Journal of Convex Analysis
Volume 19 (2012), No. 1, 249–279

Finitely Well-Positioned Sets

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Received: October 18, 2010
Revised manuscript received: February 8, 2011

We introduce and study finitely well-positioned sets, a class of asymptotically "narrow" sets that generalize the well-positioned sets recently investigated by Adly, Ernst and Théra in [1] and [3], as well as the plastering property of Krasnoselskii [13].

Keywords: Convex analysis, asymptotic cones, recession cones, plastering property

1991 Mathematics Subject Classification: 65K, 90C

1. Introduction

The analysis of asymptotic properties of sets and functions plays an important role in the study of existence of optima for noncoercive functions on normed spaces, that is, functions with unbounded level sets. For instance, in [17] we have introduced finitely well-positioned sets, a class of suitably asymptotically "narrow" sets that generalize the well-positioned sets recently investigated by Adly, Ernst and Théra in [1] and [3]. Using this notion, in [17] we have established necessary and sufficient conditions for optima of noncoercive functionals on reflexive spaces.

In the present paper we investigate in depth this class of sets and their relationships with well-positioned sets, as well as with Krasnoselskii's plastering property, as defined in [13]. We will also show that our notion is closely related to some notions of asymptotic compactness recently introduced in the literature.

Specifically, following [13] we say that a set C of a normed space V *allows plastering* if a uniformly positive continuous linear functional exists over C, that is, if there is $0 \neq x^* \in V^*$ such that $\langle x^*, x\rangle \geq \|x\|$ for all $x \in C$, where V^* is the topological dual of V. By defining the Bishop-Phelps cone $K_{x^*} = \{x : \langle x^*, x\rangle \geq \|x\|\}$, a set C allows plastering if and only if $C \subseteq K_{x^*}$ for some $x^* \in V^*$. For example, the positive cone of L^1 allows plastering, while that of L^p with $p > 1$ does not.

Though C can be any set, closed convex cones are the natural domain for this concept. In fact, a set allows plastering if and only if its closed conical hull does.[1]

[1]We refer to [13] for other properties related to this notion.

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

Recently, articles [1] and [3] have reconsidered this property under a different name. In particular, they call a set C *well-positioned* if there are $x_0 \in V$ and $x^* \in V^*$ such that $\langle x^*, x - x_0 \rangle \geq \|x - x_0\|$ for all $x \in C$. This amounts to require that the translated set $C - \{x_0\}$ allows plastering. Equivalently, $C \subseteq x_0 + K_{x^*}$. Many nice properties of well-positioned convex sets in reflexive spaces are studied in [1] and [3].

In this paper we study the following natural generalization of well-positioned sets, whose interest will become apparent through the properties addressed in the present paper.

Definition 1.1. A set C is said to be finitely well-positioned if $C \subseteq \bigcup_{i=1}^{n} C_i$, where each C_i is well-positioned.[2]

A set C is well-positioned if and only if its closed convex hull is, while C is finitely well-positioned if and only its closure is. This is a first basic difference between well-positioned sets and our generalization that shows that while closed convex sets are the relevant class of sets for well-positionedness, this is no longer the case for finite well-positionedness, whose interest goes beyond convex sets. As the paper will show, this natural generalization actually turns out to extend substantially the scope of well-positionedness.

The class of finitely well-positioned sets is obviously wider than that of the sets that are merely well-positioned. For instance, we will see momentarily that finite-dimensional vector subspaces (and so all their subsets) are finitely well-positioned, but not well-positioned. Our extension thus becomes relevant in infinite dimensional spaces, as discussed at length in the paper.

The paper is organized as follows. Section 3 establishes some of the main properties of finite well-positionedness. In particular, Theorem 3.6 characterizes this property in reflexive spaces in terms of asymptotic directions by showing that a set C is finitely well-positioned when there are no unbounded sequences $\{x_n\} \subseteq C$ such that $x_n / \|x_n\| \rightharpoonup 0$. This characterization implies that, as already mentioned, finite dimensional sets are finitely well-positioned. But, more importantly, it shows that finitely well-positioned sets feature in reflexive spaces a key convergence property of finite dimensional sets. This property will play an important role in many results of the paper and shows that finite well-positionedness can be viewed as a notion with a strong finite dimensional flavor.

Section 4 relates finite well-positionedness with some notions of asymptotic compactness studied by several authors. We show their equivalence in reflexive normed spaces. Section 5 studies in more detail convex sets by extending earlier results that papers [1] and [3] have proved for well-positioned sets.

Scalar functions feature some well known relevant sets, such as epigraphs and sublevel sets. Through them in Section 6 we introduce and study finitely well-positioned functions and other related concepts. They extend the notion of well-positioned functions introduced by [1] and [3].

[2]Since $C = \bigcup_{i=1}^{n} (C_i \cap C)$ and each $C_i \cap C$ is well-positioned, in this definition we could have equivalently used the equality.

Finally, notice that some of our results can be generalized by considering dual pairs $\langle V, W\rangle$, with V normed. Here the Bishop-Phelps cones are $K_w = \{x : \langle w, x\rangle \geq \|x\|\}$ with $w \in W$. The two cases $\langle V, V^*\rangle$ and $\langle V^*, V\rangle$ give rise to interesting results. For convenience in the paper we focus on $\langle V, V^*\rangle$, that is, on the weak topology. Though for brevity we omit details, most of the results that we establish in the paper hold also for the dual pair $\langle V^*, V\rangle$, typically under the assumption that V is separable.

2. Notation and Preliminaries

Unless otherwise specified, throughout the paper we consider subsets of a normed vector space V with norm $\|\cdot\|$. We denote by B_V its unit ball $\{x \in V : \|x\| \leq 1\}$ and by $S_V = \{x \in V : \|x\| = 1\}$ its unit sphere. The pairing of V with its topological dual V^* is usually denoted by $\langle x^*, x\rangle$, with $x \in V$ and $x^* \in V^*$. Norm convergence of a sequence will be denoted by $x_n \to x$, while the familiar notation $x_n \rightharpoonup x$ indicates weak convergence.

Given a function $f : V \to \overline{\mathbb{R}}$, we denote by $(f \leq \lambda)$ the sublevel set $\{x \in V : f(x) \leq \lambda\}$. A function $f : V \to (-\infty, \infty]$ is:

(i) *sw-lower semicontinuous* if all $(f \leq \lambda)$ are sw-closed;[3]
(ii) *lower semicontinuous* if all $(f \leq \lambda)$ are norm closed;
(iii) *coercive* if there is a nonempty sublevel set $(f \leq \lambda)$ that is norm bounded.[4]

We denote by $\operatorname{ri} C$ the relative interior of a convex set C, that is, the interior of C relative to $\overline{\operatorname{aff}}\, C$, the closed affine hull of C. In infinite dimensional spaces some other weaker notions may be adopted (see [9]). To avoid any ambiguity we will often write $\operatorname{ri}_H C$ to denote the interior of C relative to a linear space H containing C.

Throughout the paper K_{x^*} will denote the cone $\{x : \langle x^*, x\rangle \geq \|x\|\}$. Since $K_{\lambda x^*} \supseteq K_{x^*}$ for $\lambda > 1$, it is not restrictive to assume $\|x^*\| > 1$. This guarantees that K_{x^*} is sufficiently "large". In this case, K_{x^*} has nonempty interior because the open cone $\{x : \langle x^*, x\rangle > \|x\|\}$ is nonempty.

We will need a few asymptotic notions for sets. The *(weak) asymptotic cone* C_∞ is defined by

$$C_\infty = \left\{x \in V : \exists t_n \to \infty \text{ and } \{x_n\} \subseteq C \text{ such that } \frac{x_n}{t_n} \rightharpoonup x\right\}.$$

It is well-known that C_∞ reduces to the *recession cone*

$$R_C = \{y \in V : x + ty \in C \text{ for any } x \in C \text{ and all } t \geq 0\}$$

when C is closed and convex. The *lineality space* L_C of C is

$$L_C = \{y \in V : x + ty \in C \text{ for any } x \in C \text{ and all } t \in \mathbb{R}\}.$$

[3]We use "sw" in place of sequentially weakly for short.
[4]This notion is weaker than the standard one. Our definition implies that $f + \chi_{(f\leq\lambda)}$ is coercive in the usual sense (at least in the reflexive case).

Clearly, $L_C = R_C \cap (-R_C) = R_C \cap R_{-C}$.

The following modification of the asymptotic cone, with normalized asymptotic directions, will play an important role in the paper

$$B_C = \left\{ d \in V : \exists \{x_n\} \subseteq C \text{ with } \|x_n\| \to \infty \text{ and } \frac{x_n}{\|x_n\|} \rightharpoonup d \right\}.$$

The vectors $d \in B_C$ are not always of norm one, with possibly $d = 0$.

Clearly, $B_C = \emptyset$ when C is bounded, while $B_C \neq \emptyset$ when C is an unbounded set of a reflexive space. Moreover, $\operatorname{cone} B_C \subseteq C_\infty$ since $B_C \subseteq C_\infty$. Next we collect some other useful technical properties on asymptotic cones that we will need in the paper.

Proposition 2.1.

(*i*) *$C_\infty = \operatorname{cone} B_C$ if $B_C \neq \emptyset$ and $C_\infty = \{0\}$ if $B_C = \emptyset$;*
(*ii*) *B_C is weakly closed, provided V is either reflexive or with a separable dual;*
(*iii*) *B_C and C_∞ are sw-closed;*
(*iv*) *if $x_n/t_n \rightharpoonup d \neq 0$, with $t_n \to \infty$ and $\{x_n\} \subseteq C$, then there is an unbounded subsequence $\{x_{n_k}\}$ and a scalar $\lambda > 0$ such that $x_{n_k}/\|x_{n_k}\| \rightharpoonup \lambda d$.*

Proof. (*i*) Let $0 \neq d \in C_\infty$. This implies $x_n/t_n \rightharpoonup d$, with $x_n \in C$ and $t_n \to +\infty$. We know that $\liminf_n \|x_n\|/t_n \geq \|d\| > 0$. Hence, a subsequence $\{x_{n_k}\}$ exists such that $\lim_k \|x_{n_k}\|/t_{n_k} = \lambda > 0$. Notice that $\|x_{n_k}\| \to \infty$. Consequently,

$$\frac{x_{n_k}}{\|x_{n_k}\|} = \frac{x_{n_k}}{t_{n_k}} \frac{t_{n_k}}{\|x_{n_k}\|} \rightharpoonup \frac{d}{\lambda} \in B_C. \tag{1}$$

Hence, $B_C \subseteq \{0\}$ implies $C_\infty = \{0\}$. In particular, if $B_C = \emptyset$, then $C_\infty = \{0\}$, while if $B_C = \{0\}$ then $C_\infty = \operatorname{cone} B_C = \{0\}$. Moreover, if $B_C \neq \emptyset$ and $B_C \neq \{0\}$, then $C_\infty \neq \{0\}$, since $C_\infty \supseteq B_C$. Consequently, $\operatorname{cone} B_C = C_\infty$ by (1).

(*ii*) Let us first prove that B_C is weakly closed if V^* is separable. Recall that $\overline{B}_C^w \subset B_V$, and pick $d \in \overline{B}_C^w$. A metric δ on V exists for which the weak topology on every norm bounded subset D of V coincides with the topology induced on D by δ (see, e.g., [5, Theorem 3.35]). This ensures the existence of a sequence $\{d_n\} \subseteq B_C$ such that $\delta(d_n, d) \to 0$. Passing to a subsequence if necessary, we can suppose $\delta(d_n, d) \leq n^{-1}$. Since $d_n \in B_C$, there is $x_n \in C$ for which $\delta(x_n/\|x_n\|, d_n) \leq n^{-1}$ and $\|x_n\| \geq n$. It follows

$$\delta\left(\frac{x_n}{\|x_n\|}, d\right) \leq \delta\left(\frac{x_n}{\|x_n\|}, d_n\right) + \delta(d, d_n) \leq \frac{2}{n}.$$

Namely, $\delta(x_n/\|x_n\|, d) \to 0$ and so $x_n/\|x_n\| \rightharpoonup d \in B_C$ with $\|x_n\| \to \infty$. Hence, B_C is weakly closed.

Suppose that V reflexive and $d \in \overline{B}_C^w$. By Day's Lemma (see [18, Lemma 2.8.5 and Corollary 2.8.7]), a sequence $d_n \rightharpoonup d$ exists with $\{d_n\} \subseteq B_C$. Hence, sequences $\{x_m^n\} \subseteq C$ exist such that $x_m^n/\|x_m^n\| \rightharpoonup d_n$ and $\|x_m^n\| \to \infty$ as $m \to \infty$. Without loss of generality (wlog for short), we can suppose that $\|x_m^n\| \geq n$ for all m.

Consider the separable linear subspace $W = \overline{\operatorname{span}\{x_m^n / \|x_m^n\|\}}_{n,m}$. The $\sigma(W, W^*)$ convergence of sequences in W is equivalent to their $\sigma(V, V^*)$ convergence (see, e.g., [18, Proposition 2.5.22]). Since W^* is separable, an argument similar with the previous one implies the existence of a sequence $x_{m_k}^{n_k} / \left\|x_{m_k}^{n_k}\right\| \rightharpoonup d$ as $k \to \infty$. Clearly, $\left\|x_{m_k}^{n_k}\right\| \to \infty$ because $\left\|x_{m_k}^{n_k}\right\| \geq n_k$, and so $d \in B_C$.

(iii) Similar arguments apply for spaces not necessarily reflexive. But, only the sequentially weakly closure of B_C can be proved. We omit details. As to C_∞, suppose that V^* is separable and that $d_n \rightharpoonup d$, with $d_n \in C_\infty$. Consequently there are sequences $\{x_m^n / t_m^n\}_m$ for all n such that $x_m^n / t_m^n \rightharpoonup d_n$ as $m \to \infty$. Fix $x^* \in B_{V^*}$. As $\langle x^*, d_n\rangle \to \langle x^*, d\rangle$, there is a scalar A such that $|\langle x^*, d_n\rangle| \leq A$. Fix n. As $\langle x^*, x_m^n / t_m^n\rangle \to \langle x^*, x_m^n / t_m^n\rangle$ as $m \to \infty$, there is $m = m(n)$ so that $|\langle x^*, x_m^n / t_m^n\rangle| \leq 2A$ for all n and $m \geq m(n)$. By the Banach-Steinhaus Theorem, there is a scalar K such that $\|x_m^n / t_m^n\| \leq K$ for all n and $m \geq m(n)$. Since V^* is separable, the weak-topology is metrizable on the ball $\|x\| \leq K$. Let δ be such a metric. For all k, we pick $\delta(d, d_{n_k}) < 1/k$. Moreover, we select a point $x_{m_k}^{n_k} / t_{m_k}^{n_k}$ for which $\delta\left(d_{n_k}, x_{m_k}^{n_k} / t_{m_k}^{n_k}\right) < 1/k$, $m_k \geq m(n_k)$ and $t_{m_k}^{n_k} \geq k$. This implies that $x_{m_k}^{n_k} / t_{m_k}^{n_k} \rightharpoonup d$ and thus $d \in C_\infty$. With the standard technique we can extend this argument to general spaces by considering the separable subspace $W = \overline{\operatorname{span}}\left(\{x_m^n / t_m^n\}_{n,m}\right)$.

