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Differentiabilty and Partial Hölder Continuity of Solutions of Nonlinear Elliptic Systems

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The authors continue the study of regularity properties for solutions of elliptic systems started in [19] and continued [20], proving, in a bounded open set Ω of $\mathbb{R}^n$, local differentiability and partial Hölder continuity of the weak solutions u of nonlinear elliptic systems of order $2m$ in divergence form

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha\, a^\alpha\,(x, Du) = 0.$$

Specifically, we generalize the results obtained by Campanato and Cannarsa, contained in [6], under the hypothesis that the coefficients $a^\alpha(x, Du)$ are strictly monotone with nonlinearity $q = 2$.

Keywords: Higher order nonlinear elliptic systems, divergence form, monotone coefficients, generalized Sobolev spaces, local differentiability

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1. Introduction

In this paper we investigate in an open bounded $\Omega \subset \mathbb{R}^n$ the problem of local differentiability and Hölder regularity for weak solutions u of nonlinear elliptic systems of order $2m$ in divergence form

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha\, a^\alpha\,(x, Du) = 0. \tag{1}$$

Concerning the differentiability, if $0 < \lambda < 1$ and $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ is a solution of system (1), we answer to the question of what conditions are required for the vectors $a^\alpha(x, Du)$, in order that

$$u \in H^{m+1}_{\text{loc}}(\Omega, \mathbb{R}^N). \tag{2}$$

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In this note the authors consider solutions of class $C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ because, as already known, if we take solutions $u \in H^m \cap H^{m-1,\infty}(\Omega, \mathbb{R}^N)$, it is not possible in general to ensure differentiability (2) for nonlinear elliptic systems of order $2m$ even if the vectors $a^\alpha(x, Du)$ are smooth.

A first answer to the above problem has been given in [6] where the authors prove a result of local differentiability (2) for solutions of nonlinear elliptic systems of order $2m$ with quadratic growth.

The same hypotheses used in [6] are applied to second order ($m = 1$) nonlinear *parabolic* systems of variational type by Fattorusso in 1987 in the note [8] and later by Marino and Maugeri in 1995 in [16] to extend the local differentiability by Campanato and Cannarsa from the elliptic case to the parabolic one. The goal is achieved making use of the interpolation theory in Besov spaces. Moreover, as differentiability achievements allow Campanato and Cannarsa to obtain partial Hölder continuity of the derivatives $D^\alpha u$, $|\alpha| = m$, similarly Marino and Maugeri obtain in [15] a result of partial Hölder continuity for spatial gradient of the solution to the parabolic system of second order.

We also mention the note [18] where comparable outcomes are obtained by Naumann and Wolf.

Similar results concerned with interior differentiability of weak solutions u to nonlinear parabolic systems of second order are obtained using more general hypotheses, precisely exploiting natural growth and coefficients uniformly monotone in Du, at first in [9], later the complete extension of the results contained in [6] is achieved in [17]. The crucial step in the two mentioned papers by Fattorusso, Marino and Maugeri is the use of interpolation estimates of Gagliardo-Nirenberg's type in generalized Sobolev spaces. Recently, as announced in [14], the use of interpolation inequalities allows, in [10], the authors to establish differentiability results for weak solutions of nonlinear parabolic systems of second order endowed with nonlinearity $q \in (1, 2)$. The present note can be view as an extension from second order nonlinear elliptic systems to order $2m$ of the results established by one of the authors in [11].

Thus we can see that nonlinear systems of second order in divergence form have been extensively studied, much less depth if we talk about order $2m$.

The aim of this note is to give an answer to the starting problem using as assumptions that the vectors $a^\alpha(x, Du)$, $|\alpha| = m$, are strictly monotone and endowed with nonlinearity 2.

The technique used in this note to obtain Hölder regularity is not the classic one, founded on representation formulas of solutions and their derivatives, it is based on Campanato spaces $\mathcal{L}^{p,\lambda}$. They allows us to characterize Hölder functions using integral inequality and then it is very useful to study the regularity of weak solutions of elliptic and parabolic equations and systems (see e.g. [3], [5], [7]).

We wish to recall the study made by Giusti in [12] where this technique is used and appreciated.

This paper is organized as follows. In Section 1 we set the definitions of Sobolev spaces and fractionary Sobolev spaces, as well as useful preliminary Gagliardo-

Nirenberg estimates. In Section 2 are established local differentiability results for weak solutions of (1) in four steps. The heart of the paper is paragraph 2.1, where it is proved that if $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ $(0 < \lambda < 1)$ is a weak solution of the system (1) and some useful assumptions are satisfied, then $u \in H^{m+\vartheta}_{\text{loc}}(\Omega, \mathbb{R}^N)$, $\forall \vartheta \in \left(0, \frac{\lambda}{2}\right)$, $0 < \lambda < 1$. Using this result we obtain that $u \in H^{m+\vartheta'}_{\text{loc}}(\Omega, \mathbb{R}^N)$, $\forall \vartheta' \in \left(0, \vartheta + \frac{\lambda}{2}(1-\vartheta)\right)$ and applying an iterative method we attain that

$$u \in H^{m+\vartheta}_{\text{loc}}(\Omega, \mathbb{R}^N), \quad \forall\, 0 < \vartheta < 1.$$

Therefore in paragraph 2.2 the main result (Theorem 3.5) allows us to reach the differentiability (2) and in paragraph 2.3 using it, is established partial Hölder regularity for the derivatives $D^{m+1}u$ of the system (1) (see Theorem 4.1).

2. Preliminary Tools

Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, having diameter d_Ω and boundary $\partial\Omega$, $x = (x_1, x_2, \ldots, x_n)$ denotes a generic point therein.

We say *weak solution* of the system (1) a function $u \in H^m(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \sum_{|\alpha| \leq m} (a^\alpha(x, Du) |\, D^\alpha \varphi)\, dx = 0, \quad \forall \varphi \in H^m_0(\Omega, \mathbb{R}^N) \cap H^{m-1,\infty}(\Omega, \mathbb{R}^N). \tag{3}$$

Let us set k a positive integer greater than 1, $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ respectively the scalar product and the norm in $\mathbb{R}^k$. If there is no ambiguity we omit the index k.

Let k be a nonnegative integer and $\lambda \in]0, 1]$. We denote by $C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ the subspace of $C^k(\overline{\Omega}, \mathbb{R}^N)$ of functions $u : \overline{\Omega} \longrightarrow \mathbb{R}^N$ which satisfy a Hölder condition of exponent λ, together with all their derivatives $D^\alpha u$, $|\alpha| \leq k$; if $u \in C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, then

$$\|u\|_{C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)} = \sup_\Omega \sum_{|\alpha| \leq k} \|D^\alpha u\| + \sum_{|\alpha| = k} [D^\alpha u]_{\lambda, \overline{\Omega}} \tag{4}$$

where

$$[D^\alpha u]_{\lambda, \overline{\Omega}} = \sup_{\substack{x, y \in \overline{\Omega} \\ x \neq y}} \frac{\|D^\alpha u(x) - D^\alpha u(y)\|_N}{\|x - y\|_n^\lambda} < +\infty, \quad \forall \alpha : |\alpha| = k.$$

The space $C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ is a Banach space, provided with the norm

$$\|u\|_{C^{k,\lambda}(\overline{\Omega}, \mathbb{R}^N)} = \|u\|_{C^k(\overline{\Omega}, \mathbb{R}^N)} + \sum_{|\alpha| = k} [D^\alpha u]_{\lambda, \overline{\Omega}}. \tag{5}$$

Definition 2.1 (Sobolev Spaces (see e.g. [1], [13])). Let k and j be two positive integers, $k \geq j$. If $p \in [1, +\infty[$ and $u \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$, so we set

$$|u|_{j,p,\Omega} = \left(\int_\Omega \sum_{|\alpha| = j} \|D^\alpha u\|_N^p \, dx \right)^{\frac{1}{p}}, \qquad \|u\|_{k,p,\Omega} = \left(\sum_{j=0}^k |u|^p_{j,p,\Omega} \right)^{\frac{1}{p}} \tag{6}$$

and denote respectively by $H^{k,p}(\Omega,\mathbb{R}^N)$ and $H_0^{k,p}(\Omega,\mathbb{R}^N)$ the spaces obtained as closure of $C^\infty(\overline{\Omega},\mathbb{R}^N)$ and $C_0^\infty(\Omega,\mathbb{R}^N)$ regarding the norm $\|u\|_{k,p,\Omega}$.
The spaces $H^{k,p}(\Omega,\mathbb{R}^N)$ and $H_0^{k,p}(\Omega,\mathbb{R}^N)$ are known in literature as Sobolev Spaces.

We remark that $H^{0,p}(\Omega,\mathbb{R}^N) = L^p(\Omega,\mathbb{R}^N)$, $1 \le p < +\infty$.

Let us now state some properties useful in the sequel.

We set, for $x^0 \in \mathbb{R}^n$ and $\sigma > 0$, $Q(\sigma) = Q(x^0,\sigma)$ the cube of $\mathbb{R}^n$ defined by

$$\left\{x \in \mathbb{R}^n : \left|x_i - x_i^0\right| < \sigma,\ i = 1,2,\dots,n\right\}, \tag{7}$$

we also consider $t \in (0,1)$, $\sigma > 0$, $h \in \mathbb{R}\setminus\{0\}$, where $|h| < (1-t)\sigma$.
If there is no ambiguity we only write the radius and not also the center of the cube.

Let u be a function defined in $Q(\sigma)$ in $\mathbb{R}^N$ and $x \in Q(t\sigma)$, we set

$$\tau_{i,h}u(x) = u(x + he^i) - u(x), \quad i = 1,2,\dots,n, \tag{8}$$

where $\{e^i\}_{i=1,2,\dots,n}$ is the canonic basis of $\mathbb{R}^N$.

Let us now state Nirenberg's Theorem (see [4], Chapt. I, Theorem 3.X.), useful to achieve the main result of the note.

Theorem 2.2. *If $u \in L^p(\,Q(\sigma),\mathbb{R}^N\,)$, $1 < p < +\infty$, N is a positive integer and exists $M > 0$ such that $\|\,\tau_{i,h}u\|_{0,p,Q(t\sigma)} \le M\,|h|$, $\forall\,|h| < (1-t)\sigma$, $i = 1,2,\dots,n$, then $u \in H^{1,p}(Q(t\sigma),\mathbb{R}^N)$ and*

$$\|D_i u\|_{0,p,Q(t\sigma)} \le M, \quad \forall\, i = 1,2,\dots,n.$$

Theorem 2.3 (see e.g. [4], [11]). *Let $u \in H^{1,p}(Q(\sigma),\mathbb{R}^N)$ for $1 \le p < +\infty$ and N be a positive integer. Then, for every $t \in (0,1)$ and every $h \in \mathbb{R}$, $|h| < (1-t)\sigma$, we have*

$$\|\tau_{i,h}u\|_{0,p,Q(t\sigma)} \le |h|\,\|D_i u\|_{0,p,Q(\sigma)}, \quad i = 1,2,\dots,n. \tag{9}$$

2.1. Sobolev spaces with fractionary exponent $H^{k+\vartheta,p}$.

Let Ω be an open bounded set in $\mathbb{R}^n$, $\vartheta \in (0,1)$, $p \in [1,+\infty[$ and N a positive integer.

