Journal of Convex Analysis
Volume 19 (2012), No. 1, 125–140

A Note on Some Extensions of Monotone Operators to the Bidual

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Received: December 08, 2009
Revised manuscript received: March 17, 2011

The paper has a double aim. On the one hand, it studies some monotone extensions to the bidual of monotone operators which are not of type (D), paying particular attention to the issue of representability of these extensions by means of convex functions. On the other hand, it derives a density property for maximal monotone operators that, not only are of type (D), but have their unique maximal monotone extensions to the bidual which are of type (D) as well, as it is the case for subdifferentials of lower semicontinuous convex functions.

Keywords: Monotone operator, type (D), type (BR), convex representation, bidual

2010 Mathematics Subject Classification: 47H05, 46T99, 47N10

1. Introduction

A deep study of maximal monotone operators defined on nonreflexive Banach spaces can be traced back to the pioneering work of J.-P. Gossez [4, 5, 6, 7] in the seventies of the twentieth century. In particular, he defined a well-behaved class of operators, called of type (D), which admit a unique maximal monotone extension to the bidual (though, as he showed, they are not the only ones satisfying this property). More classes were defined in the nineties by S. Simons [20, 21], S. Fitzpatrick and R. R. Phelps [3] and A. Verona and M. E. Verona [23].

Marques Alves and Svaiter [10] recently obtained a characterization of maximal monotone operators that admit a unique maximal monotone extension to the bidual (see Theorem 2.13 below, where we follow the formulation given in [11]) and in [11] proved that the class of maximal monotone operators of type (NI), introduced by S. Simons, coincides with Gossez's class of maximal monotone operators of type (D).

In the present paper we provide some contributions to a general study of monotone extensions of monotone operators to the bidual, employing some of the main results of [11]. Indeed, the literature concerning extensions to the bidual has mostly focused on maximal monotone extensions, especially in the cases when they are unique. Here, on the other hand, we consider some remarkable monotone extensions to the bidual that are neither necessarily maximal nor premaximal monotone,

ISSN 0944-6532 / $ 2.50

namely those that can be somehow studied by means of convex representations of the given operator. Anyway, this study also yields a new characterization of maximal monotone operators of type (D).

The paper is organized as follows. Section 2 sets notation and gathers basic definitions and some facts that we are using in the following. Section 3 contains the main results concerning monotone extensions of monotone operators to the bidual, focusing in particular on the problem of representability of these extensions, on their reciprocal relations and on how they are affected by different properties of the operators under consideration. Finally, Section 4 considers the implications of asking for two of these properties (strict Brønsted-Rockafellar property and being of type (D)) to be satisfied by Gossez's extension of an operator, rather than simply by the operator itself.

2. Notation and Preliminary Facts

Given a nonreflexive real Banach space X, we will denote by X^* and X^{**} its topological dual and bidual, respectively. In order to keep the notation as simple as possible, we will not distinguish explicitly between the elements of X and their image through the natural injection in X^{**}. Analogously, we will denote by $\|\cdot\|$ the norm on X, as well as the norm on X^*, or X^{**}, and by π, or $\langle\cdot,\cdot\rangle$, both the duality product on $X^{**} \times X^{***}$ and its restrictions to $X^{**} \times X^*$ or $X \times X^*$.

In the following, given two Banach spaces X, Y and a set $A \subseteq X \times Y$, we will use the notation $A^\top$ to denote the set

$$A^\top = \{(y,x) \in Y \times X : (x,y) \in A\}.$$

Similarly, given a function $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$, the function

$$f^\top : Y \times X \to \mathbb{R} \cup \{+\infty\}$$

will be such that $f^\top(y,x) = f(x,y)$ for all $(y,x) \in Y \times X$.

The remaining notation is fairly standard in Convex Analysis. We will only state explicitly the definition and the main properties of monotone operators.

Recall that a point-to-set operator $T : X \rightrightarrows X^*$ can be univocally described by means of its graph

$$\mathcal{G}(T) = \{(x,x^*) \in X \times X^* : x^* \in T(x)\},$$

the projections of which on X and X^* are the domain $\mathcal{D}(T)$ and the range $\mathcal{R}(T)$, respectively. The inverse operator $T^{-1} : X^* \rightrightarrows X$ is defined by $\mathcal{G}(T^{-1}) = \mathcal{G}(T)^\top$.

We say that a point $(x,x^*) \in X \times X^*$ is monotonically related to a set $A \subseteq X \times X^*$ if

$$\forall (y,y^*) \in A : \langle x-y, x^*-y^* \rangle \geq 0.$$

An operator $T : X \rightrightarrows X^*$ is called monotone if any point $(x,x^*) \in \mathcal{G}(T)$ is monotonically related to $\mathcal{G}(T)$ (in principle, the operator with empty graph is monotone, but in the following we will implicitly neglect this particular case). If, moreover, for any point $(x, x^*) \in X \times X^*$ that is monotonically related to $\mathcal{G}(T)$, one has $(x, x^*) \in \mathcal{G}(T)$, then T is called maximal monotone. As in [15], we call T *premaximal* monotone if it admits a unique maximal monotone extension, i.e. a unique maximal monotone operator $S : X \rightrightarrows X^*$ such that $\mathcal{G}(T) \subseteq \mathcal{G}(S)$.