(iv) This has been already proved in point (i). □

Example 2.2. The hypotheses on V in Proposition 2.1(iv) are necessary. Indeed, let $C = l_1$. Clearly, $C_\infty = l_1$ and $B_C = S_{l_1}$ thanks to the Schur property of l_1. On the other hand, S_{l_1} is sequentially weakly closed, but not weakly closed, as it is well-known that there is a net $\{x_\alpha\} \subseteq S_{l_1}$ such that $x_\alpha \rightharpoonup 0$. Notice that the space l_1 is not reflexive and its dual is not separable. □

Remark 2.3. If C is finitely well-positioned, then its asymptotic cone C_∞ is clearly finitely well-positioned. Actually, from $C \subseteq \bigcup_{i=1}^n \left(K_{x_i^*} + x_i\right)$ it follows $C_\infty \subseteq \bigcup_{i=1}^n K_{x_i^*}$. In fact, C_∞ turns out to be the reunion of a finite number of cones that allow plastering, since $C_\infty = \bigcup_{i=1}^n \left(K_{x_i^*} \cap C_\infty\right)$.

3. General Results

Consider the closed convex sets

$$K_{x^*}(m) = \{x \in V : \langle x^*, x\rangle \geq \|x\| - m\}$$

associated with K_{x^*} and $m \in \mathbb{R}$. They are all well-positioned sets (see point (iii) below). Moreover $K_{x^*}(m) \supseteq K_{x^*}$ if $m \geq 0$ and $K_{x^*}(m) \subseteq K_{x^*}$ if $m \leq 0$.

Proposition 3.1.

(i) *A cone K is well-positioned if and only if it allows plastering.*

(ii) *An unbounded set C is well-positioned if and only if the set $C \cap \{\|x\| \geq \rho\}$ allows plastering for ρ large enough.*

(iii) *C is well positioned if and only if $C \subseteq K_{x^*}(m)$ for some $x^* \in V$ and $m \in \mathbb{R}$.*

Proof. (i) We omit the simple proof.

(ii) Let C be well-positioned, i.e., $\langle x^*, x - x_0\rangle \geq \|x - x_0\|$ for all $x \in C$. Then,

$$\langle x^*, x\rangle = \langle x^*, x - x_0\rangle + \langle x^*, x_0\rangle \geq \|x\| \left(\left\| \frac{x}{\|x\|} - \frac{x_0}{\|x\|} \right\| + \left\langle x^*, \frac{x_0}{\|x\|} \right\rangle \right).$$

We have $\langle x^*, x_0/\|x\|\rangle \to 0$ and $\|x/\|x\| - x_0/\|x\|\| \to 1$, when $\|x\| \to \infty$. Therefore, $\langle x^*, x\rangle \geq (1-\eta)\|x\|$ for $\|x\| \geq \rho$ large enough. Hence, $C \cap \{\|x\| \geq \rho\}$ allows plastering.

As to the converse, suppose that $C \cap (\rho B)^c$ allows plastering, where ρB is the open ball of radius ρ. That is, $C \cap (\rho B)^c \subseteq K_{x^*}$. We can assume $\|x^*\| > 1$ so that K_{x^*} has a nonempty interior. Fix $d \in \operatorname{int} K_{x^*}$. Clearly, d is a recession direction of K_{x^*}. Hence, $K_{x^*} + \lambda d \subseteq K_{x^*}$, which implies $K_{x^*} \subseteq K_{x^*} - \lambda d$. Therefore, $C \cap (\rho B)^c \subseteq K_{x^*} - \lambda d$ for all $\lambda \geq 0$. On the other hand, $\rho B + \lambda d \subseteq K_{x^*}$ for λ large enough.[5] It follows that $\rho B \subseteq K_{x^*} - \lambda d$, and so $C \cap \rho B \subseteq K_{x^*} - \lambda d$. As $C \cap (\rho B)^c \subseteq K_{x^*} - \lambda d$, we conclude that $C \subseteq K_{x^*} - \lambda d$ if λ is large enough. Namely, C is well-positioned.

(iii) If C is well positioned, $C \subseteq K_{x^*} + x_0$. It is easy to check that $K_{x^*} + x_0 \subseteq K_{x^*}(m)$, with $m = \langle x^*, x_0\rangle - \|x_0\|$. Conversely, suppose $C \subseteq K_{x^*}(m)$. If $m \leq 0$, then $K_{x^*}(m) \subseteq K_{x^*}$ and so C allows plastering. Let $C \subseteq K_{x^*}(m)$ with $m > 0$ and consider any point x of C with $\|x\| \geq \alpha m$ and $\alpha > 1$. Then,

$$\langle x^*, x\rangle \geq \|x\| - m \geq \|x\| - \alpha^{-1}\|x\| = \alpha^{-1}(\alpha - 1)\|x\|.$$

By point (ii), C is well-positioned. □

As well known, bounded sets are well-positioned (see [1], [3], and [13]). For later reference we report this property, which is here derived from Proposition 3.1(iii).

Corollary 3.2. *Bounded sets are well-positioned.*

Proof. Suppose C is bounded, say $C \subseteq \rho B_V$ for some $\rho > 0$. Then, for any $x^* \in V^*$ it holds $\langle x^*, x\rangle \geq \alpha$ for all $x \in C$. If $x \in C$, then $\langle x^*, x\rangle \geq \alpha = \alpha - \rho + \rho \geq \alpha - \rho + \|x\|$ and so $C \subseteq K_{x^*}(\alpha - \rho)$. □

By Proposition 3.1(ii), a set C is well-positioned if $C = C_0 \cup C_1$, where C_0 is bounded and C_1 allows plastering. Hence, a set C is finitely well-positioned if $C = C_0 \cup (\bigcup_{i=1}^n C_i)$, where C_0 is bounded and each C_i allows plastering. The next useful properties are other implications of Proposition 3.1 (more general results will be seen later in the paper).

Proposition 3.3.

(i) *Let $C \subseteq H$, where H is a linear subspace of V. If C is well-positioned in H, then C is well-positioned also as subset of V.*

(ii) *Let $T : V \longrightarrow W$ be a linear isomorphism between two normed spaces. If $C \subseteq V$ is well-positioned, then its image $T(C)$ is well-positioned.*

[5] This is actually equivalent to $(\rho/\lambda)B + d \subseteq K_{x^*}$, which is true if $d \in \operatorname{int} K_{x^*}$.

This result still holds if we replace well-positioned sets with finitely well-positioned ones.

Proof. (i) Use Proposition 3.1(iii) and the Hahn-Banach Theorem.

(ii) Use Proposition 3.1(iii) and the fact that the image $T(K_{x^*})$ of the cone K_{x^*} is a cone of V_1 that allows plastering. More specifically, $T(K_{x^*}(m)) \subseteq \{y \in V_1 : \langle (T^{-1})^* x^*, y\rangle \geq \|T\|^{-1}\|y\| - m\}$. □

The next lemma is a key element for the present theory and will entail several important consequences. In reading it recall that in the applications that we have in mind C will be a portion of the unit sphere.

Lemma 3.4. *Let C be a bounded set with $0 \notin \overline{C}$. Then:*

(i) $0 \notin \overline{C}^w$ *if and only if* $C = \bigcup_{i=1}^n C_i$, *where each* C_i *allows plastering;*
(ii) $0 \notin \overline{\mathrm{co}}\, C$ *if and only if* C *allows plastering.*

If, in addition, V is reflexive or with separable dual, in (i) we can replace weak closure with sw-closure.

Proof. (i) Suppose that $C = \bigcup_{i=1}^n C_i$, where each C_i allows plastering. Let $d \in \overline{C}^w$. As $\overline{C}^w = \overline{(\bigcup_{i=1}^n C_i)}^w = \bigcup_{i=1}^n \overline{C_i}^w$, it follows that $d \in \overline{C_i}^w$ for some i. Let $x_\alpha \rightharpoonup d$ be a net with $\{x_\alpha\} \subseteq C_i$. Since C_i allows plastering, $\langle u^*, x_\alpha\rangle \geq \|x_\alpha\|$ for some $u^* \in V^*$. As $0 \notin \overline{C}$, $\|x_\alpha\| \geq \eta > 0$. Taking the limit we get $\langle u^*, d\rangle \geq \eta > 0$. Hence, $d \neq 0$ and so $0 \notin \overline{C}^w$.

Conversely, suppose that $0 \notin \overline{C}^w$. There will be a weak neighborhood of zero that does not meet C. In other words, there is $\varepsilon > 0$ and a finite sequence $\{x_i^*\}_{i=1}^n$ of elements of V^* such that $|\langle x, x_i^*\rangle| < \varepsilon$ for each i implies $x \notin C$. Equivalently, $x \notin C$ if $\langle x, \pm x_i^*\rangle < \varepsilon$ for each i. Consider the finite set $D = \{\pm x_i^* : i = 1, ..., n\}$. The above property can then be equivalently described as: for each $x \in C$ there is $u^* \in D$ such that $\langle u^*, x\rangle \geq \varepsilon$. Define the possibly empty sets $C_{u^*} = C \cap \{x : \langle u^*, x\rangle \geq \varepsilon\}$ for each $u^* \in D$. The above arguments imply $C = \bigcup_{u^* \in D} C_{u^*}$. It remains to check that every $C_{u^*} \neq \emptyset$ allows plastering. In fact, $\langle u^*, x\rangle \geq \varepsilon$ for all $x \in C_{u^*}$. Since C is bounded, $\|x\| \leq N$ for $x \in C_{u^*}$. Hence, $\langle u^*, x\rangle \geq \varepsilon = (\varepsilon/N) \cdot N \geq (\varepsilon/N)\|x\|$, which shows that C_{u^*} allows plastering.

(ii) Let $0 \notin \overline{\mathrm{co}}\, C$. By a separation argument, there is $x^* \in V^*$ such that $\langle x, x^*\rangle \geq \varepsilon > 0$ for all $x \in \overline{\mathrm{co}}\, C$. As C is bounded, say $\|x\| \leq N$ for $x \in C$, then $\langle x, x^*\rangle \geq \varepsilon = (\varepsilon/N) \cdot N \geq (\varepsilon/N)\|x\|$ for all $x \in C$, and so C allows plastering. Conversely, suppose C allows plastering. Hence, $\langle x, x^*\rangle \geq \|x\|$ for all $x \in C$. Moreover, $0 \notin \overline{C}$, and so there is a positive number ε such that $\|x\| \geq \varepsilon$ for all $x \in C$. It follows $\langle x, x^*\rangle \geq \|x\| \geq \varepsilon$ for all $x \in \overline{\mathrm{co}}\, C$. Therefore, $0 \notin \overline{\mathrm{co}}\, C$.

The proof is completed by noticing that $\overline{C}^{\mathrm{seq}\, w} = \overline{C}^w$ under our hypotheses. For, if V^* is separable, the bounded sets of V are weakly metrizable and thus $\overline{C}^{\mathrm{seq}\, w} = \overline{C}^w$. If V is reflexive, the C is relatively weakly compact. By Day's Lemma, if $d \in \overline{C}^w$, there is a sequence in C that converges weakly to d. Also in this case the desired property thus holds. □

Remark 3.5. (i) Lemma 3.4(i) is closely related to Kadets and Pelczynski's criterion. They show that in reflexive spaces the condition $0 \notin \overline{C}^w$ is equivalent to the fact that C fails to contain a basic sequence (see [4, Theorem 1.5.6] for details).

(ii) It is well-known that $\overline{S}_V^w = B_V$ holds for any infinite dimensional normed spaces. Hence $0 \in \overline{S}_V^w$. Since $V = \operatorname{cone} S_V$, infinite dimensional vector spaces are never finitely well-positioned.

The next key characterization of finitely well-positioned sets is a first notable consequence of Lemma 3.4. Observe that the condition $0 \notin B_C$ means that there is no unbounded sequence $\{x_n\} \subseteq C$ such that $x_n / \|x_n\| \rightharpoonup 0$.

Theorem 3.6. *Let $C \subseteq V$ that is either reflexive or with a separable dual. The following properties are equivalent:*

(i) $0 \notin B_C$;

(ii) *C is a finitely well-positioned set.*

When V is reflexive, (i) and (ii) are equivalent to:

(iii) *for any unbounded sequence $\{x_n\} \subseteq C$, there is a subsequence $\{x_{n_k}\}$ and a scalar sequence $\{t_k\}$ such that $t_k \to \infty$ and $x_{n_k}/t_k \rightharpoonup d \neq 0$.*

A first important consequence of this result is that finite dimensional spaces, and so all their subsets, are finitely well-positioned.[6] As remarked in the Introduction, this shows that finite well-positionedness is relevant in infinite dimensional spaces.

Proof. The result holds for a bounded set C since $B_C = \emptyset$ and, by Corollary 3.2, is well-positioned. We will thus suppose that C is unbounded.

(ii) implies (i). Suppose that $C = \bigcup_{i=1}^n C_i$, where each C_i is well-positioned and let $\|x_n\| \to \infty$ and $x_n / \|x_n\| \rightharpoonup d$. Wlog we can suppose $\{x_n\} \subseteq C_{i_0}$ for some i_0. Moreover, in view of Proposition 3.1(ii), wlog we can suppose that C_{i_0} allows plastering. Therefore, $\langle x^*, x_n \rangle \geq \|x_n\|$, i.e., $\langle x^*, x_n / \|x_n\| \rangle \geq 1$. This implies $\langle x^*, d \rangle \geq 1$ and so $d \neq 0$.

(i) implies (ii). Suppose first that the dual V^* is separable and that (i) holds. Fix a radius $\rho > 0$ and define the nonempty set $S_\rho = \{x / \|x\| : x \in C \text{ and } \|x\| \geq \rho\} \subseteq S_V$. We claim that $0 \notin \overline{S}_\rho^{\operatorname{seq} w}$ for ρ large enough. Suppose not. Then, there is a sequence $\rho_n \uparrow \infty$ such that $0 \in \overline{S}_{\rho_n}^{\operatorname{seq} w}$ for all n. Taking $n = 1$, there is a sequence $\{u_n\} \subseteq S_{\rho_1}$ for which $u_n \rightharpoonup 0$. On the other hand, $u_n = x_n^1 / \|x_n^1\|$ with $\|x_n^1\| \geq \rho_1$. By hypothesis, the sequence $\|x_n^1\|$ is necessarily bounded (otherwise, $x_n^1 / \|x_n^1\| \rightharpoonup 0$, thus contradicting ($i$)). Therefore, there is some ρ_{n_2} such that $\|x_n^1\| < \rho_{n_2}$ for all n. Iterating the same argument for the set $S_{\rho_{n_2}}$, we obtain a new sequence $\{x_n^2\}$ having the properties $\|x_n^2\| \geq \rho_{n_2}$, $x_n^2 / \|x_n^2\| \rightharpoonup 0$, and $\|x_n^2\| < \rho_{n_3}$, and so on. Consequently, we get countably many sequences $\left\{x_n^k\right\}$ for which $x_n^k / \left\|x_n^k\right\| \rightharpoonup 0$ as $n \to \infty$, and $\rho_{n_k} \leq \left\|x_n^k\right\| < \rho_{n_{k+1}}$. Since V^* is separable, the unit ball of V is weakly metrizable (see the proof of Lemma 3.4). Denote by δ such a met-

[6] See Corollary 3.19 for a slightly more general result.

ric. For all k there is an element $x^k_{n_k}/\left\|x^k_{n_k}\right\|$ of the sequence $\left\{x^k_n/\left\|x^k_n\right\|\right\}_n$ such that $\delta\left(x^k_{n_k}/\left\|x^k_{n_k}\right\|,0\right) < 1/k$. Therefore, by construction, for the the sequence $\left\{x^k_{n_k}/\left\|x^k_{n_k}\right\|\right\}_k$ it holds $x^k_{n_k}/\left\|x^k_{n_k}\right\| \rightharpoonup 0$ as $k \to \infty$ and $\left\|x^k_{n_k}\right\| \to \infty$. This contradicts (i). Hence, $0 \notin \overline{S_{\overline{\rho}}}^{\operatorname{seq} w}$ for $\overline{\rho}$ sufficiently large.