Definition 2.4. We say that a function u defined in Ω having values in $\mathbb{R}^N$ belongs to $H^{\vartheta,p}(\Omega,\mathbb{R}^N)$ if $u \in L^p(\Omega,\mathbb{R}^N)$ and is finite

$$|u|^p_{\vartheta,p,\Omega} = \int_\Omega dx \int_\Omega \frac{\|u(x)-u(y)\|_N^p}{\|x-y\|_n^{n+\vartheta p}}\,dy.$$

Definition 2.5. If k is a nonnegative integer, we mean for $H^{k+\vartheta,p}(\Omega,\mathbb{R}^N)$ the subspace of $H^{k,p}(\Omega,\mathbb{R}^N)$ of functions $u \in H^{k,p}(\Omega,\mathbb{R}^N)$ such that

$$D^\alpha u \in H^{\vartheta,p}(\Omega,\mathbb{R}^N), \quad \forall\,\alpha : |\alpha| = k.$$

We stress that $H^{k+\vartheta,p}(\Omega,\mathbb{R}^N)$ is a Banach space equipped with the following norm

$$\|u\|_{k+\vartheta,p,\Omega}=\left(\|u\|^p_{k,p,\Omega}+\sum_{|\alpha|=k}|D^\alpha u|^p_{\vartheta,p,\Omega}\right)^{\frac{1}{p}}.$$

The result below is used recurrently throughout the paper (see the proof in [2], Lemma II.3).

Theorem 2.6. *If $u\in L^2(Q(3\sigma),\mathbb{R}^N)$ and, for $\vartheta\in(0,1)$, is finite*

$$\sum_{i=1}^n\int_{-2\sigma}^{2\sigma}\frac{dh}{|h|^{1+2\vartheta}}\int_{Q(\sigma)}\|\tau_{i,h}\,u(x)\|^2\,dx,$$

then $u\in H^\vartheta(Q(\sigma),\mathbb{R}^N)$ and

$$|u|^2_{\vartheta,Q(\sigma)}\le c(n)\sum_{i=1}^n\int_{-2\sigma}^{2\sigma}\frac{dh}{|h|^{1+2\vartheta}}\int_{Q(\sigma)}\|\tau_{i,h}\,u(x)\|^2\,dx\,.$$

We mention the following interpolation inequality, fundamental for the sequel of the work (see e.g. [6], Appendix, Lemma 1).

Theorem 2.7. *If $u\in H^{1+\vartheta}(Q(\sigma),\mathbb{R}^N)$, for $0<\vartheta<1$, then*

$$|u|_{1,Q(\sigma)}\le c(n,\vartheta)\left\{\left(\sum_{i=1}^n|D_i\,u|^2_{\vartheta,Q(\sigma)}\right)^{\frac{1}{2(1+\vartheta)}}\|u\|^{\frac{\vartheta}{1+\vartheta}}_{0,Q(\sigma)}+\sigma^{-1}\|u\|_{0,Q(\sigma)}\right\}.$$

Theorem 2.8 ([6], Appendix, Lemma 2). *Let us consider $u\in H^{1+\vartheta}(Q(\sigma),\mathbb{R}^N)$, for $0<\vartheta<1$, then*

$$\sum_{i=1}^n\|D_i\,u-(D_i\,u)_{Q(\sigma)}\|^2_{0,Q(\sigma)}\le c(n,\vartheta)\left(\sum_{i=1}^n|D_i\,u|^2_{\vartheta,Q(\sigma)}\right)^{\frac{1}{1+\vartheta}}\|\,u-u_{Q(\sigma)}\|^{\frac{2\vartheta}{1+\vartheta}}_{0,Q(\sigma)}.$$

Theorem 2.9 ([6], Lemma I.3). *Let us set $\Omega,\Omega_1,\Omega_2,\dots,\Omega_m$ $m+1$ bounded open sets of $\mathbb{R}^n$ such that $\bigcup_{k=1}^m\Omega_k=\Omega$, σ and ϑ two positive real numbers, $\vartheta<1$ and $u\in H^\vartheta(\Omega_k,\mathbb{R}^N)$, for every $k=1,2,\dots,m$. Then, there exists a positive constant $c(\vartheta,\sigma)$ such that*

$$|u|^2_{\vartheta,\Omega}\le c(\vartheta,\sigma)\left\{\|u\|^2_{0,\Omega}+\sum_{k=1}^m\int_{\Omega_{k,\sigma}\cap\Omega}dx\int_{\Omega_k}\frac{\|u(x)-u(y)\|^2}{\|x-y\|^{n+2\vartheta}}\,dy\right\},$$

where $\Omega_{k,\sigma}$, $k=1,2,\dots,m$, is the set of points of $\mathbb{R}^n$ away from $\overline{\Omega_k}$ less than σ.

Theorem 2.10 (see [6], Teorema 2.I). *If $u\in H^{1+\vartheta}(Q(\sigma),\mathbb{R}^N)\cap C^{0,\lambda}(\overline{Q(\sigma)},\mathbb{R}^N)$, $0<\vartheta\le 1$ and $0<\lambda\le 1$. Then, for every $t>0$ and every $i=1,2,\dots,n$, we have*

$$\text{mis}\left\{x\in Q(\sigma):\left\|D_iu(x)-(D_i\,u)_{Q(\sigma)}\right\|>t\right\}$$
$$\le c^q(n,\vartheta)\frac{\sum_{j=1}^n|D_j\,u|^{\frac{q}{1+\vartheta}}_{\vartheta,Q(\sigma)}\cdot[u]^{\frac{q\vartheta}{1+\vartheta}}_{\lambda,\overline{Q(\sigma)}}}{t^q}$$

where $q = \frac{2(1+\vartheta)n}{n-2\vartheta\lambda}$. *Specifically*

$$D_i\, u \in L^p(Q(\sigma), \mathbb{R}^N), \quad \forall\, 1 \leq p < q,$$

and is true the following inequality

$$\int_{Q(\sigma)} \left\| D_i\, u - (D_i\, u)_{Q(\sigma)} \right\|^p \, dx$$

$$\leq c(\vartheta, n, p, q)(\text{mis}\, Q(\sigma))^{1-\frac{p}{q}} \, [\, u\,]_{\lambda, Q(\sigma)}^{\frac{p\vartheta}{1+\vartheta}} \sum_{j=1}^{n} |D_j\, u|_{\vartheta, Q(\sigma)}^{\frac{p}{1+\vartheta}} .$$

3. Local differentiability in H^{m+1}spaces

Let us set m, N positive integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_n$ the order of α. We denote by $\mathcal{R}$ the Cartesian product

$$\mathcal{R} = \prod_{|\alpha| \leq m} \mathbb{R}^N_\alpha$$

and $p = \{p^\alpha\}_{|\alpha| \leq m}$, $p^\alpha \in \mathbb{R}^N$, the generic point of $\mathcal{R}$. If $p \in \mathcal{R}$, we set $p = (p', p'')$ where $p' = \{p^\alpha\}_{|\alpha| < m} \in \mathcal{R}' = \prod_{|\alpha| < m} \mathbb{R}^N_\alpha$, $p'' = \{p^\alpha\}_{|\alpha| = m} \in \mathcal{R}'' = \prod_{|\alpha| = m} \mathbb{R}^N_\alpha$, and

$$\|p\|^2 = \sum_{|\alpha| \leq m} \|p^\alpha\|^2_N, \qquad \|p'\|^2 = \sum_{|\alpha| < m} \|p^\alpha\|^2_N, \qquad \|p''\|^2 = \sum_{|\alpha| = m} \|p^\alpha\|^2_N.$$

We consider, as usual,

$$D_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n; \qquad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}.$$

Let us consider the following differential nonlinear variational system of order $2m$:

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha\, a^\alpha\, (x, Du) = 0 \tag{10}$$

where $a^\alpha(x, p) = a^\alpha(x, p', p'')$ are functions of $\Lambda = \Omega \times \mathcal{R}$ in $\mathbb{R}^N$, satisfying the following conditions:

(a) for every α and for every $p \in \mathcal{R}$, the function $x \longrightarrow a^\alpha(x, p)$, defined in Ω having values in $\mathbb{R}^N$, is measurable in x;

(b) for every α and for every $x \in \Omega$, the function $p \longrightarrow a^\alpha(x, p)$, defined in $\mathcal{R}$ having values in $\mathbb{R}^N$, is continuous in p;

(c) for every α, such that $|\alpha| < m$, for every $(x, p', p'') \in \Omega \times \mathcal{R}$, with $\|p'\|_N \leq K$, we have:

$$\|a^\alpha(x, p', p'')\| \leq M(K) \left(|f^\alpha(x)| + \sum_{|\alpha| = m} \|p^\alpha\|^2_N \right) = M(K) \left(|f^\alpha(x)| + \|p''\|^2 \right),$$

where $f^\alpha \in L^1(\Omega)$;

(d) for every $x \in \Omega, \forall y \in Q\left(x, \frac{1}{\sqrt{n}} d_x\right)$, $\forall p', q' \in \mathcal{R}'$, where $\|p'\|, \|q'\| \leq K$ and for every $p'' \in \mathcal{R}''$, we have:
$$\|a(x, p', p'')\| \leq M(K)\,(1 + \|p''\|)\,,$$
$$\|a(x, p', p'') - a(y, q', p'')\| \leq M(K)\,(\|x - y\| + \|p' - q'\|)\,(1 + \|p''\|)\,;$$
where $a\,(x, p) \equiv (a^\alpha(x, p))_{|\alpha|=m}$ and $d_x = \operatorname{dist}(\{x\}, \partial\Omega) > 0$.

(e) for every $(x, p') \in \Omega \times \mathcal{R}'$, the functions $p'' \longrightarrow a^\alpha(x, p', p'')$, $|\alpha| = m$, are strictly monotone with non-linearity $q = 2$, so that there exist two positive constants $M(K)$ and $\nu(K)$ such that $\forall (x, p') \in \Omega \times \mathcal{R}'$, with $\|p'\| \leq K$, and $\forall p'', q'' \in \mathcal{R}''$, we obtain:
$$\|a(x, p', p'') - a(x, p', q'')\| \leq M(K)\,\|p'' - q''\|\,,$$
$$(a(x, p', p'') - a(x, p', q'')|p'' - q'') \geq \nu(K)\,\|p'' - q''\|^2\,.$$

Remark 3.1. We point out that the assumptions (a)–(e) are more general than the one used by Campanato and Cannarsa in [6].

Let us now state the local fractional differentiability results.

Theorem 3.2. *If* $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (10) *and the assumptions* (a)–(e) *are satisfied, then*
$$u \in H^{m+\vartheta}_{\text{loc}}(\Omega, \mathbb{R}^N), \quad \forall \vartheta \in \left(0, \frac{\lambda}{2}\right), \tag{11}$$
moreover, for every cube $Q(4\sigma) \subset\subset \Omega$, *we have the following inequality*
$$|D''u|^2_{\vartheta, Q(\sigma)} \leq c(\nu, K, U, \vartheta, \lambda, \sigma, m, n)\left(1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(4\sigma)} + |u|^2_{m,Q(4\sigma)}\right), \tag{12}$$
where $|D''u|^2_{\vartheta,Q(\sigma)} = \sum_{|\alpha|=m} |D^\alpha u|^2_{\vartheta,Q(\sigma)}$, $K = \sup_{\overline{\Omega}} \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)}$.

