

A Note on Touching Cones and Faces

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Received: November 10, 2010

Revised manuscript received: March 25, 2011

We study touching cones of a (not necessarily closed) convex set in a finite-dimensional real Euclidean vector space and we draw relationships to other concepts in Convex Geometry. *Exposed faces* correspond to *normal cones* by an antitone lattice isomorphism. *Poonems* generalize the former to *faces* and the latter to *touching cones*, these extensions are non-isomorphic, though. We study the behavior of these lattices under projections to affine subspaces and intersections with affine subspaces. We prove a theorem that characterizes exposed faces by assumptions about touching cones. For a convex body K the notion of conjugate face adds an isotone lattice isomorphism from the exposed faces of the polar body K° to the normal cones of K . This extends to an isomorphism between faces and touching cones.

Keywords: Convex set, exposed face, normal cone, poonem, face, touching cone, projection, intersection

1991 Mathematics Subject Classification: 52A10, 52A20, 47L07

1. Introduction

The term of *touching cone* has first appeared in 1993 when Schneider used it to conjecture¹ in Section 6.6 of [15] equality conditions for the Aleksandrov-Fenchel inequality. This inequality, established in 1937, is really a system of quadratic inequalities between several *convex bodies*, i.e. compact convex subsets of a finite-dimensional real Euclidean vector space $(\mathbb{E}, \langle \cdot, \cdot \rangle)$. A very special case is the isoperimetric inequality in dimension two that states that the area A and the boundary length l of a two-dimensional convex body satisfy $4\pi A \leq l^2$ with equality if and only if the convex body is a disk.

Initially we were trying to improve our understanding of projections of state spaces. These convex bodies, motivated in Section 1.3, are examples where the notion of touching cone is the same as normal cone. We are not aware of further attention to touching cones in the literature. So in Section 1.2 we take the opportunity and collect evidence of their significance in Convex Geometry:

1. Touching cones arise from normal cones in an analogous way as faces arise from exposed faces.

¹All of these conjectures are still open.

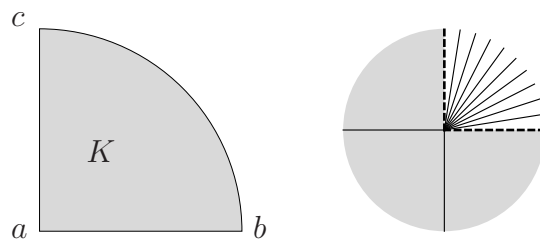


Figure 1.1: The closed quarter disk K (left) with its normal cones (right) sketched in the unit disk. Proper normal cones of K are: three quadrants at the faces $\{a\}$, $\{b\}$ and $\{c\}$, two rays at the faces $[a, b]$ and $[a, c]$ and a family of rays at the one-point faces of the arc from b to c other than $\{b\}$ or $\{c\}$. The two dashed rays are touching cones but not normal cones of K .

2. The pair of exposed face and face changes its role with the pair of normal cone and touching cone when projection to an affine subspace is replaced by intersection with an affine subspace.
3. If K is a convex body, there is a lattice isomorphism. The faces of the polar body correspond to the touching cones of K by taking positive hulls.
4. Touching cones can detect the exposed faces which are intersections of coatoms.
5. Touching cones relate to a special smoothness in dimension two.

1.1. Preliminaries

Our analysis uses the frame of Lattice Theory, see e.g. Birkhoff [3], which is well-known in Convex Geometry, see e.g. Loewy and Tam [11] and the references therein. A mapping $f : X \rightarrow Y$ between two partially ordered sets (posets) (X, \leq) and (Y, \leq) is *isotone* if for all $x, y \in X$ such that $x \leq y$ we have $f(x) \leq f(y)$. The mapping f is *antitone* if for all $x, y \in X$ such that $x \leq y$ we have $f(x) \geq f(y)$. A lattice \mathcal{L} is a partially ordered set (\mathcal{L}, \leq) where the *infimum* $x \wedge y$ and *supremum* $x \vee y$ of each two elements $x, y \in \mathcal{L}$ exist. All lattices appearing in this article are *complete*, i.e. for an arbitrary subset $S \subset \mathcal{L}$ the infimum $\bigwedge S$ and the supremum $\bigvee S$ exist. The reason is that elements x, y in these lattices are convex subsets of \mathbb{E} where a relation $x \leq y$ and $x \neq y$ always implies a dimension step $\dim(x) < \dim(y)$ (so \mathcal{L} has finite length and must be complete). In particular \mathcal{L} has a *smallest element* 0 and a *greatest element* 1. A *coatom* of \mathcal{L} is an element $x \in \mathcal{L}$ not 1 such that $y \geq x$ and $y \neq x$ implies $y = 1$ for all $y \in \mathcal{L}$.

