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Regularity Modulus of Intersection Mappings. Application to the Stability of Equations via Splitting into Inequalities*

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This paper is firstly concerned with the modulus of metric regularity of intersection mappings. We consider a finite collection of set-valued mappings and analyze the relationship between the regularity moduli of these mappings (specifically, the maximum of them) and the regularity modulus of the associated intersection mapping. As an application we derive the Lipschitz modulus of the feasible set mapping associated with linear systems of (possibly) infinitely many linear inequalities and finitely many equations. Previously we characterize the metric regularity of such systems. Specifically, we consider an intersection mapping which obeys the strategy of splitting equations into inequalities, and then we apply preliminary results for inequality systems.

Keywords: Metric regularity, linear systems, intersection mappings, linear regularity

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1. Introduction

Let us consider the finite family of set-valued mappings

$$\{\Psi_i : X \rightrightarrows Y;\ i = 1, ..., p\},$$

with $p \in \mathbb{N}$, X and Y being extended metric spaces endowed with extended distances d_X and d_Y (i.e., d_X and d_Y are allowed to take the value $+\infty$). We also consider

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the *intersection mapping* $\Psi : X \rightrightarrows Y$ given by

$$\Psi(x) = \bigcap_{i=1}^{p} \Psi_i(x), \quad x \in X. \tag{1}$$

In this paper, for a particular intersection mapping associated with a system of (possibly) infinitely many linear inequalities and a finite amount of linear equations, we aim to prove that the (metric) regularity modulus of the intersection mapping equals the maximum of the individual moduli. This will allow us to approach the quantitative stability of linear systems containing equations by applying their counterparts for linear inequality systems, established in [4].

For the sake of completeness, next we recall some well-known regularity notions. Ψ is said to be *metrically regular* at/around $(\overline{x}, \overline{y}) \in \operatorname{gph}\Psi$ (the graph of Ψ) if there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$, and a constant $\kappa \geq 0$, such that, for all $x \in U$ and all $y \in V$,

$$d_X\left(x, \Psi^{-1}(y)\right) \leq \kappa d_Y(y, \Psi(x)). \tag{2}$$

Observe that in this case, provided that U and V are open sets, Ψ is also metrically regular at any $(x, y) \in (U \times V) \cap \operatorname{gph}\Psi$; i.e., metric regularity *at* the point implies this property *around* the point. Here we adopt the usual convention $d_X(x, \varnothing) = +\infty$, and hence the metric regularity of Ψ at $(\overline{x}, \overline{y})$ entails $\Psi^{-1}(y) \neq \varnothing$ for y close enough to $\overline{y}$. The metric regularity of Ψ is known to be equivalent to the *Aubin property* (also called *pseudo-Lipschitz* or *Lipschitz-like*) of its inverse mapping Ψ^{-1} (given by $x \in \Psi^{-1}(y) \Leftrightarrow y \in \Psi(x)$). The reader is addressed to the books of Klatte and Kummer [11], Mordukhovich [13] and Rockafellar and Wets [15] for a comprehensive development of these notions, among other topics of variational analysis.

The infimum of constants κ in (2), as $U \times V$ shrinks to $(\overline{x}, \overline{y})$, provides a quantitative measure of the stability of Ψ around $(\overline{x}, \overline{y}) \in \operatorname{gph}\Psi$. This infimum is called *modulus of metric regularity* (or *regularity modulus*, for short) of Ψ at $(\overline{x}, \overline{y})$, denoted by $\operatorname{reg}\Psi(\overline{x} \mid \overline{y})$, and coincides with the so-called *exact Lipschitzian bound* (or *Lipschitz modulus*) of Ψ^{-1} at $(\overline{y}, \overline{x})$. We define $\operatorname{reg}\Psi(\overline{x} \mid \overline{y}) = +\infty$ when Ψ is not metrically regular at $(\overline{x}, \overline{y})$. For motivation and theoretical studies about this concept, see the works of Dontchev, Lewis and Rockafellar [7] and Ioffe [10]. See also papers [2], [4], [5], [9], among others, for the study of metric regularity and the associated modulus of semi-infinite constraint systems. Specifically, [2] deals with linear systems including equations, where only continuous perturbations of the right hand side (over a compact Hausdorff index set) are considered. In contrast, in the present paper neither continuity nor compactness is assumed, and both sides of the system are subject to perturbations. Moreover, the present paper follows the intersection mapping methodology introduced in [5].

We are concerned with linear semi-infinite systems, in $\mathbb{R}^n$, of the form

$$\sigma := \{a_t' x \geq b_t,\ t \in T;\ a_s' x = b_s,\ s \in S\}, \tag{3}$$

where T is arbitrary (possibly infinite) and S is a finite non-empty set, with $T \cap S = \varnothing$. The functions $t \mapsto a_t \in \mathbb{R}^n$ and $t \mapsto b_t \in \mathbb{R}$, for $t \in T \cup S$, are also arbitrary.

In this paper, elements of $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$, so that $y'x$ represents the usual inner product in $\mathbb{R}^n$. It is well-known that the stability of system (3) cannot be straightforwardly derived through the stability of the 'split system'

$$\{a_t'x \geq b_t,\ t \in T;\ a_s'x \geq b_s,\ \ -a_s'x \geq -b_s,\ s \in S\} \tag{4}$$

(where split inequalities are allowed to be perturbed independently from each other). For instance, if we just take any $s_0 \in S$ and perturb the corresponding split inequalities as $a_{s_0}'x \geq b_{s_0} + \varepsilon$ and $-a_{s_0}'x \geq -b_{s_0} + \varepsilon$, for any $\varepsilon > 0$, we obtain an inconsistent system. For further analysis on this aspect, the reader is addressed to [3].

