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A Note on Extended Characterization of Generalized Trade-Off Directions in Multiobjective Optimization

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We consider five basic optimality principles for a general multiobjective optimization problem with convex and nonconvex partial objectives. We introduce a generalization of the concept of trade-off directions defining them via some optimal surface of appropriate cones. In both cases, we link optimality and generalized trade-off directions by deriving geometrical optimality conditions in terms of appropriate cones. We scrutinize similarities and differences between the cases. Combining newly proven facts with some previously known results, we derive four general patterns reflecting structural properties and interconnections of the considered optimality principles. Additionally, we provide extended characterization of optimality for some peculiar cases.

Keywords: Multiobjective optimization, optimality principles, generalized trade-offs, tangent, normal and contingent cones, convex and nonconvex optimization

1. Introduction

In multiobjective optimization, we typically deal with conflicting (competitive) objectives, i.e. the improvement in one of them is associated with deterioration in another of them. The competition between objectives takes place because of limited resources or other constraints restricting solution feasibility. Then the overall goal in multiobjective optimization is to find a compromise between several conflicting objectives which is best-fit to the needs of a decision maker. This compromise is usually refered to as an optimality principle. Various mathematical definitions of the optimality principle can be derived in several different ways depending on

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the needs of the solution approaches used. In multiobjective optimization, there are tens various optimality principles, however among them only few which are actually used in practice, since the use of one definition may be advantageous to the other due to computational complexity reasons.

The usage of trade-offs as a tool containing essential information about compromise have been originated from the series of papers (see e.g. [21] and [22]), where certain scalarizing functions were used to define the concept. Another approach, proposed in [7] and [8], consists in generating solution satisfying some pre-specified bounds on trade-off information by means of a scalarizing function. In [6], the concept of trade-offs has been generalized for convex (including nondifferentiable) problems into a cone of trade-off directions, which was defined as a Pareto optimal surface of a contingent (tangent) cone located at the point considered.

In the case of convex optimization, the choice of the cone of feasible directions as well as contingent cone is the most natural one according to [19]. In the case of nonconvex optimization, the main obstacle comes up from the fact the two above-mentioned cones may lose convexity. Giving up convexity naturally means that we need local instead of global analysis. As suggested in [2], two additional types of cones are proven to be helpful - tangent cone and cone of local feasible directions. The last two possess the guaranteed property of convexity, and hence they can be used to overcome some difficulties which appear in nonconvex optimization. However in nonconvex case, tangent cones do not necessarily represent the shape of the set considered even locally and the relation to trade-off directions is vague. Therefore to define trade-off directions in nonconvex case, we must use nonconvex contingent cones as it was suggested originally in [9] for smooth problems and later generalized for not necessarily differentiable problems in [13, 14].

The aim of this paper is to link trade-off directions and optimality by deriving optimality conditions in terms of appropriate cones for both convex and nonconvex cases. The paper is organized as follows. In Section 2, we formulate a general multiobjective problem and introduce five most common optimality principles. For every optimality principle considered, we define generalized trade-off directions as optimal surface of some appropriate cones in Section 3. The next Section presents the main results showing interrelation between optimal solutions and corresponding generalized trade-off directions. The results are summarized in four patterns: two of them for convex case (Subsection 4.1) and the other two for nonconvex case (Subsection 4.2). Section 5 provides extended characterization of optimality and trade-offs for some peculiar cases. The paper is concluded in Section 6, where the differences and similarities between two cases are analyzed.

2. Problem formulation and preliminaries

We consider general multiobjective optimization problems of the following form:

$$\min_{x \in S} \; \{f_1(x), f_2(x), \dots, f_k(x)\},$$

where $f_i : \mathbf{R}^n \to \mathbf{R}$ for all $i \in I_k := \{1, \dots, k\}$ are *objective functions*. The *decision vector* x belongs to the nonempty *feasible set* $S \subset \mathbf{R}^n$. The image of the feasible set is denoted by $Z \subset \mathbf{R}^k$. Elements of Z are termed *objective vectors* and they are denoted by $z = f(x) = (f_1(x), f_2(x), \ldots, f_k(x))^T$, so $Z := f(S)$. Additionally, we assume $f_i : \mathbf{R}^n \to \mathbf{R}$ be continuous for all $i \in I_k$.

The Minkowski sum of two sets A_1 and A_2 is defined by $A_1 + A_2 = \{a_1 + a_2 \mid a_1 \in A_1,\ a_2 \in A_2\}$. The interior, closure, convex hull and complement of a set A are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{conv} A$ and A^C, respectively.

A set C is a *cone* if $\lambda x \in C$ whenever $x \in C$ and $\lambda > 0$. We denote the positive orthant of $\mathbf{R}^k$ by $\mathbf{R}^k_+ = \{d \in \mathbf{R}^k \mid d_i \geq 0$ for every $i \in I_k\}$. The positive orthant is also known as *standard ordering cone*. The negative orthant $\mathbf{R}^k_-$ is defined respectively. Note, that $\mathbf{R}^k_-$ and $\mathbf{R}^k_+$ are closed convex cones.

In what follows, the notation $z < y$ for $z, y \in \mathbf{R}^k$ means that $z_i < y_i$ for every $i \in I_k$ and, correspondingly, $z \leq y$ stands for $z_i \leq y_i$ for every $i \in I_k$.

Simultaneous optimization of several objectives for multiobjective optimization problem is not a straightforward task. Contrary to the the single objective case, the concept of optimality is not unique in multiobjective cases.

Below we give traditional definitions of five well-known and most fundamental principles of optimality (see e.g. [3], [11]).

Weak Pareto Optimality. An objective vector $z^* \in Z$ is *weakly Pareto optimal* if there does not exist another objective vector $z \in Z$ such that $z_i < z^*_i$ for all $i \in I_k$.

Pareto optimality or efficiency. An objective vector $z^* \in Z$ is *Pareto optimal* or *efficient* if there does not exist another objective vector $z \in Z$ such that $z_i \leq z^*_i$ for all $i \in I_k$ and $z_j < z^*_j$ for at least one index j.

Proper Pareto Optimality. An objective vector $z^* \in Z$ is *properly Pareto optimal* if there exists no objective vector $z \in Z$ such that $z \in C$ for some convex cone C, $\mathbf{R}^k_- \setminus \{0\} \subset \operatorname{int} C$, attached to z^*.

Strong Efficiency. An objective vector $z^* \in Z$ is *strongly Pareto optimal* if for all $i \in I_k$ there exists no objective vector $z \in Z$ such that $z_i < z^*_i$ or in other words $z^* \in Z$ optimizes all objectives $z_i,\ i \in I_k$.

Lexicographic Optimality. An objective vector $z^* \in Z$ is *lexicographically optimal* if for all other objective vector $z \in Z$ one of the following two conditions holds:
1) $z = z^*$
2) $\exists\, i \in I_k :\ (z^*_i < z_i) \wedge (\forall j \in I_{i-1} :\ z^*_j = z_j)$, where $I_0 = \emptyset$.