Given a convex subset $C \subset \mathbb{E}$ we explain the concepts of normal cone, exposed face and face. The *normal cone* of C at $x \in C$ is the set of vectors $u \in \mathbb{E}$, that do not make for any $y \in C$ an acute angle with the vector from x to y . We put $N(C, x) := \{u \in \mathbb{E} : \langle u, y - x \rangle \leq 0 \text{ for all } y \in C\}$. The *relative interior* $\text{ri}(C)$ of C is the interior of C with respect to the affine span $\text{aff}(C)$ of C . The *relative boundary* of C is $\text{rb}(C) := C \setminus \text{ri}(C)$. The *normal cone* of any non-empty convex subset $F \subset C$ is well-defined (see Section 4) as the normal cone of any $x \in \text{ri}(F)$. We put $N(C, F) := N(C, x)$. E.g. the normal cone of C is the orthogonal complement of the translation vector space $\text{lin}(C)$ of $\text{aff}(C)$ and further Examples are shown in Figure 1.1. The normal cone of the empty set is $N(C, \emptyset) := \mathbb{E}$. This and $\text{lin}(C)^\perp$

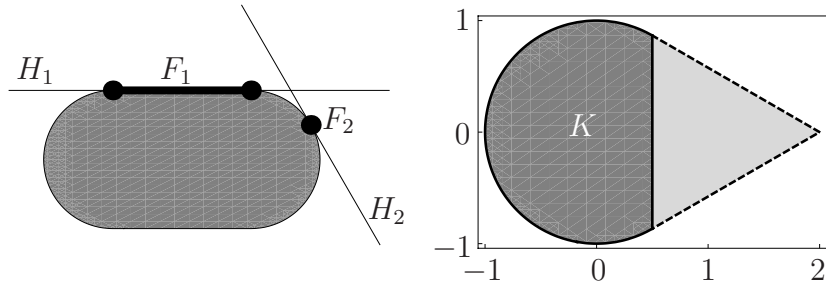


Figure 1.2: The stadium (left) consists of a square with two half-disks attached on opposite sides. The supporting hyperplane H_i defines the exposed face F_i for $i = 1, 2$. The two extreme points of F_1 are non-exposed faces. The truncated disk K (right) is the closed unit ball in \mathbb{R}^2 with the segment $x > \frac{1}{2}$ missing. The polar body K° of K is the union of K with the bright closed triangle.

are the *improper normal cones*, all other normal cones are *proper normal cones* and both together form the *normal cone lattice* $\mathcal{N}(C)$. The normal cone lattice is a complete lattice ordered by inclusion with the intersection as the infimum (see Prop. 4.8).

A *supporting hyperplane* of C is any affine hyperplane H in \mathbb{E} , such that $C \setminus H$ is convex and $C \cap H$ is non-empty. An *exposed face* of C is the intersection of C with a supporting hyperplane. An example is shown in Figure 1.2, left. In addition \emptyset and C are exposed faces called *improper exposed faces*. All other exposed faces are *proper exposed faces*. The set of exposed faces is the *exposed face lattice* $\mathcal{F}_\perp(C)$. This is a complete lattice ordered by inclusion and with the intersection as the infimum (see Prop. 3.8). If C has at least two points, then we have an antitone lattice isomorphism (see Prop. 4.7)

$$\mathcal{N}(C) : \mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C), \quad F \mapsto \mathcal{N}(C, F). \tag{1}$$

Two examples of this isomorphism are sketched in Figure 1.3 in columns two and three. The isomorphism does not require that C is closed or bounded. We can write the isomorphism (1) and its inverse in the form (18), i.e. for proper exposed faces F and proper normal cones N of C we have

$$F \mapsto \bigcap_{x \in F} \mathcal{N}(C, x) = \mathcal{N}(C, y) \quad \text{for any } y \in \text{ri}(F),$$

$$N \mapsto \bigcap_{u \in N \setminus \{0\}} F_\perp(C, u) = F_\perp(C, v) \quad \text{for any } v \in \text{ri}(N) \setminus \{0\}.$$

The *closed segment* between $x, y \in \mathbb{E}$ is $[x, y] := \{(1 - \lambda)x + \lambda y \mid \lambda \in [0, 1]\}$, the *open segment* between $x, y \in \mathbb{E}$ is $]x, y[:= \{(1 - \lambda)x + \lambda y \mid \lambda \in (0, 1)\}$. A *face* of C is a convex subset F of C , s.t. whenever for $x, y \in C$ the open segment $]x, y[$ intersects F , then the closed segment $[x, y]$ is included in F . An *extreme point* is the element of a zero-dimensional face. The faces \emptyset and C are *improper faces*, all other faces are *proper faces*. The set of all faces of C is the *face lattice* of C denoted by $\mathcal{F}(C)$. It is easy to show that arbitrary intersections of faces are faces, so $\mathcal{F}(C)$ is a complete lattice ordered by inclusion and with the intersection as the infimum.