Note that system (3) can be identified with the element

$$\sigma \equiv \begin{pmatrix} a_t \\ b_t \end{pmatrix}_{t \in T \cup S} \tag{5}$$

of the *parameter space* $(\mathbb{R}^{n+1})^{T \cup S}$. The *feasible set mapping*, $\mathcal{F} : (\mathbb{R}^{n+1})^{T \cup S} \rightrightarrows \mathbb{R}^n$, is given by

$$\mathcal{F}(\sigma) := \{x \in \mathbb{R}^n \mid a_t'x \geq b_t,\ t \in T;\ a_s'x = b_s,\ s \in S\}. \tag{6}$$

With respect to this mapping, our interest is focused on computing the regularity modulus of $\mathcal{G} := \mathcal{F}^{-1}$ –in other words, the exact Lipschitzian bound of $\mathcal{F}$–, taking advantage of the previous knowledge about systems of inequalities only (developed in [4]), and by using the technique introduced in [5] related to the analysis of intersection mappings.

Specifically, associated with each $\gamma = (\gamma_s)_{s \in S} \in \{-1, 1\}^S$ we consider

$$\mathcal{G}_\gamma := \mathcal{F}_\gamma^{-1}, \tag{7}$$

where $\mathcal{F}_\gamma : (\mathbb{R}^{n+1})^{T \cup S} \rightrightarrows \mathbb{R}^n$ is defined by

$$\mathcal{F}_\gamma(\sigma) := \{x \in \mathbb{R}^n \mid a_t'x \geq b_t,\ t \in T;\ \gamma_s a_s'x \geq \gamma_s b_s,\ s \in S\}, \tag{8}$$

providing an appropriate representation of $\mathcal{F}$ and $\mathcal{G}$ as an intersection mappings: From (6), (7) and (8) we immediately have, for all $x \in \mathbb{R}^n$ and all $\sigma \in (\mathbb{R}^{n+1})^{T \cup S}$,

$$\mathcal{F}(\sigma) = \bigcap_{\gamma \in \{-1,1\}^S} \mathcal{F}_\gamma(\sigma), \qquad \mathcal{G}(x) = \bigcap_{\gamma \in \{-1,1\}^S} \mathcal{G}_\gamma(x). \tag{9}$$

Roughly speaking, each $\gamma \in \{-1, 1\}^S$ provides a sign choice $+$ or $-$ for each equation $a_s'x = b_s$, giving rise to one of the two split inequalities $\pm a_s'x \geq \pm b_s$. We will come back to this in Section 4.

The structure of the paper is as follows: Section 2 gathers some definitions and preliminary results. Section 3 presents the general relationship between the regularity modulus of the intersection mapping Ψ and the regularity moduli of the Ψ_i's, introducing two suitable conditions which are not restrictive in our framework of interest.

From this starting point (Proposition 3.1), Section 4 focuses on the setting of systems (3) and analyzes the fulfillment of the previous conditions (Proposition 4.3), providing a characterization of the metric regularity of $\mathcal{G}$ (Proposition 4.4) and our aimed equality (Theorem 4.6) between $\operatorname{reg}\mathcal{G}(\overline{x} \mid \overline{\sigma})$ and the maximum of the regularity moduli $\operatorname{reg}\mathcal{G}_\gamma(\overline{x} \mid \overline{\sigma}_\gamma)$ (see (10) below), and hence a formula to compute it in terms only of the nominal feasible point $\overline{x}$ and the coefficients of our nominal system $\overline{\sigma}$.

Remark about notation: In the running text we will call "systems" to the elements of the form (5); i.e., each system is identified with its coefficients. Associated with σ as in (5) and $\gamma \in \{-1,1\}^S$, we shall denote as σ_γ the element of $(\mathbb{R}^{n+1})^{T\cup S}$ whose t-coordinate is $\binom{a_t}{b_t}$ for $t \in T$ and whose s-coordinate is $\binom{\gamma_s a_s}{\gamma_s b_s}$ for $s \in S$. Usually we will regard σ as a system containing equations, in the form (3), whereas σ_γ will be regarded as a system containing inequalities only, specifically

$$\sigma_\gamma = \{a_t' x \geq b_t,\ t \in T;\ \gamma_s a_s' x \geq \gamma_s b_s,\ s \in S\}. \tag{10}$$

According to this comment –and abusing the notation–, the strong Slater condition has a different meaning for σ and σ_γ. On the one hand we say that σ (containing equations) satisfies the Strong Slater condition if $\{a_s,\ s \in S\}$ is linearly independent –hence $|S| \leq n$– and there exist $\widehat{x} \in \mathbb{R}^n$ and $\rho > 0$ such that $a_t'\widehat{x} \geq b_t + \rho$ for all $t \in T$ and $a_s'\widehat{x} = b_s$ for all $s \in S$. On the other hand, σ_γ (with inequalities only) satisfies the strong Slater condition if there exists $\widehat{x} \in \mathbb{R}^n$ (called strong Slater point for σ_γ) such that $\inf_{t\in T}(a_t'\widehat{x} - b_t) > 0$ and $\min_{s\in S}(\gamma_s a_s'\widehat{x} - \gamma_s b_s) > 0$.

Nevertheless, from a formal point of view both types of systems -containing equations or not- belong to the same parameter space, and what really indicates if a system contains equations or not is the feasible set mapping acting on the system: according to (6) and (8), respectively, $\mathcal{F}$ is the feasible set mapping of a system containing equations and $\mathcal{F}_\gamma$ is the feasible set mapping of a system of inequalities only, and the same criterion applies to the inverse mappings $\mathcal{G}$ and $\mathcal{G}_\gamma$.

2. Preliminaries

The analysis of the regularity modulus developed in [5, Section 3] (for the more general framework of convex systems) makes use of the following concepts, introduced in that paper. The first one is the property of *equirregularity* of the family $\{\Psi_\gamma,\ \gamma \in \Gamma\}$, where $\Gamma \neq \varnothing$ is an arbitrary index set. Note that $\operatorname{gph}\Psi = \bigcap_{\gamma\in\Gamma} \operatorname{gph}\Psi_\gamma$.

