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A Characterization of the Solution Set of Pseudoconvex Extremum Problems

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Pseudomonotone$_*$ single–valued functions were introduced in [9] and it was proved that the gradient of a differentiable pseudoconvex function is pseudomonotone$_*$. In the same paper this concept was extended in a natural way to multivalued maps but, to date, there is no result that relates multivalued pseudomonotone$_*$ maps to the subdifferential of locally Lipschitz pseudoconvex functions. In this paper, we give a nonsmooth Lipschitz pseudoconvex function whose subdifferential is not pseudomonotone$_*$ in the sense of [9]. Besides such a characterization was achieved in [10] using a weaker definition of pseudomonotonicity$_*$. Exploiting this weaker concept, we provide a characterization of the solution set of pseudoconvex programs.

Keywords: Paramonotone map, pseudomonotone$_*$ map, pseudoconvex programs

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1. Introduction: paramonotone maps

Let $\mathbb{X}$ be a real Banach space, $\mathbb{X}^*$ be its topological dual space and $\langle \cdot, \cdot \rangle$ denote the canonical pairing between $\mathbb{X}$ and $\mathbb{X}^*$. A multivalued map $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is said to be *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for all $x, y \in D(T)$ and $x^* \in T(x)$, $y^* \in T(y)$, where $D(T)$ is the domain of T, i.e.

$$D(T) = \{x \in \mathbb{X} \ : \ T(x) \neq \emptyset\}.$$

In order to ensure convergence of several methods for variational inequalities, maps with a slight stronger property were considered for the first time in [3].

Definition 1.1. The multivalued map $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is *paramonotone* if

i) it is monotone,
ii) for every $x, y \in D(T)$ and $x^* \in T(x)$, $y^* \in T(y)$ such that $\langle x^* - y^*, x - y \rangle = 0$ then $x^* \in T(y)$ and $y^* \in T(x)$.

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The term "paramonotone" was introduced in [7] but the prefix "para" is somehow misleading: the paramonotone maps are more than monotone. In [9] this property was named *monotonicity*$^+$ but we prefer to follow [7] since paramonotonicity is commonly used in literature. The main motivation for the introduction of the paramonote maps lies in the analysis of interior point algorithm for variational inequalities. Indeed such maps possess a cutting plane property that ensures the convergence of several methods as Korpolevich–type method with Bregman distances [7], a perturbation method for saddle point problems [12], an outer approximation method in a reflexive Banach space [5], and also proximal point methods with either Bregman distance [4] or φ–divergences [1]. Even if the class of the paramonotone maps is more restricted than monotone, it's worth noting that is larger than the class of strictly monotone maps, and it includes the subdifferentials of proper convex functions (see [11]). This last property allows us to present a new simple proof of a result given in [6] concerning the solution set of a convex optimization problem with nonsmooth objective function.

We recall the following classical notation. Let $f : \mathbb{X} \longrightarrow (-\infty, +\infty]$ be a proper lower semicontinuous convex function, the *subdifferential* of f at $x \in \operatorname{dom} f$ is the (possibly empty) set

$$\partial f(x) = \{x^* \in \mathbb{X}^* \ : \ f(y) \geq f(x) + \langle x^*, y - x\rangle, \ \forall y \in \mathbb{X}\};$$

given a nonempty closed and convex set C and $x \in C$, the *normal cone* to C at x is

$$N(x, C) = \{x^* \in \mathbb{X}^* \ : \ \langle x^*, y - x\rangle \leq 0, \ \forall y \in C\}.$$

Moreover if we denote by $\delta(\cdot, C)$ the indicator function of the closed and convex set C, i.e. the function such that $\delta(x, C) = 0$ if $x \in C$ and $\delta(x, C) = +\infty$ otherwise, then $\partial\delta(x, C) = N(x, C)$ for all $x \in C$: therefore the multivalued map $N(\cdot, C)$ is paramonotone.

Now consider the following minimization problem

$$\min\{f(x) \ : \ x \in C\} \tag{1}$$

with $C \subseteq \mathbb{X}$ and $f : \mathbb{X} \longrightarrow (-\infty, +\infty]$ a given proper function and denote by S its solution set. The well–known minimum principle affirms that if C is convex and f is convex and continuous at $\bar{x} \in C$, then $\bar{x} \in S$ if and only if

$$\partial f(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset.$$

The next result shows that this intersection is not only different from the empty set but it is also constant on the solution set.