A solution is Pareto optimal if improvement in some objectives can only be obtained at the expense of some other objective(s) (see e.g. [3], [11]). The set of weakly Pareto optimal solutions contains the Pareto optimal solutions together with solutions where all the objectives cannot be improved simultaneously (see, e.g. [3], [11]). The set of improperly Pareto optimal solutions represents a set of efficient points with certain abnormal or irregular properties. Henceforth we use only one of the possible concepts of proper efficiency, which is according to Henig [5]. This concept uses a convex cone, which interior part must contain an inverse of standard ordering cone, to prohibit tradeoffs towards directions within the cone. Strong efficiency is generally referred to the solutions which deliver optimality to each objective. Despite feasibility of such solutions is rare, they provide an important information on the lower bound for each objective in the Pareto optimal set. On the other hand, lexicographic optimality principle is generally applied to the situation where objectives have no equal importance anymore but ordered according to their significance.

Next we define the five sets of efficient solutions by using appropriate ordering cones. It is trivial to verify that the definitions of optimality and efficiency formulated above are equivalent to those following below.

Definition 2.1 (see e.g. [3, 10, 11]). The *globally weakly Pareto optimal set* is

$$GWP(Z) := \left\{z \in Z \mid (z + \operatorname{int} \mathbf{R}^k_-) \cap Z = \emptyset\right\};$$

the *globally Pareto optimal set* is

$$GPO(Z) := \left\{z \in Z \mid (z + \mathbf{R}^k_- \setminus \{0\}) \cap Z = \emptyset\right\};$$

the *globally properly Pareto optimal set* is defined as

$$GPP(Z) := \{z \in Z \mid (z + C \setminus \{0\}) \cap Z = \emptyset\}$$

for some convex cone C such that $\mathbf{R}^k_- \setminus \{0\} \subset \operatorname{int} C$;
the *globally strongly efficient set* is

$$GSE(Z) := \left\{z \in Z \mid (z + (\mathbf{R}^k_+)^C) \cap Z = \emptyset\right\};$$

and the *globally lexicographically optimal set* is

$$GLO(Z) := \left\{z \in Z \mid (z + (C^k_{\text{lex}})^C) \cap Z = \emptyset\right\},$$

where the *lexicographic cone* is

$$C^k_{\text{lex}} := \{0\} \cup \left\{d \in \mathbf{R}^k \mid \exists\, i \in I_k \text{ such that } d_i > 0 \text{ and } d_j = 0 \;\forall\, j < i\right\}.$$

Let $B(x;\varepsilon)$ be *an open ball* with radius $\varepsilon > 0$ and center $x \in \mathbf{R}^n$. The corresponding local concepts are defined in the following. Naturally, in a convex case, local and global concepts are equal.

Definition 2.2 (see e.g. [3, 10, 11]). The *locally weakly Pareto optimal set* with $z = f(x) \in Z$ is given as

$$LWP(Z) := \bigcup_{\delta>0} \left\{z \in Z \mid (z + \operatorname{int} \mathbf{R}^k_-) \cap Z \cap f(B(x;\delta)) = \emptyset\right\};$$

the *locally Pareto optimal set* as

$$LPO(Z) := \bigcup_{\delta>0} \left\{z \in Z \mid (z + \mathbf{R}^k_- \setminus \{0\}) \cap Z \cap f(B(x;\delta)) = \emptyset\right\};$$

the *locally properly Pareto optimal set* as

$$LPP(Z) := \bigcup_{\delta>0} \{z \in Z \mid (z + C \setminus \{0\}) \cap Z \cap f(B(x;\delta)) = \emptyset\}$$

for some convex cone C such that $\mathbf{R}^k_- \setminus \{0\} \subset \operatorname{int} C$;
the *locally strongly efficient set* with $z = f(x)$ is defined as

$$LSE(Z) := \bigcup_{\delta>0} \left\{ z \in Z \mid (z + (\mathbf{R}^k_+)^C) \cap Z \cap f(B(x;\delta)) = \emptyset \right\};$$

and the *locally lexicographically optimal set* with $z = f(x)$ is

$$LLO(Z) := \bigcup_{\delta>0} \left\{ z \in Z \mid (z + (C^k_{\text{lex}})^C) \cap Z \cap f(B(x;\delta)) = \emptyset \right\}.$$

It is evident that we have the following relationships between the different optimality principles – see Figures 2.1 and 2.2.

$$\begin{array}{ccccccc} GSE(Z) & \subset & GPP(Z) & \subset & GPO(Z) & \subset & GWP(Z) \\ \cap & & \cap & & \cap & & \cap \\ LSE(Z) & \subset & LPP(Z) & \subset & LPO(Z) & \subset & LWP(Z) \end{array}$$

Figure 2.1: Collection of the relationships between local and global strong, weak, proper Pareto optimality and efficiency.

$$\begin{array}{ccccccc} GSE(Z) & \subset & GLO(Z) & \subset & GPO(Z) & \subset & GWP(Z) \\ \cap & & \cap & & \cap & & \cap \\ LSE(Z) & \subset & LLO(Z) & \subset & LPO(Z) & \subset & LWP(Z) \end{array}$$

Figure 2.2: Collection of the relationships between local and global efficiency, strong efficiency, lexicographic and weak Pareto optimality.

3. Generalized trade-off directions

The concept of trade-offs in multiobjective optimization is a key point to define compromise between conflicting objectives. It can be used to describe solutions which linearly approximate the feasible region and which are mutually non-dominated with respect to the given optimality principle. The trade-off directions can be used in many algorithms which requires specifying directions which may lead fast to the solution that is most preferred by the decision maker (see e.g. [1], [11]). Next we define several geometrical basic cones (see e.g. [19]).

Definition 3.1. The *contingent cone* of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is defined as

$$K_z(Z) := \left\{ d \in \mathbf{R}^k \mid \ \textit{there exist } t_j \searrow 0 \textit{ and } d_j \to d \textit{ such that } z + t_j d_j \in Z \right\}.$$

Definition 3.2. The *cone of globally feasible directions* of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is denoted by

$$D_z(Z) := \left\{ d \in \mathbf{R}^k \mid \ \text{there exists } t > 0 \text{ such that } z + td \in Z \right\}.$$

The following definition provides regularity condition for Z at $z \in Z$.

Definition 3.3. The set Z is called *regular* at $z \in Z$ if $D_z(Z) = K_z(Z)$.

The definitions of contingent cones $K_z(Z)$ and cones of globally feasible directions $D_z(Z)$ are equally valid for both convex and nonconvex sets. Note, however, that the cone convexity, which holds for convex sets, is not guaranteed in nonconvex case. Since the contingent cones linearly approximates the shape of the feasible set, equally well in both convex (global approximation) and nonconvex (local approximation) cases, it can be used to define the generalized trade-off directions. A (weakly) Pareto surface of the contingent cone serves for that purposes.

In nonconvex case, the cone of feasible directions $D_z(Z)$ does not describe the shape of Z locally. Thus, we introduce a cone of locally feasible directions, which reflects the shape of Z locally (see e.g. [15]).

Definition 3.4. The *cone of locally feasible directions* of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is denoted by

$$F_z(Z) := \left\{ d \in \mathbf{R}^k \mid \text{ there exists } t > 0 \text{ such that } z + \tau d \in Z \text{ for all } \tau \in (0, t] \right\}.$$

The following definition provides local regularity condition for Z at $z \in Z$.

Definition 3.5. The set Z is called *locally regular* at $z \in Z$ if $F_z(Z) = K_z(Z)$.

For nonconvex cases, Clarke [2] defined a convex tangent cone in the following way.