To any monotone operator $T : X \rightrightarrows X^*$ one can associate the so-called Fitzpatrick family of T, i.e. the set

$$\begin{aligned}\mathcal{H}_T = \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} &: h \text{ is lower semicontinuous} \\ &\text{and convex, } h \geq \pi \text{ and } h(x, x^*) = \langle x, x^*\rangle\ \forall (x, x^*) \in \mathcal{G}(T)\}.\end{aligned}$$

If T is maximal monotone, this set admits a minimum, the Fitzpatrick function

$$\varphi_T(x, x^*) = \langle x, x^*\rangle - \inf_{(y,y^*)\in\mathcal{G}(T)} \langle x - y, x^* - y^*\rangle = \sup_{(y,y^*)\in\mathcal{G}(T)} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\}$$

and a maximum, given by the function

$$\sigma_T(x, x^*) = \text{cl conv } (\pi + \delta_{\mathcal{G}(T)})(x, x^*)$$

for all $(x, x^*) \in X \times X^*$. As pointed out in [10], if we consider the inverse of T as taking values in X^{**}, i.e. $T^{-1} : X^* \rightrightarrows X^{**}$, we have

$$\begin{aligned}\varphi_{T^{-1}}(x^*, x^{**}) &= \sup_{(y^*,y^{**})\in\mathcal{G}(T^{-1})} \{\langle x^{**}, y^*\rangle + \langle y^{**}, x^*\rangle - \langle y^{**}, y^*\rangle\} \\ &= \sup_{(y,y^*)\in\mathcal{G}(T)} \{\langle x^{**}, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\} \\ &= \sup_{(y,y^*)\in X\times X^*} \{\langle x^{**}, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle - \delta_{\mathcal{G}(T)}(y, y^*)\} \\ &= \sigma_T^*(x^*, x^{**})\end{aligned} \tag{1}$$

for all $(x^*, x^{**}) \in X^* \times X^{**}$.

Note that φ_T and σ_T can also be defined in the case when T is a monotone operator that is not maximal. Anyway, in general, while $\sigma_T \in \mathcal{H}_T$, φ_T will not majorize the duality product on $X \times X^*$ and consequently $\varphi_T \notin \mathcal{H}_T$. Moreover, in principle, none of the two functions will characterize the operator (while this is the case when T is maximal monotone), since they can be equal to the duality product also at points which do not belong to $\mathcal{G}(T)$.

This fact gives rise to another useful definition. If $T : X \rightrightarrows X^*$ is a monotone operator, according to [15] we say that it is *representable* if there exists a lower semicontinuous convex function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $h \geq \pi$ on $X \times X^*$ and $h = \pi$ exactly on $\mathcal{G}(T)$, i.e. $h(x, x^*) = \langle x, x^*\rangle$ if and only if $(x, x^*) \in \mathcal{G}(T)$.

Remark 2.1. As a consequence of [15, Theorem 5], given a lower semicontinuous convex function $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $f \geq \pi$ on $X \times X^*$, the set

$$\{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x, x^*\rangle\}$$

is the graph of a representable monotone operator (with f as a convex representation of it).

At the end of Section 3, we will also consider a family of convex functions associated to a monotone operator which is bigger than the Fitzpatrick family (but coincides with it when the operator is maximal monotone), namely

$$\mathcal{K}_T = \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : h \text{ is lower semicontinuous, convex and } \varphi_T \leq h \leq \sigma_T\}.$$

Remark 2.2. Note that, given a monotone operator $T : X \rightrightarrows X^*$, the inclusion $\mathcal{H}_T \subseteq \mathcal{K}_T$ holds, since, for any $h \in \mathcal{H}_T$, one has $\varphi_T \leq h \leq \sigma_T$ (the second inequality is trivial, while the first inequality follows for instance from [14, Theorem 7]). Furthermore, for all $h \in \mathcal{K}_T$ and $(x, x^*) \in \mathcal{G}(T)$, one has $h(x, x^*) = \langle x, x^* \rangle$, given that $\varphi_T = \sigma_T = \pi$ on $\mathcal{G}(T)$.

In this paper we will be interested in considering extensions of a (maximal) monotone operator $T : X \rightrightarrows X^*$ to the bidual, i.e. operators $S : X^{**} \rightrightarrows X^*$ such that $\mathcal{G}(T) \subseteq \mathcal{G}(S)$ (via the natural inclusion of X in its bidual). Properly speaking, according to our definition, we would not be allowed to say that any of these extensions is (maximal) monotone, unless we considered it as if its range were defined in X^{***}; anyway, since we are only interested in extending the domain of T, therefore, when, with a slight abuse of terminology, we call S (maximal) monotone, we will mean that $S^{-1} : X^* \rightrightarrows X^{**}$ is (maximal) monotone.

In particular, we will study properties of the following two extensions, whose origin can be traced back to the papers of Gossez.

Definition 2.3. Let $T : X \rightrightarrows X^*$ be a monotone operator.

(a) $\overline{T} : X^{**} \rightrightarrows X^*$ is the operator such that, for all $(x^{**}, x^*) \in X^{**} \times X^*$, $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$ if and only if there exists a bounded net (x_α, x^*_α) in $\mathcal{G}(T)$ that converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$.

(b) $\widetilde{T} : X^{**} \rightrightarrows X^*$ is the operator with graph

$$\mathcal{G}(\widetilde{T}) = \{(x^{**}, x^*) \in X^{**} \times X^* : \ \forall (y, y^*) \in \mathcal{G}(T) \ \langle x^{**} - y, x^* - y^* \rangle \geq 0\}.$$

Proposition 2.4. *Let $T : X \rightrightarrows X^*$ be a monotone operator.*

(a) *$\overline{T}$ and $\widetilde{T}$ are extensions of T.*

(b) *$\overline{T}$ is a monotone operator and $\mathcal{G}(\overline{T})$ is contained in the graph of any maximal monotone extension of T to the bidual.*

(c) *$\mathcal{G}(\widetilde{T})$ contains the graph of any monotone extension of T to the bidual. Therefore, in particular, $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widetilde{T})$.*

Proof. (a) Given an arbitrary point $(x, x^*) \in \mathcal{G}(T)$, taking $(x_\alpha, x^*_\alpha) = (x, x^*)$ for any α in an ordered set A, we trivially obtain that $(x, x^*) \in \mathcal{G}(\overline{T})$.

On the other hand, the monotonicity of T implies that $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(y, y^*) \in \mathcal{G}(T)$. Thus $(x, x^*) \in \mathcal{G}(\widetilde{T})$.