Theorem 1.2 ([6]). *Let C be a closed and convex set and f be a continuous and convex function. Let $\bar{x} \in S$ be a fixed solution and*

$$\overline{S} = \{x \in C \ : \ \partial f(x) \cap -N(x, C) = \partial f(\bar{x}) \cap -N(\bar{x}, C)\};$$

then $S = \overline{S}$.

Proof. Put $T(z) = \partial f(z) \cap -N(z, C)$; since $\operatorname{dom} T = S$ the inclusion $\overline{S} \subseteq S$ is trivial. For the converse we observe that T is a paramonotone map. Indeed the monotonicity of T trivially derives from the monotonicity of ∂f while the paramonotonicity follows from the fact that $-N(\cdot, C)$ also satisfies property ii). Given $x \in S$, and fixed $x^* \in T(x)$ and $\bar{x}^* \in T(\bar{x})$ we have

$$0 = f(\bar{x}) - f(x) \geq \langle x^*, \bar{x} - x \rangle \geq 0$$

and

$$0 = f(x) - f(\bar{x}) \geq \langle \bar{x}^*, x - \bar{x} \rangle \geq 0.$$

Adding the two inequalities we have $\langle x^* - \bar{x}^*, x - \bar{x} \rangle = 0$; hence $\bar{x}^* \in T(x)$ and $x^* \in T(\bar{x})$ which implies $x \in \overline{S}$. $\square$

A generalization of paramonotone maps was firstly studied in [9]. The authors introduced the concept of pseudomonotonicity$_*$ both for single–valued and for multivalued maps in order to obtain weaker sufficient conditions ensuring the convergence of cutting plane methods. Moreover they proved that a differentiable function is pseudoconvex if and only if its gradient is pseudomonotone$_*$. Nevertheless, unlike in the differentiable case, there was no result that relates multivalued pseudomonotone$_*$ maps to subdifferentials of nonsmooth pseudoconvex functions. In the next section we define a Lipschitz function which is pseudoconvex with respect to the Clarke subdifferential, but its subdifferential does not fulfill the definition of pseudomonotonicity$_*$ introduced in [9].

Subsequently, Hadjisavvas and Schaible introduced in [10] a weaker definition of multivalued pseudomonotone$_*$ maps. In light of our counterexample this new definition is more appropriate. Indeed they not only showed that these maps possess the cutting plane property but they proved the characterization of a locally Lipschitz pseudoconvex function by means of the pseudomonotonicity$_*$ of its Clarke subdifferential. This equivalence allows us to formulate in Section 3 a simple characterization of the solution set of a pseudoconvex nonsmooth minimization problem over a convex set. This result is a generalization of Theorem 1.2.

2. The concept of s–pseudomonotonicity$_*$

As already mentioned in the introduction, a generalization of the concept of paramonotonicity for single–valued maps was firstly introduced in [9].

Definition 2.1. The single-valued map $F : \mathbb{X} \longrightarrow \mathbb{X}^*$ is *pseudomonotone$_*$* if

i) it is pseudomonotone, i.e. for all $x, y \in \mathbb{X}$ such that $\langle F(x), y - x \rangle \geq 0$ then $\langle F(y), x - y \rangle \leq 0$,

ii) for every $x, y \in \mathbb{X}$ such that $\langle F(x), y - x \rangle = 0$ and $\langle F(y), x - y \rangle = 0$ then there exists $k > 0$ such that $F(x) = kF(y)$.

It is clear that every paramonotone map is also pseudomonotone$_*$. The importance of the concept of pseudomonotonicity$_*$ is double. First of all it is, in some way, a minimal condition ensuring that the cutting plane property for variational inequalities holds. Moreover the gradient of a differentiable pseudoconvex function is pseudomonotone$_*$. In the same paper, the authors extended in a natural way the definition of pseudomonotonicity$_*$ to multivalued maps. Since in the next section we will employ a weaker and more appropriate concept of pseudomonotonicity$_*$ due to Hadjisavvas, and Schaible [10] we call this s–pseudomonotonicity (where the prefix "s" stands for strong).