Definition 2.1. The family $\{\Psi_\gamma,\ \gamma \in \Gamma\}$ is said to be *equirregular* at $(\overline{x}, \overline{y}) \in \operatorname{gph}\Psi$ if $\sup_{\gamma\in\Gamma} \operatorname{reg}\Psi_\gamma(\overline{x} \mid \overline{y}) < +\infty$ and, for every $\alpha > \sup_{\gamma\in\Gamma} \operatorname{reg}\Psi_\gamma(\overline{x} \mid \overline{y})$, there exist neighborhoods U_α of $\overline{x}$ and V_α of $\overline{y}$ verifying

$$d\left(x, \Psi_\gamma^{-1}(y)\right) \leq \alpha d\left(y, \Psi_\gamma(x)\right), \quad \forall\, x \in U_\alpha, \forall\, y \in V_\alpha,\ \forall\, \gamma \in \Gamma.$$

The key point of the previous definition is the fact that U_α and V_α do not depend on γ. One immediately checks that any finite family of metrically regular mappings at $(\overline{x}, \overline{y})$ is equirregular at this point.

The second concept we are interested in is the linear regularity property of a family of set-valued mappings. This concept was introduced in [5] and, as mentioned there, it was inspired in its counterpart for a family of sets (see, e.g., [12, p. 113]; see also [1] for the analysis of bounded linear regularity property among other topics as error bounds). Since we are interested in the linear regularity of the family $\left\{\Psi_\gamma^{-1},\ \gamma\in\Gamma\right\}$, we set $\Phi_\gamma := \Psi_\gamma^{-1}$ (therefore, $\Phi := \Psi^{-1} = \bigcap_{\gamma\in\Gamma}\Phi_\gamma$), and give the definition directly for $\{\Phi_\gamma,\ \gamma\in\Gamma\}$ without loss of generality.

Definition 2.2. The family $\{\Phi_\gamma,\ \gamma\in\Gamma\}$ is said to be *linearly regular* at $(\overline{y},\overline{x})\in \operatorname{gph}\Phi$ *with constant* κ if there exist neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x,\Phi\left(y\right)\right)\leq\kappa\sup_{\gamma\in\Gamma}d\left(x,\Phi_\gamma\left(y\right)\right),\quad\forall\ (x,y)\in U\times V. \tag{11}$$

It is obvious that linear regularity with constant κ implies linear regularity with any constant $\kappa'\geq\kappa$. We say that $\{\Phi_\gamma,\ \gamma\in\Gamma\}$ is *linearly regular* when some constant $\kappa\geq 0$ verifying (11) exists.

We have the following general result, where we recall that Γ is arbitrary, possibly infinite:

Theorem 2.3 ([5, Thm. 1]). *If* $\left\{\Phi_\gamma := \Psi_\gamma^{-1},\ \gamma\in\Gamma\right\}$ *is linearly regular at* $(\overline{y},\overline{x})\in\operatorname{gph}\Phi$ *with constant* κ *and* $\{\Psi_\gamma,\ \gamma\in\Gamma\}$ *is equirregular at* $(\overline{x},\overline{y})$, *then*

$$\operatorname{reg}\Psi\left(\overline{x}\mid\overline{y}\right)\leq\kappa\sup_{\gamma\in\Gamma}\operatorname{reg}\Psi_\gamma\left(\overline{x}\mid\overline{y}\right).$$

(In particular, Ψ *is metrically regular at* $(\overline{x},\overline{y})$ *.)*

The reader can immediately derive from known facts (see, e.g., [8, Thm. 6.1(vii)]) that for $\sigma\in\left(\mathbb{R}^{n+1}\right)^{T\cup S}$, $\gamma\in\{-1,1\}^S$, and a point $x\in\mathcal{F}_\gamma\left(\sigma\right)$, mapping $\mathcal{G}_\gamma$ is metrically regular at (x,σ) if and only if σ_γ satisfies the strong Slater condition.

Remark 2.4. Formally the feasible set mapping considered in [8, Thm. 6.1(vii)], denoted here by $\widetilde{\mathcal{F}}$, assigns to each $\sigma\in\left(\mathbb{R}^{n+1}\right)^{T\cup S}$ the feasible set

$$\widetilde{\mathcal{F}}\left(\sigma\right):=\left\{x\in\mathbb{R}^n\mid a_t'x\geq b_t,\ t\in T\cup S\right\}.$$

So, our $\mathcal{F}_\gamma$ is nothing else but $\widetilde{\mathcal{F}}\circ i_\gamma$, where $i_\gamma:\left(\mathbb{R}^{n+1}\right)^{T\cup S}\longrightarrow\left(\mathbb{R}^{n+1}\right)^{T\cup S}$, given by $i_\gamma\left(\sigma\right)=\sigma_\gamma$, is an involutive isometry in $\left(\mathbb{R}^{n+1}\right)^{T\cup S}$. One immediately checks that the metric regularity property of $\mathcal{G}_\gamma$ at (x,σ) is equivalent to the same property for $\widetilde{\mathcal{G}}:=\widetilde{\mathcal{F}}^{-1}$ at (x,σ_γ). The key facts are that i_γ maps the open ball centered at σ with radius r onto the open ball centered at σ_γ with the same radius, and

$$d\left(\sigma,\mathcal{G}_\gamma\left(x\right)\right)=d\left(\sigma_\gamma,\widetilde{\mathcal{G}}\left(x\right)\right).$$

Hereafter we use the notation

$$C\left(\sigma_\gamma\right):=\operatorname{conv}\left\{\begin{pmatrix}a_t\\b_t\end{pmatrix},\ t\in T;\ \begin{pmatrix}\gamma_s a_s\\\gamma_s b_s\end{pmatrix},\ s\in S\right\}, \tag{12}$$

where $\operatorname{conv} A$ stands for the convex hull of a set A.

Next, we will appeal to the analogous result to the one we are looking for, when we have only inequalities. Here $\operatorname{cl} A$ stands for the closure of $A \subset \mathbb{R}^k, k \in \mathbb{N}$, and we also appeal to the dual norm of $\|\cdot\|$, given by $\|v\|_* := \max\{v'x : \|x\| \leq 1\}$, with $v \in \mathbb{R}^k$, $k \in \mathbb{N}$.