(b) Notice that, for any $(x^{**}, x^*) \in X^{**} \times X^*$ and any bounded net (x_α, x^*_α) converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, one has

$$\lim_\alpha \langle x_\alpha, x^* \rangle = \langle x^{**}, x^* \rangle$$

and

$$0 \leq \lim_{\alpha} |\langle x_\alpha, x^*_\alpha - x^* \rangle| \leq \lim_{\alpha} \|x_\alpha\| \|x^*_\alpha - x^*\| = 0,$$

as a consequence of the hypotheses on the net (x_α, x^*_α). Hence

$$\lim_{\alpha} \langle x_\alpha, x^*_\alpha \rangle = \lim_{\alpha} (\langle x_\alpha, x^* \rangle + \langle x_\alpha, x^*_\alpha - x^* \rangle) = \langle x^{**}, x^* \rangle. \tag{2}$$

To show that $\overline{T}$ is monotone, consider first an arbitrary $(u^{**}, u^*) \in \mathcal{G}(\overline{T})$. Given a bounded net (u_β, u^*_β) in $\mathcal{G}(T)$ converging to (u^{**}, u^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, for all $(v, v^*) \in \mathcal{G}(T)$, taking (2) into account, one has

$$\langle u^{**} - v, u^* - v^* \rangle = \lim_{\beta} \langle u_\beta - v, u^*_\beta - v^* \rangle \geq 0,$$

since any point (u_β, u^*_β) belongs to $\mathcal{G}(T)$ and T is a monotone operator.

Therefore, given any $(x^{**}, x^*), (y^{**}, y^*) \in \mathcal{G}(\overline{T})$ and a bounded net (x_α, x^*_α) in $\mathcal{G}(T)$ converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, one also obtains

$$\langle x^{**} - y^{**}, x^* - y^* \rangle = \lim_{\alpha} \langle x_\alpha - y^{**}, x^*_\alpha - y^* \rangle \geq 0,$$

so that $\overline{T}$ is monotone.

Finally, let $S : X^{**} \rightrightarrows X^*$ be a maximal monotone extension of T to the bidual. Then, for all $(y^{**}, y^*) \in \mathcal{G}(S)$ and $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$, given a bounded net (x_α, x^*_α) in the graph of T converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, one has

$$\langle x^{**} - y^{**}, x^* - y^* \rangle = \lim_{\alpha} \langle x_\alpha - y^{**}, x^*_\alpha - y^* \rangle \geq 0,$$

since S is monotone and, being S an extension of T, $(x_\alpha, x^*_\alpha) \in \mathcal{G}(S)$ for all α. From the previous inequality, because of the maximality of S, it follows $(x^{**}, x^*) \in \mathcal{G}(S)$.

(c) Let $S : X^{**} \rightrightarrows X^*$ be a monotone extension of T to the bidual. Then, for all $(x^{**}, x^*) \in \mathcal{G}(S)$ and for all $(y, y^*) \in \mathcal{G}(T)$, it holds $\langle x^{**} - y, x^* - y^* \rangle \geq 0$, since $\mathcal{G}(T) \subseteq \mathcal{G}(S)$ and S is monotone. Then, by definition of $\widetilde{T}$, $\mathcal{G}(S) \subseteq \mathcal{G}(\widetilde{T})$.
The fact that $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widetilde{T})$ follows from (b). □

Remark 2.5. $\widetilde{T}$ is not necessarily monotone, even in the case in which T is maximal monotone. Indeed, there exist maximal monotone operators that have several maximal monotone extensions to the bidual [7]. Given two different extensions of T, say $T_1, T_2 : X^{**} \rightrightarrows X^*$, there exist $(x^{**}, x^*) \in \mathcal{G}(T_1)$ and $(y^{**}, y^*) \in \mathcal{G}(T_2)$ that are not monotonically related, since T_1 and T_2 are maximal monotone and do not coincide. On the other hand, Proposition 2.4(c) implies that $(x^{**}, x^*), (y^{**}, y^*) \in \mathcal{G}(\widetilde{T})$. Thus $\widetilde{T}$ is not monotone.

The following definition restates, in terms of the extensions that we have just presented, the definition of the class of monotone operators of type (D) due to Gossez [6] (see [17] for a detailed account).

Definition 2.6. Let $T : X \rightrightarrows X^*$ be a monotone operator. We say that T is of type (D) if $\overline{T} = \widetilde{T}$.

Theorem 2.7 ([10, Theorem 1.1]). *Let X be a Banach space and $T : X \rightrightarrows X^*$ a maximal monotone operator of type (D), which is equivalent to*

$$(\sigma_T)^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Then

(a) $\widetilde{T} : X^{**} \rightrightarrows X^*$ *is the unique maximal monotone extension of T to the bidual;*
(b) $(\sigma_T)^* = \varphi_{\widetilde{T}^{-1}}$*;*
(c) *for all $h \in \mathcal{H}_T$,*

$$h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$
$$h^* \in \mathcal{H}_{\widetilde{T}^{-1}};$$

(d) *T satisfies the strict Brønsted-Rockafellar property.*

Recall that the strict Brønsted-Rockafellar property was introduced in [8] and can be defined as follows.

Definition 2.8. Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. We say that T satisfies the strict Brønsted-Rockafellar property when, for all η, ε such that $0 < \varepsilon < \eta$ and for all $(x, x^*) \in X \times X^*$, if

$$\inf_{(y,y^*) \in \mathcal{G}(T)} \langle x - y, x^* - y^* \rangle \geq -\varepsilon,$$

then, for any $\lambda > 0$, there exists $(x_\lambda, x^*_\lambda) \in \mathcal{G}(T)$ such that

$$\|x - x_\lambda\| < \lambda, \qquad \|x^* - x^*_\lambda\| < \frac{\eta}{\lambda}.$$

We will use the following terminology, introduced by Simons [22, Definition 36.13] for maximal monotone operators. Anyway, we will not require maximality, in principle.

Definition 2.9. Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. We call T of type (BR), if it satisfies the strict Brønsted-Rockafellar property.

Thus, according to Theorem 2.7, any maximal monotone operator of type (D) is of type (BR).

In the following we will employ the main results of the recent relevant paper of Marques Alves and Svaiter [11].