Definition 2.2. A multivalued map $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is *s–pseudomonotone$_*$* if

i) it is pseudomonotone, i.e. for all $x, y \in D(T)$ and $x^* \in T(x)$ such that $\langle x^*, y - x\rangle \geq 0$ then $\langle y^*, x - y\rangle \leq 0$ for all $y^* \in T(y)$,

ii) for every $x, y \in D(T)$ and $x^* \in T(x)$, $y^* \in T(y)$ such that $\langle x^*, y - x\rangle = 0$ and $\langle y^*, x - y\rangle = 0$ then there exists $k > 0$ such that $ky^* \in T(x)$.

Hadjisavvas and Schaible [10] observed that, unlike in the differentiable case, there is no known result that relates multivalued s–pseudomonotone$_*$ maps to suitable subdifferentials of nonsmooth pseudoconvex functions. In spite of this attractiveness, they proposed a more appropriate definition (that we introduce in the next section). Nevertheless they did not present an example in support of their conjecture. Aim of this section is to confirm their presupposition. We need some definitions. Given a locally Lipschitz function $f : \mathbb{X} \longrightarrow (-\infty + \infty]$ the *Clarke directional derivative* of f at $x \in \mathrm{dom}(f)$ along the direction $v \in \mathbb{X}$ is

$$f^\circ(x, v) = \limsup_{(x',t)\to(x,0^+)} \frac{f(x' + tv) - f(x')}{t}$$

and the *Clarke subdifferential* at x is

$$\partial^\circ f(x) = \{x^* \in \mathbb{X}^* \ : \ f^\circ(x, v) \geq \langle x^*, v\rangle, \ \forall v \in \mathbb{X}\}.$$

A locally Lipschitz function f is said *pseudoconvex* with respect to the Clarke subdifferential ∂° if for every $x \in \mathrm{dom}(f)$, $y \in \mathbb{X}$ and $x^* \in \partial^\circ f(x)$ such that $\langle x^*, y - x\rangle \geq 0$ we have $f(y) \geq f(x)$. In [13] the authors established that a locally Lipschitz function is pseudoconvex if and only if its Clarke subdifferential is pseudomonotone. The following example shows that in general the subdifferential fails to be s–pseudomonotone$_*$.

Example 2.3. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} |x_1| + |x_2| - 1, & \text{if } |x_1| \geq 1 \\ \left(\dfrac{|x_1| - 1 + \sqrt{(1 - |x_1|)^2 + 4|x_1 x_2|}}{2x_1}\right)^2, & \text{if } 0 < |x_1| < 1 \text{ and } |x_2| < 1 \\ x_2^2, & \text{if } x_1 = 0 \text{ and } |x_2| < 1 \\ |x_2|, & \text{if } |x_1| < 1 \text{ and } |x_2| \geq 1 \end{cases}$$

Clearly f is a nonnegative continuous function and $f(x_1, x_2) = 0$ if and only if (x_1, x_2) belongs to the segment $\sigma = [-1, 1] \times \{0\}$. Therefore σ is the set of all minimizers of f. Since f is symmetric with respect to the two axes, we limit ourselves to study the main properties of f on the first orthant $\mathbb{R}^2_+$ only.

Lipschitzianity of f. The first orthant can be divided in three disjointed boxes with the following interior part: $S = (0,1) \times (0,1)$, $R_1 = (0,1) \times (1,+\infty)$ and $R_2 = (1,+\infty) \times (0,+\infty)$. In each open box the function is differentiable with

$$\nabla f(x_1,x_2) = \begin{cases} \left(\dfrac{N(x_1,x_2)(N(x_1,x_2)-2x_1x_2)}{2x_1^3\sqrt{(1-x_1)^2+4x_1x_2}}, \dfrac{N(x_1,x_2)}{x_1\sqrt{(1-x_1)^2+4x_1x_2}} \right), & \text{if } (x_1,x_2) \in S \\ (0,1), & \text{if } (x_1,x_2) \in R_1 \\ (1,1), & \text{if } (x_1,x_2) \in R_2 \end{cases}$$