Definition 3.6. The *tangent cone* of a set $Z \subset \mathbf{R}^k$ at $z \in Z$ is given by the formula

$$\begin{aligned} T_z(Z) := \big\{ d \in \mathbf{R}^k \mid \ & \text{for all } t_j \searrow 0 \text{ and } z_j \to z \text{ with } z_j \in Z, \\ & \text{there exists } d_j \to d \text{ with } z_j + t_j d_j \in Z \big\}. \end{aligned}$$

The *normal cone* of Z at $z \in Z$ is the polar cone of the tangent cone, that is,

$$N_z(Z) := T_z(Z)^\circ = \left\{ y \in \mathbf{R}^k \mid y^T d \le 0 \text{ for all } d \in T_z(Z) \right\}.$$

Due to polarity and tangent cone convexity, the cone $N_z(Z)$ is always convex and contains zero.

The following basic relations can be derived from the definitions of the concepts used and from [15], [20].

Lemma 3.7. *For the cones $K_z(Z)$, $D_z(Z)$, $T_z(Z)$ and $F_z(Z)$ we have the following*

a) $K_z(Z)$ *and* $T_z(Z)$ *are closed and* $T_z(Z)$ *is convex.*
b) $0 \in K_z(Z) \cap D_z(Z) \cap T_z(Z) \cap F_z(Z)$.
c) $Z \subset z + D_z(Z)$.
d) $\operatorname{cl} F_z(Z) \subset K_z(Z) \subset \operatorname{cl} D_z(Z)$.
e) $T_z(Z) \subset K_z(Z)$.
f) *If* Z *is convex, then* $\operatorname{cl} F_z(Z) = T_z(Z) = K_z(Z) = \operatorname{cl} D_z(Z)$. *Moreover* $F_z(Z) = D_z(Z)$.

Note that, under convexity assumption, for any $z \in Z$ we have $\operatorname{cl} F_z(Z) = K_z(Z)$, i.e. local regularity defines a bit stronger requirement on a local structure of a set than the convexity assumption. At the same time local regularity does not necessarily imply that $\operatorname{cl} D_z(Z) = K_z(Z)$, the condition which is guaranteed under convexity assumption.

In addition to previous lemma, later we will use the following technical result which is easy to prove.

Lemma 3.8. *Let C be a convex cone. Then $\lambda x + \mu y \in C$ for all $x,\ y \in C$ and $\lambda,\ \mu > 0$.*

Even though contingent cones are generally nonconvex, their convexity is guaranteed under special circumstances.

Definition 3.9. The set Z is called *tangentially regular* at $z \in Z$ if $T_z(Z) = K_z(Z)$.

Trivially, we can see that e.g. convex sets are always tangentially regular. Note that in order to formulate some of optimality conditions we use four different assumptions about structural properties of Z - convexity, tangent regularity, regularity and local regularity. In general all these are different and does not directly imply each others. The interconnection between the four regularity assumptions are presented in Figure 3.1.

$$\begin{array}{ccc} \operatorname{cl} D_z(Z) = K_z(Z) & \Leftarrow & \text{Regularity} \\ \Uparrow & & \\ \text{Convexity} & \Rightarrow & \text{Tangent Regularity} \\ \Downarrow & & \\ \operatorname{cl} F_z(Z) = K_z(Z) & \Leftarrow & \text{Local Regularity} \end{array}$$

Figure 3.1: Interconnection between various types of regularity.

The sets of generalized trade-off directions can be defined as follows

Definition 3.10. The set of *generalized trade-off directions* is defined as:

- in case of *weak Pareto optimality*:

$$G_W(Z) := GWP(K_z(Z));$$

- in case of *Pareto optimality (efficiency)*:

$$G_P(Z) := GPO(K_z(Z));$$

- in case of *strong efficiency*:

$$G_S(Z) := GSE(K_z(Z));$$

- in case of *lexicographic optimality*:

$$G_L(Z) := GLO(F_z(Z)).$$

It is easy to see that in convex case $GLO(F_z(Z)) = GLO(D_z(Z))$ and $GSE(K_z(Z)) = GSE(D_z(Z))$. These results follow directly from the definitions and Lemma 3.7.

Notice that, since two solutions are considered to be mutually lexicographically nondominated if they have the same objective vectors, we have to use the cone of locally feasible directions in the definition of the set of generalized trade-off directions in case with lexicographic optimality. Indeed, the set of generalized trade-off directions in case with local lexicographic optimality is either empty or one point 0 (origin of $F_z(Z)$) only, so it becomes indifferent if $F_z(Z)$ is closed or open, what is not true in cases with other types of local optimality.

The following proposition specifies relations between generalized trade-offs for different optimality principles.

Proposition 3.11. *The following inclusions are true*

$$G_S(Z) \subset G_L(Z) \subset G_P(Z) \subset G_W(Z).$$

Proof. The chain $G_L(Z) \subset G_P(Z) \subset G_W(Z)$ follows straightforward from the definition of generalized trade-offs and relations specified in Figure 2.2. So, it remains to prove that $G_S(Z) \subset G_L(Z)$. From the definition of $G_S(Z)$, it is clear that $G_S(Z) = \emptyset$ or $G_S(Z) = \{0\}$. If $G_S(Z) = \emptyset$, there is nothing to prove. So suppose that $G_S(Z) = \{0\}$. Then $0 \in GSE(K_z(Z))$ as well as $0 \in GLO(K_z(Z))$. Since according to Lemma 3.7 (d), $\operatorname{cl} F_z(Z) \subset K_z(Z)$, we have $0 \in GLO(F_z(Z))$, i.e. $0 \in G_L(Z)$. Thus we get the result. □

4. Main results

4.1. Convex case

Here we formulate and prove the basic results concerning relations between optimality and corresponding set of generalized trade-off directions in convex case.

Proposition 4.1. *Let Z be convex. Then the following statements are true.*

1. $G_S(Z) = \{0\}$ *if and only if*

$$K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z).$$

2. $G_P(Z) \neq \emptyset$ *if and only if*

$$K_z(Z) \cap \mathbf{R}^k_- \backslash \{0\} = \emptyset.$$

3. *If $G_P(Z) \neq \emptyset$, then $D_z(Z) \cap \mathbf{R}^k_- \backslash \{0\} = \emptyset$. The sufficiency holds under the assumption that Z is regular.*
4. $G_W(Z) \neq \emptyset$ *if and only if*

$$D_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset.$$

5. $G_L(Z) = \{0\}$ *if and only if*

$$D_z(Z) \cap C^k_{lex} = D_z(Z).$$

Proof. First, we give the proof of statement *1.* Obviously, we have

$$K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z) \iff (\mathbf{R}^k_+)^C \cap K_z(Z) = \emptyset,$$

which is in turn equivalent to $0 \in GSE(K_z(Z)) = G_S(Z)$. The last holds if and only if $G_S(Z) = \{0\}$ (see [18], Theorem 4), and hence we proved the statement.