Theorem 2.5 ([4, Thm. 1]). *Given any $\gamma = (\gamma_s)_{s\in S} \in \{-1,1\}^S$ and any $\overline{\sigma} \equiv \binom{\overline{a}_t}{\overline{b}_t}_{t\in T\cup S}$ with $\{\overline{a}_t,\ t \in T\}$ bounded, consider $(\overline{x}, \overline{\sigma}) \in \operatorname{gph} \mathcal{G}_\gamma$ and assume that $\mathcal{G}_\gamma$ is metrically regular at $(\overline{x}, \overline{\sigma})$. Then*

(i) If $\overline{x}$ is a strong Slater point for $\overline{\sigma}_\gamma$, then

$$\operatorname{reg} \mathcal{G}_\gamma (\overline{x} \mid \overline{\sigma}) = 0.$$

(ii) If $\overline{x}$ is not a strong Slater point for $\overline{\sigma}_\gamma$, then $\left\{u \in \mathbb{R}^n \mid \binom{u}{u'\overline{x}} \in \operatorname{cl} C(\overline{\sigma}_\gamma)\right\}$ is nonempty and compact. Moreover,

$$\operatorname{reg} \mathcal{G}_\gamma (\overline{x} \mid \overline{\sigma}) = \left\| \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} \right\|_* \max \left\{ \frac{1}{\|u\|_*} \middle| \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \operatorname{cl}(C(\overline{\sigma}_\gamma)) \right\}.$$

Proof. The proof follows from [4, Thm. 1], taking Remark 2.4 into account. □

3. Regularity modulus for finite intersection mappings

In this section, we isolate two conditions to be required to the intersection mapping Ψ defined in (1) in order to guarantee the aimed equality between $\operatorname{reg} \Psi (\overline{x} \mid \overline{y})$ and the maximum of $\operatorname{reg} \Psi_i (\overline{x} \mid \overline{y})$, $1 \leq i \leq p$. These conditions are the following:

(L1) For each $(x, y) \in X \times Y$ there exists $i_0 \in \{1, ..., p\}$ such that

$$d_X\left(x, \Psi^{-1}(y)\right) = d_X\left(x, \Psi_{i_0}^{-1}(y)\right). \tag{13}$$

(L2) For each sequence $\{(x^r, y^r)\} \subset X \times Y$ converging to $(\overline{x}, \overline{y}) \in \operatorname{gph} \mathcal{G}$ and each $i_0 \in \{1, ..., p\}$, there exists another sequence $\{(\widetilde{x}^r, \widetilde{y}^r)\} \subset X \times Y$ also converging to $(\overline{x}, \overline{y})$ such that, for all r,

$$d_X\left(\widetilde{x}^r, \Psi^{-1}(\widetilde{y}^r)\right) \geq d_X\left(x^r, \Psi_{i_0}^{-1}(y^r)\right) \tag{14}$$

and

$$d_Y\left(\widetilde{y}^r, \Psi(\widetilde{x}^r)\right) \leq d_Y\left(y^r, \Psi_{i_0}(x^r)\right). \tag{15}$$

We will see in Section 4 that these conditions are not restrictive in our setting of systems σ (see (3)), and their associated intersection mappings $\mathcal{F}$ and $\mathcal{G}$. Therefore, we will obtain the aimed relation through the following result:

Proposition 3.1. *Given $(\overline{x}, \overline{y}) \in \operatorname{gph} \Psi$, the following statements hold:*

(i) If condition (L1) is satisfied, then

$$\operatorname{reg} \Psi (\overline{x} \mid \overline{y}) \leq \max_{i=1,\ldots,p} \operatorname{reg} \Psi_i (\overline{x} \mid \overline{y}).$$

(ii) *If condition* (L2) *is satisfied, then*

$$\operatorname{reg}\Psi\left(\overline{x}\mid\overline{y}\right)\geq\max_{i=1,\ldots,p}\operatorname{reg}\Psi_i\left(\overline{x}\mid\overline{y}\right).$$

Proof. (i) According to Definition 2.2, condition (L1) means the *linear regularity* of $\left\{\Psi_1^{-1},...,\Psi_p^{-1}\right\}$ with constant $\kappa=1$, at every point in $\operatorname{gph}\Psi^{-1}$. In the nontrivial case when $\max_{i=1,\ldots,p}\operatorname{reg}\Psi_i\left(\overline{x}\mid\overline{y}\right)<+\infty$, taking into account that *equirregularity* (see Definition 2.1) is always held for a finite family of metrically regular mappings, we can obtain the aimed inequality appealing to Theorem 2.3. (In fact, we could adapt directly to the present framework the proof of the theorem presented in [5, Thm. 1].)

(ii) By contradiction, let us assume that

$$\operatorname{reg}\Psi\left(\overline{x}\mid\overline{y}\right)<\alpha<\max_{i=1,\ldots,p}\operatorname{reg}\Psi_i\left(\overline{x}\mid\overline{y}\right). \tag{16}$$

Then, $\alpha<\operatorname{reg}\Psi_{i_0}\left(\overline{x}\mid\overline{y}\right)$ for some $i_0\in\{1,...,p\}$, which entails the existence of $\{x^r\}$ and $\{y^r\}$ converging to $\overline{x}$ and $\overline{y}$ such that

$$d_X\left(x^r,\Psi_{i_0}^{-1}\left(y^r\right)\right)>\alpha d_Y\left(y^r,\Psi_{i_0}\left(x^r\right)\right),\quad\text{for all } r.$$

Now condition (L2) straighforwardly yields, for all r,

$$d_X\left(\widetilde{x}^r,\Psi^{-1}\left(\widetilde{y}^r\right)\right)>\alpha d_Y\left(\widetilde{y}^r,\Psi\left(\widetilde{x}^r\right)\right),$$

in contradiction with the first inequality of (16). $\square$

Remark 3.2. Condition (L1) is usually not fulfilled when we are dealing with a linear constraint system and consider as components of the intersection mapping the mappings associated with each single constraint (equality or inequality) of the system. Just think about a nominal situation $\{x_1\geq 0,\ x_2\geq 0\}$, with $x=(x_1,x_2)\in\mathbb{R}^2$. In order to ensure (L1) we should "enlarge" the system with new constraints (convex combinations of the original ones in the case when we deal with inequalities only). See [2, Lemma 2.1] or [5, Theorem 2] for more details.