By Lemma 3.4(i), $S_{\overline{\rho}} = \bigcup_{i=1}^n C_i$ where each C_i allows plastering. On the other hand, we have $\{x \in C : \|x\| \geq \overline{\rho}\} \subseteq \operatorname{cone} S_{\overline{\rho}} = \operatorname{cone} \bigcup_{i=1}^n C_i = \bigcup_{i=1}^n \operatorname{cone} C_i$. Clearly each $\operatorname{cone} C_i$ allows plastering, and thus

$$C \subseteq (C \cap \overline{\rho} B) \cup \bigcup_{i=1}^n \operatorname{cone} C_i.$$

We conclude that C is finitely well-positioned.

Suppose now that V is reflexive. The proof proceeds in a similar way until the construction of the sequences $\left\{x^k_n\right\}$. Now consider the linear space $W = \overline{\operatorname{span}\{x^k_n\}_{n,k}}$, which is a separable subspace of V. The subspace W is reflexive and its dual W^* is separable. Moreover the $\sigma(W, W^*)$ convergence of sequences in W is equivalent to their $\sigma(V, V^*)$ convergence (see, e.g., [18, Proposition 2.5.22]). Therefore, by using the existing metric on $B_V \cap W$ we can extract a sequence $\left\{x^k_{n_k}\right\}_k$ such that $x^k_{n_k} \rightharpoonup 0$ and $\left\|x^k_{n_k}\right\| \to \infty$. This leads to a contradiction and the proof proceeds as in the previous case.

We prove that (iii) is equivalent to (i) provided V is reflexive. Assume (i) and let $\{x_n\}$ be an unbounded sequence. Consider the sequence $x_n/\|x_n\|$. As V is reflexive, there is a convergent subsequence $x_{n_k}/\|x_{n_k}\| \rightharpoonup d$ and d does not vanish by (i).

Now assume (iii) and let $\{x_n\} \subseteq C$ with $\|x_n\| \to \infty$ and $x_n/\|x_n\| \rightharpoonup d$. By (iii) there is a subsequence $\{x_{n_k}\}$ and a sequence $\{t_k\}$ for which $t_k \to \infty$ and $x_{n_k}/t_k \rightharpoonup d_1 \neq 0$. By Proposition 2.1$(iv)$, there is a subsequence $\left\{x_{n_{k_r}}\right\}$ for which $x_{n_{k_r}}/\left\|x_{n_{k_r}}\right\| \rightharpoonup \lambda d_1 \neq 0$. Hence $x_n/\|x_n\| \rightharpoonup d \neq 0$ and (i) holds. □

In Theorem 3.6 condition (ii) implies (i) without any assumption on V. Something more can be actually said.

Proposition 3.7. *If C is finitely well-positioned, then $0 \notin \overline{B_C}^w$ and there is no net $\{x_\alpha\} \subseteq C$ such that $x_\alpha/\|x_\alpha\| \rightharpoonup 0$ and $\|x_\alpha\| \to \infty$.*

Similarly, it can be proved that $0 \notin \overline{\operatorname{co}} B_C$, provided C is well-positioned.

Proof. Let $C \neq \emptyset$ be finitely well-positioned. As remarked above, Theorem 3.6 implies $0 \notin B_C$ even without the assumption of reflexivity on the space. By Proposition 2.1(iii), B_C is norm closed, and so $0 \notin \overline{B}_C$. Moreover, B_C is norm bounded and $B_C \subseteq C_\infty$. On the other hand, if C is finitely well-positioned, C_∞ is a reunion of finitely many sets that allow plastering (see Remark 2.3). The same is true for its subset B_C. Therefore, by applying Lemma 3.4(i) to the set B_C, we get the desired result $0 \notin \overline{B}^w_C$.

Now suppose that $\{x_\alpha\} \subseteq C$, with $\alpha \in \Lambda$, is a net such that $x_\alpha/\|x_\alpha\| \rightharpoonup d$ and $\|x_\alpha\| \to \infty$. Since C is finitely well positioned, $C \subseteq \bigcup_{i=1}^n K_{x^*_i}(m_i)$. Set

$\Lambda_i = \left\{\alpha \in \Lambda : x_\alpha \in K_{x_i^*}(m_i)\right\}$. Clearly, for some $\bar{i} \in \{1, 2, ..., n\}$ the restriction $\alpha \in \Lambda_{\bar{i}}$ is a subnet of Λ. Hence, $x_\alpha / \|x_\alpha\| \rightharpoonup d$ and $\|x_\alpha\| \to \infty$ $(\alpha \in \Lambda_{\bar{i}})$. From $\left\langle x_\alpha, x_{\bar{i}}^* \right\rangle \geq \|x_\alpha\| - m_{\bar{i}}$ for $\alpha \in \Lambda_{\bar{i}}$, by dividing by $\|x_\alpha\|$ and taking the limit, we get $\left\langle d, x_{\bar{i}}^* \right\rangle \geq 1$, which implies $d \neq 0$. □

Bounded sets have trivial asymptotic cones and, by Corollary 3.2, are well-positioned. The next remarkable consequence of Theorem 3.6 shows that under finite well-positionedness the converse holds. That is, finite well-positionedness turns out to be the property that characterizes bounded sets among the sets that have trivial asymptotic cones.

Corollary 3.8. *A subset C of a reflexive space V is bounded if and only if C is finitely well-positioned and $C_\infty = \{0\}$.*

Proof. We prove the "if" part, the converse being trivial. Let C be finitely well-positioned with $C_\infty = \{0\}$. Suppose per contra that C is unbounded. As V is reflexive, $B_C \neq \emptyset$. Hence, $B_C = C_\infty = \{0\}$. By Theorem 3.6, this is a contradiction. □

A consequence of this characterization is that in reflexive spaces, unbounded convex sets that are linearly bounded are never finitely well-positioned.

Example 3.9. Corollary 3.8 may fail if V is not reflexive. Consider the positive cone l_1^+ of l_1. Let $e = (1, 1, ...) \in l_\infty$. For any $x \in l_1^+$, $\langle x, e\rangle = \sum_{i=1}^\infty x_i = \|x\|$. Hence l_1^+ allows plastering. Set $C = \{x \in l_1 : 0 \leq x_i \leq \alpha_i \text{ for each } i\}$, with $\sum_{i=1}^\infty \alpha_i = \infty$. The closed and convex set C is unbounded because $x^n = (\alpha_1, ..., \alpha_n, 0, 0, ...) \in C$ for all n and $\|x^n\| = \sum_{i=1}^n \alpha_i \to \infty$. It is also linearly bounded, and so $C_\infty = \{0\}$, since for each $0 \neq x \in l_1^+$ it holds $tx \notin C$ for $t > 0$ large enough. On the other hand, $C \subseteq l_1^+$ allows plastering. Notice that $B_C = \emptyset$. □

Example 3.10. Whether a given set is well-positioned depends on the considered dual pair. The positive cone l_1^+ of l_1 is well-positioned with respect to the pair $\langle l_1, l_\infty\rangle$, but not with respect to $\langle l_1, c_0\rangle$. For, if $\{e_n\} \subseteq l_1^+$ is the units' sequence, we have $e_n \overset{w^*}{\rightharpoonup} 0$. By Theorem 3.6 or Proposition 3.7, l_1^+ is not finitely well-positioned. □

The next result is a further consequence of Lemma 3.4. Unlike Theorem 3.6, no assumption is made on V. It is a general version of [3, Lemma 2.1]. Set

$$S_\rho = \left\{\frac{x}{\|x\|} : x \in C \text{ and } \|x\| \geq \rho\right\}.$$

Proposition 3.11.

(*i*) *C is well-positioned if and only if $0 \notin \overline{\text{co}}\, S_\rho$ for ρ large enough.*

(*ii*) *C is finitely well-positioned if and only if $0 \notin \overline{S}_\rho^w$ for ρ large enough.*

Proof. (*i*) Suppose that C is well-positioned. By Proposition 3.1(*ii*), $C \cap \{\|x\| \geq \rho\}$ allows plastering for some ρ. Hence, $\langle x^*, x\rangle \geq \|x\|$ for all $x \in C$ and $\|x\| \geq \rho$.

Namely, $\langle x^*, x/\|x\|\rangle \geq 1$. We conclude that $u \in S_\rho$ implies $\langle x^*, u\rangle \geq 1$. Clearly $0 \notin \overline{\mathrm{co}}\, S_\rho$.

Conversely, if $0 \notin \overline{\mathrm{co}}\, S_\rho$, by Lemma 3.4(*ii*) S_ρ allows plastering. Hence $\mathrm{cone}\,(S_\rho) \supseteq C \cap \{\|x\| \geq \rho\}$ allows plastering. By Proposition 3.1, C allows plastering.

(*ii*) Let C be finitely well-positioned. By Proposition 3.1(*ii*), $C \cap \{\|x\| \geq \rho\}$ is a finite reunion of sets that allow plastering, when ρ is sufficiently large. Therefore, there are nonzero $\{x_i^*\}_{i=1}^n \subseteq V^*$ such that $x \in C \cap \{\|x\| \geq \rho\}$ implies $\langle x_i^*, x\rangle \geq \|x\|$ for some i. Hence, for all $u \in S_\rho$, $\langle x_i^*, u\rangle \geq 1$ for some i. Consequently, if $v \in V$ is any point such that $|\langle x_i^*, v\rangle| \leq 1/2$ for $i = 1, 2, .., n$, then $v \notin S_\rho$. On the other hand, the set of points: $|\langle x_i^*, x\rangle| \leq 1/2$ for all i, is a weak neighborhood of 0. Hence, $0 \notin \overline{S}_\rho^w$.

Conversely, suppose $0 \notin \overline{S}_\rho^w$ for some ρ. Lemma 3.4 implies that S_ρ is the union of finitely many sets that allow plastering. Thus, $C \cap \{\|x\| \geq \rho\} \subseteq \mathrm{cone}\, S_\rho$, where $\mathrm{cone}\, S_\rho$ is the finite union of sets that allow plastering. This is enough to conclude that C is finitely well-positioned. □

Corollary 3.12. *Suppose that V is reflexive or has a separable dual. A set $C \subseteq V$ is finitely well-positioned if and only if the same holds for $C \cap W$, for all closed and separable subspaces W of V.*

Proof. Suppose that all $C \cap W$ are finitely well-positioned and that, per contra, C is not. Let $\rho_m \uparrow \infty$. By Proposition 3.11, we have $0 \in \overline{S}_{\rho_m}^w$ for all m. By the assumptions on the space V, we have $0 \in \overline{S}_{\rho_m}^{\mathrm{seq}\, w}$. Hence, there are sequences $\{x_n^m\}$ such that $\|x_n^m\| \geq \rho_m$ and $x_n^m/\|x_n^m\| \rightharpoonup 0$, as $n \to \infty$, for each m. Set $W = \overline{\mathrm{span}\,\{x_n^m\}}$. W is a closed and separable subspace. By construction, $C \cap W$ is not finitely well-positioned, which leads to a contradiction. □

3.1. Polyhedral Cuts

We present a useful criterion for finite well-posedness based on the boundedness of sets' slices. Specifically, given a set C, a finite set of nonzero functionals $D = \{x_i^*\}_{i=1}^n \subseteq V^*$, and scalars $T = \{t_i\}_{i=1}^n$, the set

$$C\,(D, T) = \{x \in C : \langle x_i^*, x\rangle \leq t_i \text{ for each } i\}$$

is the *slice* of C determined by D and T.

Definition 3.13. A set C is said to have a (bounded) polyhedral cut if there is a set of nonzero functionals $D = \{x_i^*\}_{i=1}^n \subseteq V^*$ such that, for each collection $T = \{t_i\}_{i=1}^n$ of scalars, the slices $C\,(D, T)$ are either empty or bounded.

The next result extends the idea behind [3, Lemma 2.2].

Proposition 3.14. *A convex set C is finitely well-positioned if and only if it has a polyhedral cut.*

The proof of this result rests on couple of lemmas of some independent interest. The first one shows that one direction holds even without convexity.

Lemma 3.15. *Finitely well-positioned sets have a polyhedral cut.*

Proof. Let C be finitely well-positioned. By Proposition 3.1(*ii*), there is $D = \{x_i^*\}_{i=1}^n \subseteq V^* \setminus \{0\}$ and scalars $\{m_i\}_{i=1}^n$ such that $x \in C$ implies $\langle x_i^*, x\rangle \geq \|x\| - m_i$ for some i. Hence, if $\langle x_i^*, x\rangle \leq t_i$ for all i, it follows that $\|x\| \leq \max_i (m_i + t_i)$ for all $x \in C$. Consequently, $C(D, T)$ are bounded or empty for all $T = \{t_i\}_{i=1}^n$. $\square$

Next we show that the other direction in Proposition 3.14 holds for a convex set even if there is only a single nonempty and bounded slice, provided it satisfies a Slater-type condition.

Lemma 3.16. *A convex set C is finitely well-positioned if there is a nonempty and bounded slice $C(D,T)$ of C such that, for some $\overline{x} \in C(D,T)$, it holds $\langle x_i^*, \overline{x}\rangle < t_i$ for all i.*

Proof. Suppose that C is convex, that $C(D,T)$ is nonempty and bounded, and that $\langle x_i^*, \overline{x}\rangle < t_i$ for all i and some $\overline{x} \in C(D;T)$. By translation, wlog we can set $\overline{x} = 0$. Hence, $0 \in C(D,T)$ with all $t_i > 0$ in T, and $C(D,T) \subseteq \eta B_V$ for some $\eta > 0$. Observe that, if $x \in C(D,T)$ and $\langle x_i^*, x\rangle = t_i$ for some i, it follows

$$\left\langle x_i^*, \frac{x}{\|x\|} \right\rangle = \frac{t_i}{\|x\|} \geq \frac{t_i}{\eta}. \tag{2}$$

Pick now any point $x \in C \cap \{\|\cdot\| \geq \eta + \varepsilon\}$. Clearly, there is some $x_i^* \in D$ for which $\langle x_i^*, x\rangle > t_i$. Consider the nonempty subset $\Gamma \subseteq D$ for which $\langle x_i^*, x\rangle > t_i$ and choose $x_j^* \in \Gamma$ such that $\langle x_j^*, x\rangle / t_j \geq \langle x_i^*, x\rangle / t_i$ for all $x_i^* \in \Gamma$. As C is convex and $0 \in C$, the points $\lambda x \in C$ with $\lambda \in [0,1]$. Hence, there is a scalar λ_0 such that $\langle x_j^*, \lambda_0 x\rangle = t_j$. Clearly, $\langle x_i^*, \lambda_0 x\rangle \leq t_i$ for all $i \neq j$. Therefore, $\lambda_0 x \in C(D,T)$ and $\langle x_j^*, \lambda_0 x\rangle = t_j$. By (2), $\langle x_j^*, x/\|x\|\rangle \geq t_j/\eta$. To conclude, for all $x \in C \cap \{\|\cdot\| \geq \eta + \varepsilon\}$ there is $x_j^* \in D$ such that $\langle x_j^*, x\rangle \geq (t_j/\eta)\|x\|$. That is, C is finitely well-positioned. $\square$

Proof of Proposition 3.14. In view of the previous two lemmas, it is enough to observe that if a nonempty set $C(D;T)$ does not satisfy the Slater condition $\langle x_i^*, x\rangle < t_i$, the set $C(D; T+\varepsilon)$ does, where $T + \varepsilon = \{t_i + \varepsilon\}_{i=1}^n$. $\square$

Corollary 3.17. *A convex set C is well-positioned if and only if it has a polyhedral cut with D singleton.*

Example 3.18. Convexity in Proposition 3.14 is needed. Let $f(x) = \sqrt{\|x\|}$ be defined on an infinite dimensional normed space V. Its non-convex epigraph $\operatorname{epi} f \subseteq V \times \mathbb{R}$ has a polyhedral cut. For instance, if we consider in $V \times \mathbb{R}$ the linear functional $(0,1)$, the slice $\{(x,\lambda) \in \operatorname{epi} f : \lambda \leq t\}$ is bounded in $V \times \mathbb{R}$ for every t. Nevertheless, $\operatorname{epi} f$ is not finitely well-positioned, as will be shown later in the paper (see Example 6.2). $\square$

Observe that Proposition 3.14 can be formulated in a slightly different equivalent way by saying that a convex set C is finitely well-positioned if and only if there is

$x^* \in V^*$ and $t \in \mathbb{R}$ such that the slice $\{x \in C : \langle x^*, x\rangle \le t\}$ is finitely well-positioned and $\langle x^*, \bar{x}\rangle < t$ for some $\bar{x} \in C$.