where $N(x_1,x_2) = x_1 - 1 + \sqrt{(1-x_1)^2 + 4x_1x_2}$. Clearly f is Lipschitzian on the closure of R_1 and R_2 respectively. On S, since $\frac{\partial f}{\partial x_2}(x_1,\cdot)$ is increasing, we have

$$0 \leq \frac{\partial f}{\partial x_2}(x_1,x_2) \leq \frac{x_1 - 1 + \sqrt{(1-x_1)^2+4x_1}}{x_1\sqrt{(1-x_1)^2+4x_1}} = \frac{2}{1+x_1} \leq 2$$

and therefore

$$0 \leq \frac{\partial f}{\partial x_1}(x_1,x_2) \leq \frac{N(x_1,x_2) - 2x_1x_2}{x_1^2}.$$

Moreover, since the function $x_2 \mapsto N(x_1,x_2) - 2x_1x_2$ assumes its maximum at $x_2 = \frac{2-x_1}{4}$ we deduce

$$\frac{\partial f}{\partial x_1}(x_1,x_2) \leq \frac{1}{2}.$$

Hence the gradient of f is bounded on S and then f is Lipschitzian on the closure of S too. In conclusion f is Lipschitzian over all the space.

Quasiconvexity of f. Fixed a level $\alpha > 0$, it is easy to show that

$$\{(x_1,x_2) \in \mathbb{R}^2_+ \ : \ f(x_1,x_2) \leq \alpha\} = \{(x_1,x_2) \in \mathbb{R}^2_+ \ : \ x_2 \leq r_\alpha(x_1)\}$$

where

$$r_\alpha(x_1) = \begin{cases} (\alpha - \sqrt{\alpha})x_1 + \sqrt{\alpha}, & \text{if } x_1 \in [0,1] \\ -x_1 + 1 + \alpha, & \text{if } x_1 \in (1,+\infty) \end{cases}$$

when $\alpha \leq 1$ and

$$r_\alpha(x_1) = \begin{cases} \alpha, & \text{if } x_1 \in [0,1] \\ -x_1 + 1 + \alpha, & \text{if } x_1 \in (1,+\infty) \end{cases}$$

when $\alpha \geq 1$. Hence the contour lines are convex polygons.

Pseudoconvexity of f. Now we show that all the points of $\mathbb{R}^2_+$ except σ are not stationary points. This is trivial for all the points where f is differentiable. Now we evaluate the Clarke subdifferential in the other points. We recall that if Ω_f denotes the set of all the points where f is differentiable, Rademacher's theorem affirms that

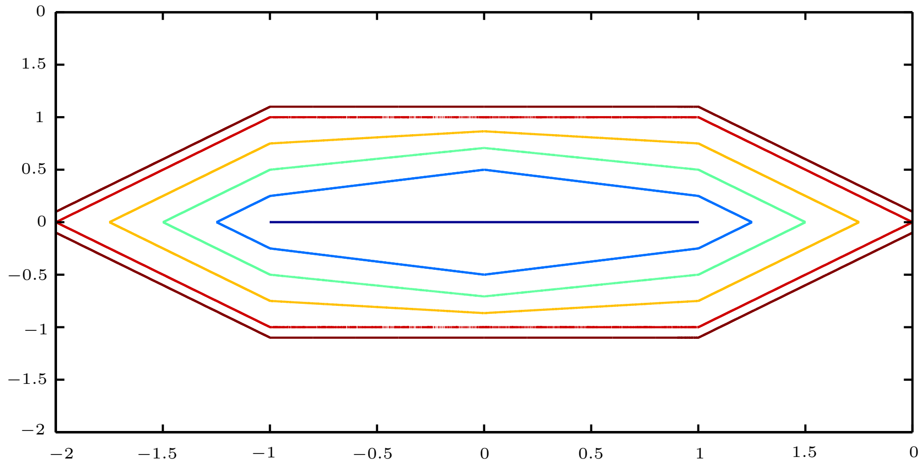

Figure 2.1: Contour lines of f.