Remark 3.3. For completeness purposes, we mention that the appearance of the inequalities of Proposition 3.1 brings the formula of [13, Theorem 4.22] to mind, although the latter is focused on a different operation. Specifically, Theorem 4.22 of [13] provides, under suitable assumptions, the regularity modulus for compositions of general set-valued mappings, say $G:X\rightrightarrows Y$ and $F:Y\rightrightarrows Z$, X,Y and Z being Asplund spaces. There, $\operatorname{reg}F\circ G\left(\overline{x},\overline{z}\right)$, with $\left(\overline{x},\overline{z}\right)\in\operatorname{gph}F\circ G$, is bounded from above by the maximum of the products $\operatorname{reg}G\left(\overline{x},\overline{y}\right)\operatorname{reg}F\left(\overline{y},\overline{z}\right)$ as $\overline{y}\in G\left(\overline{x}\right)\cap F^{-1}\left(\overline{z}\right)$.

4. Application to linear systems containing equations

The goal of this section is to compute the regularity modulus of $\mathcal{G}=\mathcal{F}^{-1}$, where $\mathcal{F}$ is given in (6), at the nominal pair $\left(\overline{x},\overline{\sigma}\right)\in\operatorname{gph}\mathcal{G}$, with $\overline{\sigma}:=\binom{\overline{a}_t}{\overline{b}_t}_{t\in T\cup S}$.

Along this section, we consider $X := \mathbb{R}^n$ and $Y := (\mathbb{R}^{n+1})^{T\cup S}$. We assume that Y is endowed with the uniform convergence topology via the extended distance

$$d_Y(\sigma, \widetilde{\sigma}) := \sup_{t\in T\cup S} \left\{ \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} - \begin{pmatrix} \widetilde{a}_t \\ \widetilde{b}_t \end{pmatrix} \right\| \right\},$$

where $\sigma := \begin{pmatrix} a_t \\ b_t \end{pmatrix}_{t\in T\cup S}$ and $\widetilde{\sigma} := \begin{pmatrix} \widetilde{a}_t \\ \widetilde{b}_t \end{pmatrix}_{t\in T\cup S}$. We also need to require the norm considered in $\mathbb{R}^{n+1}$ to verify

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \left\| \begin{pmatrix} a \\ -b \end{pmatrix} \right\| \text{ for all } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1}. \tag{17}$$

Any p-norm, but not any norm (see [14, Thm. 15.2]), verifies this condition. Assuming (17) with respect to the norm in $\mathbb{R}^{n+1}$, we have

$$\left\| \begin{pmatrix} a \\ b_1 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} a \\ b_2 \end{pmatrix} \right\| \text{ whenever } |b_1| \leq |b_2|, \tag{18}$$

which comes directly from the fact that $\binom{a}{b_1}$ is a convex combination of $\binom{a}{b_2}$ and $\binom{a}{-b_2}$. In this situation, in $\mathbb{R}^n$ we consider the norm given by

$$\|a\| := \left\| \begin{pmatrix} a \\ 0 \end{pmatrix} \right\| \text{ for all } a \in \mathbb{R}^n. \tag{19}$$

For simplicity, we denote by $\|\cdot\|$ both norms in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. It is easy to check that properties (17), (18) and (19) also verify for the dual norms in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, both denoted by $\|\cdot\|_*$.

When no confusion arises, we shall use the same notation d for the distance d_X (coming from the norm of $\mathbb{R}^n$) and for the extended distance d_Y.

The following technical lemmas will yield that (L1) and (L2) hold for $\Gamma = \{-1,1\}^S$ and $\mathcal{G}_\gamma$ and $\mathcal{F}_\gamma$, with $\gamma \in \Gamma$, playing the role of Ψ_γ and Φ_γ, respectively.

Lemma 4.1. *Let $\sigma \in (\mathbb{R}^{n+1})^{T\cup S}$ with $\mathcal{F}(\sigma) \neq \varnothing$, and pick $x^\gamma \in \mathcal{F}_\gamma(\sigma)$ for each $\gamma \in \{-1,1\}^S$. Then,*

$$\mathcal{F}(\sigma) \cap \operatorname{conv}\left\{ x^\gamma \mid \gamma \in \{-1,1\}^S \right\} \neq \varnothing.$$

Proof. Let us prove this lemma by induction on the cardinality $|S|$.

For $|S| = 1$, writing $S = \{s\}$, $\gamma \equiv \gamma_s \in \{-1,1\}$ we have

$$x_+ := x^1 \text{ verifies } a_t' x_+ \geq b_t,\ t \in T;\ a_s' x_+ \geq b_s;$$

$$x_- := x^{-1} \text{ verifies } a_t' x_- \geq b_t,\ t \in T;\ -a_s' x_- \geq -b_s.$$

Then, it is clear that there must exist $\widetilde{x} \in \operatorname{conv}\{x_+, x_-\}$ verifying $a_s'\widetilde{x} = b_s$ and then $\widetilde{x} \in \mathcal{F}(\sigma)$ (inequalities indexed by T are trivially satisfied).