Theorem 2.10 ([11, Lemma 4.1]). *Let X be a Banach space and $f : X \times X^* \to \overline{\mathbb{R}}$ be a lower semicontinuous proper convex function. Then*

$$f^{**}(x^{**}, x^*) = \liminf_{(y,y^*) \to (x^{**}, x^*)} f(y, y^*), \quad \forall (x^{**}, x^*) \in X^{**} \times X^*,$$

where the lim inf *is taken over all nets in $X \times X^*$ converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$.*

Theorem 2.11 ([11, Theorem 4.2]). *Let X be a Banach space and $f : X \times X^* \to \overline{\mathbb{R}}$ be a lower semicontinuous proper convex function. Then, for any $(x^{**}, x^*) \in X^{**} \times X^*$ there exists a bounded net $\{(z_i, z_i^*)\}_{i\in I}$ in $X \times X^*$ which converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and*

$$f^{**}(x^{**}, x^*) = \lim_{i\in I} f(z_i, z_i^*).$$

The third result we will refer to is not stated explicitly, but is embedded in the proof of [11, Theorem 4.4].

Theorem 2.12 ([11, Proof of Theorem 4.4]). *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator of type (BR). For any $h \in \mathcal{H}_T$ and $(x^{**}, x^*) \in X^{**} \times X^*$, if $h^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$, then there exists a bounded net (x_α, x_α^*) in $\mathcal{G}(T)$ converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$.*

We will also need the following property, originally proved in [10], where $\delta_{\mathcal{G}(T)}$ stands for the indicator function of $\mathcal{G}(T)$.

Theorem 2.13 ([11, Theorem 4.5]). *Let X be a Banach space. If $T : X \rightrightarrows X^*$ is maximal monotone and has a unique maximal monotone extension to the bidual, then one of the following conditions holds:*

(a) *T is of type (D);*
(b) *T is affine and non-enlargeable, that is $\varphi_T = \pi + \delta_{\mathcal{G}(T)}$ and $\mathcal{H}_T = \{\varphi_T\}$.*

Finally, in the last section, we will use the following result from [18]. To this end, we introduce the following notation. Given two vector spaces A and B, for $i = 1, 2$ define the reflection $\varrho_i : A \times B \to A \times B$,

$$(a, b) \mapsto \varrho_i(a, b) = \begin{cases} (-a, b), & \text{if } i = 1 \\ (a, -b), & \text{if } i = 2 \end{cases}$$

and the projections $p_1 : A \times B \to A$, $(a, b) \mapsto a$, and $p_2 : A \times B \to B$, $(a, b) \mapsto b$. Moreover, given a convex function $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, we set

$$(\mathcal{T}_{(w,w^*)} f)(x, x^*) = f(x + w, x^* + w^*) - (\langle x, w^* \rangle + \langle w, x^* \rangle + \langle w, w^* \rangle)$$

for all $(w, w^*), (x, x^*) \in X \times X^*$.

Theorem 2.14 ([18, Theorem 4.6]). *Let X be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D) and $u, v \in X$. If, for all $u^* \in X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u - v, u^*)] \quad \textit{is a closed subspace of } X \times X^*$$

and

$$\bigcup_{\lambda>0} \lambda[p_1 \operatorname{dom} \mathcal{T}_{(u,u^*)} h - p_1 \operatorname{dom} \mathcal{T}_{(v,0_{X^*})} k] \quad \textit{is a closed subspace of } X,$$

then $\operatorname{cl}(\mathcal{R}(S(\,\cdot + u) + T(\,\cdot + v))) = X^*$.

3. Monotone Extensions to the Bidual

For any monotone operator $T : X \rightrightarrows X^*$, along with $\overline{T}$ and $\widetilde{T}$, we will consider a family of extensions of T to the bidual which is generated by the Fitzpatrick family of T.

Definition 3.1. Let X be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $h \in \mathcal{H}_T$. Then we denote by $\widehat{T}_h : X^{**} \rightrightarrows X^*$ the operator with graph

$$\mathcal{G}(\widehat{T}_h) := \{(x^{**}, x^*) \in X^{**} \times X^* : h^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle\}.$$

The following theorem states that $\widehat{T}_h$ is a representable monotone extension of T to the bidual and considers the relations holding between $\overline{T}$, $\widehat{T}_h$ and $\widetilde{T}$, both in the sense of graph inclusion and with respect to the Fitzpatrick functions (as $\overline{T}$ and $\widehat{T}_{\sigma_T}$ are concerned).

Theorem 3.2. *Let X be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $h \in \mathcal{H}_T$. Then:*

(a) *for all $(x^{**}, x^*) \in X^{**} \times X^*$, one has $h^{**}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle$;*

(b) *$\widehat{T}_h^{-1} : X^* \rightrightarrows X^{**}$ is a representable monotone operator, with $\left(h^{**\top}\right)|_{X^* \times X^{**}}$ as a convex representation and such that $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_h) \subseteq \mathcal{G}(\widetilde{T})$;*

(c) *$\varphi_{\overline{T}^{-1}} = \varphi_{\widehat{T}_{\sigma_T}^{-1}} = \sigma_T^*$.*

Proof. (a) Let $(x^{**}, x^*) \in X^{**} \times X^*$. Given a bounded net (x_α, x_α^*) in $X \times X^*$ converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology and such that $h(x_\alpha, x_\alpha^*)$ converges to $h^{**}(x^{**}, x^*)$ (the existence of such a net is guaranteed by Theorem 2.11), one has

$$h^{**}(x^{**}, x^*) = \lim_\alpha h(x_\alpha, x_\alpha^*) \geq \lim_\alpha \langle x_\alpha, x_\alpha^* \rangle = \langle x^{**}, x^* \rangle,$$

where the inequality follows from $h \geq \pi$ (being $h \in \mathcal{H}_T$), while the latter equality is given by (2).

(b) Since $(h^{**\top})_{|X^* \times X^{**}}$ is a lower semicontinuous convex function and, by (a), $(h^{**\top})_{|X^* \times X^{**}} \geq \pi$, then, according to Remark 2.1, $\widehat{T}_h^{-1}$ is a representable monotone operator, with the function $(h^{**\top})_{|X^* \times X^{**}}$ as a convex representation.

Moreover, since $h^{**} = h$ on $X \times X^*$ and $h \in \mathcal{H}_T$, then, for any $(x, x^*) \in \mathcal{G}(T)$, $h^{**}(x, x^*) = \langle x, x^* \rangle$, i.e. $(x, x^*) \in \widehat{T}_h$. Thus $\widehat{T}_h$ is an extension of T to the bidual.