After Theorem 3.6 we observed that finite dimensional subspaces are finitely well-positioned. Something more is proved in the next result.

Corollary 3.19.

(i) *Vector subspaces $H \subseteq V$ are finitely well-positioned if and only if they are finite dimensional. In this case,* $\dim H+1$ *is the least number of cones allowing plastering that cover H.*

(ii) *More generally, C is not finitely well-positioned if* $\mathrm{ri}_H^w\, C \neq \emptyset$, *with* $H = \overline{\mathrm{span}\, C}$ *and* $\dim H = \infty$.

Proof. (i) We already observed (see Remark 3.5) that H is not finitely well-positioned when $\dim H = \infty$. Therefore, it remains to prove the second part of the statement. By Proposition 3.3(ii), wlog we can consider $H = \mathbb{R}^n$. For convenience, $\mathbb{R}^n$ will be endowed with the sup-norm; i.e., $\|x\|_\infty = \max_{i=1,..,n} |x_i|$. We first show that $\mathbb{R}^n$ can be covered by the $n+1$ allowing plastering cones $\langle x_i^*, x\rangle \ge \varepsilon \|x\|_\infty$, $i = 1, ..., n$, and $\langle y^*, x\rangle \ge \varepsilon \|x\|_\infty$, where $0 < \varepsilon \le n^{-1}$, $\langle x_i^*, x\rangle = x_i$, and $\langle y^*, x\rangle = -\sum_{i=1}^n x_i$.

Suppose per contra that there is $\bar{x}$, with $\|\bar{x}\|_\infty = 1$, such that $\bar{x}_i < \varepsilon$ for all i and $\sum_{i=1}^n \bar{x}_i > -\varepsilon$. Set $P = \{i : \bar{x}_i \ge 0\}$. If $j \in P$, then $|\bar{x}_j| < \varepsilon \le n^{-1} \le 1$. Therefore, if $\mathrm{card}\,(P) = n$, we get $\|\bar{x}\|_\infty < 1$, a contradiction. Suppose then that $\mathrm{card}\,(P) \le n-1$. For every $j \notin P$ we have

$$0 > \bar{x}_j > -\varepsilon + \sum_{i\neq j} (-\bar{x}_i) \ge -\varepsilon + \sum_{i\in P} (-\bar{x}_i) \ge -\varepsilon - \mathrm{card}\,(P)\,\varepsilon \ge -n\varepsilon \ge -1.$$

Once again we get $\|\bar{x}\|_\infty < 1$, a contradiction.

It remains to prove that $n+1$ is the least number of cones allowing plastering that cover $\mathbb{R}^n$. Let $\langle x_i^*, x\rangle \ge \|x\|_\infty$, with $i = 1, ..., m$, be a family of cones covering $\mathbb{R}^n$. By the same argument used in the proof of Lemma 3.15, the slices $C\,(D, T)$ are either empty or bounded with $D = \{x_i^*\}_{i=1}^m$ and $T = \{t_i\}_{i=1}^m$. In particular, the cone $K = \{x \in \mathbb{R}^n : \langle x_i^*, x\rangle \le 0 \text{ for all } i\}$ is bounded. A cone is bounded if and only if it is trivial. Hence, $K = \{0\}$. Consider the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ given by $Ax = (\langle x_i^*, x\rangle)_{i=1}^m$. Clearly, $K = A^{-1}\left(\mathbb{R}_-^m\right)$. If $m \le n$, we have $K \neq \{0\}$. Therefore it must be $m \ge n+1$ and the proof of (i) is complete.

(ii) Suppose C is finitely well-positioned. By Proposition 3.3(i), C is finitely well-positioned in H. If $\overline{x} \in \mathrm{ri}_H^w\, C$, by the arguments used in the proof of Theorem 4.4 to show that (i) implies (v), there is a weak neighborhood $U \subseteq H$ of $\overline{x}$ that is norm bounded. By translation, we obtain a weak neighborhood V of 0 that is norm bounded. Consequently, there exist functionals $\{x_i^*\}_{i=1}^m$ and $\varepsilon > 0$ for which $|\langle x_i^*, x\rangle| < \varepsilon$ for all i implies $x \in V$. Hence, $\bigcap_{i=1}^m \ker x_i^* \subseteq V$. If y^* is linearly independent of $\{x_i^*\}_{i=1}^m$, there is a point $\overline{x} \in [\bigcap_{i=1}^m \ker x_i^*] \setminus \ker y^*$. Hence $\overline{x} \neq 0$ and $n\overline{x} \in V$ for all n. This contradicts the fact that V is norm bounded. $\square$

Corollary 3.19(i) implies that the hypotheses on V made in Theorem 3.6 are needed. For, consider any infinite dimensional subspace H of a space with the Schur property (e.g., l_1). Clearly, $0 \notin B_H$, but H is not finitely well-positioned. Notice that, inter alia, we get the well-known result that infinite dimensional spaces with the Schur property are not reflexive and their duals are not separable.

4. Asymptotic Compactness

In the literature several notions have been recently introduced to describe the asymptotic behavior of unbounded sets. Here we compare some of them with the notions that we introduced in the paper. Though many of them have been formulated for sets in Hausdorff topological vector spaces, here we consider normed spaces endowed with the weak topology.

We begin with the notion of weak asymptotic compactness due to Dedieu [10] and Zălinescu [20].

Definition 4.1. A set C is weakly asymptotically compact if there is $\varepsilon > 0$ and a weak neighborhood U of the origin such that $[0,\varepsilon]\, C \cap U$ is relatively weakly compact.

The next notion is due to Luc [14].

Definition 4.2. A set C has the weak CB property if a bounded set A exists such that $\overline{\text{cone}}\,(C \setminus A)$ has a weakly compact base.[7]

Luc and Penot [16] and Luc [15] recently introduced a weaker concept.

Definition 4.3. A set C is weakly recessively compact if, for any unbounded net $\{x_\alpha\}_\alpha \subseteq C$, there are a subnet $\{x_\beta\}_\beta$ and scalars t_β such that $\lim_\beta t_\beta = \infty$ and $x_\beta / t_\beta \rightharpoonup d \neq 0$.

Finally, recall that a set C is *locally weakly compact* if each $x \in C$ has a weak neighborhood that is relatively weakly compact.

The following "omnibus" result establishes the equivalences among finite well-positionedness and the asymptotic concepts just introduced.[8]

Theorem 4.4. *Let C be a subset of a reflexive space V. Consider the following properties:*

(i) *C is finitely well-positioned;*
(ii) *C is weakly asymptotically compact;*
(iii) *C has the weak CB property;*
(iv) *C is weakly recessively compact;*

[7] That is, $\overline{\text{cone}}\,(C \setminus A) = \text{cone}\, B$, where B is a weakly compact set that does not contain the origin.
[8] In other settings the various asymptotic concepts fail to coincide. In particular, recessive compactness seems to be the natural extension of the finite well-positionedness beyond the cases $\langle V, V^* \rangle$ and $\langle V^*, V \rangle$. To this end one should use the general version of Definition 4.3 for Hausdorff topological vector space given by [15].

(v) *C is locally weakly compact.*

Then,

$$(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv) \Longrightarrow (v).$$

If, in addition, C is convex, then they are all equivalent.

Proof. (i) implies (ii). It is easy to check that $[0, \varepsilon_0]\, K_{x^*}(m) \subseteq K_{x^*}(\varepsilon_0 m)$ holds for all $K_{x^*}(m)$ with $m \geq 0$. Therefore, if U is the weak neighborhood $U = \{x : |\langle x^*, x\rangle| \leq 1\}$ of 0, then

$$x \in [0, \varepsilon_0]\, K_{x^*}(m) \cap U \Rightarrow \|x\| \leq 1 + \varepsilon_0 m.$$

Namely, $K_{x^*}(m)$ is weakly asymptotically compact. If we now consider any finitely well-positioned set $C \subseteq \bigcup_{i=1}^n K_{x^*}(m_i)$, with $m_i \geq 0$, it follows $[0, \varepsilon_0]\, C \subseteq \bigcup_{i=1}^n [0, \varepsilon_0]\, K_{x^*}(m_i)$. By the neighborhood $U = \{x : |\langle x_i^*, x\rangle| \leq 1,\ i = 1, .., n\}$, the set $[0, \varepsilon_0]\, C \cap U$ is norm bounded.

(ii) implies (iv) and (iii) implies (iv) by ([15, Proposition 2.2]).

(iv) implies (i). Let C be recessively weakly compact. Suppose per contra that C is not finitely well-positioned. By Theorem 3.6 there is an unbounded sequence $\{x_n\} \subseteq C$ such that $x_n/\|x_n\| \rightharpoonup 0$. As C is recessively weakly compact, there is a subnet $\{x_\beta\}_\beta \subseteq \{x_n\}_n$ and a net $\{t_\beta\}_\beta$ such that $\lim_\beta t_\beta = \infty$ and $x_\beta/t_\beta \rightharpoonup d \neq 0$. Hence,

$$\frac{x_\beta}{t_\beta} = \frac{x_\beta}{\|x_\beta\|} \cdot \frac{\|x_\beta\|}{t_\beta} \rightharpoonup d.$$

As a weakly convergent sequence is norm bounded, it follows $0 \leq \|x_\beta\|/t_\beta \leq L$. Passing to a subnet we get $\|x_\gamma\|/t_\gamma \to \lambda$. Moreover, $\lambda \neq 0$, since otherwise $x_\beta/t_\beta \rightharpoonup 0$. Consequently,

$$\frac{x_\gamma}{\|x_\gamma\|} = \frac{x_\gamma}{t_\gamma} \cdot \frac{t_\gamma}{\|x_\gamma\|} \rightharpoonup \frac{1}{\lambda} d \neq 0,$$

a contradiction because $x_\gamma/\|x_\gamma\|$ is a subnet of $x_n/\|x_n\|$ and thus $x_\gamma/\|x_\gamma\| \rightharpoonup 0$.

(i) implies (iii). Notice first that if C is a cone that allows plastering, then C satisfies CB with $A = \emptyset$. Actually, if $C \subseteq K_{x^*}$, then $B = \{x : \langle x^*, x\rangle = 1\} \cap C$ is a weak compact base of C.

Now, let C be finitely well-positioned. By Proposition 3.1(ii), $C \cap \{\|x\| \geq \rho\} = \bigcup_{i=1}^n C_i$, where each C_i allows plastering. Set $A = \{\|x\| \leq \rho\}$. Therefore, $\overline{\text{cone}}\,(C \setminus A) = \overline{\text{cone}}\,(\bigcup_{i=1}^n C_i) = \bigcup_{i=1}^n \overline{\text{cone}}\, C_i$. We have seen that each $\overline{\text{cone}}\, C_i$ has a compact base B_i. Hence $\overline{\text{cone}}\, C_i = \text{cone}\, B_i$. It follows that $\overline{\text{cone}}\,(C \setminus A) = \bigcup_{i=1}^n \text{cone}\, B_i = \text{cone}\,(\bigcup_{i=1}^n B_i)$, as desired.

(i) implies (v). Let C be finitely well-positioned. By Proposition 3.14, there exists $D = \{x_i^*\}_{i=1}^n \subseteq V^*$ that determines a polyhedral cut for C. Let $\bar{x} \in C$. The weak neighborhood of $\bar{x}$

$$C \cap \{|\langle x_i^*, x - \bar{x}\rangle| \leq \varepsilon \text{ for each } i\}$$

is norm bounded. Thus, it is relatively weakly compact if V is reflexive.

(v) implies (i) if C is convex. Pick $\bar{x} \in C$. There is a weak neighborhood of $\bar{x}$ that is relatively weakly compact, and so is bounded. That means that there is $\varepsilon > 0$ and $\{x_i^*\}_{i=1}^n$ such that $\|x\| \leq \eta$ if $x \in C$ and $|\langle x_i^*, \bar{x} - x\rangle| \leq \varepsilon$ for all i. That is, the functionals $\{x_i^*\}_{i=1}^n \cup \{-x_i^*\}_{i=1}^n$ determine a polyhedral cut for C. By Proposition 3.14, C is finitely well-positioned. □

Notice that in view of Proposition 3.6(iii) and of the equivalence between (i) and (iv), in Definition 4.3 it is enough to consider sequences in place of nets when the space is reflexive.

5. Convex Sets

This section is mainly devoted to convex sets. The next lemma provides some characterizations of convex well-positioned sets. Point (iii), due to [3, Proposition 2.1], says that closed convex sets are well-positioned if and only if they are finitely well-positioned and do not contain any line. For the sake of brevity we omit the proof.

Lemma 5.1. *Let V be a reflexive space. Then:*

(i) *A closed convex cone K allows plastering if and only if is pointed and $0 \notin B_K$.*[9]

(ii) *A closed convex cone K allows plastering if and only if the set $K \cap \{\langle x^*, x\rangle = 1\}$ is bounded for every strictly positive functional x^* on K.*[10]

(iii) *A closed convex set C is well-positioned set if and only if $L_C = \{0\}$ and $0 \notin B_C$.*

Next we characterize the convex sets that are finitely well-positioned.

Proposition 5.2. *Let C be a convex subset of a Banach space V. Consider the following conditions:*

(i) *C is finitely well-positioned;*

(ii) *there is a projection $P : V \to V$ with finite codimensional range such that its image $P(C)$ is well-positioned;*

(iii) *there is a finite dimensional subspace L of V such that $\pi(C)$ is well-positioned, where $\pi : V \to V/L$.*

Then,

$$(ii) \Longleftrightarrow (iii) \Longrightarrow (i).$$

If, in addition, C is closed and V is reflexive, then they are all equivalent. In particular, $C = L_C \oplus C_1$ with C_1 well-positioned.

Notice that the direct sum $C = L_C \oplus C_1$ means that there exists a closed complementary vector space M to L_C that contains C_1 (i.e., $V = L_C \oplus M$ and $C_1 \subseteq M$).