Assume that the statement of the lemma holds for $|S| = m-1$, and let us prove it for $|S| = m$. For simplicity write $S = \{1, ..., m\}$ and denote the elements $\gamma \in \{-1,1\}^S$ as $\gamma = (\gamma_1, ..., \gamma_m)$, with $\gamma_i \in \{-1,1\}$ for all i. Consider the set $\widetilde{S} = \{1, ..., m-1\}$ and, for $\gamma = (\gamma_1, ..., \gamma_m) \in \{-1,1\}^S$, let us denote $\widetilde{\gamma} := (\gamma_1, ..., \gamma_{m-1}) \in \{-1,1\}^{\widetilde{S}}$, $x_+^{\widetilde{\gamma}} := x^\gamma$ if $\gamma_m = 1$ and $x_-^{\widetilde{\gamma}} := x^\gamma$ if $\gamma_m = -1$. In other words, $\gamma = (\widetilde{\gamma}, \gamma_m)$, $x_+^{\widetilde{\gamma}} = x^{(\widetilde{\gamma},1)}$, $x_-^{\widetilde{\gamma}} = x^{(\widetilde{\gamma},-1)}$.

Then, under the current induction assumption, there exist

$$\widetilde{x}_+ \in \operatorname{conv}\left\{x_+^{\widetilde{\gamma}} \mid \widetilde{\gamma} \in \{-1,1\}^{\widetilde{S}}\right\},$$
$$\widetilde{x}_- \in \operatorname{conv}\left\{x_-^{\widetilde{\gamma}} \mid \widetilde{\gamma} \in \{-1,1\}^{\widetilde{S}}\right\},$$

verifying

$$a_t'\widetilde{x}_+ \geq b_t,\ t \in T;\ a_s'\widetilde{x}_+ = b_s,\ s = 1, ..., m-1;\ a_m'\widetilde{x}_+ \geq b_m, \quad \text{and}$$
$$a_t'\widetilde{x}_- \geq b_t,\ t \in T;\ a_s'\widetilde{x}_- = b_s,\ s = 1, ..., m-1;\ -a_m'\widetilde{x}_- \geq -b_m.$$

Therefore there exist a convex combination of $\widetilde{x}_+$ and $\widetilde{x}_-$, say

$$\widetilde{x} \in \operatorname{conv}\left\{x^\gamma \mid \gamma \in \{-1,1\}^S\right\},$$

verifying $a_m'\widetilde{x} = b_m$. Moreover $\widetilde{x}$ trivially verifies the remaining relations $a_t'\widetilde{x} \geq b_t$, $t \in T$; $a_s'\widetilde{x} = b_s$, $s = 1, ..., m-1$, and therefore

$$\widetilde{x} \in \mathcal{F}(\sigma) \cap \operatorname{conv}\left\{x^\gamma \mid \gamma \in \{-1,1\}^S\right\},$$

as we aimed to prove. □

Condition (i) in the following lemma comes straighforwardly from [6, Lem. 10] (taking Remark 2.4 into account), and condition (ii) extends the previous one to the framework of systems including equations. The proof of this second condition is analogous to the original version and so it is omitted here. We denote by $[\alpha]_+ := \max\{0, \alpha\}$ the positive part of $\alpha \in \mathbb{R}$.

Lemma 4.2. *Given* $\sigma = \begin{pmatrix} a_t \\ b_t \end{pmatrix}_{t \in T \cup S} \in (\mathbb{R}^{n+1})^{T \cup S}$ *and* $x \in \mathbb{R}^n$*, one has:*

(i) *For each* $\gamma \in \{-1,1\}^S$,

$$d(\sigma, \mathcal{G}_\gamma(x)) = \left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\|_*^{-1} \sup\left\{[b_t - a_t'x]_+,\ t \in T;\ [\gamma_s b_s - \gamma_s a_s'x]_+,\ s \in S\right\};$$

(ii)

$$d(\sigma, \mathcal{G}(x)) = \left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\|_*^{-1} \sup\left\{[b_t - a_t'x]_+,\ t \in T;\ |b_s - a_s'x|,\ s \in S\right\}.$$

Now we can show that conditions (L1) and (L2) are not restrictive in our current setting.

Proposition 4.3. *Mappings $\mathcal{G}_\gamma$, $\gamma \in \{-1,1\}^S$ defined in* (7) *and* (8) *verify conditions* (L1) *and* (L2) *(see* (13), (14) *and* (15)*).*

Proof. Let us prove (L1). Take any $(x,\sigma) \in \mathbb{R}^n \times (\mathbb{R}^{n+1})^{T\cup S}$ and let us see that there exists some $\overline{\gamma} \in \{-1,1\}^S$ such that

$$d(x, \mathcal{F}(\sigma)) = d(x, \mathcal{F}_{\overline{\gamma}}(\sigma)).$$

From the definition of $\mathcal{F}$, we only need to prove the inequality '$\leq$'. Reasoning by contradiction assume that, for each $\gamma \in \{-1,1\}^S$,

$$d(x, \mathcal{F}(\sigma)) > d(x, \mathcal{F}_\gamma(\sigma)). \tag{20}$$

Since $\mathcal{F}_\gamma(\sigma)$ is closed, the distance $d(x, \mathcal{F}_\gamma(\sigma))$ is attained at some $x^\gamma \in \mathcal{F}_\gamma(\sigma)$. Applying then Lemma 4.1 we can take some

$$\widetilde{x} \in \mathcal{F}(\sigma) \cap \operatorname{conv}\left\{x^\gamma \mid \gamma \in \{-1,1\}^S\right\}$$

and writing it as

$$\widetilde{x} = \sum_{\gamma\in\{-1,1\}^S} \alpha_\gamma x^\gamma,$$

where $\alpha_\gamma \geq 0$ for each $\gamma \in \{-1,1\}^S$ and $\sum_{\gamma\in\{-1,1\}^S} \alpha_\gamma = 1$, we obtain the contradiction

$$\begin{aligned} d(x, \mathcal{F}(\sigma)) \leq\ & \|x - \widetilde{x}\| \leq \sum_{\gamma\in\{-1,1\}^S} \alpha_\gamma \|x - x^\gamma\| \\ =\ & \sum_{\gamma\in\{-1,1\}^S} \alpha_\gamma d(x, \mathcal{F}_\gamma(\sigma)) < \sum_{\gamma\in\{-1,1\}^S} \alpha_\gamma d(x, \mathcal{F}(\sigma)) \\ =\ & d(x, \mathcal{F}(\sigma)), \end{aligned}$$

where the strict inequality comes from assumption (20).