Then we give the proof of statement *2.* Indeed, $K_z(Z) \cap \mathbf{R}^k_- \backslash \{0\} = \emptyset$ is true if and only if $0 \in GPO(K_z(Z))$, i.e. $GPO(K_z(Z)) \neq \emptyset$. Now it remains to show that if $G_P(Z) \neq \emptyset$, then $0 \in G_P(Z)$. Assume that there exists $y \in G_P(Z)$. Then we get $K_z(Z) \cap (y + \mathbf{R}^k_- \backslash \{0\}) = \emptyset$. Obviously (due to sharpness of $K_z(Z)$ in convex case), we have $K_z(Z) \cap (0 + \mathbf{R}^k_- \backslash \{0\}) = \emptyset$, i.e. $0 \in G_P(Z)$.

Now we prove statement *3.* Since $D_z(Z) \subset K_z(Z)$ the necessity follows from the previous statement. If Z is regular, then from $D_z(Z) \cap \mathbf{R}^k_- \backslash \{0\} = \emptyset$ it follows that $K_z(Z) \cap \mathbf{R}^k_- \backslash \{0\} = \emptyset$, thus $G_P(Z) \neq \emptyset$.

We proceed with the proof of statement *4.* In convex case $K_z(Z) = \operatorname{cl} D_z(Z)$, so $D_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset$ is equivalent to (see Theorem 3, [12]) $K_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset$, which is true if and only if $0 \in GWP(K_z(Z))$, i.e. $G_W(Z) \neq \emptyset$.

Finally, we show that statement *5.* is true. Indeed, $D_z(Z) \cap C^k_{lex} = D_z(Z)$ is equivalent to $D_z(Z) \cap (C^k_{lex})^C = \emptyset$, which is true if and only if $0 \in GLO(D_z(Z))$, which is equivalent to $G_L(Z) = \{0\}$ (see Theorem 5, [18]), because in convex case $D_z(Z) = F_z(Z)$ (see Lemma 3.7 (f)). □

The following proposition can be obtained in convex case.

Proposition 4.2. *Let Z be convex. Then the following statements are true.*

1. $G_S(Z) = \{0\}$ *if and only if*

$$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-.$$

2. $G_P(Z) \neq \emptyset$ *if and only if*

$$N_z(Z) \cap \operatorname{int} \mathbf{R}^k_- \neq \emptyset.$$

3. *If* $N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset$, *then* $G_W(Z) \neq \emptyset$.

Proof. First, we prove statement *1.* From Proposition 4.1 (statement *1.*), $G_S(Z) = \{0\}$ if and only if $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$. Using the properties of the polar cone (see Theorem 16.4.2 in [19], as well as Lemma 2.1(b) and Lemma 3.1(a) of [16]), we get

$$(K_z(Z) \cap \mathbf{R}^k_+)^\circ = \operatorname{cl}(K_z(Z)^\circ + (\mathbf{R}^k_+)^\circ) = \operatorname{cl}(N_z(Z) + \mathbf{R}^k_-) = N_z(Z) + \mathbf{R}^k_-.$$

Furthermore, $N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-$ if and only if $\mathbf{R}^k_- \subset N_z(Z)$. So we need to prove that $N_z(Z) + \mathbf{R}^k_- = N_z(Z)$ if and only if $\mathbf{R}^k_- \subset N_z(Z)$. If $N_z(Z) + \mathbf{R}^k_- = N_z(Z)$, since $0 \in N_z(Z)$, we have $\mathbf{R}^k_- \subset N_z(Z)$. On the other hand, if $\mathbf{R}^k_- \subset N_z(Z)$, then $N_z(Z) + \mathbf{R}^k_- \subseteq N_z(Z)$, (if $a, b \in N_z(Z)$, then $a + b \in N_z(Z)$), and since $N_z(Z) \subseteq N_z(Z) + \mathbf{R}^k_-$ is always true, this ends the proof. □

Statement *2.* follows directly from Proposition 4.1 (statement *2.*), Lemma 3.7(f) and definition of the normal cone.

Finally we show that statement *3.* is true. Indeed, we have $K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$, which is equivalent to (see Theorem 3 in [12]) $D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$, and from this with Proposition 4 it follows that $G_W(Z) \neq \emptyset$. Let us assume that $y \in N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\}$. If there exists $x \in K_z(Z) \cap \text{int } \mathbf{R}^k_-$, then $y^T x > 0$, and this is in contradiction with the definition of the polar cone. □

The following basic result gives conditions connecting various optimality and generalized trade-offs in convex case.

Theorem 4.3. *Let Z be convex. Then the following statements are true.*

1. $G_S(Z) = \{0\}$ *if and only if* $z \in GSE(Z)$.
2. $G_P(Z) \neq \emptyset$ *if and only if* $z \in GPP(Z)$.
3. *If* $G_P(Z) \neq \emptyset$, *then* $z \in GPO(Z)$. *The sufficiency holds under assumption that Z is regular.*
4. $G_W(Z) \neq \emptyset$ *if and only if* $z \in GWP(Z)$.
5. $G_L(Z) = \{0\}$ *if and only if* $z \in GLO(Z)$.

Proof. Statement *1.* follows directly from Proposition 4.1 (statement *1.*) and the fact that $z \in GSE(Z)$ if and only if $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$ (see Theorem 3.1, [16]).

Statement *2.* follows directly from Proposition 4.1 (statement *2.*) and the fact that $z \in GPP(Z)$ if and only if $K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$ (see Theorem 2.1, [16]).

Statement *3.* follows directly from Proposition 4.1 (statement *3.*) and the fact that $z \in GPO(Z)$ if and only if $D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$ (see Theorem 2.1, [16]).

Statement *4.* follows directly from Proposition 4.1 (statement *4.*) and the fact that $z \in GWP(Z)$ if and only if $D_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$ (see Theorem 2.1, [16]).

Statement *5.* follows directly from Proposition 4.1 (statement *5.*) and the fact that $z \in GLO(Z)$ if and only if $D_z(Z) \cap C^k_{lex} = D_z(Z)$ (see Theorem 4.1, [16]). □

4.2. Nonconvex case

In this section we additionally assume $f(B(x;\varepsilon))$ to be open for all $x \in S$ and $\varepsilon > 0$. Here we formulate and prove the basic results concerning relations between optimality and corresponding set of generalized trade-off directions in nonconvex case.

Proposition 4.4. *The following statements are true.*

1. $G_S(Z) = \{0\}$ *if and only if* $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$.
2. *If* $K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$, *then* $G_P(Z) \neq \emptyset$. *The sufficiency holds under assumption that Z is tangentially regular.*
3. *If* $F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$, *then* $G_P(Z) \neq \emptyset$ *under assumption that Z is locally regular. The sufficiency holds under assumption that Z is both tangentially and locally regular.*
4. $G_W(Z) \neq \emptyset$ *if and only* $K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$.
5. $G_L(Z) = \{0\}$ *if and only if* $F_z(Z) \cap C^k_{lex} = F_z(Z)$.

Proof. The proof of statement *1.* is analogous to the proof of statement *1.* in Proposition 4.1.

Now we prove statement *2.* The necessity follows similarly to convex case. As far as sufficiency is concerned, let us assume that $G_P(Z) \neq \emptyset$. Then from the definition of $G_P(Z)$ it follows that there exists $y \in GPO(K_z(Z))$. If there exists $x \in K_z(Z) \cap \mathbf{R}^k_- \backslash\{0\}$, using that $K_z(Z)$ is a convex cone because of the tangent regularity and Lemma 3.8, we get: $y + x \in K_z(Z)$. But $y + x \leq y$, and since $x \neq 0$, there exists i, such that $y_i + x_i < y_i$, i.e. y is not Pareto optimal, and thus $y \notin GPO(K_z(Z))$, and this is a contradiction.

