}{2}M(t,\|u_n(t)\|)+\frac{1}{2}M(t,\|v_n(t)\|)\right]dt \\
&\quad + \int_{H_{n3}} \left[M\left(t,\frac{1}{2}\|u_n(t)\|+\frac{1}{2}\|v_n(t)\|-\frac{1}{2}\|u_n(t)+v_n(t)\|\right)\right]dt \\
&= \int_{H_{n1}\cup H_{n2}} \delta_0 \left[\frac{1}{2}M(t,\|u_n(t)\|)+\frac{1}{2}M(t,\|v_n(t)\|)\right]dt \\
&\quad + \int_{H_{n3}} \left[M\left(t,\frac{1}{2}\|u_n(t)\|+\frac{1}{2}\|v_n(t)\|-\|u(t)\|\right)\right]dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 \left[M\left(t,\frac{1}{2}\|u_n(t)\|+\frac{1}{2}\|v_n(t)\|\right)\right]dt+\int_{H_{n3}} M\left(t,\frac{7}{8}\eta_0\right)dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 M\left(t,\frac{1}{2}\|u_n(t)-v_n(t)\|\right)dt+\int_{H_{n3}} M\left(t,\frac{7}{8}\eta_0\right)dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 M(t,\sigma_0)dt+\int_{H_{n3}} M\left(t,\frac{7}{8}\eta_0\right)dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 M(t,K)dt+\int_{H_{n3}} M(t,K)dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 a dt+\int_{H_{n3}} a dt \\
&\geq \int_{H_{n1}\cup H_{n2}} \delta_0 a dt+\int_{H_{n3}} \delta_0 a dt \\
&= \delta_0 a\cdot \mu H_n \geq \delta_0 a \cdot \frac{3}{8}\varepsilon_0 > 0.
\end{aligned}
$$

This implies that $\frac{1}{2}\|u_n\|+\frac{1}{2}\|v_n\|-\|u_n+v_n\| = \frac{1}{2}\|u_n\|+\frac{1}{2}\|v_n\|-\|2u\| \nrightarrow 0$, a contradiction. Hence for any $\sigma > 0$ and $\varepsilon > 0$ there exists N such that

$$\mu\{t\in T : \|u_n(t)-u(t)\| \geq \sigma\} < \varepsilon$$

whenever $n \geq N$. By the Riesz theorem, there exists subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that $u_{n_k}(t) \to u(t)(n\to\infty)$ μ-a.e. on T. By the convexity of M, we have

$$\frac{1}{2}M(t,\|u_{n_k}(t)\|)+\frac{1}{2}M(t,\|u(t)\|)-M\left(t,\frac{1}{2}\|u_{n_k}(t)-u(t)\|\right)\geq 0$$

for μ-a.e. $t \in T$. Therefore, by the Fatou Lemma, we obtain the following

$$\begin{aligned}
&\rho_M(u)\\
&= \int_T \lim_{k\to\infty} \left[\frac{1}{2}M(t, \|u_{n_k}(t)\|) + \frac{1}{2}M(t, \|u(t)\|) - M\left(t, \frac{1}{2}\|u_{n_k}(t) - u(t)\|\right)\right] dt\\
&\leq \liminf_{k\to\infty} \int_T \left[\frac{1}{2}M(t, \|u_{n_k}(t)\|) + \frac{1}{2}M(t, \|u(t)\|) - M\left(t, \frac{1}{2}\|u_{n_k}(t) - u(t)\|\right)\right] dt\\
&= \rho_M(u) - \limsup_{k\to\infty} \int_T \rho_M\left(\frac{1}{2}(u_{n_k} - u)\right),
\end{aligned}$$

which implies that $\rho_M(\frac{1}{2}(u_{n_k} - u)) \to 0$ as $k \to \infty$. By Lemma 1.2, we have $\|u_{n_k} - u\| \to 0$ as $k \to \infty$. Then $\|u_n - u\| \to 0$ as $n \to \infty$, i.e, $\|u_n - v_n\| \to 0$ as $n \to \infty$. Hence $L_M(X)$ is midpoint locally uniformly rotund. $\square$

Corollary 2.4. *Let $L_M(X)$ be Musielak-Orlicz-Bochner function spaces endowed with the Luxemburg norm, Then the following statements are equivalent.*

(*1*) $L_M(X)$ *is midpoint locally uniformly rotund if and only if* $L_M(X)$ *is rotund;*
(*2*) X *is midpoint locally uniformly rotund.*

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