It is easy to show $\mathcal{F}(C) \supset \mathcal{F}_\perp(C)$. A face which is not an exposed face will be called a *non-exposed* face, see e.g. Figure 1.2, left.

1.2. Observations about touching cones

We introduce touching cones according to our results in Theorem 7.4. A *touching cone* of C is any non-empty face of a normal cone of C . An example is shown in Figure 1.1. The improper normal cones $\text{lin}(C)^\perp$ and \mathbb{E} are touching cones called *improper touching cones*, all other touching cones are *proper touching cones*. These together form the *touching cone lattice* denoted by $\mathcal{T}(C)$. This is a complete lattice ordered by inclusion and with the intersection as the infimum. One has $\mathcal{T}(C) \supset \mathcal{N}(C)$.

1.2.1. Analogy in creation of touching cones and faces

There is an analogy between touching cone and face if we use the concept of *poonem* that Grünbaum [6] applies for a closed convex subset of \mathbb{E} . In finite dimension *poonem* is equivalent to *face*. We define a *poonem* of a convex subset $C \subset \mathbb{E}$ as a subset P of C s.t. there exist subsets F_0, F_1, \dots, F_k of C with $F_0 = P$, $F_k = C$ and F_{i-1} is an exposed face of F_i for $i = 1, \dots, k$. Every poonem is a face because a face of a face of C is a face of C . The converse is also true: given a proper face F of C , the smallest exposed face $\text{sup}_\perp(F)$ containing F is a proper exposed face of C by Lemma 4.6, so $\dim(\text{sup}_\perp(F)) < \dim(C)$. By induction F is a poonem of C . We have unified extensions

$$\begin{aligned}\mathcal{F}_\perp(C) \subset \mathcal{F}(C) &= \{\text{poonems of elements in } \mathcal{F}_\perp(C)\}, \\ \mathcal{N}(C) \subset \mathcal{T}(C) &= \{\text{non-empty poonems of elements in } \mathcal{N}(C)\}.\end{aligned}$$

As $\mathcal{F}(C)$ is the set of poonems of C , a more systematic definition would consider poonems of proper elements or of coatoms of $\mathcal{F}_\perp(C)$ and of $\mathcal{N}(C)$. In any case we can see that the concepts of *exposed face*, *normal cone* and *poonem* suffice to define *face* and *touching cone* in a unified way.

1.2.2. Compatibility with projection and intersection

We introduce Schneider's (equivalent) definition of touching cone: If $v \in \mathbb{E}$ is non-zero and the exposed face $F := F_\perp(C, v)$ is non-empty, then the face $T(C, v)$ of the normal cone $N(C, F)$ that contains v in its relative interior, is called a touching cone; $\text{lin}(C)^\perp$ and \mathbb{E} are touching cones by definition.

Let $\mathbb{A} \subset \mathbb{E}$ be an affine subspace, by $\pi_{\mathbb{A}}(C)$ we denote the orthogonal projection of C to \mathbb{A} . If $v \in \text{lin}(\mathbb{A})$ and $T(C, v)$ is a normal cone of C , then $T(\pi_{\mathbb{A}}(C), v)$ is a normal cone of $\pi_{\mathbb{A}}(C)$. This is proved in Section 6 by a new characterization of normal cones. Exposed faces of C however may project to non-exposed faces of $\pi_{\mathbb{A}}(C)$.

Dually, exposed faces are preserved under intersection of C with \mathbb{A} . But for some $v \in \text{lin}(\mathbb{A})$ the cone $T(C, v)$ may be a normal cone of C while $T(C \cap \mathbb{A}, v)$ is not a normal cone of $C \cap \mathbb{A}$. Example 7.8 discusses these aspects.