The inclusion $\mathcal{G}(\widehat{T}_h) \subseteq \mathcal{G}(\widetilde{T})$ is obvious, given that $\widetilde{T}$ contains the graph of any monotone extension of T to the bidual (Proposition 2.4(c)).

On the other hand, for any $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$, given a bounded net (x_α, x_α^*) in $\mathcal{G}(T)$ converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, we have

$$\langle x^{**}, x^* \rangle = \lim_\alpha \langle x_\alpha, x_\alpha^* \rangle = \lim_\alpha h(x_\alpha, x_\alpha^*) = \lim_\alpha h^{**}(x_\alpha, x_\alpha^*) \geq h^{**}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle,$$

where the first inequality is a consequence of Theorem 2.10, while the second one follows from item (a). Thus $h^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$ and $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_h)$.

(c) Taking (1) into account, since $\overline{T}$ is an extension of T and given that, by (b) (recalling that $\sigma_T \in \mathcal{H}_T$), $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_{\sigma_T})$, then, for all $(x^*, x^{**}) \in X^* \times X^{**}$,

$$\begin{aligned}
\sigma_T^*(x^*, x^{**}) &= \sup_{(y,y^*)\in\mathcal{G}(T)} \{\langle x^{**}, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\} \\
&\le \sup_{(y^{**},y^*)\in\mathcal{G}(\overline{T})} \{\langle x^{**}, y^*\rangle + \langle y^{**}, x^*\rangle - \langle y^{**}, y^*\rangle\} \\
&= \sup_{(y^{**},y^*)\in\mathcal{G}(\overline{T})} \{\langle x^{**}, y^*\rangle + \langle y^{**}, x^*\rangle - \sigma_T^{**}(y^{**}, y^*)\} \\
&\le \sigma_T^{***}(x^*, x^{**}) = \sigma_T^*(x^*, x^{**}).
\end{aligned}$$

Thus, as

$$\sup_{(y^{**},y^*)\in\mathcal{G}(\overline{T})} \{\langle x^{**}, y^*\rangle + \langle y^{**}, x^*\rangle - \langle y^{**}, y^*\rangle\} = \varphi_{\overline{T}^{-1}}(x^*, x^{**}),$$

we obtain $\varphi_{\overline{T}^{-1}} = \sigma_T^*$.
Substituting $\mathcal{G}(\widehat{T}_{\sigma_T})$ for $\mathcal{G}(\overline{T})$ in the previous inequalities, one proves that $\varphi_{\widehat{T}^{-1}_{\sigma_T}} = \sigma_T^*$ as well. $\square$

With respect to item (c) of the previous theorem, we can draw the following consequence. In principle, one could conceive a generalization of the class of maximal monotone operators of type (D) given by the family of maximal monotone operators $T : X \rightrightarrows X^*$ such that $\overline{T}$ is maximal monotone, though not necessarily equal to $\widetilde{T}$. Anyway, this actually turns out not to be a broader class, according to the following corollary.

Corollary 3.3. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then $\overline{T}$ is maximal monotone if and only if T is of type (D).*

Proof. If T is of type (D), then $\overline{T}$ coincides with $\widetilde{T}$, by definition. Therefore, as a consequence of Proposition 2.4, $\overline{T}$ is maximal monotone.

Vice versa, if $\overline{T}$ is a maximal monotone operator, i.e. if $\overline{T}^{-1}$ is maximal monotone, then the Fitzpatrick function $\varphi_{\overline{T}^{-1}} = \sigma_T^*$ (Theorem 3.2(c)) majorizes the duality product on $X^* \times X^{**}$, implying that T is of type (D), according to Theorem 2.7. $\square$

With respect to item (b) of Theorem 3.2, a natural question is to determine when the inclusions considered either hold as equalities, or are strict. The following corollary provides a partial answer to this question.

Corollary 3.4. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator of type (BR). Then, for all $h \in \mathcal{H}_T$, $\overline{T} = \widehat{T}_h$. Thus $\overline{T}^{-1}$ is a representable monotone operator and $\{(h^{**\top})_{|X^*\times X^{**}} : h \in \mathcal{H}_T\}$ is a collection of convex representations of $\overline{T}^{-1}$.*

Proof. It is an immediate consequence of Theorem 2.12 and item (b) of Theorem 3.2. $\square$

As a consequence of the preceding corollary, we have the following characterization of the property of being of type (D) for maximal monotone operators.

Proposition 3.5. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then the following statements are equivalent:*

(a) *T is of type (D);*
(b) *for all $h \in \mathcal{H}_T$, $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$;*
(c) *there exists $h \in \mathcal{H}_T$ such that $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$.*

Proof. $(a) \Longrightarrow (b)$ It is a consequence of Theorem 3.2 and of the definition of type (D).

$(b) \Longrightarrow (c)$ Obvious.

$(c) \Longrightarrow (a)$ Let $h \in \mathcal{H}_T$ be such that $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$. As a consequence of Theorem 3.2(b), $\widetilde{T}$ is monotone, hence maximal monotone, by Proposition 2.4(c). Thus T has a unique maximal monotone extension to the bidual and, by Theorem 2.13, it is either of type (D) or non enlargeable. In both cases, T is of type (BR). Indeed, maximal monotone operators of type (D) are of type (BR) according to Theorem 2.7, while non enlargeable operators are trivially of type (BR), since, for any $\varepsilon > 0$,

$$\inf_{(y,y^*)\in\mathcal{G}(T)} \langle x-y, x^*-y^*\rangle \geq -\varepsilon \quad \Leftrightarrow \quad \varphi_T(x,x^*) \leq \langle x,x^*\rangle + \varepsilon \quad \Leftrightarrow \quad (x,x^*) \in \mathcal{G}(T).$$

Then, by Corollary 3.4, $\mathcal{G}(\overline{T}) = \mathcal{G}(\widehat{T}_h)$, implying $\mathcal{G}(\overline{T}) = \mathcal{G}(\widetilde{T})$, i.e. T is of type (D). $\square$

Notice that we need maximality of T only to prove the implication $(c) \Longrightarrow (a)$.