Proof. (ii) implies (i). $P_1 = I - P$ is a projection with finite-dimensional range. Set $L = P_1(V)$. Since L is finitely well-positioned, by Proposition 3.14 there exists

[9]A cone K is pointed if $K \cap -K = \{0\}$.

[10]That is, $\langle x^*, x\rangle > 0$ for all $0 \neq x \in K$ (see [12, Theorem 2.7] and [13]).

$D = \{x_i^*\}_{i=1}^n \subseteq V^* \setminus \{0\}$ that determines a polyhedral cut for L. Likewise, by Corollary 3.17, since $P(C)$ is well-positioned there is a functional y^* such that $\langle y^*, x\rangle \leq t$ for all $t \in \mathbb{R}$ is bounded or empty.

We claim that the set of functionals $\{x_i^* \circ P_1\}_{i=1}^n \cup \{y^* \circ P\}$ determines a polyhedral cut for C. By the decomposition $x = P_1 x + Px$ for all $x \in V$, it actually holds

$$\begin{aligned} & \{x \in C : \langle x_i^* \circ P_1, x\rangle \leq t_i \text{ and } \langle y^* \circ P, x\rangle \leq \tau\} \\ = \ & \{x \in P_1(C) : \langle x_i^*, x\rangle \leq t_i\} + \{x \in P(C) : \langle y^*, x\rangle \leq \tau\} \\ \subseteq \ & \{x \in L : \langle x_i^*, x\rangle \leq t_i\} + \{x \in P(C) : \langle y^*, x\rangle \leq \tau\}. \end{aligned}$$

From this it easily follows that $\{x_i^* \circ P_1\}_{i=1}^n \cup \{y^* \circ P\}$ is a polyhedral cut and, in turn, that C is finitely well-positioned.

The equivalence between (*ii*) and (*iii*) easily follows from the fact that $M \simeq V/L$ if $V = M \oplus L$ and V is complete.

Finally, suppose that C is closed and finitely well-positioned, with V reflexive. As the linear space L_C is finitely well-positioned, by Corollary 3.19 we have $\dim L_C < \infty$. Hence, it is complemented in V. That is, there is a closed subspace M of V for which $L_C \oplus M = V$. Therefore, it holds the decomposition $C = L_C \oplus (C \cap M)$. Since $C \cap M \subseteq C$, it is finitely well-positioned. Moreover, the lineality space of $C \cap M$ is clearly trivial. By Lemma 5.1(*iii*), $C \cap M$ is well-positioned. Clearly, $P(C) = C \cap M$, where we denote by P the projection with range M. □

5.1. Dual Properties

Now we study convex sets by using dual properties. To this end, we need some standard notation. The *negative polar cone* M^- of a set M is given by

$$M^- = \{x^* : \langle x^*, x\rangle \leq 0 \text{ for each } x \in M\}.$$

If A and B are subsets of V and V^* respectively, we define the *annihilators* by the formulas

$$A^\perp = \{x^* \in V^* : \langle x^*, x\rangle = 0 \text{ for each } x \in A\}$$
$${}^\perp B = \{x \in V : \langle x^*, x\rangle = 0 \text{ for each } x^* \in B\}.$$

The *support functional* σ_C of a convex set C is given by $\sigma_C(x^*) = \sup\{\langle x^*, x\rangle : x \in C\}$. The domain of σ_C is called the *barrier cone* $b(C)$ of C; i.e., $b(C) = \{x^* : \sigma_C(x^*) < \infty\}$.

As well-known, $b(C)^- = C_\infty$ if C is closed and convex. By the Bipolar Theorem, $\overline{b(C)}^{w^*} = C_\infty^-$. This is equivalent to $\overline{b(C)} = C_\infty^-$ when V is reflexive.

In the next result the well-positioned set C is not necessarily convex. The result for convex sets is due to [3].

Proposition 5.3. *A set C is well-positioned if and only if* $\operatorname{int} b(C) \neq \emptyset$. *Moreover, when V is reflexive:*

(*i*) $\operatorname{int} C_\infty^- = \operatorname{int} b(C)$;

(ii) *the functionals* $-x^* + \delta_C$ *are coercive for all* $x^* \in \operatorname{int} C_\infty^-$;
(iii) *the functionals* $x^* \in \operatorname{int} C_\infty^-$ *attain the* sup *when* C *is sw-closed;*
(iv) $\overline{\mathrm{co}}\, C_\infty = [\overline{\mathrm{co}}\, C]_\infty$.

Proof. Suppose that C is well-positioned. Hence, $C \subseteq K_{x^*}(m)$. Thus, $\sigma_C \leq \sigma_{K_{x^*}(m)}$ and so $b(C) \supseteq b(K_{x^*}(m))$. Let us prove that $b(K_{x^*}(m)) \supseteq -x^* + B_{V^*}$, so that $b(C)$ has nonempty interior. Let $u^* \in B_{V^*}$. Clearly $\langle u^*, x\rangle \leq \|x\|$. Hence, $\langle x^*, x\rangle \geq \|x\| - m$ implies $\langle x^*, x\rangle \geq \langle u^*, x\rangle - m$. Namely, $m \geq \langle -x^* + u^*, x\rangle$ for all $x \in K_{x^*}(m)$. That is, $-x^* + u^* \in b(K_{x^*}(m))$, and so $b(K_{x^*}(m)) \supseteq -x^* + B_{V^*}$.

As to the converse, observe that $\sigma_C = \sigma_{\overline{\mathrm{co}}\, C}$ and that C is well-positioned if and only if $\overline{\mathrm{co}}\, C$ is. In particular, $b(C) = b(\overline{\mathrm{co}}\, C)$. Hence, $\operatorname{int} b(\overline{\mathrm{co}}\, C) \neq \emptyset$ if $\operatorname{int} b(C) \neq \emptyset$. In this way we can apply the arguments of [3, Theorem 2.1] and infer that $\overline{\mathrm{co}}\, C$ is well-positioned. In turn this implies that C is well-positioned.

(ii) Set $\lambda_1 < \sup_{x\in C} \langle x^*, x\rangle$ and consider the set of points $x \in C$ such that $\langle x^*, x\rangle \geq \lambda_1$. It follows $\lambda_1 \leq \langle x^*, x\rangle \leq -\varepsilon \|x\| + m$. Namely, $\|x\| \leq \varepsilon^{-1}(m - \lambda_1)$. The functional $-x^* + \delta_C$ is thus coercive.

(iii) It follows from (ii) and the reflexivity of V.

(i) As C is well-positioned, $C \subseteq K_{x^*} + x_0$. It follows $C_\infty \subseteq K_{x^*}$. Consequently, $K_{x^*}^- \subseteq C_\infty^-$. On the other hand, $K_{x^*}^- \supseteq -x^* + B_{V^*}$. Hence, $\operatorname{int} C_\infty^- \neq \emptyset$.

We claim that $\operatorname{int} C_\infty^- \subseteq b(C)$. Fix $x^* \in \operatorname{int} C_\infty^-$. We can suppose that C is unbounded, otherwise the claim is trivial. There are $\eta > 0$ and $\varepsilon > 0$ such that $x \in C$ and $\|x\| \geq \eta \Rightarrow \langle -x^*, x\rangle \geq \varepsilon \|x\|$. Suppose per contra that there are two scalar sequences $\eta_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$, as well as a sequence $\{x_n\}_n \subseteq C$, for which $\|x_n\| \geq \eta_n$ and $\langle -x^*, x_n\rangle < \varepsilon_n \|x_n\|$. Clearly, $\|x_n\| \to \infty$. As V is reflexive, passing to a subsequence, we have $x_n / \|x_n\| \rightharpoonup d$. This implies $\langle x^*, d\rangle \geq 0$. Clearly, $\langle x^*, d\rangle = 0$ since $d \in C_\infty$ and $x^* \in C_\infty^-$.

Since $x^* \in \operatorname{int} C_\infty^-$, we have $\langle x^* + u^*, d\rangle \leq 0$ for all $u^* \in \varepsilon B_{V^*}$. Namely, $\langle u^*, d\rangle \leq 0$, which implies $d = 0$. But, this is a contradiction because C is well positioned. Hence, $\langle -x^*, x\rangle \geq \varepsilon \|x\|$ over C and $\|x\| \geq \eta$. Clearly, this means $\langle -x^*, x\rangle \geq \varepsilon \|x\| - m$ for all $x \in C$ and for some m. Consequently, $\langle x^*, x\rangle \leq -\varepsilon \|x\| + m \leq m$ that implies $\sup_{x\in C} \langle x^*, x\rangle < \infty$ and $x^* \in b(C)$ and the claim is proved.

To complete the proof of point (i), suppose first that C is closed and convex. We have proved that $\operatorname{int} C_\infty^- \subseteq b(C)$. Hence, $\operatorname{int} C_\infty^- \subseteq b(C) \subseteq \overline{b(C)} = C_\infty^-$. This implies that $\operatorname{int} b(C) = \operatorname{int} C_\infty^-$.

Suppose now that C is any well-positioned set and define $D = \overline{co}(C)$. Clearly, $C_\infty \subseteq D_\infty$ and $C_\infty^- \supseteq D_\infty^-$. For what has been proved, we can write

$$\operatorname{int} b(C) = \operatorname{int} b(D) = \operatorname{int} D_\infty^- \subseteq \operatorname{int} C_\infty^- \subseteq \operatorname{int} b(C).$$

Hence point (i) holds.

(iv) Notice that we just proved that $\operatorname{int} D_\infty^- = \operatorname{int} C_\infty^-$. Hence,

$$\overline{\operatorname{int} D_\infty^-} = \overline{\operatorname{int} C_\infty^-} \Rightarrow D_\infty^- = C_\infty^-.$$

By the Bipolar Theorem, $D_\infty = \overline{co}(C_\infty)$, as desired. $\square$

Example 5.4. Proposition 5.3(i) may fail when V is not reflexive. Consider the well-positioned closed convex set $C = \{x \in l_1 : 0 \leq x_i \leq 1 \text{ for each } i\}$. We have $C_\infty^- = l_\infty$ since $C_\infty = \{0\}$ (see Example 3.9). It is not difficult to check that $b(C) = \{x \in l_\infty : x^+ \in l_1\}$. Clearly, $\operatorname{int} b(C) \neq \emptyset$, while $\operatorname{int} b(C) \subseteq b(C) \subset l_\infty = \operatorname{int} C_\infty^-$. Observe that even Proposition 5.3(ii) fails. Consider for instance the functional $-e = (-1, -1, \ldots) \in \operatorname{int} b(C)$. Clearly, $e + \delta_C$ is not coercive. □

We give a few corollaries of the characterization of well-positionedness established in Proposition 5.3.

Corollary 5.5. *A cone K allows plastering if and only if* $\operatorname{int} K^- \neq \emptyset$.

Proof. Observe that $\sigma_K = \delta_{K^-}$. Thus, $b(K) = K^-$. Propositions 3.1(i) and 5.3 conclude the proof. □

This is the simplest criterion to check whether a cone allows plastering. For instance, the positive cone L_+^p does not allow plastering for $p > 1$ since $(L_+^p)^- = -L_+^q$ has empty interior, unless it is finite dimensional.

By Proposition 5.3, $\operatorname{int} C_\infty^- \neq \emptyset$ when C is well-positioned. However, in general this is not a characterizing property. For instance, if C is unbounded and linearly bounded, $C_\infty = \{0\}$. Hence $C_\infty^- = V$ but C is not well-positioned. Next we show that among finitely well-positioned sets, the property $\operatorname{int} C_\infty^- \neq \emptyset$ indeed characterizes sets that are well-positioned.

Corollary 5.6. *A finitely well-positioned set C in a reflexive space is well-positioned if and only if* $\operatorname{int} C_\infty^- \neq \emptyset$.

Proof. If C is well-positioned, Proposition 5.3 implies $\operatorname{int} C_\infty^- = \operatorname{int} b(C) \neq \emptyset$.

Let us prove the converse implication. Suppose $x^* \in \operatorname{int} C_\infty^-$. We will use an argument similar to that of Proposition 5.3. We claim that there are $\eta > 0$ and $\varepsilon > 0$ such that $x \in C$ and $\|x\| \geq \eta$ implies $\langle -x^*, x\rangle \geq \varepsilon \|x\|$. Suppose not. There are then two sequences $\eta_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$, as well as a sequence $\{x_n\} \subseteq C$, for which $\|x_n\| \geq \eta_n$ and $\langle -x^*, x_n\rangle < \varepsilon_n \|x_n\|$. Clearly, $\|x_n\| \to \infty$. Hence, $\langle x^*, x_n / \|x_n\|\rangle > -\varepsilon_n$. Since the space is reflexive, passing to a subsequence, we have $x_n / \|x_n\| \rightharpoonup d \in C_\infty$. Consequently, $\langle x^*, d\rangle \geq 0$. This implies $\langle x^*, d\rangle = 0$ since $x^* \in C_\infty^-$. Moreover, from $x^* \in \operatorname{int} C_\infty^-$, $\langle x^* + u^*, d\rangle \leq 0$ for all $u^* \in \varepsilon B_{V^*}$. Hence, $\langle u^*, d\rangle \leq 0$ for every $u^* \in B_{V^*}$. That is, $d = 0$. We have obtained $x_n / \|x_n\| \rightharpoonup 0$, which contradicts the fact that C is finitely well-positioned. Therefore, the claim is true and C is well-positioned. □

We now extend the characterization established in Proposition 5.3 from well-positioned sets to finitely well-positioned ones, though under stronger assumptions.

Proposition 5.7. *A closed and convex set C of a reflexive space V is finitely well-positioned if and only the following two conditions hold:*

(i) $Q \equiv \overline{b(C) - b(C)}$ *has finite codimension in* V^**;*

(ii) $\operatorname{ri}_Q b(C) \neq \emptyset$.

In this case, $\mathrm{ri}_Q\, b(C) = \mathrm{ri}_Q\, C_\infty^-$ *and* $Q = L_C^\perp = b(C) - b(C) = C_\infty^- - C_\infty^-$.

Since $b(C)$ is a cone, Q is the minimal closed affine space containing $b(C)$; that is, $Q = \overline{\mathrm{aff}}\, b(C)$.

Proof. Let C be a closed and finitely well-positioned set. By Proposition 5.2, $C = L \oplus C_1$, where $L = L_C$, C_1 is a well-positioned set contained into a closed subspace M, $V = L \oplus M$, and $\dim L = n$.

It is well-known that $V = L \oplus M$ implies $V^* = M^\perp \oplus L^\perp$ and $M^\perp \simeq L^*$, $L^\perp \simeq M^*$ (all these properties are true for any Banach space V). Notice that $\dim M^\perp = n$ and thus $L^\perp$ has finite codimension. From the obvious relation

$$\langle x_1^* \oplus x_2^*, x_1 \oplus x_2 \rangle = \langle x_1^*, x_1 \rangle + \langle x_2^*, x_2 \rangle,$$

with $x_1^* \oplus x_2^* \in M^\perp \oplus L^\perp$ and $x_1 \oplus x_2 \in L \oplus M$, it follows that

$$\sigma_C(x_1^* \oplus x_2^*) = \sigma_L(x_1^*) + \sigma_{C_1}(x_2^*)$$

where σ_L and σ_{C_1} denote the support functionals of L and C_1 as subsets of L and M, respectively. Clearly, x_1^* and x_2^* in the arguments of σ_L and σ_{C_1} denote the restrictions of x_1^* to L and x_2^* to M, respectively.