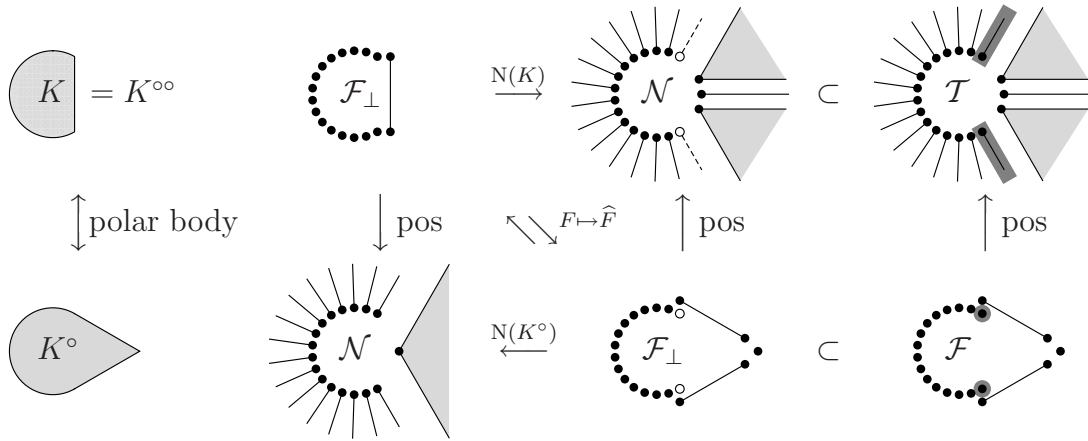


Figure 1.3: A finite sketch of proper lattice elements, empty circles denote deleted points, dashed lines denote deleted lines. Lattices belong to the convex body to their left, we have $\mathcal{F}(K) = \mathcal{F}_\perp(K)$ and $\mathcal{T}(K^\circ) = \mathcal{N}(K^\circ)$. In both rows there is an antitone isomorphism between exposed faces and normal cones (between columns two and three). The positive hull operator pos defines three isotone isomorphisms between rows one and two. Touching cones that are not normal cones and non-exposed faces are highlighted by a dark background (right column). The antitone isomorphism of the conjugate face is $F \mapsto \widehat{F}$.

1.2.3. A lattice isomorphism for convex bodies

We consider a convex body $K \subset \mathbb{E}$ with at least two points and with the origin in the interior, $0 \in \text{int}(K)$. The *polar body*

$$K^\circ := \{u \in \mathbb{E} \mid \langle u, x \rangle \leq 1 \text{ for all } x \in K\}$$

is a convex body with $0 \in \text{int}(K^\circ)$, an example is shown in Figure 1.2, right. Given a subset $S \subset \mathbb{E}$, the *positive hull* $\text{pos}(S)$ of S is the set of all finite *positive combinations* of elements of S , i.e. an element $x \in \mathbb{E}$ belongs to $\text{pos}(S)$ if and only if there is $k \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ with $\lambda_i \geq 0$ and $s_i \in S$ for $i = 1, \dots, k$ such that $x = \sum_{i=1}^k \lambda_i s_i$ (we have $0 \in \text{pos}(S)$). In Section 8 we establish isotone lattice isomorphisms

$$\begin{aligned} \mathcal{F}_\perp(K^\circ) &\rightarrow \mathcal{N}(K), & F &\mapsto \text{pos}(F), \\ \mathcal{F}(K^\circ) &\rightarrow \mathcal{T}(K), & F &\mapsto \text{pos}(F). \end{aligned} \tag{2}$$

The inverse isomorphism is given for a proper touching cone $T \in \mathcal{T}(K)$ by $T \mapsto \text{ri}(K^\circ) \cap T$. We think that (2) underlines (in the case of convex bodies) that the notion of touching cone is as fundamental as face. An example of the lattice isomorphisms is shown in Figure 1.3.

Following Remark 7.3 for a convex body K we have the partition of \mathbb{E} into the relative interiors of touching cones $\neq \mathbb{E}$. Denoting $T(K, u)$ the touching cone with the vector $u \in \mathbb{E} \setminus \{0\}$ in its relative interior, we have the partition

$$\mathbb{E} \setminus \{0\} = \dot{\bigcup}_{u \in \mathbb{E} \setminus \{0\}} \text{ri}(T(K, u)).$$

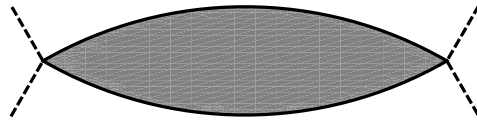


Figure 1.4: This intersection of two closed disks has touching cones which are not normal cones (dashed rays). Still, all proper faces are coatoms.

This is reminiscent of the partition of the *metric projection* (see e.g. Schneider [15])

$$\mathbb{E} = \dot{\bigcup}_{x \in K} (x + N(K, x)).$$

The partition of $\mathbb{E} \setminus \{0\}$ reminds us also of the partition of K° into the relative interiors of its faces (10). We have the following analogy:

Partition of $\text{rb}(K^\circ)$ in relative interiors of proper faces.	Partition of $\mathbb{E} \setminus \{0\}$ in relative interiors of proper touching cones of K .
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1.2.4. Coatoms of the face lattice