Remark 3.6. Corollary 3.4 and Proposition 3.5 imply that, for any maximal monotone operator T and any $h \in \mathcal{H}_T$:

(a) if T admits a unique maximal monotone extension to the bidual but it is not of type (D), then $\mathcal{G}(\overline{T}) = \mathcal{G}(\widehat{T}_h) \subsetneq \mathcal{G}(\widetilde{T})$;
(b) the relations $\mathcal{G}(\overline{T}) \subsetneq \mathcal{G}(\widehat{T}_h)$ and $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$ cannot hold simultaneously.

When T is a monotone operator that is not of type (BR), we don't know if the inclusion $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_h)$ is proper or not. If the former case is true, we cannot approximate arbitrary points of $\mathcal{G}(\widehat{T}_h)$ by means of bounded nets in $\mathcal{G}(T)$ converging in the $\sigma(X^{**}, X^*) \otimes$ norm topology. Anyway, there is a family of monotone extensions to the bidual for which we can recover a weaker approximation result by means of Proposition 3.8 below, as a consequence of Theorem 2.11, based on a proof similar to that of Theorem 2.12 with the strict Brønsted-Rockafellar property replaced by the usual Brønsted-Rockafellar property of subdifferentials. Specifically, Proposition 3.8 will give an approximation result for the set

$$\operatorname{fix} \partial h^* := \{(x^*, x^{**}) \in X^* \times X^{**} : \ h^*(x^*, x^{**}) + h^{**}(x^{**}, x^*) = 2\langle x^{**}, x^*\rangle\}$$

by means of $\mathcal{G}(\partial h)^\top$, for all $h \in \mathcal{K}_T$.

The interest of this set of fixed points can be motivated if we refer to the case of maximal monotone operators defined on reflexive Banach spaces, since, in this particular setting, $\mathcal{K}_T = \mathcal{H}_T$ and $\operatorname{fix}\partial h^* = \operatorname{fix}\partial h = \mathcal{G}(T)$, where

$$\operatorname{fix}\partial h := \{(x, x^*) \in X \times X^* : \ h(x, x^*) + h^*(x^*, x) = 2\langle x, x^*\rangle\}$$

and the equality $\operatorname{fix}\partial h = \mathcal{G}(T)$ follows from the fact that, in this case, $h, h^* \geq \pi$ and $\mathcal{G}(T) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x, x^*\rangle\} = \{(x, x^*) \in X \times X^* : h^*(x^*, x) = \langle x, x^*\rangle\}$ (see for instance [1]).

Consider first the following properties of the operator $F_h : X^{**} \rightrightarrows X^*$, defined by

$$\mathcal{G}(F_h) = (\operatorname{fix}\partial h^*)^\top.$$

Proposition 3.7. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. Then, for all $h \in \mathcal{K}_T$, F_h is an extension of T to the bidual and F_h^{-1} is a representable monotone operator.*

Proof. Let $h \in \mathcal{K}_T$. Since $\varphi_T \leq h \leq \sigma_T$, one has

$$\sigma_T^* \leq h^* \leq \varphi_T^*.$$

Notice that, for any $(x, x^*) \in \mathcal{G}(T)$,

$$\begin{aligned}
\varphi_T^*(x^*, x) &= \sup_{(y,y^*)\in X\times X^*} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \varphi_T(y, y^*)\} \\
&= \sup_{(y,y^*)\in X\times X^*} \left\{\langle x, y^*\rangle + \langle y, x^*\rangle - \sup_{(z,z^*)\in\mathcal{G}(T)} \{\langle y, z^*\rangle + \langle z, y^*\rangle - \langle z, z^*\rangle\}\right\} \\
&= \sup_{(y,y^*)\in X\times X^*} \left\{\langle x, y^*\rangle + \langle y, x^*\rangle + \inf_{(z,z^*)\in\mathcal{G}(T)} \{-\langle y, z^*\rangle - \langle z, y^*\rangle + \langle z, z^*\rangle\}\right\} \\
&\leq \sup_{(y,y^*)\in X\times X^*} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, x^*\rangle - \langle x, y^*\rangle + \langle x, x^*\rangle\} \\
&= \langle x, x^*\rangle.
\end{aligned}$$

Thus

$$\langle x, x^*\rangle = (\varphi_T)^\top(x^*, x) = \sigma_T^*(x^*, x) \leq h^*(x^*, x) \leq \varphi_T^*(x^*, x) \leq \langle x, x^*\rangle,$$

implying $h^*(x^*, x) = \langle x, x^*\rangle$. Since, moreover, $h^{**}(x, x^*) = h(x, x^*) = \langle x, x^*\rangle$, then F_h is an extension of T to the bidual.

Finally, by Fenchel inequality, the lower semicontinuous convex function

$$\frac{h^* + \left(h^{**\top}\right)|_{X^*\times X^{**}}}{2}$$

majorizes the duality product. Therefore, according to Remark 2.1, F_h^{-1} is a representable monotone operator. $\square$

As a consequence, for all $h \in \mathcal{K}_T$,

$$\mathcal{G}(T) \subseteq \mathcal{G}(F_h) \subseteq \mathcal{G}(\widetilde{T}).$$

When $h = \sigma_T$, the first inclusion can be refined, yielding

$$\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(F_{\sigma_T}) \subseteq \mathcal{G}(\widetilde{T}),$$

since, for all $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$, one has $\sigma_T^*(x^*, x^{**}) = \varphi_{\overline{T}^{-1}}(x^*, x^{**}) = \langle x^{**}, x^* \rangle$ and $\sigma_T^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$, by items (c) and (b), respectively, of Theorem 3.2.

Proposition 3.8. *Let X be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $h \in \mathcal{K}_T$. For any $(x^{**}, x^*) \in \mathcal{G}(F_h)$, there exists a bounded net $((x_\alpha, x_\alpha^*), (y_\alpha^*, y_\alpha^{**}))$ in $\mathcal{G}(\partial h)$ such that (x_α, x_α^*) converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and $(y_\alpha^*, y_\alpha^{**})$ converges to (x^*, x^{**}) in the norm topology of $X^* \times X^{**}$.*

Proof. By Theorem 2.11, there exists a bounded net (z_α, z_α^*) in $X \times X^*$ converging to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and such that $\lim_\alpha h(z_\alpha, z_\alpha^*) = h^{**}(x^{**}, x^*)$. Since $h^{**}(x^{**}, x^*)$ is finite, we can choose (z_α, z_α^*) such that the net $(h(z_\alpha, z_\alpha^*))$ is bounded.
Therefore, taking Fenchel inequality into account, one can define

$$\varepsilon_\alpha^2 := h(z_\alpha, z_\alpha^*) + h^*(x^*, x^{**}) - \langle z_\alpha, x^* \rangle - \langle x^{**}, z_\alpha^* \rangle \geq 0,$$

where ε_α is bounded and, since $(x^{**}, x^*) \in \mathcal{G}(F_h) = (\operatorname{fix} \partial h^*)^\top$, $\lim_\alpha \varepsilon_\alpha^2 = 0$.