On the other hand, $\sigma_L(x_1^*) = \infty$ unless $x_1^* = 0$. Hence, $b(C) \subseteq L^\perp$ and so $b(C) - b(C) \subseteq L^\perp$. Now, as C_1 is well-positioned in M, Proposition 5.3 implies that $\mathrm{int}\, b(C_1) \neq \emptyset$ on the space M^*. That is, $\mathrm{ri}_{L^\perp}\, b(C) \neq \emptyset$. This also implies that $b(C) - b(C) = L^\perp$. To check the other properties, observe that $C_\infty = L \oplus (C_1)_\infty$ and $C_\infty^- = \{0\} \oplus (C_1)_\infty^-$. By Proposition 5.3($i$), $\mathrm{ri}_{L^\perp}\, b(C) = \mathrm{ri}_{L^\perp}\, C_\infty^-$. This concludes the first part of the proof by setting $Q = L^\perp$.

As to the converse, suppose that C satisfies (i) and (ii). Since Q is closed and has finite codimension, V^* admits the decomposition $V^* = N \oplus Q$ where $N \subseteq V^*$ has dimension n. By reflexivity, $V = {}^\perp Q \oplus {}^\perp N$ with ${}^\perp Q \simeq N^*$ and ${}^\perp N \simeq Q^*$. Suppose per contra that C is not finitely well-positioned. As ${}^\perp Q$ is finite-dimensional, there are $n+1$ functionals $\{x_i^* \oplus 0\}_{i=1}^{n+1}$ in $N \oplus Q$ that determine a polyhedral cut for ${}^\perp Q$. Pick now any element $0 \oplus y^* \in \mathrm{ri}_Q\, b(C)$, with $y^* \neq 0$. Then, the collection $\{x_i^* \oplus 0\}_{i=1}^{n+1} \cup \{0 \oplus (-y^*)\}$ would not determine a polyhedral cut. That is, for some scalars $\{t_i\}_{i=1}^{n+1} \cup \{\tau\}$ the slice would be unbounded. On the other hand, the slice is contained within

$$\{x \in P_1(C) : \langle x_i^*, x \rangle \le t_i \ \forall i\} + \{x \in P_2(C) : \langle y^*, x \rangle \ge -\tau\}$$

where $P_1 : V \to {}^\perp Q$ and $P_2 : V \to {}^\perp N$ are the canonical projections. As $P_1(C) \subseteq {}^\perp Q$, the first set is bounded by construction, and so the set $\{x \in P_2(C) : \langle y^*, x \rangle \ge -\tau\}$ would be unbounded. By the Banach-Steinhaus Theorem, there is a sequence $\{x_n\}_n \subseteq P_2(C)$ and a functional $0 \oplus z^*$ such that $\langle z^*, x_n \rangle \ge n$. On the other hand, since $0 \oplus y^* \in \mathrm{ri}_Q\, b(C)$, $0 \oplus y^* + \lambda(0 \oplus z^*) \in b(C)$ for $\lambda > 0$ small enough. But, $\langle y^* + \lambda z^*, x_n \rangle \to \infty$ and so $\sigma_C(y^* + \lambda z^*) = +\infty$, a contradiction. We conclude that C is finitely well-positioned. $\square$

The property of the barrier cone in Proposition 5.7 is closely related with compact epi-lipschitzianity.[11]

Proposition 5.8. *Let C be a closed and convex set in a reflexive space. If C is finitely well-positioned, then its barrier cone $b(C)$ is compactly epi-Lipschitzian. The converse implication holds provided $b(C)$ is closed.*

Proof. Suppose that C is finitely well-positioned. Using the notation of the previous proposition, it holds $C = L \oplus C_1$, where $L = L_C$, $C_1 \subseteq M$ is well-positioned, and $V = L \oplus M$. We have $b(C) \subseteq L^{\perp} \simeq M^*$. Moreover, there is a nonempty open set U of M^* for which $U \subseteq b(C)$. Let $\{e_k\}_{k=1}^n$ be a basis of L^* and define the convex polytope $\Sigma = \sum_{k=1}^n [-1,1]\, e_k$. Clearly, there is a nonempty open set V in L^* for which $V \subseteq \Sigma$. Hence, $U \times V \subseteq b(C) + \Sigma$. But, $U \times V$ is a nonempty open set of V^*, and so $\operatorname{int}(b(C) + \Sigma) \neq \emptyset$. By [8, Proposition 2.10], the convex cone $b(C)$ is compactly epi-Lipschitzian.

As to the converse, suppose $b(C)$ is a closed and compactly epi-Lipschitzian set. Thanks to [8, Theorem 2.5(vii)] and to Proposition 5.7, the set C is finitely well-positioned. □

6. Functions

A function $f : V \to \overline{\mathbb{R}}$ is

(i) *well-positioned* if its epigraph $\operatorname{epi} f \subseteq V \times \mathbb{R}$ is well-positioned;[12]
(ii) *finitely well-positioned* if its epigraph $\operatorname{epi} f \subseteq V \times \mathbb{R}$ is finitely well-positioned;
(iii) *quasi finitely well-positioned* if all its nonempty sublevel sets $(f \leq \lambda)$ are finitely well-positioned;
(iv) *semi finitely well-positioned* if there is a sublevel set $(f \leq \lambda)$, with $\lambda > \inf f$, that is finitely well-positioned.

Clearly, property (i) implies (ii) and (iii) implies (iv). To see that (ii) implies (iii) is enough to consider the equality $\operatorname{epi} f \cap \{\lambda = \bar{\lambda}\} = (f \leq \bar{\lambda}) \times \{\bar{\lambda}\}$.

The next examples show that in general these implications do not have a converse. However, Theorem 6.7 will show that properties (ii)–(iv) are equivalent for convex functions.

Example 6.1. The convex function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(t) = t$ is finitely well-positioned, but not well-positioned. □

Example 6.2. Let $f(x) = \sqrt{\|x\|}$ be defined over an infinite dimensional reflexive space. It is quasi finitely well-positioned since all nonempty sublevel sets are bounded. However, it is not finitely well-positioned. For, take an unbounded sequence $\{x_n\}$ with $x_n / \|x_n\| \rightharpoonup 0$. Clearly, $\sqrt{\|x\|} / \|x_n\| \to 0$. By Lemma 6.5 below, f is not finitely well-positioned. The function $f \wedge 1$ is a simple example of a semi finitely well-positioned that is not quasi finitely well-positioned. □

[11] The notion of compactly epi-Lipschitzian sets in locally convex topological spaces is due to Borwein and Strojwas in [7]. We are grateful to the referee for drawing our attention to this point.
[12] Well-positioned functions are proper (their epigraphs would otherwise contain a line).

The following lemma, whose simple proof is omitted, will be useful in deriving the results of this section. Notice that $V \times \mathbb{R}$ is endowed with the norm $\|(x, \lambda)\| = \|x\| + |\lambda|$.

Lemma 6.3. *Let $\{(x_n, \lambda_n)\}_n \subseteq V \times \mathbb{R}$. Then*

$$\frac{(x_n, \lambda_n)}{\|x_n\| + |\lambda_n|} \rightharpoonup 0 \Longleftrightarrow \frac{x_n}{\|x_n\|} \rightharpoonup 0 \text{ and } \frac{\lambda_n}{\|x_n\|} \to 0.$$

Next we establish a full characterization of finitely well-positioned functions.

Theorem 6.4. *Let V be reflexive or with separable dual. A function $f : V \to \overline{\mathbb{R}}$ is finitely well-positioned if and only if there is no unbounded sequence $\{x_n\} \subseteq \operatorname{dom} f$ such that $x_n / \|x_n\| \rightharpoonup 0$ and either $f(x_n) / \|x_n\| \to 0$ or $f(x_n) \downarrow -\infty$.*

The proof relies on couple of lemmas.

Lemma 6.5. *Let V be reflexive or with separable dual. A function f bounded from below is finitely well-positioned if and only if there is no unbounded sequence $\{x_n\} \subseteq \operatorname{dom} f$ such that $x_n / \|x_n\| \rightharpoonup 0$ and $f(x_n) / \|x_n\| \to 0$.*

Proof. Setting $f - \inf f$, wlog we can assume $f \geq 0$. Suppose that the claimed conditions hold and that, per contra, f is not finitely well-positioned. There is a sequence $(x_n, \lambda_n) \in \operatorname{epi} f$ such that $\|x_n\| + |\lambda_n| \to \infty$ and $(x_n, \lambda_n) / (\|x_n\| + |\lambda_n|) \rightharpoonup 0$. By Lemma 6.3, $x_n / \|x_n\| \rightharpoonup 0$ and $\lambda_n / \|x_n\| \to 0$. The sequence $\|x_n\|$ cannot be bounded. Otherwise, $|\lambda_n|$ would be bounded and thus $\|x_n\| + |\lambda_n|$ cannot go to infinity. Therefore, we can suppose $\|x_n\| \to \infty$. As $0 \leq f(x_n) \leq \lambda_n$, it follows that $f(x_n) / \|x_n\| \to 0$, a contradiction.

As to the converse, assume f is finitely well-positioned and that there is a sequence $\{x_n\}$ such that $\|x_n\| \to \infty$, $x_n / \|x_n\| \rightharpoonup 0$ and $f(x_n) / \|x_n\| \to 0$. Consider the points $(x_n, f(x_n)) \in \operatorname{epi} f$. We have $(x_n, f(x_n)) / (\|x_n\| + |f(x_n)|) \rightharpoonup 0$ and $\|x_n\| + |f(x_n)| \to \infty$. This implies that f is not finitely well-positioned. □

Lemma 6.6. *Let V be reflexive or with separable dual. A function $f : V \to \overline{\mathbb{R}}$ is finitely well-positioned if and only if it is quasi finitely well-positioned and there is no unbounded sequence $\{x_n\} \subseteq \operatorname{dom} f$ such that $x_n / \|x_n\| \rightharpoonup 0$, $f(x_n) / \|x_n\| \to 0$ and $f(x_n) \uparrow \sup f$.*

Proof. The conditions are clearly necessary. Let us prove their sufficiency. Suppose per contra that f is not finitely well-positioned under the two claimed conditions. Then, there is a sequence $(x_n, \lambda_n) \in \operatorname{epi} f$ such that $x_n / \|x_n\| \rightharpoonup 0$ and $\lambda_n / \|x_n\| \to 0$. Notice that necessarily $\|x_n\| \to \infty$. We consider separately two cases.

(i) $\limsup_{n\to\infty} f(x_n) < \sup f$. This implies that there is a subsequence $\{x_{n_k}\} \subseteq (f \leq \lambda)$ for some $\lambda < \sup f$. Hence, $(f \leq \lambda)$ would not be finitely well-positioned.

(ii) $\limsup_{n\to\infty} f(x_n) = \sup f$. In this case a subsequence can be extracted for which $f(x_n) \uparrow \sup f$. Clearly, $f(x_n) / \|x_n\| \to 0$ and this contradicts the hypothesis. □

Proof of Theorem 6.4. The conditions are necessary. For, suppose f is finitely well-positioned. From $f \leq |f|$ it follows that $\operatorname{epi}|f| \subseteq \operatorname{epi} f$. Hence, $|f|$ is finitely well-positioned. Suppose that a sequence exists as claimed in the statement for which $f(x_n) / \|x_n\| \to 0$; i.e., $|f(x_n)| / \|x_n\| \to 0$. By Lemma 6.5, $|f|$ would not be finitely well-positioned. If $f(x_n) \downarrow -\infty$, $\{x_n\} \in (f \leq \lambda)$, and so f would not be quasi finitely well-positioned. This concludes the proof of necessity.

As to the converse, let us first prove that f is quasi finitely well-positioned. Suppose not. Then, $(f \leq \lambda_0)$ is not finitely well-positioned for some λ_0. There is consequently an unbounded sequence $\{x_n\}$ such that $x_n / \|x_n\| \rightharpoonup 0$ and $f(x_n) \leq \lambda_0$. Two cases are possible.

(i) $\liminf_{n\to\infty} f(x_n) = -\infty$. In this case there is a subsequence such that $f(x_{n_k}) \downarrow -\infty$. But, this leads to a contradiction.

(ii) $\liminf_{n\to\infty} f(x_n) > -\infty$. In this case the sequence is bounded from below, i.e., $k \leq f(x_n) \leq \lambda_0$. This implies $f(x_n) / \|x_n\| \to 0$ and we get again a contradiction.

Thus, f is quasi finitely well-positioned. Suppose that f is not finitely well-positioned. By Lemma 6.6, there is an unbounded sequence such that $x_n / \|x_n\| \rightharpoonup 0$, $f(x_n) \uparrow \sup f$, and $f(x_n) / \|x_n\| \to 0$, a contradiction. The sufficiency part of the proof is completed. □

The next result shows that the three classes of functions introduced at the beginning of the section through finite well-positionedness are equivalent for convex functions.

Theorem 6.7. *A convex function $f : V \to \overline{\mathbb{R}}$ is finitely well-positioned if and only if is semi finitely well-positioned.*

The proof of this theorem relies on some lemmas.

Lemma 6.8. *Let $f : V \to \overline{\mathbb{R}}$ be convex and $\lambda_1, \lambda_2 > \inf f$. A sublevel set $(f \leq \lambda_1)$ is unbounded if and only if $(f \leq \lambda_2)$ is. Moreover, $B_{(f\leq\lambda_1)} = B_{(f\leq\lambda_2)}$.*

Proof. Set $\lambda_1 > \lambda_2 > \inf f$. This implies $(f \leq \lambda_2) \subseteq (f \leq \lambda_1)$ and $B_{(f\leq\lambda_2)} \subseteq B_{(f\leq\lambda_1)}$. Hence $(f \leq \lambda_1)$ is unbounded if $(f \leq \lambda_2)$ is. Suppose $(f \leq \lambda_1)$ is unbounded and let $x_n \in (f \leq \lambda_1)$ with $\|x_n\| \to \infty$. As $\lambda_2 > \inf f$, there is $v \in V$ with $f(v) = \lambda_2 - \varepsilon$. Consider the sequence $(1-\alpha)v + \alpha x_n$ where $\alpha \in (0,1)$. By convexity,

$$f((1-\alpha)v + \alpha x_n) \leq (1-\alpha) f(v) + \alpha f(x_n) \leq (1-\alpha)(\lambda_2 - \varepsilon) + \alpha\lambda_1.$$

We can thus pick $\overline{\alpha} \in (0,1)$, so that $y_n = (1-\overline{\alpha})v + \overline{\alpha}x_n \in (f \leq \lambda_2)$. Hence, $(f \leq \lambda_2)$ is unbounded. Let $d \in B_{(f\leq\lambda_1)}$; i.e., $x_n / \|x_n\| \rightharpoonup d$, $f(x_n) \leq \lambda_1$. If y_n is the above sequence of points, then

$$\frac{y_n}{\|y_n\|} = \frac{(1-\overline{\alpha})v}{\|y_n\|} + \frac{\overline{\alpha}\|x_n\|}{\|y_n\|}\frac{x_n}{\|x_n\|}.$$

On the other hand,

$$\frac{\overline{\alpha}\|x_n\|}{\|y_n\|} = \frac{\overline{\alpha}\|x_n\|}{\|(1-\overline{\alpha})v + \overline{\alpha}x_n\|} = \left\|\frac{(1-\overline{\alpha})v}{\overline{\alpha}\|x_n\|} + \frac{x_n}{\|x_n\|}\right\|^{-1}$$

that goes to 1 as $n \to \infty$. Hence $y_n / \|y_n\| \rightharpoonup d$, and so $B_{(f \le \lambda_1)} = B_{(f \le \lambda_2)}$. $\square$