We explain for a general convex subset $C \subset \mathbb{E}$ that touching cones can characterize exposed faces in terms of coatoms in $\mathcal{F}_\perp(C)$. We recall that a coatom F of $\mathcal{F}_\perp(C)$ does not need to satisfy the dimension equation $\dim(F) + 1 = \dim(C)$, see e.g. F_2 in Figure 1.2, left. Since intersections of exposed faces are exposed, any intersection of coatoms in $\mathcal{F}_\perp(C)$ is an exposed face. A sufficient condition for the converse is proved in Thm. 7.10:

Theorem. *Let F be a proper exposed face of C where every touching cone included in the normal cone $N(C, F)$ is a normal cone. Then F is an intersection of coatoms of $\mathcal{F}_\perp(C)$.*

Figure 1.4 shows that there is no converse to the theorem. Examples are discussed after the remark below. A main argument to the theorem is *Minkowski's theorem* (a convex body is the convex hull of its extreme points) applied to a section of a normal cone. Another argument is the isomorphism (1). If we consider convex bodies, then the isomorphism (2) turns the theorem into an equivalent form, which more obviously follows from Minkowski's theorem (see Section 8).

Remark 1.1 (Exposed faces in dimension two). In dimension $\dim(C) = 2$ every non-exposed face of C is the endpoint of a unique one-dimensional face of C .

We prove this claim. All one-dimensional faces of C are coatoms of $\mathcal{F}_\perp(C)$ (as $\text{sup}_\perp(F)$ is proper for a proper face F). One dimension below, a point x of C may belong to $i = 0, 1, 2$ one-dimensional faces of C and exactly for $i = 0, 2$ the set $\{x\}$ is an intersection of coatoms of C . So a proper exposed face F of C is not the intersection of coatoms of $\mathcal{F}_\perp(C)$ if and only if $F = \{x\}$ where x is the endpoint of a unique one-dimensional face of C .

If in addition the assumptions of the above theorem hold for C , then non-exposed faces F are characterized by the conditions $F = \{x\}$ where x is the endpoint of a unique one-dimensional face of C .

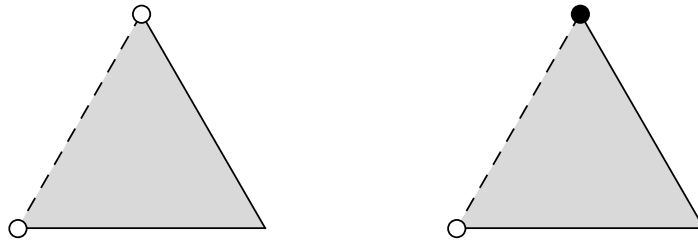


Figure 1.5: Empty circles denote deleted points, dashed lines denote deleted lines. The left triangle has three proper touching cones, all of which are normal cones. Accordingly every proper exposed face of the triangle is an intersection of coatoms. If the top vertex is added (right) one normal cone is added but two touching cones are added. The top vertex is not an intersection of coatoms.

An example with $\mathcal{N}(C) = \mathcal{T}(C)$ is the polar body K° (mouse shape) of the truncated disk. Further examples of $\mathcal{N}(C) = \mathcal{T}(C)$ are the state space discussed in Example 7.8. Examples of $\mathcal{N}(C) \subsetneq \mathcal{T}(C)$ that do not have the characterization of Remark 1.1 are the quarter disk in Figure 1.1 and the truncated disk K in Figure 1.3. Two convex set, which are not closed, are discussed in Figure 1.5.

1.2.5. Smoothness in dimension two

There is a special smoothness issue in dimension two. This holds for a general convex subset $C \subset \mathbb{R}^2$ if $\mathcal{N}(C) = \mathcal{T}(C)$, examples are listed in the previous paragraph. It would be interesting to see how smoothness generalizes into higher dimensions (where however coatoms of $\mathcal{F}_\perp(C)$ can have small dimension). A boundary point x of C is *singular*, if C has two linearly independent normal vectors at x .

The smoothness property, given $\dim(C) = 2$ and $\mathcal{T}(C) = \mathcal{N}(C)$, is that every singular point $x \in C$ is the intersection to two distinct boundary segments of C : If $x \in C$ is singular then the normal cone of C at x has two distinct boundary rays t_1, t_2 , which are touching cones of C by definition. By assumption t_1 is a normal cone of C , so it is the normal cone at a boundary point $y_1 \neq x$ of C . It follows that the segment $[x, y_1]$ is a boundary segment of C . The same arguments applied to t_2 show $\{x\} = [x, y_1] \cap [x, y_2]$ (If the intersection was a segment, then $\dim(C) \leq 1$ by (15)(iv)).