Statement *3.* follows directly. Indeed, local regularity means that $F_z(Z) = K_z(Z)$, with this assumption the result follows from the previous statement.

In statement *4.*, necessity is also straightforward. Indeed, $K_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset$ is true if and only if $0 \in GWP(K_z(Z))$, i.e. $G_W(Z) \neq \emptyset$. As far as sufficiency is concerned, let us assume that $G_W(Z) \neq \emptyset$. Then from the definition of $G_W(Z)$ it follows that there exists $y \in GWP(K_z(Z))$. Suppose that there exists $x \in K_z(Z) \cap \operatorname{int} \mathbf{R}^k_-$. Obviously, multiplying by an arbitrary large $\mu > 0$, we can guarantee that $\mu \cdot x < y$ (i.e. $\mu \cdot x_i < y_i$ for all indices i) and at the same time $\mu \cdot x \in K_z(Z)$, i.e. y is not weakly Pareto optimal, and thus $y \notin GWP(K_z(Z))$, so the last is a contradiction.

To prove the last statement it is enough to notice that $F_z(Z) \cap C^k_{lex} = F_z(Z)$ is equivalent to $F_z(Z) \cap (C^k_{lex})^C = \emptyset$, which is true if and only if $0 \in GLO(F_z(Z))$. From the last it follows that $G_L(Z) \neq \emptyset$. Now it remains to show, that if $y \in F_z(Z)$, $y \neq 0$, then $y \notin G_L(Z)$. Indeed, if $y \in F_z(Z)$, $y \neq 0$, then $(y + (C^k_{lex})^C) \cap F_z(Z) \neq \emptyset$ and then $y \notin G_L(Z)$ □

The following corollaries can be obtained in nonconvex case.

Proposition 4.5. *The following statements are true.*

1. *If $G_S(Z) = \{0\}$, then*
$$N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_-.$$
The sufficiency holds under assumption that Z is tangentially regular.

2. *Let Z be tangentially regular. Then $G_P(Z) \neq \emptyset$ if and only if*
$$N_z(Z) \cap \operatorname{int} \mathbf{R}^k_- \neq \emptyset.$$

3. *If $N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset$, then $G_W(Z) \neq \emptyset$ under assumption that Z is tangentially regular.*

Proof. The proof of statement *1.* is similar to the proof of statement *1.* in Proposition 4.2 using tangent regularity in case of sufficiency. For the necessity part, we do not need this assumption, since from Proposition 4.4 (statement *1.*) we know that if $G_S(Z) = \{0\}$ then $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$, thus $T_z(Z) \subset K_z(Z) \subset \mathbf{R}^k_+$. We know that if C_1, C_2 are convex cones such that $C_1 \subset C_2$, then $C_2^\circ \subset C_1^\circ$. Thus from $T_z(Z) \subset \mathbf{R}^k_+$ it follows that $\mathbf{R}^k_- \subset N_z(Z)$.

The proof of statement *2.* is identical to the proof of statement *2.* in Proposition 4.2 using that $K_z(Z) = T_z(Z)$.

To prove statement *3.*, we notice that $K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$ together with statement *4.* in Proposition 4.4 imply $G_W(Z) \neq \emptyset$. Let us assume that $y \in N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\}$. If there exists $x \in K_z(Z) \cap \text{int } \mathbf{R}^k_-$, then $y^T x > 0$, and this is in contradiction with the definition of the polar cone. □

The following basic results give conditions connecting various optimality and generalized trade-offs in nonconvex case.

Theorem 4.6. *The following statements are true.*

1. *If $z \in LSE(Z)$, then $G_S(Z) = \{0\}$.*
2. *If $z \in LPP(Z)$, then $G_P(Z) \neq \emptyset$. The sufficiency holds under assumption that Z is tangentially regular.*
3. *If $z \in LPO(Z)$, then $G_P(Z) \neq \emptyset$ under assumption that Z is locally regular. The sufficiency holds under assumption that Z is both locally and tangentially regular.*
4. *If $z \in LWP(Z)$, then $G_W(Z) \neq \emptyset$.*
5. *If $z \in GLO(Z)$, then $G_L(Z) = \{0\}$.*

Proof. First we prove statement *1.* Indeed, if $z \in LSE(Z)$, then $K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z)$. (for the proof see Theorem 3 of [17]). Now taking into account statement *1.* of Proposition 4.4, we get the correctness of statement *1.*

Statement *2.* follows directly from Proposition 4.4 (statement *2.*) and the fact that $z \in LPP(Z)$ if and only if $K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$ (see Theorem 2, [13]).

Now we show the correctness of statement *3.* Indeed, if $F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$ and Z is locally regular at z, then $z \in LPO(Z)$ (for the proof see Theorem 2 of [17]). Now taking into account statement *3.* of Proposition 4.4, we get the result.

Statement *4.* follows directly form statement *4.* of Proposition 4.4 and the fact that if $z \in LWP(Z)$, then $K_z(Z) \cap \text{int } \mathbf{R}^k_- = \emptyset$ (see Theorem 7, [12]).

Finally, we show the correctness of statement *5.* Indeed, if $z \in LLO(Z)$, then $F_z(Z) \cap C^k_{\text{lex}} = F_z(Z)$ (see Theorem 5 of [17]). Now taking into account statement *5.* of Proposition 4.4, we get the statement. □

The results related to the four optimality concepts involving proper Pareto optimality and different cones are collected in Table 6.1, in the case if Z is convex, and in Table 6.3, otherwise. The results related to the four optimality concepts involving lexicographic optimality and different cones are collected in Table 6.2, in the case if Z is convex, and in Table 6.4, otherwise. In these tables tangent regularity is noted by $*$, local regularity is noted by $**$, and regularity is noted by $***$.

Now we shortly analyze the similarity and difference between the results in two cases - convex and nonconvex. As it can be seen from the results above, some optimality conditions, which are necessary and sufficient in convex case, are transformed into necessary but not sufficient ones for corresponding local optimality in nonconvex case. For example, $G_W(Z) \neq \emptyset$, being a necessary and sufficient condition in convex case for weak Pareto optimality, becomes only necessary condition for local weak Pareto optimality. The loss of sufficiency can be explained by the fact that the above-mentioned conditions use the contingent cone, which may have "bad" directions towards no feasibility in nonconvex case due to the loss of own property of being convex. The same situation is also true with the conditions $G_S(Z) = \{0\}$ and $G_L(Z) = \{0\}$. Here the loss of sufficiency can be explained by the fact that in nonconvex case the cone of locally feasible directions similar to the contingent cone may also have "bad" directions towards no feasibility which prohibits sufficiency of the corresponding optimality condition to be proven in general case. Imposing tangent regularity helps to sustain sufficiency in the nonconvex cases with Pareto and proper Pareto optimality, however it is not useful with the other three optimality principles.

5. Characterization of generalized trade-offs

Next we show how the concept of generalized trade-offs can be characterized by normal cones and vectors. Before we need some lemmas.