Lemma 6.9. *Let $f : V \to \overline{\mathbb{R}}$ be convex. If $\left(f \le \bar{\lambda}\right)$, with $\bar{\lambda} > \inf f$, is finitely well-positioned, then all its nonempty sublevel sets $(f \le \lambda)$ are finitely well-positioned. Specifically, if*

$$\left(f \le \bar{\lambda}\right) \subseteq \bigcup_{i=1}^{n} K_{x_i^*}(m_i), \tag{3}$$

then, for each λ there are scalars $\varepsilon_i(\lambda) > 0$ and $m_i(\lambda)$ such that, for some $i = 1, ..., n$,

$$x \in (f \le \lambda) \Longrightarrow \langle x_i^*, x \rangle \ge \varepsilon_i(\lambda) \|x\| - m_i(\lambda). \tag{4}$$

Proof. The property is trivially true for the level sets $(f \le \lambda)$ with $\lambda \le \bar{\lambda}$. Therefore, it is enough to study the case $\lambda > \bar{\lambda}$. As $\bar{\lambda} > \inf f$, there is a point $\bar{x}$ for which $f(\bar{x}) < \bar{\lambda}$. Pick now scalars $\{t_i\}$ so that $\langle x_i^*, \bar{x} \rangle < t_i$ and consider the convex function $\varphi(x) = \bigvee_{i=1}^{n} (\langle x_i^*, x \rangle - t_i) \vee \left(f(x) - \bar{\lambda}\right)$. Clearly, $\varphi(\bar{x}) < 0$. Therefore, $(\varphi \le 0) \ne \emptyset$. Moreover, $(\varphi \le 0) = \bigcap_i \{\langle x_i^*, x \rangle \le t_i\} \cap \left(f \le \bar{\lambda}\right)$. By (3), $(\varphi \le 0)$ is bounded. By Lemma 6.8, all sublevel sets $(\varphi \le h)$ are bounded for each $h > 0$. On the other hand, the level $(\varphi \le h)$ is given by the slice $\bigcap_i \{\langle x_i^*, x \rangle \le t_i + h\} \cap \left(f \le \bar{\lambda} + h\right)$, which satisfies the Slater condition in that $\bar{x} \in \left(f \le \bar{\lambda} + h\right)$ and $\langle x_i^*, \bar{x} \rangle < t_i + h$ for all i. By Proposition 3.14, $\left(f \le \bar{\lambda} + h\right)$ is finitely well-positioned and (4) holds. $\square$

Proof of Theorem 6.7. Suppose that f is semi finitely well-positioned and that $(f \le \lambda) \subseteq \bigcup_{i=1}^{n} K_{x_i^*}(m_i)$ holds. Consider the set of $n+2$ functionals $D = \{(x_i^*, 0)\}_{i=1}^{n} \cup \{(0,1), (0,-1)\}$ on $V \times \mathbb{R}$. By Lemma 6.9, it is easy to see that $\operatorname{epi} f\,(D, T)$ is a polyhedral cut. By Proposition 3.14, $\operatorname{epi} f$ is finitely well-positioned. The converse is trivial. $\square$

A result similar to Theorem 6.7 does not hold in general for well-positioned functions, as the example $\varphi(t) = t$ shows. However, if the functions are bounded below the following analogous result holds.

Proposition 6.10. *Let V be reflexive or with separable dual. A convex and bounded below function $f : V \to \overline{\mathbb{R}}$ is well-positioned if and only if a sublevel set $(f \le \lambda)$, with $\lambda > \inf f$, is well-positioned.*

Proof. Suppose first that f is lower semicontinuous. By Theorem 6.7, f is finitely well-positioned. By Lemma 5.1 it suffices to show that $\operatorname{epi} f$ does not contain any line. It is easy to see that the unique feasible directions of these lines are $(v, 0)$ since, otherwise, the function would not be bounded from below. This implies that $f(x_0 + tv) \le \lambda$ for all t. Hence, $v \in L_{(f \le \lambda)}$. This contradicts the fact that $(f \le \lambda)$ is well-positioned.

If f is not lower semicontinuous, we easily obtain the same result by considering its lower semicontinuous hull $\bar{f} \le f$. $\square$

The results on support functionals derived in the previous sections can be easily translated into results on Fenchel conjugates through the relation $f^*(x^*) =$

$\sigma_{\operatorname{epi} f}(x^*, -1)$. In this vein, next we characterize well-positioned and finitely well-positioned functions through properties of their Fenchel conjugates. Point (i) is due to [3, Proposition 3.1].

Proposition 6.11. *Let $f : V \to \overline{\mathbb{R}}$ be a proper lower semicontinuous and convex function on a reflexive space V. Then,*

(i) *f is well-positioned if and only if $\operatorname{int} \operatorname{dom} f^* \neq \emptyset$;*

(ii) *f is finitely well-positioned if and only if $\operatorname{ri} \operatorname{dom} f^* \neq \emptyset$ and $\overline{\operatorname{aff}}(\operatorname{dom} f^*)$ has finite codimension in V^*.*

Proof. (i) See [3, Proposition 3.1]. As to (ii), suppose first that f is finitely well-positioned and $f \geq 0$. As $\operatorname{epi} f \subseteq V \times \mathbb{R}_+$, the lineality space of $\operatorname{epi} f$ is contained in V. Let L be such a finite-dimensional space and let $V = L \oplus M$ be the usual decomposition. This implies that f is constant over L and, by Proposition 5.2, $f(x_1 \oplus x_2) = \varphi(x_2)$ for $x_1 \oplus x_2 \in L \oplus M$, where φ is a well-positioned function over M. The Fenchel conjugate is

$$f^*(x_1^* \oplus x_2^*) = \begin{cases} \varphi^*(x_2^*) & \text{if } x_1^* = 0 \\ +\infty & \text{if } x_1^* \neq 0 \end{cases}$$

where $x_1^* \oplus x_2^* \in M^\perp \oplus L^\perp$. Hence, $\operatorname{dom} f^* \subseteq L^\perp$. Since $f \geq 0$, we have $0 \in \operatorname{dom} f^*$ and so $\overline{\operatorname{aff}}(\operatorname{dom} f^*) \subseteq L^\perp$. Since φ is well-positioned, $\operatorname{ri} \operatorname{dom} f^* \neq \emptyset$ and $\operatorname{aff} \operatorname{dom} f^* = L^\perp$. The proof is thus complete for $f \geq 0$.

Let f be now any finitely well-positioned function. As f is lower semicontinuous and proper, $f \geq y^* + \gamma$. Hence, $f_1 = f - y^* - \gamma \geq 0$ and $y^* \in \operatorname{dom} f^*$. By applying the previous result, we easily obtain that $\operatorname{aff} \operatorname{dom} f^* = y^* + L^\perp$ and $\operatorname{ri} \operatorname{dom} f^* \neq \emptyset$.

As to the converse, suppose that f satisfies (i) and (ii). Notice that (ii) implies $\overline{\operatorname{aff}}(\operatorname{dom} f^*) = \operatorname{aff}(\operatorname{dom} f^*) = Q$. If $x^* \in \operatorname{ri}_Q \operatorname{dom} f^*$, $(x^*, -1)$ is interior with respect to the affine space $Q \times \{-1\}$. Hence, $(x^*, -1)$ is interior with respect to $Q \times \mathbb{R}$. So, $(x^*, -1) \in \operatorname{ri} b(\operatorname{epi} f)$ and $\operatorname{aff} b(\operatorname{epi} f) = Q \times \mathbb{R}$. By Proposition 5.7, $\operatorname{epi} f$ is finitely well-positioned. $\square$

We close with a noteworthy coercitivity property.

Proposition 6.12. *Let $f : V \to \overline{\mathbb{R}}$ be a proper lower semicontinuous and convex function on a reflexive space V.*

(i) *If f is well-positioned, then $f - x^*$ is coercive for all $x^* \in \operatorname{int} \operatorname{dom} f^*$;*

(ii) *If f is finitely well-positioned, then $f - x^*$ is semicoercive for all $x^* \in \operatorname{ri} \operatorname{dom} f^*$.*

Proof. (i) By Proposition 5.3, the functional $-(x^*, -1) + \delta_{\operatorname{epi} f}$ is coercive when $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. This implies that for a fixed scalar $\bar{\lambda}$, the set $\{\langle -x^*, x\rangle + \lambda \leq \bar{\lambda}\} \cap \{f(x) \leq \lambda\}$ is bounded in $V \times \mathbb{R}$. In particular, it is bounded for $f(x) = \lambda$. That is, $\langle -x^*, x\rangle + f(x) \leq \bar{\lambda}$ is bounded. Namely, $f - x^*$ is coercive.

(ii) Wlog set $x^* = 0$. As $0 \in \operatorname{dom} f^*$, $\inf f > -\infty$. In view of the proof of Proposition 6.11, we have $f(x_1 \oplus x_2) = \varphi(x_2)$ with φ a well-positioned function over M and $V = L \oplus M$. As $0 \in \operatorname{ri} \varphi^*$, by Proposition 6.12$(i)$ φ is coercive over

M. Hence, f is semicoercive. If $\bar{x} \in M$ is a minimum of φ, then $\bar{x} + L$ is a set of minimizers of f. □

7. Applications

7.1. Intersections

Given a chain of nonempty sets $\{C_n\}$, with $C_{n+1} \subseteq C_n$ for all n, following Bertsekas and Tseng [6] define the asymptotic cones

$$\{C_n\}_\infty = \left\{ d \in V : \exists t_n \to \infty \text{ and } x_n \in C_n \text{ such that } \frac{x_n}{t_n} \rightharpoonup d \right\}$$

and

$$B(\{C_n\}) = \left\{ d \in V : \exists x_n \in C_n \text{ such that } \|x_n\| \to \infty \text{ and } \frac{x_n}{\|x_n\|} \rightharpoonup d \right\}$$

When $C_n = C$ for all n, we get back to the usual asymptotic objects, that is, $\{C_n\}_\infty = C_\infty$ and $B(\{C_n\}) = B_C$.

Most of the properties in Proposition 2.1 remain unchanged for this generalization to chains. For instance, cone $B(\{C_n\}) = \{C_n\}_\infty$. The following relation is key (see [17, Proposition 26]):

$$B(\{C_n\}) = \bigcap_{n=1}^{\infty} B_{C_n}.$$

It implies, inter alia, that $B(\{C_n\})$ is weakly compact and nonempty when V is reflexive and the sets C_n are unbounded.

Definition 7.1. Given a chain $\{C_n\}$ and $d \in \{C_n\}_\infty$, we say that $\{C_n\}$ retracts along d if, for any sequence $y_k = x_{n_k} \in C_{n_k}$ such that $y_k/t_k \rightharpoonup d$ with $t_k \to \infty$, there exists a subsequence $\{z_r\}$, with $z_r = y_{k_r}$, and a bounded sequence $\alpha_r > 0$ such that $z_r - \alpha_r d \in C_{n_{k_r}}$ for all r.

The chain $\{C_n\}$ is called *retractive* if it retracts along all $d \in \{C_n\}_\infty$. When $C_n = C$ for all n, we get similar definitions for a fixed set C, with $\{C_n\}_\infty = C_\infty$.

Any chain trivially retracts along 0. Hence, $\{C_n\}_\infty = \{0\}$ is a simple sufficient condition for $\{C_n\}$ to be retractive. The next lemma gives an equivalent condition of retractivity in terms of $B(\{C_n\})$.

Lemma 7.2. *A chain $\{C_n\}$ is retractive if and only if for each $d \in B(\{C_n\})$ and for any unbounded sequence $y_k = x_{n_k} \in C_{n_k}$ such that $y_k/\|y_k\| \rightharpoonup d$, there is a subsequence $\{z_r\}$, with $z_r = y_{k_r}$, and a bounded sequence $\alpha_r > 0$ such that $z_r - \alpha_r d \in C_{n_{k_r}}$ for all r.*

Proof. We prove the "if," the converse being trivial. The result is trivially true when $d = 0$. Therefore, take a sequence $y_k = x_{n_k} \in C_{n_k}$ with $y_k/t_k \rightharpoonup d \neq 0$

and $t_k \to \infty$. By the usual argument, there exists a subsequence y_{k_r} for which $\|y_{k_r}/t_{k_r}\| \to \lambda \neq 0$. Hence

$$\frac{y_{k_r}}{\|y_{k_r}\|} = \frac{y_{k_r}}{t_{k_r}} \cdot \frac{t_{k_r}}{\|y_{k_r}\|} \rightharpoonup \frac{d}{\lambda},$$

and so $\lambda^{-1}d \in B(\{C_n\})$. Therefore, there is a subsequence $\{y_s\} \subseteq y_{k_r}$ and a sequence of scalars $\{\alpha_s\}$ such that $y_s - \lambda^{-1}\alpha_s d \in C_s$, as desired. $\square$

The next result is an infinite dimensional extension of [6, Proposition 1].

Theorem 7.3. *A retractive chain of nonempty sw-closed and finitely well-positioned sets of a reflexive space V has nonempty intersection.*

In [17] we derived this result as a consequence of more general results. Here we give a direct proof. We need a geometrical fact. Given a cone $K = \bigcup_{i=1}^n K_{x_i^*}$, we introduce in V the following equivalent norm, with $\alpha > 1$,

$$\|x\|_1 = \|x\| + \alpha \sum_{i=1}^{n} |\langle x_i^*, x\rangle| . \tag{5}$$

The next property is proved in [17, Lemma 11].

Lemma 7.4. *If $x \in K \cap \{\|x\|_1 \leq 1\}$, then $\|x\| \leq (1+\alpha)^{-1}$.*

Proof of Theorem 7.3. Let $\{C_n\}$ be a chain of finitely well-positioned sets. We can assume that C_n is unbounded (otherwise the result is trivial by the reflexivity of the space). By Proposition 3.1, there is some $\rho > 0$ such that $C_n \cap \{\|x\| \geq \rho\} \subseteq K = \bigcup_{i=1}^n K_{x_i^*}$ for all n. We thus renorm the space with the equivalent norm (5).

Set $x_n \in \arg\min_{x \in C_n} \|x\|_1$ for all n. Notice that $\arg\min_{x\in C_n} \|x\|_1 \neq \emptyset$ under our hypotheses. If the sequence $\{x_n\}$ is bounded, by taking a subsequence, $x_n \rightharpoonup x_0 \in C_n$ for all n. Therefore, $x_0 \in \bigcap_{n=1}^\infty C_n$ and the theorem is true. It remains to show that the sequence $\{x_n\}$ is bounded.

As $\|x_n\|_1 \leq \|x_{n+1}\|_1$, we have $\|x_n\|_1 \to \infty$ if the sequence is unbounded. Under our assumption, there is a subsequence $\{y_k\}$ of $\{x_n\}$, with $y_k = x_{n_k} \in C_{n_k}$, such that $y_k / \|y_k\|_1 \rightharpoonup d \neq 0$. Accordingly, $\|d\|_1 \leq 1$, and so $\|d\| \leq (1+\alpha)^{-1}$ by Lemma 7.4.