1.3. Projections of state spaces

Our motivation to study touching cones lies in Information Theory, see Amari and Nagaoka [2]. Analysis takes place in the convex body of *state space* $\mathbb{S}(n)$. This is a convex body in the algebra $\text{Mat}(\mathbb{C}, n)$ of complex $n \times n$ -matrices. In fact $\mathbb{S}(n)$ consists of all positive semi-definite matrices (i.e. being self-adjoint and without negative eigenvalues) that have trace one. We have $\mathcal{T}(\mathbb{S}(n)) = \mathcal{N}(\mathbb{S}(n))$ and $\mathcal{F}(\mathbb{S}(n)) = \mathcal{F}_\perp(\mathbb{S}(n))$, see Example 6.2. In Example 7.8 we discuss orthogonal projections P of $\mathbb{S}(n)$ to vector spaces L , they too satisfy $\mathcal{T}(P) = \mathcal{N}(P)$. These projections are connected to information manifolds called *exponential families*, see e.g. Knauf and Weis [10].

We ask if a finite-dimensional convex set C is *stable*, which means that for any $0 \leq d \leq \dim(C)$ the union of faces F of C with $\dim(F) \leq d$ is a closed set (see Papadopoulou [13]). It is well-known that $\mathbb{S}(n)$ is stable. Is P also stable? This would have consequences for the topology of exponential families.

Another question is about non-exposed faces of P and their behavior if L varies in a Grassmannian manifold of subspaces. This question may be related to continuity properties of information measures, see [10]. It is likely to be accessible by Convex Algebraic Geometry (as studied by Henrion, Rostalski, Sturmfels and others) because P is polar to an affine section of $\mathbb{S}(n)$, see [7, 12, 16]. On the other hand, the faces of P correspond to the touching cones of the affine section, which is an affine algebraic set.

2. Posets and lattices

We introduce lattices and cite two fundamental assertions about lattices.

Definition 2.1. A *partially ordered set* or *poset* (X, \leq) is a set X with a binary relation \leq , such that for all $x, y, z \in X$ we have $x \leq x$ (reflexive), $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetric) and $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive); $y \geq x$ is used instead of $x \leq y$.

A mapping $f : X \rightarrow Y$ between two posets (X, \leq) and (Y, \leq) is *isotone*, if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$ for any $x_1, x_2 \in X$. The mapping f is *antitone* if $x_1 \leq x_2$ implies $f(x_2) \leq f(x_1)$.

In a poset (X, \leq) , a *lower bound* of a subset $S \subset X$ is an element $x \in X$ such that $x \leq s$ for all $s \in S$. An *infimum* of S is a lower bound x of S such that $y \leq x$ for every lower bound y of S . Dually, an *upper bound* of a subset $S \subset X$ is an element $x \in X$ such that $s \leq x$ for all $s \in S$. A *supremum* of S is an upper bound x of S such that $x \leq y$ for every upper bound y of S . We may write $S = \{s_\alpha\}_{\alpha \in I}$ for an index set I . In case of existence, the infimum of S is unique and is denoted by $\bigwedge S$ or by $\bigwedge_{\alpha \in I} s_\alpha$, likewise the supremum of S is denoted by $\bigvee S$ or by $\bigvee_{\alpha \in I} s_\alpha$ in case of existence.

If (X, \leq) has a smallest element 0 , then an element $x \in X$ not 0 is an *atom* of X if for all $y \leq x$ in X with $y \neq x$ we have $y = 0$. If (X, \leq) has a greatest element 1 , then an element $x \in X$ not 1 is a *coatom* of X if for all $y \geq x$ in X with $y \neq x$ we have $y = 1$.

A *lattice* $(\mathcal{L}, \leq, \wedge, \vee)$ is a poset (\mathcal{L}, \leq) , such that for any two elements $x, y \in \mathcal{L}$ the infimum $x \wedge y := \bigwedge\{x, y\}$ and the supremum $x \vee y := \bigvee\{x, y\}$ exist. A lattice $(\mathcal{L}, \leq, \wedge, \vee)$ is *complete* if every subset X of \mathcal{L} has an infimum and a supremum. We denote a complete lattice by $(\mathcal{L}, \leq, \wedge, \vee, 0, 1)$ with 0 the smallest and 1 the greatest element of \mathcal{L} . A lattice $(\mathcal{L}, \leq, \wedge, \vee)$ is *modular* if for all elements $x, y, z \in \mathcal{L}$ the *modular law* is true:

$$x \leq z \quad \text{implies} \quad x \vee (y \wedge z) = (x \vee y) \wedge z. \quad (3)$$

The partial ordering of \mathcal{L} restricts to subsets. We call $X \subset \mathcal{L}$ a *sublattice* of \mathcal{L} if for all $x, y \in X$ the infimum $x \wedge y$ and the supremum $x \vee y$ (calculated in \mathcal{L}) belong to X .

Remark 2.2. Birkhoff has proved in [3], Lemma 1 on page 24, that an isotone bijection between two lattices with isotone inverse is a lattice isomorphism.