Lemma 5.1. *If $Z = z + C$, where C is a cone, then $F_z(Z) = C$ (in other words $Z = z + F_z(Z)$).*

Proof. Suppose $d \in F_z(Z)$. Then there exists $t > 0$ such that $z + \tau d \in Z = z + C$ for all $\tau \in (0, t]$. It implies that $\tau d \in C$, and hence $d \in C$. The last implies $F_z(Z) \subseteq C$. Now suppose $d \in C$. Choose $t := 1$. Then $z + \tau d \in z + C = Z$ for all $\tau \in (0, 1]$. It implies that $d \in F_z(Z)$, i.e. $C \subseteq F_z(Z)$. □

Lemma 5.2. *If $Z = z+C$, where C is a cone, then* $\operatorname{cl} F_z(Z) = K_z(Z) = \operatorname{cl} D_z(Z) = \operatorname{cl} C$.

Proof. From Lemma 3.7, it follows that $\operatorname{cl} F_z(Z) \subseteq K_z(Z) \subseteq \operatorname{cl} D_z(Z)$. Let $d \in D_z(Z)$, then there exists $d_j \to d$ such that $d_j \in D_z(Z)$ for all j. It implies that there exists $t_j > 0$ such that $z + t_j d_j \in Z = z + C$, i.e. $t_j d_j \in C$ for all j. Hence $d_j \in C$ for all j. Since $Z = z + C$, by Lemma 5.1 we have $d_j \in F_z(Z)$ for all j. Therefore $d_j \to d \in \operatorname{cl} F_z(Z)$ for all j. The last implies $\operatorname{cl} D_z(Z) \subseteq \operatorname{cl} F_z(Z)$, and finally taking into account Lemma 5.1, we deduce $\operatorname{cl} F_z(Z) = K_z(Z) = \operatorname{cl} D_z(Z) = \operatorname{cl} C$. The last ends the proof.

Theorem 5.3. *If $Z = z + C$, where C is a closed cone, then*

$$GPP(Z) = LPP(Z) = LPO(Z) = GPO(Z).$$

Proof. Evidently,

$$\begin{array}{ccc} GPP(Z) & \subseteq & LPP(Z) \\ \cap\!| & & \cap\!| \\ GPO(Z) & \subseteq & LPO(Z) \end{array}.$$

We first show that $LPO(Z) \subseteq LPP(Z)$. Indeed, if $z \in LPO(Z)$, then by Theorem 8 in [12] $F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$. Hence by Lemma 5.1 $C \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$, and since C is closed, $\operatorname{cl} C \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$. Again applying Lemma 5.1 we get $\operatorname{cl} F_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$. By Lemma 3.7, we deduce $K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$, and finally by Theorem 6 in [12] we get $z \in LPP(Z)$.

Next we show that $LPP(Z) \subseteq GPP(Z)$. Indeed, if $z \in LPP(Z)$, then by Theorem 6 in [12] $K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$, and hence by Lemma 5.2 $\operatorname{cl} D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset$. Finally, according to Theorem 1 in [17], we conclude $z \in GPP(Z)$.

Combined all what has been proven above we obtain that $GPP(Z) = LPP(Z) = LPO(Z) = GPO(Z)$. This ends the proof.

Theorem 5.4. *Let Z be tangentially regular. If $d \in G_P(Z)$, then $d^T y \leq 0$ for all $y \in N_z(Z)$ and $d^T y = 0$ for some $y \in N_z(Z) \cap \operatorname{int} \mathbf{R}^k_-$.*

Proof. Let $d \in G_P(Z)$. Using tangent regularity, we get

$$G_P(Z) = GPO(K_z(Z)) = GPO(T_z(Z)) \subseteq T_z(Z).$$

Hence, $d^T y \leq 0$ for all $y \in N_z(Z)$. For any $\lambda \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ consider the following problem

$$\min_{\delta \in T_z(Z)} \lambda^T \delta. \tag{1}$$

Using consequently tangent regularity and the fact that $T_z(Z)$ is a closed cone, it follows from Theorem 5.3 that

$$d \in G_P(Z) = GPO(T_z(Z)) = GPP(T_z(Z)),$$

i.e. d is properly Pareto optimal with respect to $T_z(Z)$. Since $T_z(Z)$ is convex, due to the classical result of Theorem 2 in [4], there exists $\lambda > 0$ such that d is the solution of (1). Then it follows that $\lambda^T d \geq 0$, otherwise the problem (1) would have an unbounded solution, since $T_z(Z)$ is unbounded. Thus we get $0 \leq \lambda^T d \leq \lambda^T \delta$ for all $\delta \in T_z(Z)$. Since $0 \in T_z(Z)$, we get $0 \leq \lambda^T d \leq \lambda^T \cdot 0 = 0$, i.e. $\lambda^T d = 0$. By choosing $y := -\lambda$, we get $d^T y = 0$, and $y \in N_z(Z) \cap \operatorname{int} \mathbf{R}^k_-$. □

Note that the result of Theorem 5.4 is stronger than the result of Theorem 5 in [13]. The following theorem states similar result for weakly Pareto trade-offs.

Theorem 5.5. *Let Z be tangentially regular. If $d \in G_W(Z)$, then $d^T y \leq 0$ for all $y \in N_z(Z)$ and $d^T y = 0$ for some $y \in N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\}$.*

Proof. We follow the proof scheme of Theorem 5.4 performing minor changes reflecting the specific of the case considered. Let $d \in G_W(Z)$. Using tangent regularity, we get

$$G_W(Z) = GWP(K_z(Z)) = GWP(T_z(Z)) \subseteq T_z(Z).$$

Hence, $d^T y \leq 0$ for all $y \in N_z(Z)$. For any $\lambda \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ consider again problem (1). Since d is weakly Pareto optimal with respect to $T_z(Z)$ and $T_z(Z)$ is convex, due to the classical result connecting the weakly Pareto optimal solutions and the solutions of weighted sum scalarization (see e.g. Proposition 3.10 in [3]), there exists $\lambda \geq 0$, $\lambda \neq 0$ such that d is the solution of (1). Then it follows that $\lambda^T d \geq 0$, otherwise the problem (1) would have an unbounded solution, since $T_z(Z)$ is unbounded. Thus we get $0 \leq \lambda^T d \leq \lambda^T \delta$ for all $\delta \in T_z(Z)$. As a remark, notice also that $\lambda^T d = \lambda^T \delta$ for all $\delta \in T_z(Z)$ with $\delta_i = k \cdot d_i$ for any collection of indices i such that $\lambda_i = 0$ and $\delta_i = d_i$ for the remaining i, where k is an arbitrary multiplier. Since $0 \in T_z(Z)$, we get $0 \leq \lambda^T d \leq \lambda^T \cdot 0 = 0$, i.e. $\lambda^T d = 0$. By choosing $y := -\lambda$, we get $d^T y = 0$, and $y \in N_z(Z) \cap \mathbf{R}^k_-$. □

As noted in Theorem 5.4 and Theorem 5.5, generalized trade-off directions can be characterized with the help of normal vectors in a tangentially regular case. In [13] (see Theorem 6 therein), some optimality conditions related to normal vectors were obtained in the case of local proper Pareto optimality. Here we present one more result of that kind related to local weak Pareto optimality.