Theorem 3.3. *If* $u \in H^{m+\vartheta}(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \vartheta, \lambda < 1$, *is a weak solution of the system* (10), *the assumptions* (a), (b), (d), (e) *and the condition* (c) *with* $f^\alpha \in L^{\frac{q}{2}}(\Omega)$, $q = \frac{2(1+\vartheta)n}{n-2\vartheta\lambda}$, *are true, we have*
$$u \in H^{m+\vartheta'}_{\text{loc}}(\Omega, \mathbb{R}^N), \quad \forall \vartheta' \in \left(0, \vartheta + \frac{\lambda}{2}(1 - \vartheta)\right), \tag{13}$$
and, for every cube $Q(4\sigma) \subset\subset \Omega$, *we have the following inequality*
$$\begin{aligned} &|D''u|^2_{\vartheta',Q(\sigma)} \\ &\leq c(\nu, K, U, \vartheta, \vartheta', \lambda, \sigma, m, n)\left\{1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(4\sigma)}\right)^{1+\vartheta} \right. \\ &\qquad \left. + |u|^2_{m,Q(4\sigma)} + |D''u|^2_{\vartheta,Q(4\sigma)}\right\}, \end{aligned} \tag{14}$$
where $|D''u|^2_{\vartheta,Q(\sigma)} = \sum_{|\alpha|=m} |D^\alpha u|^2_{\vartheta,Q(\sigma)}$, $K = \sup_{\overline{\Omega}} \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)}$.

Applying an iterative method, we have the following result.

Theorem 3.4. *If* $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (10), *the hypotheses* (a), (b), (d), (e) *and the condition* (c) *with* $f^\alpha \in L^{\frac{2n}{n-2\lambda}}(\Omega)$ *are verified, then*

$$u \in H^{m+\vartheta}_{\text{loc}}(\Omega, \mathbb{R}^N), \quad \forall \vartheta : 0 < \vartheta < 1. \tag{15}$$

Moreover, for every cube $Q(\sigma) \subset\subset Q(\sigma_0) \subset\subset \Omega$, *we have*

$$\begin{aligned} &|D''u|^2_{\vartheta,Q(\sigma)} \\ &\leq c(\nu, K, U, \vartheta, \lambda, \sigma, \sigma_0, m, n)\left\{1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(\sigma_0)}\right)^{1+\vartheta} + |u|^2_{m,Q(\sigma_0)}\right\}, \end{aligned} \tag{16}$$

where $K = \sup_{\overline{\Omega}} \|D'u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)}$.
Moreover

$$u \in H^{m,4}_{\text{loc}}(\Omega, \mathbb{R}^N). \tag{17}$$

Let us now apply the previous local differentiability properties in $H^{m+\vartheta}(\Omega, \mathbb{R}^N)$, $0 < \vartheta < 1$ to reach the main objective of this note.

Theorem 3.5 (Main result). *If* $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$, *is a weak solution of the system* (10) *satisfying the hypotheses* (a), (b), (d), (e) *and, for* $f^\alpha \in L^{\frac{2n}{n-2\lambda}}(\Omega)$ *assumption* (c), *then*

$$u \in H^{m+1}_{\text{loc}}(\Omega, \mathbb{R}^N) \tag{18}$$

and, for every cube $Q(4\sigma) \subset\subset \Omega$, *the following inequality is true*

$$\begin{aligned} &|u|^2_{m+1,Q(\sigma)} \\ &\leq c(\nu, K, U, \lambda, \sigma, m, n)\left(1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(4\sigma)}\right)^2 + |u|^2_{m,Q(4\sigma)} + |u|^4_{m,4,Q(4\sigma)}\right), \end{aligned} \tag{19}$$

where $K = \sup_{\overline{\Omega}} \|u\|$ *and* $U = \|u\|_{C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)}$.

Proof of Theorem 3.1. Let us choose $x_0 \in \Omega$ and a generic cube $Q(4\sigma) = Q(x^0, 4\sigma) \subset\subset \Omega$, let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ a *cut-off* function having the following properties:

$$\begin{aligned} 0 \leq \psi \leq 1 \ \text{ in } \mathbb{R}^n, \qquad & \psi = 1 \ \text{ in } Q(\sigma), \\ \psi = 0 \ \text{ in } \mathbb{R}^n \setminus Q(2\sigma), \qquad & \|D\psi\| \leq \frac{k}{\sigma} \ \text{ in } \mathbb{R}^n. \end{aligned} \tag{20}$$

Let us also consider $i \leq n$ a positive integer, h a real number, $|h| < \sigma$, and let us also set

$$\varphi = \tau_{i,-h}\left(\psi^{2m} \tau_{i,h}\, u\right), \tag{21}$$

it follows that $\varphi \in H_0^m(\Omega, \mathbb{R}^N) \cap H^{m-1,\infty}(\Omega, \mathbb{R}^N)$. From (3), written for this "test function" φ, it follows

$$\int_\Omega \sum_{|\alpha|=m} \left(\tau_{i,h} a^\alpha(x, Du) \,|\, D^\alpha\left(\psi^{2m}\tau_{i,h}\, u\right)\right) dx \tag{22}$$
$$= -\sum_{|\alpha|<m} \int_\Omega \left(a^\alpha(x, Du) \,|\, \tau_{i,-h} D^\alpha\left(\psi^{2m}\tau_{i,h}\, u\right)\right) dx.$$

On the other hand, for every α such that $|\alpha| = m$ and for a.e. $x \in Q(2\sigma)$, it follows:

$$\begin{aligned}
&\tau_{i,h} a^\alpha(x, Du(x)) = \tau_{i,h} a^\alpha(x, D'u(x), D''u(x)) \\
&= a^\alpha(x+he^i, D'u(x+he^i), D''u(x+he^i)) - a^\alpha(x, D'u(x), D''u(x)) \\
&= a^\alpha(x+he^i, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) - a^\alpha(x, D'u(x), D''u(x)) \\
&= \big[a^\alpha(x+he^i, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) - \\
&\qquad - a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x))\big] \\
&\qquad + [a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) - \\
&\qquad - a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x))] \\
&\qquad + [\, a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x)) - a^\alpha(x, D'u(x), D''u(x))] \,.
\end{aligned}$$

Regarding in mind that

$$D^\alpha(\psi^{2m}\tau_{i,h}\, u) = \psi^{2m}\tau_{i,h}D^\alpha u + 2m\psi^{2m-1}\left(D^\alpha\psi\right)\tau_{i,h}u,$$

formula (22) becomes:

$$\begin{aligned}
&\int_\Omega \psi^{2m} \sum_{|\alpha|=m} (a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x)) \\
&\qquad -a^\alpha(x, D'u(x), D''u(x)) \,|\, \tau_{i,h}D^\alpha u)\; dx \\
&= -2m \int_\Omega \psi^{2m-1} \sum_{|\alpha|=m} (a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x)) \\
&\qquad -a^\alpha(x, D'u(x), D''u(x)) \,|\, (D^\alpha\psi)\,\tau_{i,h}u) dx \\
&\quad - \int_\Omega \sum_{|\alpha|=m} (a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) \\
&\qquad -a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x)) \,|\, D^\alpha\left(\psi^{2m}\tau_{i,h}\, u\right))\; dx \\
&\quad - \int_\Omega \sum_{|\alpha|=m} \left(a^\alpha(x+he^i, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x))\right. \\
&\qquad \left. -a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x)) \,|\, D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right) dx \\
&\quad - \sum_{|\alpha|<m} \int_\Omega \left(a^\alpha(x, D'u, D''u) \,|\, \tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\, u\right)\right)\; dx.
\end{aligned} \tag{23}$$

Using hypotheses (e) we can minimize the first member of (23), as follows

$$\begin{aligned}
&\int_\Omega \psi^{2m} \sum_{|\alpha|=m} (a^\alpha(x, D'u(x),\ D''u(x) + \tau_{i,h} D''u(x)) \\
&\qquad -a^\alpha(x, D'u(x), D''u(x)) \,|\, \tau_{i,h} D^\alpha u)\ dx \\
&= \int_{Q(2\sigma)} \psi^{2m}\, (a(x, D'u(x), D''u(x) + \tau_{i,h}\, D''u(x)) \\
&\qquad -a(x, D'u(x), D''u(x)) \,|\, \tau_{i,h} D''u)\ dx \\
&\geq \nu \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2\, dx,
\end{aligned}$$

then we obtain

$$\nu \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2\, dx \leq A + B + C + D, \tag{24}$$

where

$$\begin{aligned}
A = &-2m \int_\Omega \psi^{2m-1} \sum_{|\alpha|=m} (a^\alpha(x, D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \\
&-a^\alpha(x, D'u(x), D''u(x)) \,|\, (D^\alpha \psi)\ \tau_{i,h} u)\, dx,
\end{aligned} \tag{25}$$

$$\begin{aligned}
B = &-\int_\Omega \sum_{|\alpha|=m} \big(a^\alpha(x + he^i, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \\
&-a^\alpha(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \,|\, D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\big)\, dx,
\end{aligned} \tag{26}$$

$$\begin{aligned}
C = &-\int_\Omega \sum_{|\alpha|=m} (a^\alpha(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \\
&-a^\alpha(x, D'u(x), D''u(x)\ +\ \tau_{i,h}\, D''u(x)) \,|\, D^\alpha \left(\psi^{2m} \tau_{i,h}\, u\right))\ dx,
\end{aligned} \tag{27}$$

$$D = -\sum_{|\alpha|<m} \int_\Omega \left(a^\alpha\, (x, D'u(x), D''u)\ |\, \tau_{i,-h}\, D^\alpha \left(\psi^{2m} \tau_{i,h}\ u\right)\right) dx. \tag{28}$$

Let us estimate the terms A, B, C and D.
Applying hypothesis (e) and the properties of the function ψ, we have

$$\begin{aligned}
|A| &\leq 2m \int_{Q(2\sigma)} \psi^{2m-1} \sum_{|\alpha|=m} \|a^\alpha(x, D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \\
&\qquad -\ a^\alpha\, (x, D'u(x), D''u(x))\, \|\, \big\|\, (\, D^\alpha\, \psi\,)\, \tau_{i,h}\, u\ \big\|\, dx \\
&\leq c(\,K,\ m\,) \int_{Q(2\sigma)} \psi^{2m-1} \sum_{|\alpha|=m} |D^\alpha \psi|\, \|\tau_{i,h} D''u\|\, \|\tau_{i,h} D'u\|\ dx \\
&\leq c\,(K, \sigma, m, n) \int_{Q(2\sigma)} \psi^{2m-1}\, \|\tau_{i,h} D''u\| \|\tau_{i,h} D'u\|\ dx \\
&\leq c\,(K, \sigma, m, n) \int_{Q(2\sigma)} \psi^{m}\ \|\tau_{i,h} D''u\|\, \|\tau_{i,h} D'u\|\ dx.
\end{aligned}$$