Definition 2.3. A property of subsets of a set M is a *closure property* when (i) M has the property, and (ii) any intersection of subsets having the given property itself has this property.

Remark 2.4. Birkhoff has proved in [3], Corollary on page 7, that those subsets \mathcal{M} of any set M which have a given closure property form a complete lattice. The ordering on \mathcal{M} is given by inclusion. The infimum of $\{M_\alpha\}_{\alpha \in I} \subset \mathcal{M}$ is the intersection $\bigwedge_{\alpha \in I} M_\alpha = \bigcap_{\alpha \in I} M_\alpha$ and the supremum is $\bigvee_{\alpha \in I} M_\alpha = \bigcap \{\widetilde{M} \in \mathcal{M} \mid \forall \alpha \in I : M_\alpha \subset \widetilde{M}\}$.

3. Faces and exposed faces

We introduce faces and exposed faces of a convex set and their lattice structure. Klingenberg [9] may be consulted for the background in affine geometry. Let $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real Euclidean vector space. We recommend a monograph such as Rockafellar or Schneider [14, 15] for an introduction to convex sets.

Definition 3.1 (Convexity). The *convex hull* $\text{conv}(C)$ of a subset $C \subset \mathbb{E}$ consists of all *convex combinations* of elements of C , i.e. $x \in \text{conv}(C)$ if and only if there is $k \in \mathbb{N}$ and for $i = 1, \dots, k$ there are $\lambda_i \in \mathbb{R}$ with $\lambda_i \geq 0$ and $\sum_{j=1}^k \lambda_j = 1$ and there are $x_i \in C$ such that $x = \sum_{j=1}^k \lambda_j x_j$. We understand $\text{conv}(\emptyset) = \emptyset$. The subset $C \subset \mathbb{E}$ is *convex*, if $x, y \in C$ implies $[x, y] \subset C$, which is the same as $C = \text{conv}(C)$. A *convex body* is a closed and bounded convex set. If we drop the condition of $\sum_{i=1}^n \lambda_i = 1$ then we speak of a *positive combination* and we denote the set of positive combinations of C by $\text{pos}(C)$ (and $\text{pos}(\emptyset) = \{0\}$). A *convex cone* is a non-empty convex subset C of \mathbb{E} where $x \in C$ and $\lambda \geq 0$ imply $\lambda x \in C$, which is the same as $C = \text{pos}(C)$.

According to Rockafellar [14] §2 the convex hull of C is the smallest convex subset of \mathbb{E} containing C . It is a closure property that a subset $C \subset \mathbb{E}$ is convex, i.e. \mathbb{E} is convex and arbitrary intersections of convex subsets are convex. Hence, Remark 2.4 ensures that the convex subsets of \mathbb{E} are the elements of a complete lattice ordered by inclusion and $\text{conv}(C)$ is the intersection of all convex subsets of \mathbb{E} that include C . Closure properties are important also for face lattices.

Definition 3.2 (Face lattice). If $C \subset \mathbb{E}$ is a convex subset, then a convex subset $F \subset C$ is a *face* of C if for all $x, y \in C$ the non-empty intersection $[x, y] \cap F$ implies $[x, y] \subset F$. The empty set \emptyset and C are *improper* faces, all other faces of C are *proper*. A face of the form $\{x\}$ for $x \in C$ is called an *extreme point* of C . The set of faces of C will be denoted by $\mathcal{F}(C)$ and will be called the *face lattice* of C .

If $C \subset \mathbb{E}$ is a convex subset then the intersection of any family of faces of C is a face of C . In other words, the property face is a closure property. Thus, by Remark 2.4

the face lattice

$$(\mathcal{F}(C), \subset, \cap, \vee, \emptyset, C) \quad (4)$$

is a complete lattice ordered by inclusion and the infimum is the intersection. The smallest element of $\mathcal{F}(C)$ is \emptyset , the greatest is C . We cite Schneider [15], Chap. 1, for two fundamental theorems. *Carathéodory's theorem* says if $C \subset \mathbb{E}$ and $x \in \text{conv}(C)$, then x is a convex combination of affinely independent points of C . *Minkowski's theorem* says that every convex body is the convex hull of its extreme points.

Definition 3.3 (Relative interior). If $C \subset \mathbb{E}$ then the *affine hull* of C , denoted by $\text{aff}(C)$ is the smallest affine subspace of \mathbb{E} that contains C . The interior of C with respect to the relative topology of $\text{aff}(C)$ is the *relative interior* $\text{ri}(C)$ of C . The complement $\text{rb}(C) := C \setminus \text{ri}(C)$ is the *relative boundary* of C . If $C \subset \mathbb{E}$ is convex and non-empty then the *vector space* of C is defined as the translation vector space of $\text{aff}(C)$,

$$\text{lin}(C) := \{x - y \mid x, y \in \text{aff}(C)\}. \quad (5)$$

We define the *dimension* $\dim(C) := \dim(\text{lin}(C))$ and $\dim(\emptyset) = -1$.