Ω_f is of full Lebesgue measure and the Clarke subdifferential can be equivalently expressed as

$$\partial^\circ f(x) = \operatorname{conv}\left\{x^* \in \mathbb{R}^2 \ : \ x^* = \lim_{n\to+\infty} \nabla f(x_n),\ x_n \in \Omega_f \text{ and } x_n \to x\right\}$$

where the notation "conv" indicates the convex hull. Using this formula we are able to compute the subdifferential of f:

$$\partial^\circ f(x_1,x_2) = \begin{cases} [2x_2^3 - 2x_2^2, 2x_2^2 - 2x_2^3] \times \{2x_2\}, & \text{if } x_1 = 0 \text{ and } x_2 \in (0,1) \\ \{0\} \times [1, 2(1+x_1)^{-1}], & \text{if } x_1 \in [0,1) \text{ and } x_2 = 1 \\ [0,1] \times \{1\}, & \text{if } x_1 = 1 \text{ and } x_2 \in [1,+\infty) \\ [\sqrt{x_2} - x_2, 1] \times \{1\}, & \text{if } x_1 = 1 \text{ and } x_2 \in (0,1) \end{cases}$$

Observe that there are no stationary points out of σ. This property and the quasiconvexity of f ensure that f is pseudoconvex as proved in [2, Theorem 4.1].

The map $\partial^\circ f$ is not s–pseudomonotone$_*$. First of all we show that f is differentiable at the origin with $\nabla f(0,0) = (0,0)$; indeed

$$\begin{aligned} 0 &\le \liminf_{(x_1,x_2)\to(0,0)} \frac{f(x_1,x_2)}{\sqrt{x_1^2+x_2^2}} \\ &\le \limsup_{(x_1,x_2)\to(0,0)} \frac{f(x_1,x_2)}{\sqrt{x_1^2+x_2^2}} \\ &\le \lim_{(x_1,x_2)\to(0,0)} \frac{\left(|x_1| - 1 + \sqrt{(1-|x_1|)^2 + 4|x_1x_2|}\right)^2}{4x_1^2\sqrt{x_1^2+x_2^2}} = 0. \end{aligned}$$

Moreover f is not differentiable at $(1,0)$ and $(0,1) \in \partial^\circ f(1,0)$. Indeed it is sufficient to consider the sequence of points $\{(x_n, \frac{1-x_n}{4})\}$ with $x_n < 1$ and converging to 1; then

$$\lim_{n\to+\infty} \nabla f\left(x_n, \frac{1-x_n}{4}\right) = (1,0).$$

Therefore, choosing the points $(0,0)$ and $(1,0)$ and the elements $(0,0) = \nabla f(0,0)$ and $(0,1) \in \partial^\circ f(1,0)$ we have

$$\langle (0,0), (1,0) - (0,0) \rangle = 0 = \langle (0,1), (0,0) - (1,0) \rangle$$

but there doesn't exist $k > 0$ such that $(0,1) = k\nabla f(0,0)$.

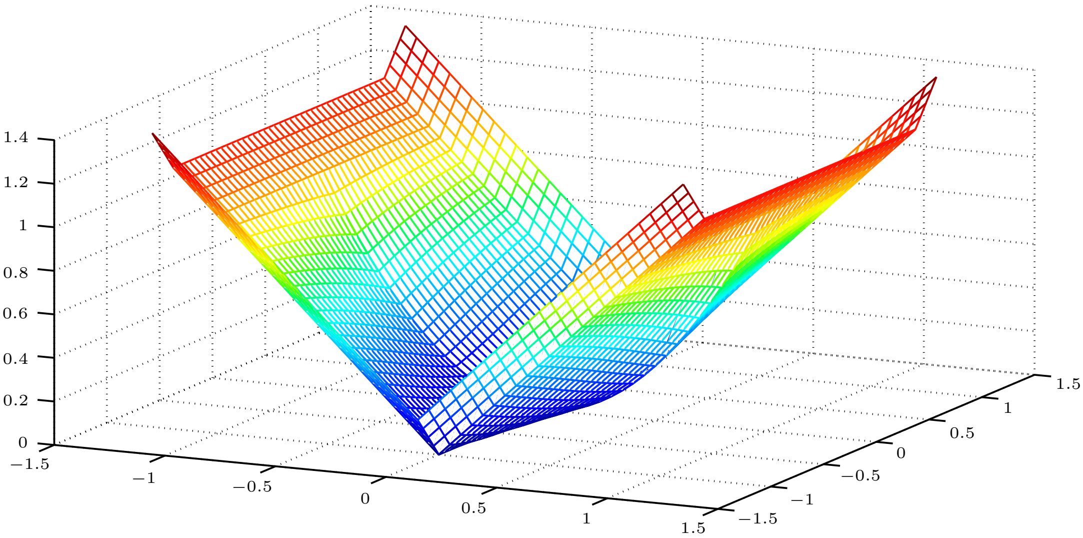

Figure 2.2: Graph of f.