Now let us establish (L2). Take any sequence $\{(x^r, \sigma^r)\}_{r\in\mathbb{N}} \subset \mathbb{R}^n \times (\mathbb{R}^{n+1})^{T\cup S}$ converging to $(\overline{x}, \overline{\sigma}) \in \operatorname{gph} \mathcal{G}$ and pick arbitrarily $\gamma \in \{-1,1\}^S$. We must prove that there exists $\{(\widetilde{x}^r, \widetilde{\sigma}^r)\} \subset \mathbb{R}^n \times (\mathbb{R}^{n+1})^{T\cup S}$ also converging to $(\overline{x}, \overline{\sigma})$ such that, for all r,

$$d(\widetilde{x}^r, \mathcal{F}(\widetilde{\sigma}^r)) \geq d(x^r, \mathcal{F}_\gamma(\sigma^r)) \tag{21}$$

and

$$d(\widetilde{\sigma}^r, \mathcal{G}(\widetilde{x}^r)) \leq d(\sigma^r, \mathcal{G}_\gamma(x^r)). \tag{22}$$

Writing $\sigma^r := \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}_{t\in T\cup S}$ we take, for all r, $\widetilde{x}^r := x^r$ and define

$$\widetilde{\sigma}^r := \begin{pmatrix} a_t^r \\ \widetilde{b}_t^r \end{pmatrix}_{t\in T\cup S},$$

with

$$\widetilde{b}_t^r := \begin{cases} b_t^r, & \text{for all } t \in T; \\ b_t^r, & \text{if } t \in S \text{ and } \gamma_t \left(b_t^r - (a_t^r)' x^r\right) > 0; \\ (a_t^r)' x^r, & \text{if } t \in S \text{ and } \gamma_t \left(b_t^r - (a_t^r)' x^r\right) \leq 0. \end{cases}$$

Inequality (21) comes from the fact that

$$\mathcal{F}\left(\widetilde{\sigma}^r\right) \subset \mathcal{F}_\gamma\left(\sigma^r\right).$$

Let us prove (22). Applying Lemma 4.2 we have

$$\begin{aligned} d\left(\widetilde{\sigma}^r, \mathcal{G}\left(\widetilde{x}^r\right)\right) &= \frac{1}{\left\|\binom{x^r}{-1}\right\|_*} \sup_{t \in T;\, s \in S} \left\{ \left[b_t^r - (a_t^r)' x^r\right]_+ ; \left|\widetilde{b}_s^r - (a_s)' x^r\right| \right\} \\ &= d\left(\sigma^r, \mathcal{G}_\gamma\left(x^r\right)\right), \end{aligned}$$

where the last equality comes from

$$\begin{aligned} \left|\widetilde{b}_s^r - (a_s^r)' x^r\right| &= \left|\gamma_s \left(\widetilde{b}_s^r - (a_s^r)' x^r\right)\right| \\ &= \left\{ \begin{matrix} \gamma_s \left(b_s^r - (a_s^r)' x^r\right), & \text{if } \gamma_s \left(b_s^r - (a_s^r)' x^r\right) > 0, \\ 0, & \text{if } \gamma_s \left(b_s^r - (a_s^r)' x^r\right) \leq 0 \end{matrix} \right\} \\ &= \left[\gamma_s \left(b_s^r - (a_s^r)' x^r\right)\right]_+ . \end{aligned}$$

So we have shown that (22) may be obtained as an equality indeed. □

Now we are ready to combine Propositions 3.1 and 4.3 in order to obtain, in Theorem 4.6, our aimed expression for the regularity modulus of $\mathcal{G}$ (i.e., the Lipschitz modulus of $\mathcal{F}$) under the assumption of the boundedness of the left-hand side system coefficients and the metric regularity of each $\mathcal{G}_\gamma$, $\gamma \in \{-1, 1\}^S$. Before this, we clarify the meaning of the latter assumption in the next proposition, which characterizes the metric regularity of $\mathcal{G}$. The reader can observe that equivalence $(ii) \Leftrightarrow (iii)$ in the following proposition is a known fact in the context of *continuous* perturbations (with respect to the index in a compact index set) of the *right-hand side* of the system (see, e.g., [9, Thms. 1 and 2] or [2, Thm. 2.1]).

Proposition 4.4. *Given* $\overline{\sigma} \equiv \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}_{t \in T \cup S} \in \left(\mathbb{R}^{n+1}\right)^{T \cup S}$ *and* $\overline{x} \in \mathcal{F}\left(\overline{\sigma}\right)$, *the following are equivalent:*

(i) $\mathcal{G}_\gamma$ *is metrically regular at* $(\overline{x}, \overline{\sigma})$, *for all* $\gamma \in \{-1, 1\}^S$;

(ii) $\mathcal{G}$ *is metrically regular at* $(\overline{x}, \overline{\sigma})$.

If, moreover, $\{\overline{a}_t,\ t \in T\}$ *is bounded, the previous conditions are also equivalent to the following one:*

(iii) $\overline{\sigma}$ *satisfies the* strong Slater condition.

Proof. $(i) \Leftrightarrow (ii)$ is a consequence of Propositions 3.1 and 4.3.

$(ii) \Rightarrow (iii)$. Under (ii), the linear independence of $\{\overline{a}_s,\ s \in S\}$ is obvious, since otherwise an arbitrarily small perturbation of equalities in $\overline{\sigma}$ could yield inconsistent systems (i.e., with no feasible solutions), contradicting (ii). On the other hand, if there would not exist $\widehat{x}$ as in (iii), then all members of the family of systems $\{\sigma^r\}_{r\in\mathbb{N}}$ given by

$$\sigma^r = \left\{ \overline{a}_t' x \geq \overline{b}_t + \frac{1}{r},\ t \in T;\ \overline{a}_s' x = \overline{b}_s,\ s \in S \right\}$$

would be inconsistent, which again contradicts (ii).