In what follows, we assume the feasible region to be of the form

$$S = \{x \in \mathbb{R}^n \mid (g_1(x), g_2(x), \ldots, g_m(x))^T \leq 0\},$$

where each $g_i : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous. The *Cottle constraint qualification* is valid at $\hat{x}$ if either $g_s(\hat{x}) < 0$ for all $s = 1, \ldots, m$, or $0 \notin \operatorname{conv} \{\partial g_s(\hat{x}) \mid g_s(\hat{x}) = 0\}$. For any given $y \in \mathbb{R}^k_-$, denote $I := \{1, 2, ..., k\} = I_<(y) \cup I_=(y)$, where $I_<(y) := \{i | \; y_j < 0\}$ and $I_=(y) := \{i | \; y_j = 0\}$.

Theorem 5.6. *Let $\hat{z} = f(\hat{x}) \in LWP(Z)$ and let $\hat{x} \in S$ satisfy the Cottle constraint qualification. If there exists $y \in N_{\hat{z}}(Z) \cap \mathbf{R}^k_- \backslash \{0\}$, then there exists $0 \leq \lambda \in \mathbf{R}^k$, $\lambda \neq 0$, and $0 \leq \mu \in \mathbf{R}^m$ such that $\mu_s g_s(\hat{x}) = 0$ for every $s = 1, \ldots, m$, and*

$$0 \in \sum_{i \in I_<(y)} -\frac{\lambda_i}{y_i} \partial f_i(\hat{x}) + \sum_{j \in I_=(y)} \lambda_j \partial f_j(\hat{x}) + \sum_{s=1}^{m} \mu_j \partial g_j(\hat{x}).$$

Proof. As a guideline we follow the proof scheme of Theorem 6 in [13]. Since $\hat{z} \in LWP(Z)$, there exists a radius $\delta > 0$ such that $\hat{z}$ is globally weakly Pareto optimal in $Z \cap f(B(\hat{x}; \delta))$. The assumption $y \in N_{\hat{z}}(Z) \cap \mathbf{R}^k_- \backslash \{0\}$ means that $y_i \leq 0$ for every $i = 1, \ldots, k$ and $y_j < 0$ for some j. Notice that $I_<(y) \neq \emptyset$ for all $y \in N_{\hat{z}}(Z) \cap \mathbf{R}^k_- \backslash \{0\}$.

Let us define for every $x \in S$ a function

$$F(x) = \max_{i \in I} F_i(x),$$

where

$$F_i(x) = \begin{cases} -\dfrac{1}{y_i}(f_i(x) - \hat{z}_i - y_i), & i \in I_<(y); \\ f_i(x) - \hat{z}_i + 1, & i \in I_=(y). \end{cases}$$

This function attains its local minimum at $\hat{x} \in S \cap B(\hat{x}; \delta)$. If this was not the case, there would exist $x^\circ \in S \cap B(\hat{x}; \delta)$ such that

$$\begin{aligned} F(x^\circ) = \max_{i \in I} F_i(x^\circ) &= \max \left\{ \max_{i \in I_<(y)} \left[-\frac{1}{y_i}(f_i(x^\circ) - \hat{z}_i - y_i) \right], \max_{j \in I_=(y)} (f_j(x^\circ) - \hat{z}_j + 1) \right\} \\ < F(\hat{x}) &= \max \left\{ \max_{i \in I_<(y)} \left[-\frac{1}{y_i}(f_i(\hat{x}) - \hat{z}_i - y_i) \right], \max_{j \in I_=(y)} (f_j(\hat{x}) - \hat{z}_j + 1) \right\} = 1. \end{aligned}$$

This means that $-\frac{1}{y_i}(f_i(x^\circ) - \hat{z}_i - y_i) < 1$ for every $i \in I_<(y)$, and $f_j(x^\circ) - \hat{z}_j + 1 < 1$ for every $j \in I_=(y)$. In other words, $f_i(x^\circ) < \hat{z}_i$ for every $i \in I$, which contradicts the local weak Pareto optimality of $\hat{z}$.

Now we know that $F(\hat{x}) \leq F(x)$ for every $x \in S \cap B(\hat{x};\delta)$. According to necessary Karush-Kuhn-Tucker optimality conditions (see e.g. Corollary 5.3.5 in [15]) there exists $0 \leq \mu \in \mathbf{R}^m$ such that $\mu_s g_s(\hat{x}) = 0$ for every $s = 1, \ldots, m$ and

$$0 \in \partial F(\hat{x}) + \sum_{s=1}^{m} \mu_s \partial g_s(\hat{x}).$$

According to Propositions 2.3.1, 2.3.3 and 2.3.12 in [2] we have

$$\begin{aligned}
&\partial F(\hat{x}) \\
&\subset \text{conv}\left\{\partial\Big(-\frac{1}{y_i}(f_i(\hat{x}) - \hat{z}_i - y_i)\Big), \partial\Big(f_j(\hat{x}) - \hat{z}_j + 1\Big) \mid i \in I_<(y),\ j \in I_=(y)\right\} \\
&\subset \text{conv}\left\{-\frac{1}{y_i}\partial(f_i(\hat{x}) - \hat{z}_i - y_i), \partial(f_j(\hat{x}) - \hat{z}_j + 1) \mid i \in I_<(y),\ j \in I_=(y)\right\} \\
&\subset \text{conv}\left\{-\frac{1}{y_i}\partial f_i(\hat{x}), \partial f_j(\hat{x}) \mid i \in I_<(y),\ j \in I_=(y)\right\}.
\end{aligned}$$

From the definition of convex hulls we know that there exists $0 \leq \lambda \in \mathbf{R}^k$ such that $\sum_{i \in I} \lambda_i = 1$ and

$$\partial F(\hat{x}) \subset \sum_{i \in I_<(y)} -\frac{\lambda_i}{y_i}\partial f_i(\hat{x}) + \sum_{j \in I_=(y)} \lambda_j \partial f_j(\hat{x}).$$

□

6. Concluding remarks

In this paper we have generalized the concept of trade-off directions by defining them as an optimal surface of the appropriate cone. For both convex and nonconvex cases, we derived optimality conditions linking trade-offs and corresponding efficiency. We noticed that in the case of lexicographic and strong efficiency, the set of generalized trade-off directions is either empty or it contains zero vector only. This reflects the fact that these two optimality principles do not contain non-zero trade-offs, i.e. there is no meaningful compromise between objectives in these cases. Indeed, the lexicographic optimality principle involves sequential optimization, and strong efficiency is a kind of parallel optimization, whose ideas are closer to single objective than to multiple objective optimization. Despite this, non-emptiness of the set of generalized trade-offs is quite informative itself, and therefore generalized trade-offs can be seen as an alternative advance tool to describe the given optimality conditions for all five optimality principles considered in the paper.