By the Brønsted-Rockafellar property of subdifferentials, for any α there exists $((x_\alpha, x_\alpha^*), (y_\alpha^*, y_\alpha^{**})) \in \mathcal{G}(\partial h)$ such that

$$\|(x_\alpha, x_\alpha^*) - (z_\alpha, z_\alpha^*)\| \leq \varepsilon_\alpha \quad \text{and} \quad \|(y_\alpha^*, y_\alpha^{**}) - (x^*, x^{**})\| \leq \varepsilon_\alpha,$$

implying

$$\|x_\alpha - z_\alpha\| \leq \varepsilon_\alpha, \qquad \|x_\alpha^* - z_\alpha^*\| \leq \varepsilon_\alpha.$$

Thus (x_α, x_α^*) is bounded and converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, while $(y_\alpha^*, y_\alpha^{**})$ is bounded and converges to (x^*, x^{**}) in the norm topology of $X^* \times X^{**}$. □

4. Adding Properties to the Extensions

Until now, we have only considered properties of T. Now we are going to study what happens when we require $\overline{T}$ to satisfy some particular property as well. While the properties of T considered in the previous section were typically weaker than (or equivalent to) that of being of type (D), endowing $\overline{T}$ with the property of being of type (BR) will imply that T is of type (D).

Lemma 4.1. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator of type (BR). Then T is premaximal monotone and the graph of its unique maximal monotone extension is equal to $\operatorname{cl} \mathcal{G}(T)$, the closure of $\mathcal{G}(T)$ in the norm topology of $X \times X^*$.*

Proof. Let $(x_0, x_0^*) \in X \times X^*$ be monotonically related to $\mathcal{G}(T)$. Then, for all $n \in \mathbb{N} \backslash \{0\}$,

$$\inf_{(y, y^*) \in \mathcal{G}(T)} \langle x_0 - y, x_0^* - y^* \rangle \geq 0 > -\frac{1}{n+1}.$$

Thus, since T is of type (BR), there exists $(x_n, x_n^*) \in \mathcal{G}(T)$ such that $\|x_n - x_0\| < 1/\sqrt{n}$ and $\|x_n^* - x_0^*\| < 1/\sqrt{n}$. Thus (x_0, x_0^*) belongs to the closure of $\mathcal{G}(T)$ in the norm topology of $X \times X^*$. Hence, the graph of any monotone extension $S : X \rightrightarrows X^*$ of T is contained in $\operatorname{cl}\mathcal{G}(T)$.

On the other hand, if S is maximal monotone, the opposite inclusion holds as well, being

$$\mathcal{G}(T) \subseteq \mathcal{G}(S) \Rightarrow \operatorname{cl}\mathcal{G}(T) \subseteq \operatorname{cl}\mathcal{G}(S)$$

and $\operatorname{cl}\mathcal{G}(S) = \mathcal{G}(S)$, since any maximal monotone operator has a closed graph. In particular, then, T admits only one maximal monotone extension (in $X \times X^*$), defined by $\operatorname{cl}\mathcal{G}(T)$, i.e. it is premaximal monotone. □

Remark 4.2. As a consequence of the preceding lemma, any representable monotone operator of type (BR) is maximal monotone, since its graph is closed, according to [15, Proposition 8].

Definition 4.3. Let X be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. We call T of type (BR*) if $\overline{T}^{-1} : X^* \rightrightarrows X^{**}$ is of type (BR).

Proposition 4.4. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. If T is of type (BR*), then T is of type (D).*

Proof. If $\overline{T}^{-1}$ is of type (BR), then, by Lemma 4.1, $\overline{T}^{-1}$ is premaximal monotone. Thus, by Proposition 2.4 (b), T admits a unique maximal monotone extension to the bidual. As a consequence, by Theorems 2.13 and 2.7, T is of type (BR), implying that $\overline{T} = \widehat{T}_h$ for all $h \in \mathcal{H}_T$, by Corollary 3.4. Thus $\widehat{T}_h^{-1}$ is of type (BR) for all $h \in \mathcal{H}_T$. Moreover, by Theorem 3.2, $\widehat{T}_h^{-1}$ is a representable monotone operator. Therefore, according to Remark 4.2, $\widehat{T}_h^{-1}$ is maximal monotone, i.e. $\overline{T}^{-1}$ is maximal monotone. Finally, this fact is equivalent to T being of type (D), as stated in Corollary 3.3. □

In the end, we consider for $\overline{T}^{-1}$ a condition which, in principle, is stronger than that of being of type (BR), i.e. the condition of being of type (D).

Definition 4.5. Let X be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. We call T of type (D*) if $\overline{T}^{-1} : X^* \rightrightarrows X^{**}$ is maximal monotone of type (D).

Note that any maximal monotone operator of type (D*) is of type (D) as well, since, by Corollary 3.3, the maximality of $\overline{T}$ implies that T is of type (D) (the same conclusion can also be drawn as an immediate consequence of Proposition 4.4). Therefore, $\overline{T} = \widetilde{T}$.

Moreover, the subdifferential of a lower semicontinuous proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is of type (D*), given that, as is well known [4], $\overline{\partial f} = \widetilde{\partial f} = (\partial f^*)^{-1}$ and ∂f^*, being again the subdifferential of a lower semicontinuous proper convex function, is a maximal monotone operator of type (D).