This example clarifies that the concept of s–pseudomonotonicity$_*$ is too strong in order to characterize the Clarke subdifferential of a locally Lipschitz and pseudoconvex function. This fact was presupposed in [10] but, to the best of our knowledge, there was not any proof of this conjecture.

3. The pseudoconvex nonsmooth minimization problem

In [10] the authors proposed a slightly modified concept of pseudomonotonicity$_*$ for multivalued maps which is based on a particular equivalence relation between multivalued maps. In order to introduce the definition we recall some results. Given a multivalued map $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ we denote by $Z(T)$ the set of zeros of T, i.e.

$$Z(T) = \{x \in \mathbb{X} \ : \ 0 \in T(x)\}.$$

Two maps T_1, T_2 are called *equivalent* and write $T_1 \sim T_2$ if $D(T_1) = D(T_2)$, $Z(T_1) = Z(T_2)$, and for every $x \in \mathbb{X} \setminus Z(T_1)$,

$$\bigcup_{r>0} rT_1(x) = \bigcup_{s>0} sT_2(x).$$

This is an equivalence relation and for each map T we denote by

$$\hat{T}(x) = \bigcup_{S \sim T} S(x)$$

the equivalent maximum element with respect to the order defined by graph inclusion. If we consider the family of pseudomonotone maps only, it is easy to show that $\hat{T}$ is pseudomonotone and it could be expressed as

$$\hat{T}(x) = \begin{cases} N(x, L_{T,x}), & \text{if } x \in Z(T) \\ \bigcup_{r>0} rT(x), & \text{if } x \in D(T) \setminus Z(T) \\ \emptyset, & \text{if } x \notin D(T) \end{cases}$$

where

$$L_{T,x} = \{y \in \mathbb{X} \ : \ \exists y^* \in T(y) \text{ s.t. } \langle y^*, x - y\rangle = 0\}$$

and $N(x, L_{T,x})$ is the normal cone to $L_{T,x}$ at x.

Definition 3.1. A multivalued map $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is *pseudomonotone*$_*$ if

i) it is pseudomonotone,
ii) for every $x, y \in D(T)$ and $x^* \in T(x)$, $y^* \in T(y)$ such that $\langle x^*, y - x\rangle = 0$ and $\langle y^*, x - y\rangle = 0$ then $x^* \in \hat{T}(y)$ and $y^* \in \hat{T}(x)$.

Definitions 2.1 and 3.1 are equivalent for single–valued maps (see [10]). Moreover condition ii) in Definition 3.1 is less restrictive than condition ii) in Definition 2.2. Indeed, fixed $y \in L_{T,x}$, from the pseudomonotonicity of T we deduce

$$\langle x^*, y - x\rangle \le 0, \quad \forall x^* \in T(x)$$

and thus $T(x) \subseteq N(x, L_{T,x})$. Hence every s–pseudomonotone$_*$ map is pseudomonotone$_*$ too but the converse does not hold as follows from Example 2.3 and from the following result [10] which characterizes nonsmooth pseudoconvex functions with the pseudomonotonicity$_*$ of a suitable subdifferential.

Theorem 3.2. *Let $f : \mathbb{X} \longrightarrow (-\infty, +\infty]$ be a locally Lipschitz function; then f is pseudoconvex with respect to ∂° if and only if $\partial^\circ f$ is pseudomonotone$_*$.*

Going back to Example 2.3, we have shown that

$$(0, 1) \notin \bigcup_{r>0} r\nabla f(0, 0) = \{(0, 0)\}.$$

Consequently $\partial^\circ f$ is not s-pseudomonotone$_*$. Nevertheless Theorem 3.2 guaranties that $\partial^\circ f$ is pseudomonotone$_*$. In particular, since it is immediate to verify that $\sigma = L_{\partial^\circ f,(0,0)}$, then

$$(0, 1) \in \widehat{\partial^\circ f}(0, 0) = \{0\} \times \mathbb{R}.$$

In order to prove the main result of this paper we need two results. The first one was proved in [10].