Then, for every $\varepsilon > 0$, we have

$$|A| \le \varepsilon \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D'' u\right\|^2 \, dx + c(K, \sigma, m, n, \varepsilon) \int_{Q(2\sigma)} \left\|\tau_{i,h} D' u\right\|^2 \, dx, \tag{29}$$

On the other hand, using Theorem 2.3 for $p = 2$, $t = \frac{2}{3}$ and $Q(3\sigma)$ instead of $Q(\sigma)$, for every $h \in \mathbb{R}$ such that $|h| < \left(1 - \frac{2}{3}\right) 3\sigma = \sigma$, we have

$$\int_{Q(2\sigma)} \|\tau_{i,h} u\|_N^2 \, dx \le h^2 \int_{Q(3\sigma)} \|D_i u\|_N^2 \, dx, \quad i = 1, 2, \dots, n. \tag{30}$$

From (29) and (30), we have

$$|A| \le \varepsilon \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D'' u\right\|^2 \, dx + c(K, \sigma, m, n, \varepsilon)\, h^2 \int_{Q(3\sigma)} \left\|D'' u\right\|^2 \, dx, \quad \forall \varepsilon > 0. \tag{31}$$

From (d) we can majorize the term B as follows

$$\begin{aligned}
|B| \le & \int_{Q(2\sigma)} \sum_{|\alpha|=m} \big\| a^\alpha(x + he^i, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \\
& \qquad - a^\alpha(x, D'u(x) + \tau_{i,h} D'u(x), D''u(x) + \tau_{i,h} D''u(x)) \| \left\| D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\right\| dx \\
\le & \; c(K, n)\, |h| \int_{Q(2\sigma)} (1 + \|D''u\| + \|\tau_{i,h} D''u\|) \\
& \qquad \left(\frac{2mk}{\sigma} \psi^{2m-1} \|\tau_{i,h} D'u\| + \psi^{2m} \|\tau_{i,h} D''u\| \right) dx \\
\le & \; c(K, m, n)\, |h| \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \\
& \qquad + \int_{Q(2\sigma)} \left[c(K, m, n)\, |h|\, \psi^m \left(1 + \|D''u\|\right)\right] \left[\psi^m \|\tau_{i,h} D''u\|\right] dx \\
& \qquad + \int_{Q(2\sigma)} \left[c(\sigma, m) \|\tau_{i,h} D'u\|\right] \left[c(K, n)\, |h|\, \psi^m \left(1 + \|D''u\| + \|\tau_{i,h} D''u\|\right)\right] dx.
\end{aligned}$$

Then, for every $\varepsilon > 0$, we have

$$\begin{aligned}
|B| \le & \; c(K, m, n)\, |h| \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx + \varepsilon \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \\
& \qquad + c(K, m, n, \varepsilon) h^2 \int_{Q(2\sigma)} \psi^{2m} \big(1 + \|D''u\|^2\big) dx + c(\sigma, m) \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx \\
& \qquad + c(K, m, n) h^2 \int_{Q(2\sigma)} \psi^{2m} \big(1 + \|D''u\|^2 + \|\tau_{i,h} D''u\|^2\big) dx \\
= & \; \left\{ \varepsilon + c(K, m, n) \left(h^2 + |h|\right) \right\} \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \\
& \qquad + c(K, m, n, \varepsilon) h^2 \int_{Q(2\sigma)} \psi^{2m} (1 + \|D''u\|^2) dx + c(\sigma, m) \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx.
\end{aligned}$$

Using (30), we can estimate $|B|$ as follows

$$
\begin{aligned}
|B| \leq\ & \left\{\varepsilon + c(K,m,n)\left(h^2+|h|\right)\right\} \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D'' u\right\|^2 dx \\
& + c(K,\sigma,m,n,\varepsilon) h^2 \int_{Q(3\sigma)} \left(1+\left\|D''u\right\|^2\right) dx, \quad \forall \varepsilon > 0.
\end{aligned}
$$

Similarly, using (d), for the term C we have

$$
\begin{aligned}
|C| \leq\ & \int_{Q(2\sigma)} \sum_{|\alpha|=m} \left\|a^\alpha(x, D'u(x)+\tau_{i,h}D'u(x), D''u(x)+\tau_{i,h}D''u(x))\right. \\
& \qquad \left. - a^\alpha(x, D'u(x), D''u(x)+\tau_{i,h}D''u(x))\right\| \left\|D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right\| dx \\
\leq\ & c(K,n) \int_{Q(2\sigma)} \left\|\tau_{i,h}D'u\right\| \left(1+\left\|D''u\right\|+\left\|\tau_{i,h}D''u\right\|\right) \\
& \qquad \left(c(\sigma,m)\psi^{2m-1}\left\|\tau_{i,h}D'u\right\| + \psi^{2m}\left\|\tau_{i,h}D''u\right\|\right) dx \\
=\ & \int_{Q(2\sigma)} \left[\psi^m \left\|\tau_{i,h}D''u\right\|\right] \left[c(K,n)\psi^m \left\|\tau_{i,h}D'u\right\| \left(1+\left\|D''u\right\|\right)\right] dx \\
& \qquad + c(K,n) \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D'u\right\| \left\|\tau_{i,h}D''u\right\|^2 dx \\
& \qquad + \int_{Q(2\sigma)} \left[c(\sigma,m)\left\|\tau_{i,h}D'u\right\|\right] \\
& \qquad\quad \left[c(K,n)\psi^{2m-1}\left\|\tau_{i,h}D'u\right\| \left(1+\left\|D''u\right\|+\left\|\tau_{i,h}D''u\right\|\right)\right] dx.
\end{aligned}
$$

Because of $u \in C^{m-1,\lambda}(\Omega,\mathbb{R}^N)$, for every $\varepsilon > 0$, it follows

$$
\begin{aligned}
|C| \leq\ & \varepsilon \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D''u\right\|^2 dx + c(K,n,\varepsilon) \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D'u\right\|^2 \left(1+\left\|D''u\right\|^2\right) dx \\
& + c(K,U,n)\,|h|^\lambda \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D''u\right\|^2 dx + c(\sigma,m) \int_{Q(2\sigma)} \left\|\tau_{i,h}D'u\right\|^2 dx \\
& + c(K,n) \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D'u\right\|^2 \left(1+\left\|D''u\right\|^2+\left\|\tau_{i,h}D''u\right\|^2\right) dx \\
\leq\ & \left\{\varepsilon + c(K,U,n)\left(|h|^\lambda+|h|^{2\lambda}\right)\right\} \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D''u\right\|^2 dx \\
& + c(K,\sigma,m,n,\varepsilon) \int_{Q(2\sigma)} \left\|\tau_{i,h}D'u\right\|^2 dx + c(K,U,n,\varepsilon)|h|^{2\lambda} \int_{Q(2\sigma)} \left\|D''u\right\|^2 dx.
\end{aligned}
$$

Using again (30), we get

$$
\begin{aligned}
|C| \leq\ & \left\{\varepsilon + c(K,U,n)\left(|h|^\lambda+|h|^{2\lambda}\right)\right\} \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h}D''u\right\|^2 dx \\
& + c(K,U,\sigma,m,n,\varepsilon)\left(h^2+|h|^{2\lambda}\right) \int_{Q(3\sigma)} \left\|D''u\right\|^2 dx, \quad \forall \varepsilon > 0.
\end{aligned} \tag{32}
$$

Finally, let us estimate the terms D. For the hypothesis (c), we have

$$
\begin{aligned}
|D| &\leq \sum_{|\alpha|<m} \int_{Q(3\sigma)} \|a^\alpha(x, D'u, D''u)\| \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\right\| \, dx \qquad (33) \\
&\leq c(K) \sum_{|\alpha|<m} \int_{Q(3\sigma)} \left(|f^\alpha| + \|D''u\|^2\right) \left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)\right\| \, dx.
\end{aligned}
$$

On the other hand, using the hypothesis that $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, we easily obtain

$$
\left\|\tau_{i,-h} D^\alpha \left(\psi^{2m} \tau_{i,h} u\right)(x)\right\| \leq 2\, U\, |h|^\lambda, \quad \forall x \in Q(3\sigma). \qquad (34)
$$

From (33) and (34) we have

$$
|D| \leq c(K, U, m)\, |h|^\lambda \left(\sum_{|\alpha|<m} \int_{Q(3\sigma)} |f^\alpha| + \|D''u\|^2 \, dx \right). \qquad (35)
$$

From (24), (31), (32), (35), choose $\varepsilon = \frac{\nu}{12}$ we deduce that

$$
\begin{aligned}
&\nu \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \qquad (36) \\
&\leq \left\{ \frac{\nu}{4} + c(\nu, K, U, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda} \right) \right\} \cdot \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \\
&\quad + c(\nu, K, U, \sigma, m, n)(h^2 + |h|^\lambda + |h|^{2\lambda}) \int_{Q(3\sigma)} \left(1 + \sum_{|\alpha|<m} |f^\alpha| + \|D''u\|^2 \right) dx.
\end{aligned}
$$

Because of the continuity of the function $h \longrightarrow c(K, U, \sigma, m, n)(|h| + h^2 + |h|^\lambda + |h|^{2\lambda})$ in the origin, $\exists\, h_0(\nu, K, U, \lambda, \sigma, n)$, $0 < h_0 < \min\{1, \sigma\}$, such that for every $|h| < h_0$, we have

$$
c(K, U, \sigma, m, n) \left(|h| + h^2 + |h|^\lambda + |h|^{2\lambda} \right) < \frac{\nu}{4}.
$$

Let us consider, at first, that $|h| < h_0 < 1$.