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$$
\begin{array}{ccccccc}
 & & D_z(Z) \cap \mathbf{R}^k_+ = D_z(Z) & & & & \\
 & & \Updownarrow & & & & \\
z \in GSE(Z) & \Leftrightarrow & K_z(Z) \cap \mathbf{R}^k_+ = K_z(Z) & \Leftrightarrow & N_z(Z) \cap \mathbf{R}^k_- = \mathbf{R}^k_- & \Leftrightarrow & G_S(Z) = \{0\} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
z \in GPP(Z) & \Leftrightarrow & K_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & \Leftrightarrow & N_z(Z) \cap \operatorname{int} \mathbf{R}^k_- \neq \emptyset & \Leftrightarrow & G_P(Z) \neq \emptyset \\
\Updownarrow^{***} & & \Updownarrow^{***} & & \Downarrow & & \Updownarrow \\
z \in GPO(Z) & \Leftrightarrow & D_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & \Leftrightarrow & & \Leftrightarrow^{***} & G_P(Z) \neq \emptyset \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
z \in GWP(Z) & \Leftrightarrow & D_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset & \Leftrightarrow & N_z(Z) \cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset & \Leftrightarrow & G_W(Z) \neq \emptyset \\
 & & \Updownarrow & & & & \\
 & & K_z(Z) \cap \operatorname{int} \mathbf{R}^k_- = \emptyset & & & &
\end{array}
$$

Table 6.1: Collection of the relationships in convex case with proper Pareto optimality ($*$ - tangent regularity , $**$ - local regularity, $***$ - regularity).

$$
\begin{array}{ccccccc}
 & & D_z(Z)\cap\mathbf{R}^k_+ = D_z(Z) & & & & \\
 & & \Updownarrow & & & & \\
z\in GSE(Z) & \Leftrightarrow & K_z(Z)\cap\mathbf{R}^k_+ = K_z(Z) & \Leftrightarrow & N_z(Z)\cap\mathbf{R}^k_- = \mathbf{R}^k_- & \Leftrightarrow & G_S(Z)=\{0\} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
z\in GLO(Z) & \Leftrightarrow & D_z(Z)\cap C^k_{lex} = D_z(Z) & \Leftrightarrow & & \Leftrightarrow & G_L(Z)=\{0\} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
z\in GPO(Z) & \Leftrightarrow & D_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}=\emptyset & \Leftrightarrow & & \Leftrightarrow^{***} & G_P(Z)\neq\emptyset \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
z\in GWP(Z) & \Leftrightarrow & D_z(Z)\cap\mathrm{int}\,\mathbf{R}^k_- = \emptyset & \Leftrightarrow & N_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}\neq\emptyset & \Leftrightarrow & G_W(Z)\neq\emptyset \\
 & & \Updownarrow & & & & \\
 & & K_z(Z)\cap\mathrm{int}\,\mathbf{R}^k_- = \emptyset & & & &
\end{array}
$$

Table 6.2: Collection of the relationships in convex case with lexicographic optimality ($*$ - tangent regularity , $**$ - local regularity, $***$ - regularity).

$$
\begin{array}{ccccccccc}
 & & & & & & & {}^{*}\Leftrightarrow & N_z(Z)\cap\mathbf{R}^k_- = \mathbf{R}^k_- \\
D_z(Z)\cap\mathbf{R}^k_+ = D_z(Z) & \Leftrightarrow & z\in GSE(Z) & \Rightarrow & z\in LSE(Z) & \Rightarrow & K_z(Z)\cap\mathbf{R}^k_+ = K_z(Z) & & \Updownarrow_{*} \\
 & & & & & & & \Leftrightarrow & G_S(Z)=\{0\} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & & & & {}^{*}\Leftrightarrow & N_z(Z)\cap\operatorname{int}\mathbf{R}^k_- \neq\emptyset \\
\operatorname{cl} D_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}=\emptyset & \Leftrightarrow & z\in GPP(Z) & \Rightarrow & z\in LPP(Z) & \Leftrightarrow & K_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}=\emptyset & & \Updownarrow^{*}_{*} \\
 & & & & & & & {}^{*}\Leftrightarrow & G_P(Z)\neq\emptyset \\
\Updownarrow^{***} & & \Updownarrow^{***} & & \Updownarrow^{**} & & \Updownarrow^{**} & & \\
 & & & & & & & {}^{**,*}\Leftrightarrow^{**} & N_z(Z)\cap\operatorname{int}\mathbf{R}^k_- \neq\emptyset \\
D_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}=\emptyset & \Leftrightarrow & z\in GPO(Z) & \Rightarrow & z\in LPO(Z) & {}^{**}\Leftrightarrow & F_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}=\emptyset & & \Updownarrow^{*}_{*} \\
 & & & & & & & {}^{**,*}\Leftrightarrow^{**} & G_P(Z)\neq\emptyset \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & & & & {}^{*}\Leftrightarrow & N_z(Z)\cap\mathbf{R}^k_-\setminus\{0\}\neq\emptyset \\
D_z(Z)\cap\operatorname{int}\mathbf{R}^k_- = \emptyset & \Leftrightarrow & z\in GWP(Z) & \Rightarrow & z\in LWP(Z) & \Rightarrow & K_z(Z)\cap\operatorname{int}\mathbf{R}^k_- = \emptyset & & \Updownarrow_{*} \\
 & & & & & & & \Leftrightarrow & G_W(Z)\neq\emptyset
\end{array}
$$

Table 6.3: Collection of the relationships in nonconvex case with proper Pareto optimality ($*$ - tangent regularity , $**$ - local regularity, $***$ - regularity).

$$
\begin{array}{ccccccccc}
 & & & & & & & {}^{*}\Leftrightarrow & N_z(Z)\cap \mathbf{R}^k_- = \mathbf{R}^k_- \\
D_z(Z)\cap \mathbf{R}^k_+ = D_z(Z) & \Leftrightarrow & z\in GSE(Z) & \Rightarrow & z\in LSE(Z) & \Rightarrow & K_z(Z)\cap \mathbf{R}^k_+ = K_z(Z) & & \Updownarrow_* \\
 & & & & & & & \Leftrightarrow & G_S(Z)=\{0\} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
D_z(Z)\cap C^k_{\mathrm{lex}} = D_z(Z) & \Leftrightarrow & z\in GLO(Z) & \Rightarrow & z\in LLO(Z) & \Rightarrow & F_z(Z)\cap C^k_{lex} = F_z(Z) & \Leftrightarrow & G_L(Z)=\{0\} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & & & & {}^{**,*}\Leftrightarrow^{**} & N_z(Z)\cap \mathrm{int}\ \mathbf{R}^k_- \neq \emptyset \\
D_z(Z)\cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & \Leftrightarrow & z\in GPO(Z) & \Rightarrow & z\in LPO(Z) & {}^{**}\Leftrightarrow & F_z(Z)\cap \mathbf{R}^k_- \setminus \{0\} = \emptyset & & \Updownarrow^*_* \\
 & & & & & & & {}^{**,*}\Leftrightarrow^{**} & G_P(Z)\neq\emptyset \\
\Downarrow & & \Downarrow & & \Downarrow & & \Updownarrow & & \\
 & & & & & & & {}^{*}\Leftrightarrow & N_z(Z)\cap \mathbf{R}^k_- \setminus \{0\} \neq \emptyset \\
D_z(Z)\cap \mathrm{int}\ \mathbf{R}^k_- = \emptyset & \Leftrightarrow & z\in GWP(Z) & \Rightarrow & z\in LWP(Z) & \Rightarrow & K_z(Z)\cap \mathrm{int}\ \mathbf{R}^k_- = \emptyset & & \Updownarrow_* \\
 & & & & & & & \Leftrightarrow & G_W(Z)\neq\emptyset
\end{array}
$$

Table 6.4: Collection of the relationships in nonconvex case with lexicographic optimality ($*$ - tangent regularity , $**$ - local regularity, $***$ - regularity).