Lemma 3.3. *If $T : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is pseudomonotone$_*$ and $S : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a pseudomonotone map equivalent to T, then S is pseudomonotone$_*$.*

The second result is a minimum principle for the pseudoconvex minimization problem (1). For sake of completeness we give a short proof.

Lemma 3.4. *Consider the minimization problem* (1). *Suppose f is a locally Lipschitz pseudoconvex function with respect to ∂° and $\bar{x} \in C$. Then $\bar{x} \in S$ if and only if $\partial^\circ f(\bar{x}) \cap -N(\bar{x}, C)$ is a nonempty set.*

Proof. The necessary optimality condition is well known [8]. For the reverse implication, fix $\bar{x}^* \in \partial^\circ f(\bar{x}) \cap -N(\bar{x}, C)$ and $y \in C$. Since $-\bar{x}^*$ is normal to C at $\bar{x}$, then $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0$. Finally the ∂°–pseudoconvexity of f implies that $f(y) \geq f(\bar{x})$ which concludes the proof. □

Now we are able to provide a characterization of the solution set of the pseudoconvex minimization problem (1).

Theorem 3.5. *Consider the minimization problem* (1). *Suppose f is a locally Lipschitz pseudoconvex function with respect to ∂° and $\bar{x} \in S$. Define*

$$\overline{S} = \{x \in C \ : \ \widehat{\partial^\circ f}(x) \cap -N(x, C) = \widehat{\partial^\circ f}(\bar{x}) \cap -N(\bar{x}, C)\}.$$

Then $S = \overline{S}$.

Proof. If $x, \bar{x} \in S$, the necessary optimality condition implies that $\partial^\circ f(x) \cap -N(x, C)$ and $\partial^\circ f(\bar{x}) \cap -N(\bar{x}, C)$ are nonempty sets. Since $\widehat{\partial^\circ f}$ is the maximal element with respect to graph inclusion we have $\widehat{\partial^\circ f}(x) \cap -N(x, C) \neq \emptyset$ and $\widehat{\partial^\circ f}(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset$ too. Fix two elements $x^* \in \widehat{\partial^\circ f}(x) \cap -N(x, C)$ and $\bar{x}^* \in \widehat{\partial^\circ f}(\bar{x}) \cap -N(\bar{x}, C)$. Since $-x^*$ and $-\bar{x}^*$ belong to the normal cones to C in x and $\bar{x}$ respectively, we can write

$$\langle x^*, \bar{x} - x \rangle \geq 0 \quad \text{and} \quad \langle \bar{x}^*, x - \bar{x} \rangle \geq 0. \tag{2}$$

From the pseudomonotonicity of $\widehat{\partial^\circ f}$ we deduce

$$\langle \bar{x}^*, x - \bar{x} \rangle \leq 0 \quad \text{and} \quad \langle x^*, \bar{x} - x \rangle \leq 0. \tag{3}$$

and, comparing equations (2) and (3) we have

$$\langle x^*, \bar{x} - x \rangle = 0 = \langle \bar{x}^*, x - \bar{x} \rangle. \tag{4}$$

Besides, from Theorem 3.2 the subdifferential map $\partial^\circ f$ is pseudomonotone$_*$, and from Lemma 3.3 we obtain that $\widehat{\partial^\circ f}$ is pseudomonotone$_*$ too. Hence $\bar{x}^* \in \widehat{\partial^\circ f}(x)$ and $x^* \in \widehat{\partial^\circ f}(\bar{x})$. Furthermore for any $y \in C$

$$\langle \bar{x}^*, y - x \rangle = \langle \bar{x}^*, y - \bar{x} \rangle + \langle \bar{x}^*, \bar{x} - x \rangle = \langle \bar{x}^*, y - \bar{x} \rangle \geq 0.$$

where the second equality descends from (4), while the inequality is due to the fact that $\bar{x}^* \in -N(\bar{x}, C)$. Thus $\bar{x}^* \in -N(x, C)$. Analogously $x^* \in -N(\bar{x}, C)$ which concludes the first inclusion.