Recalling that $0 < \lambda < 1$ we have $h^2 + |h|^\lambda + |h|^{2\lambda} \leq 3\, |h|^\lambda$ and taking into consideration that $\psi(x) = 1$ in $Q(\sigma)$, from (36), it follows, for $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
&\frac{\nu}{2} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2 \, dx \qquad (37) \\
&\leq c(\nu, K, U, \sigma, m, n)\, |h|^\lambda \left\{ 1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)} \right\}.
\end{aligned}
$$

Otherwise if $h_0 \leq |h| < 2\sigma$, we get for $i = 1, 2, \ldots, n$,

$$\int_{Q(\sigma)} \|\tau_{i,h} D''u\|^2 \, dx \tag{38}$$
$$\leq 2 \int_{Q(\sigma)} \left\|D''u\left(x + he^i\right)\right\|^2 dx + 2 \int_{Q(\sigma)} \|D''u(x)\|^2 \, dx$$
$$\leq 2 \int_{Q(3\sigma)} \| D''u(x) \|^2 \, dx + 2 \int_{Q(\sigma)} \| D''u(x) \|^2 \, dx$$
$$\leq 4 \int_{Q(3\sigma)} \| D''u \|^2 \, dx \leq 4 \frac{|h|^\lambda}{h_0^\lambda} \int_{Q(3\sigma)} \| D''u \|^2 \, dx \leq$$
$$\leq c(\nu, K, U, \lambda, \sigma, m, n)|h|^\lambda \left\{ 1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)} \right\}$$

From (37) and (38), for every $0 < |h| < 2\sigma$, it follows that

$$\sum_{i=1}^{n} \frac{1}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h} \, D''u\|^2 \, dx \tag{39}$$
$$\leq c(\nu, K, U, \lambda, \sigma, m, n)\left\{ 1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)} \right\} \frac{1}{|h|^{1+2\vartheta-\lambda}} .$$

The hypothesis $0 < \vartheta < \frac{\lambda}{2}$ assures that $1 + 2\vartheta - \lambda < 1$, then the function of the variable h that appears in the second member of (39) is integrable in $[-2\sigma, 2\sigma]$, it implies the integrability in $[-2\sigma, 2\sigma]$ of the left term of inequality (39) and it follows

$$\sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h} \, D''u\|^2 \, dx \tag{40}$$
$$\leq c(\nu, K, U, \vartheta, \lambda, \sigma, m, n)\left\{ 1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,1,Q(3\sigma)} + |u|^2_{m,Q(3\sigma)} \right\}.$$

Finally, recalling that $u \in H^m(\Omega, \mathbb{R}^N)$, from (40) it follows that $D''u$ satisfy the hypotheses of Theorem 2.6, we can conclude that

$$D''u \in H^\vartheta(Q(\sigma), \mathcal{R}") \tag{41}$$

and

$$|D''u|^2_{\vartheta,Q(\sigma)} \leq c(n) \sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta}} \int_{Q(\sigma)} \|\tau_{i,h} D''u\|^2 \, dx, \quad \forall 0 < \vartheta < \frac{\lambda}{2}. \tag{42}$$

From (41), (42) and (40) we reach the conclusion. □

Proof of Theorem 3.2. Let us fix $x_0 \in \Omega$ and the cube $Q(4\sigma) = Q(x^0, 4\sigma) \subset\subset \Omega$. Let us also consider a positive integer $i \leq n$, and a real number h such that $|h| < \frac{\sigma}{2}$.

As in the proof of Theorem 3.2, for every $\varepsilon > 0$, we have

$$\nu \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \tag{43}$$
$$\leq \varepsilon \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx + c(K,\sigma,m,n,\varepsilon) \int_{Q(2\sigma)} \|\tau_{i,h} D' u\|^2 \, dx \ + B + C + D,$$

where ψ is the above defined *cut-off* function (see (20)) and the terms B, C and D are considered in (26)–(28).

The terms $|B|$ and $|C|$ can be estimated, $\forall \varepsilon > 0$, as follows

$$|B| \leq \left\{\varepsilon + c(K,\sigma,m,n)\left(|h| + h^2\right)\right\} \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \tag{44}$$
$$+ c(K,\sigma,m,n,\varepsilon) h^2 \int_{Q(2\sigma)} \psi^{2m} \left(1 + \|D''u\|^2\right) dx + \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx,$$

$$|C| \leq \int_{Q(2\sigma)} \left\{\varepsilon + c(K,\sigma,m,n)\left(\|\tau_{i,h} D'u\| + \|\tau_{i,h} D'u\|^2\right)\right\} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx$$
$$+ c(K,\sigma,m,n,\varepsilon) \int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx \tag{45}$$
$$+ c(K,\sigma,m,n,\varepsilon) \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx,$$

similarly to the proof of Theorem 3.2.

Recalling that $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, we have

$$\|\tau_{i,h} D'u(x)\| \leq U \ |h|^\lambda, \quad \forall x \in Q(2\sigma). \tag{46}$$

Moreover applying Theorem 2.3, for $p = 2$, $t = \frac{4}{5}$ and $Q(\frac{5}{2}\sigma)$ in replacement of $Q(\sigma)$, for every $h \in \mathbb{R}$ such that $|h| < \frac{\sigma}{2}$, we achieve

$$\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^2 \, dx \leq \ h^2 \int_{Q(\frac{5}{2}\sigma)} \|D''u\|^2 \, dx, \quad i = 1, 2, \ldots, n. \tag{47}$$

From (43)–(47) it follows

$$\nu \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx \tag{48}$$
$$\leq \{3\varepsilon + c(K,U,\sigma,m,n)(|h| + h^2 + |h|^\lambda + |h|^{2\lambda})\} \cdot \int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D''u\|^2 \, dx$$
$$+ c(K,\sigma,m,n,\varepsilon) \, h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \|D''u\|^2\right) dx$$
$$+ c(K,\sigma,m,n,\varepsilon) \ \int_{Q(2\sigma)} \psi^{2m} \| \, \tau_{i,h} D'u \, \|^2 \, \| \, D''u \, \|^2 \, dx \ + \ | \, D \, | \ , \quad \forall \varepsilon > 0.$$

As in the proof of Theorem 3.2 there exists $h_0(\nu, K, U, \lambda, \sigma, m, n)$, $0 < h_0 < \min\{1, \frac{\sigma}{2}\}$, such that

$$c(K, U, \sigma, m, n)\left(|h| + h^2 + |h|^{\lambda} + |h|^{2\lambda}\right) < \frac{\nu}{4}, \quad |h| < h_0$$

then, for $\varepsilon = \frac{\nu}{12}$ we have

$$\begin{aligned} &\frac{\nu}{2}\int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D''u\right\|^2 dx \qquad (49)\\ &\leq c(\nu, K, \sigma, m, n)h^2 \int_{Q(\frac{5}{2}\sigma)} \left(1 + \left\|D''u\right\|^2\right) dx \\ &\quad + c(\nu, K, \sigma, m, n) \int_{Q(2\sigma)} \psi^{2m} \left\|\tau_{i,h} D'u\right\|^2 \left\|D''u\right\|^2 dx + |D| . \end{aligned}$$

Let us now estimate the last two terms in (49).
Applying Theorem 2.10 we have, $\forall\, Q(\rho) = Q(x^0, \rho) \subset\subset \Omega$ and $2 \leq p < q = \frac{2(1+\vartheta)n}{n-2\vartheta\lambda}$, that

$$u \in H^{m,p}(Q(\rho), \mathbb{R}^N) \qquad (50)$$

and

$$\begin{aligned} &\int_{Q(\rho)} \left\|D''u - (D''u)_{Q(\rho)}\right\|^p dx \qquad (51)\\ &\leq c(\vartheta, \lambda, m, n, p) \left(\operatorname{mis} Q(\rho)\right)^{1-\frac{p}{q}} \left[D'u\right]_{\lambda,\overline{\Omega}}^{\frac{p\vartheta}{1+\vartheta}} \left|D''u\right|_{\vartheta,Q(\rho)}^{\frac{p}{1+\vartheta}} . \end{aligned}$$

Thus we deduce that

$$\begin{aligned} &|u|_{m,p,Q(\rho)}^{2(1+\vartheta)} = \left(\int_{Q(\rho)} \left\|D''u\right\|^p dx\right)^{\frac{2(1+\vartheta)}{p}} \qquad (52)\\ &\leq 2^{(p-1)\frac{2(1+\vartheta)}{p}} \left\{\int_{Q(\rho)} \left\|D''u - (D''u)_{Q(\rho)}\right\|^p dx + \int_{Q(\rho)} \left\|(D''u)_{Q(\rho)}\right\|^p dx\right\}^{\frac{2(1+\vartheta)}{p}} \\ &\leq c(U, \vartheta, \lambda, m, n, p) \left\{\left|D''u\right|_{\vartheta,Q(\rho)}^2 + \left\|(D''u)_{Q(\rho)}\right\|^{2(1+\vartheta)}\right\} \\ &\leq c(U, \vartheta, \lambda, m, n, p) \left\{\left|D''u\right|_{\vartheta,Q(\rho)}^2 + |u|_{m,Q(\rho)}^{2(1+\vartheta)}\right\} . \end{aligned}$$

Using interpolation inequality contained in Theorem 2.7 we derive

$$\begin{aligned} |u|_{m,Q(\rho)}^{2(1+\vartheta)} &\leq c(\vartheta, n) \left\{\left|D''u\right|_{\vartheta,Q(\rho)}^2 \|u\|_{m-1,Q(\rho)}^{2\vartheta} + \rho^{-2(1+\vartheta)} \|u\|_{m-1,Q(\rho)}^{2(1+\vartheta)}\right\} \\ &\leq c(K, \vartheta, m, n, \rho) \left\{\left|D''u\right|_{\vartheta,Q(\rho)}^2 + 1\right\} \end{aligned}$$

then, $\forall\, Q(\rho) \subset\subset \Omega$ and $\forall 2 \leq p < q = \frac{2(1+\vartheta)n}{n-2\vartheta\lambda}$, we have

$$|u|_{m,p,Q(\rho)}^{2(1+\vartheta)} \leq c(K, U, \vartheta, \lambda, m, n, p, \rho) \left\{\left|D''u\right|_{\vartheta,Q(\rho)}^2 + 1\right\} . \qquad (53)$$

By (50), the hypothesis $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ and Hölder inequality, for every $2(1+\vartheta) < p < q$, it follows

$$\int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx \tag{54}$$
$$\leq \left(\int_{Q(2\sigma)} \|D''u\|^p \, dx\right)^{\frac{2}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^{\frac{2p}{p-2}} \, dx\right)^{\frac{p-2}{p}}$$
$$= \left(\int_{Q(2\sigma)} \|D''u\|^p \, dx\right)^{\frac{2}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^{\frac{2\vartheta p}{p-2}} \|\tau_{i,h} D'u\|^{\frac{2(1-\vartheta)p}{p-2}} \, dx\right)^{\frac{p-2}{p}}$$
$$\leq \left(\int_{Q(2\sigma)} \|D''u\|^p \, dx\right)^{\frac{2}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^p \, dx\right)^{\frac{2\vartheta}{p}} \left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^{\frac{2(1-\vartheta)p}{p-2(1+\vartheta)}} \, dx\right)^{\frac{p-2(1+\vartheta)}{p}}$$

the last inequality is obtained considering that $\|\tau_{i,h} D'u\|^{\frac{2\vartheta p}{p-2}} \in L^{\frac{p-2}{2\vartheta}}(Q(2\sigma))$, $\|\tau_{i,h} D'u\|^{\frac{2(1-\vartheta)p}{p-2}} \in L^{\frac{p-2}{p-2(1+\vartheta)}}(Q(2\sigma))$, $\frac{2\vartheta}{p-2} + \frac{p-2(1+\vartheta)}{p-2} = 1$. From Theorem 2.3 for $t = \frac{4}{5}$ and $Q(\frac{5}{2}\sigma)$ in place of $Q(\sigma)$, $\forall h \in \mathbb{R}$, $|h| < \frac{\sigma}{2}$, we attain the inequality

$$\left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^p \, dx\right)^{\frac{2\vartheta}{p}} \tag{55}$$
$$\leq |h|^{2\vartheta} \left(\int_{Q(\frac{5}{2}\sigma)} \|D''u\|^p \, dx\right)^{\frac{2\vartheta}{p}}, \quad \forall p, q : 2(1+\vartheta) < p < q .$$