Summarizing, for any maximal monotone operator $T : X \rightrightarrows X^*$, the following relations hold:

$$T \text{ is a subdifferential} \Longrightarrow T \text{ is of type } (\mathrm{D}^*) \Longrightarrow T \text{ is of type } (\mathrm{BR}^*) \Longrightarrow T \text{ is of type } (\mathrm{D}).$$

It is an open question whether the converse of any of the preceding implications holds as well.

Finally, we know from [18, Corollary 3.4] that two maximal monotone operators $S, T : X \rightrightarrows X^*$ of type (D), under suitable conditions, satisfy the inclusion $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. Operators of type (D*), by means of Theorem 2.14, enable us to extend this property, in terms of a density result, to the whole of $X^{**} \times X^*$.

Theorem 4.6. *Let X be a Banach space and $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D*). For all $(w^{**}, w^*) \in X^{**} \times X^*$, if there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h^{**}|_{X^{**}\times X^*} - \varrho_1(\operatorname{dom} k^{**}|_{X^{**}\times X^*}) - (w^{**}, w^*)] \quad \text{is a closed subspace of } X^{**} \times X^*, \tag{3}$$

and

$$\bigcup_{\lambda>0} \lambda \left[p_2 \left(\operatorname{dom} \mathcal{T}_{(w^{**},w^*)} h^{**}|_{X^{**}\times X^*}\right) - p_2 \left(\operatorname{dom} k^{**}|_{X^{**}\times X^*}\right)\right] \quad \text{is a closed subspace of } X^*, \tag{4}$$

then there exists a sequence (w_n^{**}) in X^{**} converging to w^{**} and such that $(w_n^{**}, w^*) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. Therefore, in particular, $\operatorname{cl}(\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})) = X^{**} \times X^*$.

Proof. Since S and T are maximal monotone operators of type (D), then, for all $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$, setting $\tilde{h} := (h^{**\top})_{|X^*\times X^{**}}$ and $\tilde{k} := (k^{**\top})_{|X^*\times X^{**}}$, as a consequence of Theorem 3.2 we have $\tilde{h} \in \mathcal{H}_{\widetilde{S}^{-1}}$ and $\tilde{k} \in \mathcal{H}_{\widetilde{T}^{-1}}$. Thus, for any $(w^*, w^{**}) \in X^* \times X^{**}$, there exist $\tilde{h} \in \mathcal{H}_{\widetilde{S}^{-1}}$ and $\tilde{k} \in \mathcal{H}_{\widetilde{T}^{-1}}$ such that, by (3),

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} \tilde{h} - \varrho_2(\operatorname{dom} \tilde{k}) - (w^*, w^{**})]$$

is a closed subspace of $X^* \times X^{**}$ and, by (4),

$$\bigcup_{\lambda>0} \lambda[p_1 \operatorname{dom} \mathcal{T}_{(w^*,w^{**})}\tilde{h} - p_1 \operatorname{dom} \tilde{k}]$$

is a closed subspace of X^*.

Therefore, since S and T are of type (D*), by Theorem 2.14 (with X, X^*, u, u^*, v replaced by $X^*, X^{**}, w^*, w^{**}, 0_{X^*}$ and h, k, S, T replaced by $\widetilde{h}, \widetilde{k}, \widetilde{S}^{-1}, \widetilde{T}^{-1}$), one has

$$\operatorname{cl} \mathcal{R}(\widetilde{S}^{-1}(\,\cdot + w^*) + \widetilde{T}^{-1}) = X^{**}$$

for all $w^* \in X^*$.

Thus, there exist a sequence (x_n^{**}, x_n^*) in $X^{**} \times X^*$ and a sequence $(-y_n^{**})$ in X^{**} such that

$$(w^* + x_n^*, w^{**} + x_n^{**}) \in \mathcal{G}(\widetilde{S}^{-1}), \qquad (x_n^*, -y_n^{**}) \in \mathcal{G}(\widetilde{T}^{-1})$$

and $x_n^{**} - y_n^{**}$ converges to $0_{X^{**}}$ in the norm topology of X^{**}. As a consequence, one has

$$(w^{**} + x_n^{**} - y_n^{**}, w^*) = (w^{**} + x_n^{**}, w^* + x_n^*) + (-y_n^{**}, -x_n^*) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$$

and $w_n^{**} := w^{**} + x_n^{**} - y_n^{**}$ converges to w^{**} in the norm topology of X^{**}.

Since $(w^{**}, w^*) \in X^{**} \times X^*$ was chosen arbitrarily and (w_n^{**}, w^*) converges to (w^{**}, w^*) in the norm topology of $X^{**} \times X^*$, we conclude that $\mathrm{cl}\,(\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})) = X^{**} \times X^*$. $\square$

A relevant example illustrating the preceding theorem is given by the case when T is the duality mapping J, i.e. the subdifferential of the function $j : X \to \mathbb{R}$ defined by $j(x) = 1/2\|x\|^2$ for all $x \in X$. As the subdifferential of a lower semicontinuous proper convex function, J is a maximal monotone operator of type (D*) and the function $j \oplus j^* : X \times X^* \to \mathbb{R}$, defined by $(j \oplus j^*)(x, x^*) = j(x) + j^*(x^*)$ for all $(x, x^*) \in X \times X^*$, belongs to $\mathcal{H}_J$. Moreover,

$$\mathrm{dom}\,(j \oplus j^*) = X \times X^* \quad \text{and}$$
$$\mathrm{dom}\,(j \oplus j^*)^{**}|_{X^{**} \times X^*} = \mathrm{dom}\,(j^{**} \oplus j^{***})|_{X^{**} \times X^*} = X^{**} \times X^*.$$

Therefore, for any maximal monotone operator $S : X \rightrightarrows X^*$ of type (D*), setting $T = J$, the hypotheses of Theorem 4.6 are satisfied for any given $h \in \mathcal{H}_S$ and $k = j \oplus j^*$. Thus, not only (by [18, Corollary 3.4]) we have $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$, but also

$$\mathrm{cl}\,(\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})) = X^{**} \times X^*.$$

Acknowledgements. This research has been carried out while visiting the Departament d'Economia i d'Història Econòmica of the Universitat Autònoma de Barcelona. I am grateful to Juan Enrique Martínez-Legaz for helpful comments, which improved the present paper. I also thank an anonymous referee for useful suggestions.

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