By retractivity, passing to a subsequence $\{z_r\}$, we have $z_r - \alpha_r d \in C_{n_{k_r}}$ for all r, where $z_r = y_{k_r}$. Hence, $\|z_r - \alpha_r d\|_1 \geq \|z_r\|_1$; that is, $\|z_r / \|z_r\|_1 - \beta_r d\|_1 \geq 1$ with $0 < \beta_r = \alpha_r / \|z_r\|_1$. Observe that

$$\begin{aligned} \left\| \frac{z_r}{\|z_r\|_1} - \beta_r d \right\|_1 &= \left\| (1-\beta_r) \frac{z_r}{\|z_r\|_1} + \beta_r \left(\frac{z_r}{\|z_r\|_1} - d \right) \right\|_1 \\ &\leq (1-\beta_r) + \beta_r \left\| \frac{z_r}{\|z_r\|_1} - d \right\|_1 . \end{aligned}$$

If we prove that $\|z_r/\|z_r\|_1 - d\|_1 < 1$ for r large enough, then $\|z_r/\|z_r\|_1 - \beta_r d\|_1 < 1$, a contradiction. Notice that for r large enough $z_r/\|z_r\|_1 \in K \cap \{\|x\|_1 \leq 1\}$. Set $u_r = z_r/\|z_r\|_1$. By Lemma 7.4,

$$\begin{aligned}\|u_r - d\|_1 = \; & \|u_r - d\| + \alpha \sum_{i=1}^{n} |\langle x_i^*, u_r - d\rangle| \leq \|u_r\| + \|d\| + \alpha \sum_{i=1}^{n} |\langle x_i^*, u_r - d\rangle| \\ & \leq 2(1+\alpha)^{-1} + \alpha \sum_{i=1}^{n} |\langle x_i^*, u_r - d\rangle| .\end{aligned}$$

Since $\sum_{i=1}^{n} |\langle x_i^*, u_r - d\rangle| \to 0$, we have $\|u_r - d\|_1 < 1$ for r large enough, provided $\alpha > 1$. This completes the proof. $\square$

7.2. Closed Images and Algebraic Differences

Next we give a first application of Theorem 7.3.

Proposition 7.5. *Let $T : V \to W$ be a continuous linear mapping between two reflexive spaces, and $C \subseteq V$ be finitely well-positioned.*

(i) *The image $T(C)$ is sw-closed if C is sw-closed and retracts along all directions $d \in C_\infty \cap \ker T$.*

(ii) *The image $T(C)$ is finitely well-positioned if $C_\infty \cap \ker T$ is a linear space included in L_C; in this case, $T(C_\infty) = (TC)_\infty$.*

Remark 7.6. (a) The hypothesis in point (i) is fulfilled when, for instance, either C is retractive or $C_\infty \cap \ker T = \{0\}$ or $C_\infty \cap \ker T \subseteq L_C$.

(b) Notice that even for non-convex sets, L_C is defined as $L_C + C = C$, though in general L_C may not be a vector space.

Proof. (i) Suppose first that W is separable and that $y_n \in T(C)$ with $y_n \rightharpoonup \bar{y}$. We have to show that $\bar{y} \in T(C)$. The sequence is bounded, i.e., $y_n \in \rho B_W$. As W is separable, its dual is reflexive as well. Therefore, W is weakly metrizable over the bounded set ρB_W. Denote by δ such a metric. Consider the following sequence of sets in W

$$W_n = \{y \in W : \delta(y, \bar{y}) \leq \delta(y_n, \bar{y}) \text{ and } y \in \rho B_W\},$$

and the associated sets $C_n = C \cap T^{-1}(W_n)$ in V. Clearly, the sets W_n are weakly closed and thus the sets C_n are sw-closed. Moreover, the sets C_n are nonempty by construction. If their intersection is nonempty and $\bar{x} \in \bigcap_{n=1}^{\infty} C_n$, then $T(\bar{x}) = \bar{y}$. Notice that it is not restrictive to suppose $\delta(y_n, \bar{y}) \downarrow 0$ and thus $\{C_n\}$ is a chain.

Let us prove that $\{C_n\}_\infty \subseteq C_\infty \cap \ker T$. Let $d \in \{C_n\}_\infty$. Clearly, $d \in C_\infty$. Let $\{x_n\}$ be a sequence such that $\{x_n\} \subseteq C_n$, $t_n \to \infty$ and $x_n/t_n \rightharpoonup d$. Consequently, $T(x_n/t_n) \rightharpoonup T(d)$. On the other hand, $T(x_n) \in \rho B_W$ and so it is bounded. We have $T(d) = 0$ and $d \in \ker T$, and so the inclusion $\{C_n\}_\infty \subseteq C_\infty \cap \ker T$ is proved.

To apply Theorem 7.3 we must prove that the chain $\{C_n\}$ is retractive. Pick any $d \in \{C_n\}_\infty$ such that $T(d) = 0$. Let $\{x_{n_k}\}$ be a sequence for which $x_{n_k} \in C_{n_k}$, $t_{n_k} \to \infty$ and $x_{n_k}/t_{n_k} \rightharpoonup d \in C_\infty \cap \ker T$. By hypothesis, C retracts along any direction in $C_\infty \cap \ker T$. Hence $x_{n_{k_r}} - \alpha_r d \in C$ for a subsequence $\{x_{n_{k_r}}\}$ and for a scalar sequence $\{\alpha_r\}$. On the other hand, $T(x_{n_{k_r}} - \alpha_r d) = T(x_{n_{k_r}})$. It follows that $x_{n_{k_r}} - \alpha_r d \in C_{n_{k_r}}$. We have proved that $\{C_n\}$ are retractive and thus $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ by Theorem 7.3, as desired.

Suppose now that W is not separable and that $y_n \rightharpoonup \overline{y}$ with $y_n \in T(C)$. It suffices to consider the separable linear space $W_1 = \overline{\operatorname{span}}\{y_n\}_n$ and the linear mapping $\widetilde{T} : T^{-1}(W_1) \to W_1$. It is easy to see that all hypotheses hold for the set $C \cap T^{-1}(W_1)$ and so the result follows from the first part of the proof.

(ii) Set $V_0 = \ker T \cap C_\infty$. The cone C_∞ is finitely well-positioned. Therefore, the linear space $V_0 \subset C_\infty$ has finite dimension. Hence, we have $V = V_0 \oplus Z$. Let $y_n \in T(x_n)$ with $\|y_n\| \to \infty$ and $x_n \in C_n$. The sequence $x_n \in V$ admits the decomposition $x_n = x_n^0 \oplus z_n$ with $x_n^0 \in V_0$ and $z_n \in Z$. Clearly $T(x_n) = T(z_n) = y_n$. As T is continuous, the sequence z_n is unbounded. Notice also that $z_n \in C$. Actually, $z_n = x_n - x_n^0 \in C - V_0 = C$. Since C is finitely well-positioned, there is a subsequence z_{n_k} and $t_k \to \infty$ such that $z_{n_k}/t_k \rightharpoonup d_1 \neq 0$. Consequently, $y_{n_k}/t_k = T(z_{n_k}/t_k) \rightharpoonup T(d_1)$. On the other hand, $0 \neq d_1 \in Z$. Hence $d_1 \notin V_0 = \ker T \cap C_\infty$ and so $T(d_1) \neq 0$, which proves that $T(C)$ is finitely well-positioned.

As to the last statement, since the inclusion $T(C_\infty) \subseteq (TC)_\infty$ holds in general, it suffices to show that $(TC)_\infty \subseteq T(C_\infty)$. Let $y_n/t_n \rightharpoonup d$ with $y_n \in T(x_n)$, $t_n \to \infty$ and $d \neq 0$. Clearly, $\|y_n\| \to \infty$ and so there is a subsequence $\{y_{n_k}\}$ for which $y_{n_k}/\|y_{n_k}\| \rightharpoonup \lambda d$ with $\lambda \neq 0$. Applying to the sequence $\{x_{n_k}\}$ the above arguments, we get a subsequence $\{x_{n_{k_r}}\}$ such that $x_{n_{k_r}}/\tau_r \rightharpoonup d_1$ where $0 \neq d_1 \in C_\infty$. Once again there is a subsequence $\{x_l\} \subseteq \{x_{n_{k_r}}\}$ so that $x_l/\|x_l\| \rightharpoonup \mu d_1$ with $\mu \neq 0$. But, $y_l = T(x_l)$ is a subsequence of y_{n_k}; hence $\mu T(d_1) = \lambda d$ and so $T(\mu\lambda^{-1}d_1) = d$. Therefore, $(TC)_\infty \subseteq T(C_\infty)$ and the result is proved. □

A second noteworthy application of Theorem 7.3 is in providing general conditions under which the algebraic difference of two sw-closed sets is sw-closed.

Proposition 7.7. *Let C and D be two sw-closed sets of a reflexive space. The set $C - D$ is sw-closed under one of the following cases:*

(i) *C and D are finitely well-positioned and both retract along any direction in $C_\infty \cap D_\infty$;*

(ii) *C is finitely well-positioned and both C and D retract completely along any direction in $C_\infty \cap D_\infty$.*[13]

When $C_\infty \cap D_\infty = \{0\}$, the condition that both C and D retract completely along the directions in $C_\infty \cap D_\infty$ is trivially satisfied. Hence, in this case the set $C - D$ is sw-closed provided both sets are sw-closed and C is finitely well-positioned. In view of Theorem 4.4, Dieudonné [11]'s original result on differences of convex sets is thus a special case of Proposition 7.7, at least for the weak topology. We refer the reader to Adly, Ernst, and Théra [2] for more recent results for differences of

[13] That is, in Definition 7.1 it must hold $z_r - \alpha d \in C_{n_{k_r}}$ for all r and all α small enough.

closed convex sets based on the condition $C_\infty \cap D_\infty = \{0\}$.

Proof. (i) It suffices to consider the map $T : (x, y) \mapsto x - y$ from $V \times V$ to V, and to apply Proposition 7.5 to the set $T(C, D) = C - D$. We omit details.

(ii) The proof follows closely that of Proposition 7.5. Suppose first V is separable and $x_n \in C$, $y_n \in D$ with $z_n = x_n - y_n \rightharpoonup \bar{z}$. The sequence z_n is bounded, i.e., $z_n \in \rho B_V$. Denote by δ the metric on ρB_V that generates the weak topology. Consider the sequence of sets $C_n = C \cap [U_n(\bar{z}) + D]$, where

$$U_n(\bar{z}) = \{z \in V : \delta(\bar{z}, z) \leq \delta(\bar{z}, z_n), \, z \in \rho B_V\}.$$

Passing to a subsequence if necessary, we can assume that $U_{n+1}(\bar{z}) \subseteq U_n(\bar{z})$. Therefore, $\{C_n\}$ is a chain. Clearly, $C_n \neq \emptyset$ since $x_n = z_n + y_n$ and $x_n \in C \cap [U_n(\bar{z}) + D]$. Moreover, $[U_n(\bar{z}) + D]$ is sw-closed and so the sets C_n are sw-closed and finitely well-positioned. Notice further that $\{U_n(\bar{z}) + D\}_\infty = D_\infty$ and $\{C_n\}_\infty \subseteq C_\infty \cap D_\infty$. To apply Theorem 7.3 it remains to show that the chain $\{C_n\}$ is retractive. Let $x_n \in C_n$ and $x_n/t_n \rightharpoonup d$. Clearly, $d \in C_\infty \cap D_\infty$. Under our assumption there is a subsequence x_{n_k} such that $x_{n_k} - \alpha d \in C$ for α small enough. On the other hand, $x_{n_k} \in [U_{n_k}(\bar{z}) + D]$. Namely, $x_{n_k} = \xi_{n_k} + y_{n_k}$ with $\xi_{n_k} \in U_{n_k}(\bar{z})$ and $y_{n_k} \in D$. From, $x_{n_k}/t_{n_k} = \xi_{n_k}/t_{n_k} + y_{n_k}/t_{n_k}$ it follows $y_{n_k}/t_{n_k} \rightharpoonup d \in C_\infty \cap D_\infty$. By the assumptions, passing to a subsequence $\{y_s\} \subseteq \{y_{n_k}\}$ we have $y_s - \alpha d \in D$ for small α. Namely, $x_s - \alpha d = \xi_s + (y_s - \alpha d) \in U_s(\bar{z}) + D$. Hence, for small α it holds $x_s - \alpha d \in C_n$, where $\{x_s\}$ is a subsequence of $\{x_n\}$. This proves that the sequence $\{C_n\}$ is retractive. Hence, there is $\bar{x} \in \bigcap_{n=1}^\infty C_n$. This means that for all n we have $\bar{x} \in C$ and $\bar{x} \in U_n(\bar{z}) + D$. That is, $\bar{x} = \xi_n + y_n$ with $\xi_n \in U_n(\bar{z})$ and $y_n \in D$. Clearly, $\xi_n \rightharpoonup \bar{z}$. Hence, $\bar{x} - \xi_n = y_n \rightharpoonup \bar{x} - \bar{z}$. As D is sw-closed, $\bar{x} - \bar{z} = \bar{y} \in D$. Therefore, $\bar{z} = \bar{x} - \bar{y}$. □

Acknowledgements. The financial support of the European Research Council (advanced grant, BRSCDP-TEA) is gratefully acknowledged. We are greatly indebted to an anonymous referee for very helpful comments and suggestions.

References

[1] S. Adly, E. Ernst, M. Théra: Stability of the solution set of non-coercive variational inequalities, Commun. Contemp. Math. 4 (2002) 145–160.

[2] S. Adly, E. Ernst, M. Théra: On the closedness of the algebraic difference of closed convex sets, J. Math. Pures Appl. 82 (2003) 1219–1249.

[3] S. Adly, E. Ernst, M. Théra: Well-positioned closed convex sets and well-positioned closed convex functions, J. Global Optim. 29 (2004) 337–351.

[4] F. Albiac, N. J. Kalton: Topics in Banach Space Theory, Springer, Berlin (2006).

[5] C. D. Aliprantis, O. Burkinshaw: Positive Operators, Springer, New York (2006).

[6] D. P. Bertsekas, P. Tseng: Set intersection theorems and existence of optimal solutions, Math. Program. 110 (2007) 287–314.

[7] J. M. Borwein, H. M. Strojwas: Tangential approximations, Nonlinear Anal. 9 (1985) 1347–1366.

[8] J. M. Borwein, Y. Lucet, B. Mordukhovich: Compactly epi-Lipschitzian convex sets and functions in normed spaces, J. Convex Analysis 7 (2000) 375–393.

[9] J. M. Borwein, R. Goebel: Notions of relative interior in Banach spaces, J. Math. Sci., New York 115 (2003) 2542–2553.

[10] J. P. Dedieu: Critères de fermeture pour l'image d'un fermé non convexe par une multiapplication, C. R. Acad. Sci., Paris, Sér. A 287 (1978) 501–503.

[11] J. Dieudonné: Sur la séparation des ensembles convexes, Math. Ann. 163 (1966) 1–3.

[12] V. L. Klee: Separation properties of convex cones, Proc. Amer. Math. Soc. 2 (1955) 313–318.

[13] M. A. Krasnoselskii: Positive Solutions of Operator Equations, P. Noordhoff, Groningen (1964).

[14] D. T. Luc: Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems 319, Springer, Berlin (1989).

[15] D. T. Luc: Recessively compact sets: properties and uses, Set-Valued Anal. 10 (2002) 15–35.

[16] D. T. Luc, J. -P. Penot: Convergence of asymptotic directions, Trans. Amer. Math. Soc. 353 (2001) 4095–4121.

[17] M. Marinacci, L. Montrucchio: Necessary and sufficient conditions for optima in reflexive spaces, SIAM J. Optim. 21 (2011) 174–192.

[18] R. E. Megginson: An Introduction to Banach Space Theory, Springer, New York (1998).

[19] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[20] C. Zălinescu: Recession cones and asymptotically compact sets, J. Optimization Theory Appl. 77 (1993) 209–220.