Using the hypothesis $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ we deduce, for every $2(1+\vartheta) < p < q$, that

$$\left(\int_{Q(2\sigma)} \|\tau_{i,h} D'u\|^{\frac{2(1-\vartheta)p}{p-2(1+\vartheta)}} \, dx\right)^{\frac{p-2(1+\vartheta)}{p}} \leq U^{2(1-\vartheta)} |h|^{2\lambda(1-\vartheta)} [\text{mis}\, Q(2\sigma)]^{\frac{p-2(1+\vartheta)}{p}} . \tag{56}$$

From (54)–(56) we reach

$$\int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx$$
$$\leq c(U, \vartheta, n, p, \sigma) |h|^{2\vartheta + 2\lambda(1-\vartheta)} |u|^{2(1+\vartheta)}_{m,p,Q(\frac{5}{2}\sigma)}, \quad \forall\, 2(1+\vartheta) < p < q,$$

that, for $p = 1 + \vartheta + \frac{q}{2}$ and combined with (53) for $\rho = \frac{5}{2}\sigma$ and for $p = 1 + \vartheta + \frac{q}{2}$, gives

$$\int_{Q(2\sigma)} \psi^{2m} \|\tau_{i,h} D'u\|^2 \|D''u\|^2 \, dx \tag{57}$$
$$\leq c(K, U, \vartheta, \lambda, n, \sigma) |h|^{2\vartheta + 2\lambda(1-\vartheta)} \left\{ |D''u|^2_{\vartheta, Q(\frac{5}{2}\sigma)} + 1 \right\},$$

$$
\begin{aligned}
&\frac{\nu}{2}\int_{Q(2\sigma)} \psi^{2m}\left\|\tau_{i,h}\,D''u\right\|^2\,dx \qquad (58)\\
&\le c(\nu,K,\sigma,m,n)h^2\int_{Q(\frac{5}{2}\sigma)}\left(1+\left\|D''u\right\|^2\right)\,dx\\
&\qquad + c(\nu,K,U,\vartheta,\lambda,n,m,\sigma)\,|h|^{2\vartheta+2\lambda(1-\vartheta)}\left\{|D''u|^2_{\vartheta,Q(\frac{5}{2}\sigma)}+1\right\}\,+\,|\,D\,|\,.
\end{aligned}
$$

Let us focus our attention on the term D. Combining (c), (50) and Hölder inequality, for $2<p<\min(4,q)$, we carry out

$$
\begin{aligned}
|D| &\le \sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}[M(K)(|f^\alpha|+\left\|D''u\right\|^2)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^{\frac{4}{p}-1}] \qquad (59)\\
&\qquad\cdot[\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right\|^{2-\frac{4}{p}}]dx\\
&\le c(K,p)\sum_{|\alpha|<m}\left(\int_{Q(\frac{5}{2}\sigma)}\left(|f^\alpha|^{\frac{p}{2}}+\left\|D''u\right\|^p\right)\left\|\tau_{i,-h}D^\alpha(\psi^{2m}\tau_{i,h}u)\right\|^{\frac{4-p}{p}\frac{p}{2}}\,dx\right)^{\frac{2}{p}}\\
&\qquad\cdot\left(\int_{Q(\frac{5}{2}\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\right)^{\frac{p-2}{p}}\\
&= c(K,p)\sum_{|\alpha|<m}\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{p-2}\left(|f^\alpha|^{\frac{p}{2}}+\left\|D''u\right\|^p\right)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^{\frac{4-p}{2}}\,dx\right)^{\frac{2}{p}}\\
&\qquad\cdot\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{-2}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2dx\right)^{\frac{p-2}{p}}.
\end{aligned}
$$

The use of the suitable consequence of Young inequality $ab\le\varepsilon a^{1+s}+\varepsilon^{-\frac{1}{s}}b^{1+\frac{1}{s}}$, denoting with

$$
s=\frac{2}{p-2},\qquad a=\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{-2}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\right)^{\frac{p-2}{p}},
$$

$$
b=c(K,p)\left(\int_{Q(\frac{5}{2}\sigma)}|h|^{p-2}\left(|f^\alpha|^{\frac{p}{2}}+\left\|D''u\right\|^p\right)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right\|^{\frac{4-p}{2}}\,dx\right)^{\frac{2}{p}}
$$

and the hypothesis $u\in C^{m-1,\lambda}(\overline{\Omega},\mathbb{R}^N)$ allows us to have

$$
\begin{aligned}
|D| &\le \varepsilon\,|h|^{-2}\sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx+c(K,U,p,\varepsilon)\,|h|^{p-2+\lambda(2-\frac{p}{2})}\\
&\qquad\cdot\sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}\left(|f^\alpha|^{\frac{p}{2}}+\left\|D''u\right\|^p\right)\,dx,\quad\forall\varepsilon>0,\,\forall\,2<p<\min(4,q). \qquad (60)
\end{aligned}
$$

Thus we also need Theorem 2.3 to obtain

$$\varepsilon\,|h|^{-2}\sum_{|\alpha|<m}\int_{Q(\frac{5}{2}\sigma)}\left\|\tau_{i,-h}D^{\alpha}\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^{2}\,dx\le\varepsilon\int_{Q(3\sigma)}\left\|D''\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^{2}\,dx$$

$$\le 2\varepsilon\int_{Q(2\sigma)}\psi^{4m}\left\|\tau_{i,h}D''u\right\|^{2}\,dx+c(\sigma,\varepsilon)\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}D'u\right\|^{2}\,dx$$

$$\le 2\varepsilon\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}D''u\right\|^{2}\,dx+c(\sigma,\varepsilon)h^{2}\int_{Q(\frac{5}{2}\sigma)}\left\|D''u\right\|^{2}\,dx.$$

Therefore for every $\varepsilon>0$ and every $2<p<\min(4,q)$ we reach

$$\frac{\nu}{2}\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}\,D''u\right\|^{2}\,dx$$

$$\le c(\nu,K,\sigma,m,n)h^{2}\int_{Q(\frac{5}{2}\sigma)}\left(1+\left\|D''u\right\|^{2}\right)\,dx$$

$$+\,c(\nu,K,U,\vartheta,\lambda,m,n,\sigma)\;|\,h\,|^{2\vartheta+2\lambda(1-\vartheta)}\;\left\{\,|\,D''u\,|^{2}_{\vartheta,Q(\frac{5}{2}\sigma)}\,+\,1\,\right\}$$

$$+\,2\varepsilon\int_{Q(2\sigma)}\psi^{2m}\,\|\,\tau_{i,h}\,D''u\,\|^{2}\,dx\;+\;c(\sigma,\varepsilon)\,h^{2}\int_{Q(\frac{5}{2}\sigma)}\|\,D''u\,\|^{2}\,dx$$

$$+\,c(K,U,p,\varepsilon)\,|h|^{p-2+\lambda(2-\frac{p}{2})}\int_{Q(\frac{5}{2}\sigma)}\left(\left(\sum_{|\alpha|<m}|f^{\alpha}|\right)^{\frac{p}{2}}+\,\|\,D''u\,\|^{p}\right)dx.$$

Let us now set in the last inequality $\varepsilon=\frac{\nu}{8}$ and $p=2(1+\vartheta)\in\big(2,\min(4,q)\big)$. We have, for every $h:|h|<h_0\,(<1)$, that

$$\frac{\nu}{4}\int_{Q(2\sigma)}\psi^{2m}\,\|\,\tau_{i,h}\,D''u\,\|^{2}\,dx$$

$$\le c(\nu,K,\sigma,m,n)h^{2}\int_{Q(\frac{5}{2}\sigma)}\left(1+\left\|D''u\right\|^{2}\right)\,dx$$

$$+\,c(\nu,K,U,\vartheta,\lambda,m,n,\sigma)\;|\,h\,|^{\,2\vartheta+2\lambda(1-\vartheta)}\left\{\;|\;D''u\;|^{2}_{\vartheta,Q(\frac{5}{2}\sigma)}\;+\;1\right\}$$

$$+\,c(\nu,K,U,\vartheta)\,|h|^{2\vartheta+\lambda(1-\vartheta)}\int_{Q(\frac{5}{2}\sigma)}\left(\left(\sum_{|\alpha|<m}|f^{\alpha}|\right)^{1+\vartheta}+\|D''u\|^{2(1+\vartheta)}\right)dx$$

$$\le c(\nu,K,U,\vartheta,\lambda,m,n,\sigma)\;|\;h\;|^{2\vartheta+\lambda(1-\vartheta)}$$

$$\cdot\left\{1+|u|^{2}_{m,Q(\frac{5}{2}\sigma)}+|D''u|^{2}_{\vartheta,Q(\frac{5}{2}\sigma)}+|u|^{2(1+\vartheta)}_{m,2(1+\vartheta),Q(\frac{5}{2}\sigma)}+\left(\sum_{|\alpha|<m}\|f^{\alpha}\|_{0,\frac{q}{2},Q(\frac{5}{2}\sigma)}\right)^{1+\vartheta}\right\}.$$

From (53), for $|h|<h_0$, we gain

$$\sum_{i=1}^{n}\int_{Q(\sigma)}\left\|\tau_{i,h}\,D''u\right\|^{2}\,dx \tag{61}$$

$$\le c(\nu,K,U,\vartheta,\lambda,m,n,\sigma)\,|h|^{2\vartheta+\lambda(1-\vartheta)}$$

$$\left\{1+\left(\sum_{|\alpha|<m}\|f^{\alpha}\|_{0,\frac{q}{2},Q(3\sigma)}\right)^{1+\vartheta}+|u|^{2}_{m,Q(3\sigma)}+|D''u|^{2}_{\vartheta,Q(3\sigma)}\right\}.$$

The procedure if $h_0 \le |h| < 2\sigma$ is similar to the one used in the proof of Theorem 3.2.

Combining both results we obtain that (61) is true for $|h| < 2\sigma$.
Let us now choose $0 < \vartheta' < \vartheta + \frac{\lambda}{2}(1-\vartheta)$, it implies that $1 + 2\vartheta' - 2\vartheta - \lambda(1-\vartheta) < 1$ then, for every $h : 0 < |h| < 2\sigma$ is integrable in $[-2\sigma, 2\sigma]$ the second member of

$$\sum_{i=1}^{n} \frac{1}{|h|^{1+2\vartheta'}} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2\; dx \tag{62}$$
$$\le c(\nu, K, U, \vartheta, \lambda, m, n, \sigma)$$
$$\cdot \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(3\sigma)} \right)^{1+\vartheta} + |u|^2_{m,Q(3\sigma)} + |D''u|^2_{\vartheta,Q(3\sigma)} \right\} \frac{1}{|h|^{1+2\vartheta'-2\vartheta-\lambda(1-\vartheta)}}$$

and thus also the first one is integrable.