$(iii) \Rightarrow (i)$. Fix arbitrarily $\gamma \in \{-1, 1\}^S$ and let us construct a strong Slater point for the inequality system $\overline{\sigma}_\gamma$ (see comments after Theorem 2.3). Since $\{\overline{a}_s,\ s \in S\}$ is linearly independent, the system $\{\overline{a}_s' u = \gamma_s,\ s \in S\}$ has at least one solution (obviously nonzero), say $\widehat{u}$.

On the one hand, for any $\lambda > 0$ and any $t \in T$ we have

$$\overline{a}_t' \left(\widehat{x} + \lambda \widehat{u} \right) \geq \overline{b}_t + \rho - \lambda \left\| \widehat{u} \right\| \sup_{t \in T} \left\| \overline{a}_t \right\|_* .$$

Thus, by choosing $0 < \widehat{\lambda} \leq \rho / \left(2 \left\| \widehat{u} \right\| \sup_{t\in T} \left\| \overline{a}_t \right\|_* \right)$, we obtain

$$\overline{a}_t' \left(\widehat{x} + \widehat{\lambda} \widehat{u} \right) \geq \overline{b}_t + \rho/2 \ \text{ for all } t \in T.$$

On the other hand, for all $s \in S$,

$$\gamma_s \overline{a}_s' \left(\widehat{x} + \widehat{\lambda} \widehat{u} \right) = \gamma_s \overline{b}_s + \widehat{\lambda} \gamma_s^2 = \gamma_s \overline{b}_s + \widehat{\lambda}.$$

Hence, $\widehat{x} + \widehat{\lambda} \widehat{u}$ is a strong Slater point for $\overline{\sigma}_\gamma$, which entails (i). □

The following example shows that the boundedness assumption on $\{\overline{a}_s,\ s \in S\}$ is essential in the previous proposition.

Example 4.5. Consider the system in $\mathbb{R}$ given by $\overline{\sigma} = \{tx \geq -1,\ t \in \mathbb{Z};\ x = 0\}$. It is clear that $\widehat{x} := 0$ is a strong Slater point for $\overline{\sigma}$ in the sense of Proposition 4.4(iii). Since $\widehat{x}$ is indeed the only solution of $\overline{\sigma}$, it is clear that, for all $r \in \mathbb{N}$, the system $\sigma^r := \{tx \geq -1,\ t \in \mathbb{Z};\ x = 1/r\}$ is inconsistent, so that (ii) – and also (i) – fails.

In order to establish the final result, we consider the following set $E(\sigma)$, associated with σ in (3), which was introduced in [2]:

$$E(\sigma) = \left\{ \sum_{t\in T} \lambda_t \binom{a_t}{b_t} + \sum_{s \in S} \mu_s \binom{a_s}{b_s} \mid \lambda_t \geq 0,\ \mu_s \in \mathbb{R},\ \sum_{t \in T} \lambda_t + \sum_{s\in S} |\mu_s| = 1 \right\}.$$

Finally, we can obtain the aimed regularity modulus as follows:

Theorem 4.6. *Let $(\overline{x}, \overline{\sigma}) \in \operatorname{gph} \mathcal{G}$, with $\{\overline{a}_t,\ t \in T\}$ bounded, and assume that $\mathcal{G}_\gamma$ is metrically regular at $(\overline{x}, \overline{\sigma})$, for all $\gamma \in \{-1, 1\}^S$. Then*

$$\begin{aligned}
\operatorname{reg} \mathcal{G} (\overline{x} \mid \overline{\sigma}) &= \max_{\gamma \in \{-1,1\}^S} \operatorname{reg} \mathcal{G}_\gamma (\overline{x} \mid \overline{\sigma}_\gamma) \\
&= \left\| \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} \right\|_* \max \left\{ \frac{1}{\|u\|_*} \middle| \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \operatorname{cl} E (\overline{\sigma}) \right\}.
\end{aligned}$$

Proof. Due to the previous Proposition 4.3, and making use of Proposition 3.1, we have

$$\operatorname{reg} \mathcal{G} (\overline{x} \mid \overline{\sigma}) = \max_{\gamma \in \{-1,1\}^S} \operatorname{reg} \mathcal{G}_\gamma (\overline{x} \mid \overline{\sigma}_\gamma).$$

Recall that, for each $\gamma = (\gamma_s)_{s \in S} \in \{-1, 1\}^S$ and according to (10),

$$\overline{\sigma}_\gamma := \left\{ \overline{a}_t' x \geq \overline{b}_t,\ t \in T;\ \gamma_s \overline{a}_s' x \geq \gamma_s \overline{b}_s,\ s \in S \right\}.$$

In this way, $\overline{x}$ is not a strong Slater point for $\overline{\sigma}_\gamma$, since $\gamma_s \overline{a}_s' x = \gamma_s \overline{b}_s$, $\forall s \in S$; hence, appealing to Theorem 2.5 we obtain

$$\begin{aligned}
&\max_{\gamma \in \{-1,1\}^S} \operatorname{reg} \mathcal{G}_\gamma (\overline{x} \mid \overline{\sigma}_\gamma) \\
&= \left\| \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} \right\|_* \max_{\gamma \in \{-1,1\}^S} \max \left\{ \frac{1}{\|u\|_*} \middle| \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \operatorname{cl} C (\overline{\sigma}_\gamma) \right\} \\
&= \left\| \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} \right\|_* \max \left\{ \frac{1}{\|u\|_*} \middle| \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \bigcup_{\gamma \in \{-1,1\}^S} \operatorname{cl} C (\overline{\sigma}_\gamma) \right\} \\
&= \left\| \begin{pmatrix} \overline{x} \\ -1 \end{pmatrix} \right\|_* \max \left\{ \frac{1}{\|u\|_*} \middle| \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \operatorname{cl} E (\overline{\sigma}) \right\},
\end{aligned}$$

taking into account that, as the reader can easily check (recalling (12)), we have

$$E (\overline{\sigma}) = \bigcup_{\gamma \in \{-1,1\}^S} C (\overline{\sigma}_\gamma).$$

This completes the proof. □

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