For the converse let x be in $\overline{S}$. The necessary optimality condition for $\bar{x}$ implies that $\widehat{\partial^\circ f}(\bar{x}) \cap -N(\bar{x}, C) \neq \emptyset$. Then $\widehat{\partial^\circ f}(x) \cap -N(x, C) \neq \emptyset$ too. Fix $\hat{x}^*$ in $\widehat{\partial^\circ f}(x) \cap -N(x, C)$: only two different cases are conceivable.

Suppose $x \notin Z(\partial^\circ f)$. Then $\hat{x}^* \in \widehat{\partial^\circ f}(x)$ means that there exist $k > 0$ and $x^* \in \partial^\circ f(x)$ such that $\hat{x}^* = kx^*$. Since $-N(x, C)$ is a cone $x^* = k^{-1}\hat{x}^* \in -N(x, C)$. Then $\partial^\circ f(x) \cap -N(x, C) \neq \emptyset$ and the sufficiency of the first order optimality condition in Lemma 3.4 implies that $x \in S$.

Suppose $x \in Z(\partial^\circ f)$. Then $0 \in \partial^\circ f(x)$. The sufficiency of the first order optimality condition concludes the proof. □

We conclude describing the particular case when the function f is differentiable.

Theorem 3.6. *Consider the minimization problem* (1)*. Suppose f is a differentiable pseudoconvex function and $\bar{x} \in S$. Define*

$$S_k = \{x \in C \ : \ \nabla f(x) = k\nabla f(\bar{x}) \ \textit{and} \ \langle \nabla f(\bar{x}), x - \bar{x} \rangle = 0\}.$$

with $k > 0$. Then $S = \bigcup_{k>0} S_k$.

Proof. Two different cases are possible.

Suppose $\nabla f(\bar{x}) \neq 0$. Since $-\nabla f(\bar{x}) \in N(\bar{x}, C)$ then $\widehat{\nabla f}(\bar{x}) \cap -N(\bar{x}, C) = \widehat{\nabla f}(\bar{x})$ and hence

$$\overline{S} = \left\{x \in C \ : \ \widehat{\nabla f}(x) = \widehat{\nabla f}(\bar{x})\right\} = \bigcup_{k>0} S_k$$

where the equality $\langle \nabla f(\bar{x}), x - \bar{x} \rangle = 0$ descends from the minimum principle

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0,$$

from the pseudomonotonicity

$$\langle \nabla f(x), \bar{x} - x \rangle \leq 0$$

and from the fact that $\widehat{\nabla f}(x) = \widehat{\nabla f}(\bar{x})$.

Suppose $\nabla f(\bar{x}) = 0$. The set $\bigcup_{k>0} S_k$ coincides with the set of all stationary points on C and, by definition, every $x \in \overline{S}$ is a stationary point on C too. Fix an arbitrary $x \in C$ such that $\nabla f(x) = 0$; from the pseudomonotonicity$_*$ we have

$$L_{\nabla f, x} = \{y \in \mathbb{X} \ : \ \langle \nabla f(y), x - y \rangle = 0\} = \{y \in \mathbb{X} \ : \ \nabla f(y) = 0\}.$$

and the set $\widehat{\nabla f}(x) \cap -N(x, C)$ coincides with the solution set of the following inequality system

$$\begin{cases} \langle x^*, y - x \rangle \leq 0, & \forall y \in \mathbb{X} \ \text{ s.t. } \nabla f(y) = 0, \\ \langle x^*, y - x \rangle \geq 0, & \forall y \in C. \end{cases}$$

Since we can choose $y = \bar{x}$ we deduce that $\langle x^*, \bar{x} - x \rangle = 0$ and thus the system is equivalent to the following

$$\begin{cases} \langle x^*, y - \bar{x} \rangle \leq 0, & \forall y \in \mathbb{X} \ \text{ s.t. } \nabla f(y) = 0, \\ \langle x^*, y - \bar{x} \rangle \geq 0, & \forall y \in C. \end{cases}$$

This concludes the proof. □

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