It is then proved that

$$\sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta'}} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2\; dx \tag{63}$$
$$\le c(\nu, K, U, \vartheta, \vartheta', \lambda, m, n, \sigma)$$
$$\cdot \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(3\sigma)} \right)^{1+\vartheta} + |u|^2_{m,Q(3\sigma)} + |D''u|^2_{\vartheta,Q(3\sigma)} \right\},$$
$$\forall\, 0 < \vartheta' < \vartheta + \frac{\lambda}{2}(1-\vartheta).$$

Because of $u \in H^m(\Omega, \mathbb{R}^N)$ from Theorem 2.6, we have

$$D''u \in H^{\vartheta'}(Q(\sigma), \mathcal{R}")$$

and

$$|D''u|^2_{\vartheta',Q(\sigma)}$$
$$\le c(n) \sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\vartheta'}} \int_{Q(\sigma)} \|\tau_{i,h}\, D''u\|^2\; dx$$
$$\le c(\nu, K, U, \vartheta, \vartheta', \lambda, m, n, \sigma) \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q}{2},Q(3\sigma)} \right)^{1+\vartheta} + |u|^2_{m,Q(3\sigma)} + |D''u|^2_{\vartheta,Q(3\sigma)} \right\},$$

we achieve our goal. □

Proof of Theorem 3.3. Let us fix $\vartheta_0 = \frac{\lambda}{4}$ and make a point of the geometric series

$$1 + (1-\vartheta_0) + (1-\vartheta_0)^2 + \cdots + (1-\vartheta_0)^r + \cdots\cdots .$$

For $s = 0, 1, \ldots$, let us set $\vartheta_s = \vartheta_0 \sum_{r=0}^{s} (1-\vartheta_0)^r$. We achieve, for every $s = 0, 1, \ldots,$ that

i) $\vartheta_s = 1 - (1-\vartheta_0)^{s+1}$;
ii) $0 < \vartheta_s < \vartheta_{s+1} < 1$;
iii) $\vartheta_{s+1} - \vartheta_s = \vartheta_0(1-\vartheta_0)^{s+1}$;
iv) $\vartheta_{s+1} < \vartheta_s + \frac{\lambda}{2}(1-\vartheta_s)$;
v) $q_s = \frac{2(1+\vartheta_s)n}{n-2\vartheta_s\lambda} < \frac{4n}{n-2\lambda}$.

It ensure that $f^\alpha \in L^{\frac{q_s}{2}}(\Omega)$, for every $s = 0,1,2,\dots$ and every α such that $|\alpha| < m$. Due to $\lim_{s\to+\infty} \vartheta_s = 1$, fixing arbitrarily $\vartheta \in (\vartheta_0, 1)$ exists a positive integer $i = i(\vartheta,\lambda)$ such that $\vartheta_{i-1} < \vartheta \le \vartheta_i < 1$.
Additionally, from Theorem 3.2 we deduce

$$u \in H^{m+\vartheta_0}(Q(4\rho), \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{Q(4\rho)}, \mathbb{R}^N), \quad \forall Q(4\rho) \subset\subset \Omega$$

and

$$|D''u|^2_{\vartheta_0,Q(\rho)} \le c(\nu,K,U,\lambda,\rho,m,n)\left(1 + \sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(4\rho)} + |u|^2_{m,Q(4\rho)}\right). \tag{64}$$

Exploiting Theorem 3.3 for $\vartheta = \vartheta_0\, q = q_0$, $\vartheta' = \vartheta_1$ and $\Omega = Q(4\rho)$, as well as iv) and v) for $s = 0$, we have

$$u \in H^{m+\vartheta_1}(Q(\rho), \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{Q(\rho)}, \mathbb{R}^N)$$

and

$$\begin{aligned} &|D''u|^2_{\vartheta_1,Q(4^{-1}\rho)} \\ &\le c(\nu,K,U,\lambda,\rho,m,n)\left(1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{q_0}{2},Q(\rho)}\right)^{1+\vartheta_0} + |u|^2_{m,Q(\rho)} + |D''u|^2_{\vartheta_0,Q(\rho)}\right) \\ &\le c(\nu,K,U,\lambda,\rho,m,n)\left(1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(4\rho)}\right)^{1+\vartheta_0} + |u|^2_{m,Q(4\rho)}\right). \end{aligned}$$

making use i times of Theorem 3.3 we establish, $\forall Q(4\rho) \subset\subset \Omega$, so that

$$u \in H^{m+\vartheta_i}(Q(4^{-i+1}\rho), \mathbb{R}^N) \tag{65}$$

and we reach the inequality

$$\begin{aligned} &|D''u|^2_{\vartheta_i,Q(4^{-i}\rho)} \\ &\le c(\nu,K,U,\vartheta,\lambda,\rho,m,n)\left(1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,\frac{2n}{n-2\lambda},Q(4\rho)}\right)^{1+\vartheta_{i-1}} + |u|^2_{m,Q(4\rho)}\right). \end{aligned} \tag{66}$$

Let us fix arbitrarily $x_0 \in \Omega$, $Q(\sigma) = Q(x^0,\sigma) \subset\subset Q(\sigma_0) = Q(x^0,\sigma_0) \subset\subset \Omega$ and assume $\rho = \frac{\sigma_0-\sigma}{8}$. The set of cubes

$$\mathcal{F} = \left\{Q(y^0, 4^{-i-1}\rho),\ y^0 \in Q(\sigma)\right\}$$

is an open cover of $\overline{Q(\sigma)}$, let us then extract the finite cover

$$Q(y^{(1)}, 4^{-i-1}\rho),\ Q(y^{(2)}, 4^{-i-1}\rho), \dots, Q(y^{(t)}, 4^{-i-1}\rho).$$

After that, set $\Omega_k = Q(y^{(k)}, 4^{-i-1}\rho) \cap Q(\sigma),\ k = 1, 2, \dots, t,$

$$\bigcup_{k=1}^{t} \Omega_k = Q(\sigma),\ Q(y^{(k)}, 4\rho) \subset\subset Q(\sigma_0) \subset\subset \Omega,\ \ \forall k = 1, 2, \dots, t\ ,$$

from (66) and Theorem 2.9 (if $\Omega = Q(\sigma)$, $\vartheta = \vartheta_i$ and $\sigma = \frac{3\rho}{4^{i+1}}$), we have

$$|D''u|^2_{\vartheta_i, Q(\sigma)}$$
$$\le c(\nu, K, U, \vartheta, \lambda, \sigma, \sigma_0, m, n) \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0, \frac{2n}{n-2\lambda}, Q(\sigma_0)} \right)^{1+\vartheta_{i-1}} + |u|^2_{m, Q(\sigma_0)} \right\}$$

we gain (15) and (16) bearing in mind that $\vartheta_{i-1} < \vartheta \le \vartheta_i$.

To prove (17) we remark that $u \in C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, then

$$u \in H^{m+\vartheta}(Q(\sigma), \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{Q(\sigma)}, \mathbb{R}^N),\ \ \forall\, 0 < \vartheta < 1,\ \forall\, Q(\sigma) \subset\subset \Omega.$$

In addition, Theorem 2.10 ensures that

$$u \in H^{m,p}(Q(\sigma), \mathbb{R}^N),\ \ \forall\, 1 \le p < \frac{2(1+\vartheta)n}{n - 2\vartheta\lambda},\ \forall\, 0 < \vartheta < 1,\ \forall Q(\sigma) \subset\subset \Omega. \tag{67}$$

and observing that

$$\lim_{\vartheta \to 1^-} \frac{2(1+\vartheta)n}{n - 2\vartheta\lambda} = \frac{4n}{n - 2\lambda} > 4\,,$$

we assure that $\vartheta^* \in (0, 1)$ exists and is such that $\frac{2(1+\vartheta^*)n}{n-2\vartheta^*\lambda} > 4$. Let us set p^* in $\left(4, \frac{2(1+\vartheta^*)n}{n-2\vartheta^*\lambda}\right)$, from (67), we have

$$u \in H^{m,p^*}(Q(\sigma), \mathbb{R}^N),\ \ \forall Q(\sigma) \subset\subset \Omega$$

from which, because of $p^* > 4$, it follows

$$u \in H^{m,4}(Q(\sigma), \mathbb{R}^N)\,. \tag{68}$$

We end the conclusion remarking that (68) is true for every $Q(\sigma) \subset\subset \Omega$. □

Proof of Theorem 3.4. Let us consider $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ the *cut-off* function above defined in (20), $Q(4\sigma) \subset\subset \Omega$ a generic cube, $i \le n$ a positive integer and h a real number such that $|h| < \frac{\sigma}{2}$. Carrying on as in the proof of Theorem 3.2, we obtain

$$\begin{aligned}&\frac{\nu}{2}\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}D''u\right\|^2dx \\ &\le c(\nu,K,\sigma,m,n)h^2\int_{Q(3\sigma)}\left(1+\|D''u\|^2\right)dx \\ &\qquad + c(\nu,K,\sigma,m,n)\int_{Q(2\sigma)}\|\tau_{i,h}D'u\|^2\,\|D''u\|^2\,dx \\ &\qquad + c(K)\sum_{|\alpha|<m}\int_{Q(3\sigma)}(|f^\alpha|+\|D''u\|^2)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}u\right)\right\|dx.\end{aligned} \tag{69}$$

Exploiting Theorem 3.4 we can achieve that $u\in H^{m,4}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, then we can estimate the last term as follows

$$\begin{aligned}&\sum_{|\alpha|<m}\int_{Q(3\sigma)}(|f^\alpha|+\|D''u\|^2)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|\,dx \\ &\le \sum_{|\alpha|<m}\left(\int_{Q(3\sigma)}|h|^{-2}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx\right)^{\frac12}\left(\int_{Q(3\sigma)}h^2(|f^\alpha|+\|D''u\|^2)^2\,dx\right)^{\frac12} \\ &\le \frac{\varepsilon}{2}\,|h|^{-2}\sum_{|\alpha|<m}\int_{Q(3\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx \\ &\qquad + c(\varepsilon)\,h^2\sum_{|\alpha|<m}\int_{Q(3\sigma)}(|f^\alpha|^2+\|D''u\|^4)\,dx.\end{aligned}$$

Furthermore, from Theorem 2.3 (for $p=2$, $Q(\frac72\sigma)$ instead of $Q(\sigma)$ and $t=\frac67$), for every $h\in\mathbb{R}$ con $|h|<h_0$ and every $\varepsilon>0$, we have

$$\begin{aligned}&\frac{\varepsilon}{2}\,|h|^{-2}\int_{Q(3\sigma)}\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx \\ &\le \frac{\varepsilon}{2}\int_{Q(\frac72\sigma)}\left\|D''\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|^2\,dx \\ &\le \varepsilon\int_{Q(2\sigma)}\psi^{2m}\,\|\,\tau_{i,h}\,D''u\,\|^2\,dx\,+c(\sigma,\varepsilon)\int_{Q(2\sigma)}\|\,\tau_{i,h}D'u\,\|^2\,dx \\ &\le \varepsilon\int_{Q(2\sigma)}\psi^{2m}\,\|\,\tau_{i,h}\,D''u\,\|^2\,dx\,+c(\sigma,\varepsilon)\,h^2\int_{Q(3\sigma)}\|D''u\|^2\,dx.\end{aligned}$$

the last inequality follows, as before, applying Theorem 2.3 (for $p=2$, $Q(3\sigma)$ instead of $Q(\sigma)$ and $t=\frac23$). Let us now choose $\varepsilon=\frac{\nu}{4\,c(K)}$, it ensure

$$\begin{aligned}&\sum_{|\alpha|<m}\int_{Q(3\sigma)}\left(|f^\alpha|+\|D''u\|^2\right)\left\|\tau_{i,-h}D^\alpha\left(\psi^{2m}\tau_{i,h}\,u\right)\right\|\,dx \\ &\le \frac{\nu}{4c(K)}\int_{Q(2\sigma)}\psi^{2m}\left\|\tau_{i,h}D''u\right\|^2dx \\ &\qquad + c(\nu,K,\sigma,m)h^2\left\{\left(\sum_{|\alpha|<m}\|f^\alpha\|_{0,Q(3\sigma)}\right)^2+|u|^2_{m,Q(3\sigma)}+|u|^4_{m,4,Q(3\sigma)}\right\}.\end{aligned}$$

Taking into consideration the last inequality and the properties of the function ψ, from (69) we deduce

$$\int_{Q(\sigma)} \|\tau_{i,h} D'' u\|^2 \, dx \tag{70}$$
$$\leq c(\nu, K, \sigma, m, n) h^2 \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)} \right)^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)} \right\}$$
$$+ c(\nu, K, \sigma, n) \int_{Q(2\sigma)} \|\tau_{i,h} D' u\|^2 \|D'' u\|^2 \, dx.$$

Let us now estimate the last term using the Hölder inequality

$$\int_{Q(2\sigma)} \|\tau_{i,h} D' u\|^2 \|D'' u\|^2 \, dx \leq \left(\int_{Q(2\sigma)} \|\tau_{i,h} D' u\|^4 \, dx \right)^{\frac{1}{2}} \left(\int_{Q(2\sigma)} \|D'' u\|^4 \, dx \right)^{\frac{1}{2}}.$$

Then, applying Theorem 2.3 (for $p = 4$, $Q(\frac{5}{2}\sigma)$ instead of $Q(\sigma)$ and $t = \frac{4}{5}$), for every $|h| < h_0$, it follows

$$\int_{Q(2\sigma)} \|\tau_{i,h} D' u\|^2 \|D'' u\|^2 \, dx \tag{71}$$
$$\leq h^2 \|D'' u\|^2_{0,4,Q(\frac{5}{2}\sigma)} \|D'' u\|^2_{0,4,Q(2\sigma)} \leq h^2 |u|^4_{m,4,Q(3\sigma)} .$$

From (70) and (71), for every i $(1 \leq i \leq n)$ and every $|h| < h_0$, we gain the following estimate

$$\int_{Q(\sigma)} \|\tau_{i,h} \, D'' u\|^2 \, dx$$
$$\leq c(\nu, K, \sigma, m, n) \, h^2 \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)} \right)^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)} \right\}.$$

If $h_0 \leq |h| < \frac{\sigma}{2}$, as in (38), we have that

$$\int_{Q(\sigma)} \|\tau_{i,h} D'' u\|^2 \, dx \leq 4 \int_{Q(3\sigma)} \|D'' u\|^2 dx$$
$$\leq 4 \frac{h^2}{h_0^2} \int_{Q(3\sigma)} \|D'' u\|^2 dx \leq c(\nu, K, U, \lambda, \sigma, m, n) h^2 |u|^2_{m,Q(3\sigma)}$$
$$\leq c(\nu, K, U, \lambda, \sigma, n) \, h^2 \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)} \right)^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)} \right\},$$
$$\forall i = 1, 2, \ldots, n \, .$$

It is then proved, for every $|h| < \frac{\sigma}{2}$ and every $i \in \{1, 2, \ldots, n\}$, that

$$\int_{Q(\sigma)} \|\tau_{i,h}\, D'' u\|^2 \, dx$$

$$\leq c(\nu, K, U, \lambda, \sigma, m, n)\, h^2 \left\{ 1 + \left(\sum_{|\alpha|<m} \|f^\alpha\|_{0,Q(3\sigma)} \right)^2 + |u|^2_{m,Q(3\sigma)} + |u|^4_{m,4,Q(3\sigma)} \right\},$$

applying Theorem 2.2, it follows (18) and (19). □

4. Partial Hölder continuity of higher order derivatives

As application of the previous differentiability properties for solutions of system (10) we have the following result of partial Hölder continuity of derivatives of order m.

Theorem 4.1. *Let* $u \in H^m(\Omega, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$, *a weak solution of the system* (10), *are true the hypotheses* (a), (b), (d), (e), (c) *for* $f^\alpha \in L^{\frac{2n}{n-2\lambda}}(\Omega)$, $|\alpha| < m$, *and* $a^\alpha(x, Du) \in C^1(\Omega \times \mathcal{R}, \mathbb{R}^N)$ *for* $|\alpha| = m$. *Then, there exists a closed set* $\Omega_0 \subset \Omega$, *such that*

$$H_{n-q}(\Omega_0) = 0 \quad \textit{for a number } q > 2,$$
$$u \in C^{m,\gamma}(\Omega \setminus \Omega_0, \mathbb{R}^N) \quad \textit{for a suitable } \gamma \in (0,1),$$

where $H_{n-q}(\Omega_0)$ *is the* $(n-q)$*-dimensional Hausdorff measure of* Ω_0.

Proof of Theorem 4.1. Let us fix a positive number s, $s \leq n$, and assume in the definition of weak solution (3) $\varphi = D_s\,\theta$, for $\theta \in C_0^\infty(\Omega_0, \mathbb{R}^N)$, $\Omega_0 \subset\subset \Omega$, we have

$$\int_{\Omega_0} \sum_{|\alpha| \leq m} (D_s\, a^\alpha(x, Du) | D^\alpha \theta) \, dx = 0, \quad \forall\, \theta \in C_0^\infty(\Omega_0, \mathbb{R}^N)\,. \tag{72}$$

we can write the derivatives:

$$D_s\, a^\alpha(x, Du) = \frac{\partial a^\alpha}{\partial x_s} + \sum_{|\beta|<m} \sum_{k=1}^N (D_s D^\beta\, u_k) \frac{\partial a^\alpha}{\partial p_k^\beta} + \sum_{|\beta|=m} \sum_{k=1}^N (D_s\, D^\beta\, u_k) \frac{\partial a^\alpha}{\partial p_k^\beta}.$$

Applying the previous theorem we have that $u \in H^{m+1}_{\text{loc}}(\Omega, \mathbb{R}^N)$, thus we are able to write (72) as follows

$$\int_{\Omega_0} \sum_{|\alpha|=|\beta|=m} \left(A_{\alpha\beta}(x, Du) D_s\, D^\beta\, u | D^\alpha\, \theta \right) \, dx \tag{73}$$

$$= \int_{\Omega_0} \left\{ \sum_{|\alpha|=m} (G^{\alpha,s}(x, Du) | D^\alpha \theta) - \sum_{|\alpha|<m} (a^\alpha(x, Du) | D^\alpha D_s \theta) \right\} dx, \ \forall \theta \in C_0^\infty(\Omega_0, \mathbb{R}^N)$$

where $\forall \alpha, \beta : |\alpha| = |\beta| = m$,

$$A_{\alpha\beta} = \{A^{hk}_{\alpha\beta}\}\,,\ A^{hk}_{\alpha\beta} = \frac{\partial a^{\alpha}_{h}(x, Du)}{\partial p^{\beta}_{k}}, \quad h, k = 1, \dots, N \tag{74}$$

and, $\forall \alpha : |\alpha| = m$,

$$G^{\alpha s}(x, Du) = -\frac{\partial a^{\alpha}(x, Du)}{\partial x_s} - \sum_{|\beta|<m} \sum_{k=1}^{N} (D_s\, D^{\beta} u_k) \frac{\partial a^{\alpha}(x, Du)}{\partial p^{\beta}_{k}}. \tag{75}$$

Let us also assume in (73) $\theta = D_s\varphi$ with $\varphi \in C^{\infty}_{0}(\Omega_0, \mathbb{R}^N)$, summing from 1 to n respect to s, we gain that the function $u \in H^{m+1}(\Omega_0, \mathbb{R}^N) \cap C^{m-1,\lambda}(\overline{\Omega_0}, \mathbb{R}^N)$ is solution of the following quasilinear system of order $2\,(m+1)$

$$\int_{\Omega_0} \sum_{|\alpha|=|\beta|=m} \sum_{r,s=1}^{n} \left(B_{\alpha r,\beta s}(x, Du) D_s\, D^{\beta}\, u | D_r\, D^{\alpha}\, \varphi\right)\, dx \tag{76}$$

$$= \int_{\Omega_0} \sum_{|\alpha|=m} \sum_{s=1}^{n} (G^{\alpha s}(x, Du) + \delta_{\alpha s} \sum_{|\beta|<m} a^{\beta}(x, Du) | D^{\beta}\, D_s \varphi)\, dx\,, \quad \forall \varphi \in C^{\infty}_{0}(\Omega_0, \mathbb{R}^N)$$

where

$$B_{\alpha r \beta s} = \delta_{rs}\, A_{\alpha\beta}\,. \tag{77}$$

We point out that system (10) is strictly monotone but, because of $a^{\alpha} \in C^1(\Omega \times \mathcal{R}, \mathbb{R}^{\mathcal{N}})$, for $|\alpha| = m$, this condition is equivalent to that of strict ellipticity. Let us prove that the same is also true of system (76) with the same ellipticity constant ν. Indeed, thanks to (77) and (74), for every system $\{\eta^{\alpha s}\}_{\alpha, s=1,2,\dots,n}$ of vectors of $\mathbb{R}^N$, we have

$$\sum_{|\alpha|=|\beta|=m} \sum_{r,s=1}^{n} \left(B_{\alpha r,\beta s}\, \eta^{\beta s} |\, \eta^{\alpha r}\right) = \sum_{s=1}^{n} \sum_{|\alpha|=|\beta|=m} \left(A_{\alpha\beta}\, \eta^{\beta s} |\, \eta^{\alpha s}\right)$$

$$= \sum_{s=1}^{n} \sum_{|\alpha|=|\beta|=m} \sum_{h,k=1}^{N} A^{hk}_{\alpha\beta}\, \eta^{\beta s}_{h}\, \eta^{\alpha s}_{k} = \sum_{s=1}^{n} \sum_{|\alpha|=|\beta|=m} \sum_{h,k=1}^{N} \frac{\partial a^{\alpha}_{h}}{\partial p^{\beta}_{k}} \eta^{\beta s}_{h}\, \eta^{\alpha s}_{k} \geq \nu \sum_{s=1}^{n} \sum_{|\alpha|=m} \|\eta^{\alpha s}\|^2_N\,.$$

Moreover from the hypotheses (c) and (d) it follows

$$\left\| G^{\alpha s} + \delta_{\alpha s} \sum_{|\beta|<m} a^{\beta}(x, Du) \right\| \leq c(K) \left\{ 1 + \sum_{|\alpha|<m} |f^{\alpha}| + \|D'' u\|^2 \right\},$$

where $K = \sup_{\overline{\Omega}} \|D' u\|$.

Therefore, because we are exactly in the same situation studied in n. 3, Chapt. IV of [4], we get the conclusion. □

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