Let $C, D \subset \mathbb{E}$ be convex subsets. Rockafellar proves in [14], Cor. 6.6.2, the sum formula for the relative interior

$$\text{ri}(C) + \text{ri}(D) = \text{ri}(C + D). \quad (6)$$

In Thm. 6.5 he proves for the case $\text{ri}(C) \cap \text{ri}(D) \neq \emptyset$

$$\text{ri}(C) \cap \text{ri}(D) = \text{ri}(C \cap D). \quad (7)$$

If \mathbb{A} is an affine space and $\alpha : \mathbb{E} \rightarrow \mathbb{A}$ is an affine mapping, then by Thm. 6.6 in [14]

$$\alpha(\text{ri}(C)) = \text{ri}(\alpha(C)) \quad (8)$$

holds. If F is a face of C and if D is a subset of C , then by Thm. 18.1 in [14] we have

$$\text{ri}(D) \cap F \neq \emptyset \implies D \subset F. \quad (9)$$

By Thm. 18.2 in [14] C admits a partition by relative interiors of its faces

$$C = \dot{\bigcup}_{F \in \mathcal{F}(C)} \text{ri}(F). \quad (10)$$

In particular, every proper face of C is included in the relative boundary $\text{rb}(C)$ and its dimension is strictly smaller than the dimension of C . We need the following.

Lemma 3.4. *If $H \subset \mathbb{E}$ is an affine hyperplane with $0 \notin H$ and $C \subset H$ is a convex subset, then $\text{pos} : \mathcal{F}(C) \rightarrow \mathcal{F}(\text{pos}(C)) \setminus \{\emptyset\}$ is a bijection with inverse $F \mapsto C \cap F$.*

Proof. If F is a face of $\text{pos}(C)$, then F is a convex cone. So, if $F \neq \emptyset$, then $F = \text{pos}(F \cap C)$. Moreover, since $C \subset \text{pos}(C)$ the set $F \cap C$ is a face of C . This gives an injective mapping

$$\mathcal{F}(\text{pos}(C)) \setminus \{\emptyset\} \rightarrow \mathcal{F}(C), \quad F \mapsto F \cap C.$$

By (10) the relative interiors of faces F of $\text{pos}(C)$ are a partition of $\text{pos}(C)$ so the sets $\text{ri}(F) \cap C$ are a partition of C . If F is a face of $\text{pos}(C)$ where $\text{ri}(F) \cap C \neq \emptyset$ then $\text{ri}(F \cap C) = \text{ri}(F) \cap C$ by (7). This proves that the above mapping is a bijection. \square

The decomposition (10) justifies a definition:

Definition 3.5. Let $C \subset \mathbb{E}$ be a convex subset. For every $x \in C$ a unique face $F(C, x)$ of C is defined by the condition $x \in \text{ri}(F(C, x))$.

We describe suprema of faces.

Lemma 3.6. If $C \subset \mathbb{E}$ is a convex subset and $\{F_\alpha\}_{\alpha \in I}$ is a non-empty family of faces of C with $x_\alpha \in \text{ri}(F_\alpha)$ for all $\alpha \in I$, then for any $z \in \text{ri}(\text{conv}\{x_\alpha \mid \alpha \in I\})$ we have $\bigvee_{\alpha \in I} F_\alpha = F(C, z)$.

Proof. Since $z \in F(C, z)$ and since z is in the relative interior of the convex set $\text{conv}\{x_\alpha \mid \alpha \in I\}$, this convex set is included in $F(C, z)$ by (9). So all the x_α belong to $F(C, z)$. Again by (9) all the faces F_α are included in $F(C, z)$ because $x_\alpha \in \text{ri}(F_\alpha)$. Thus $F(C, z)$ is an upper bound for the family $\{F_\alpha\}_{\alpha \in I}$ and thus $\bigvee_{\alpha \in I} F_\alpha \subset F(C, z)$. Conversely we have $z \in \text{conv}\{x_\alpha \mid \alpha \in I\} \subset \bigvee_{\alpha \in I} F_\alpha$, so $F(C, z) \subset \bigvee_{\alpha \in I} F_\alpha$ by (9) because $z \in \text{ri}(F(C, z))$. \square

Some faces of C are obtained by intersection of C with a hyperplane, these are the *exposed faces*. Different to Rockafellar or Schneider [14, 15] we always include \emptyset and C to the exposed faces in order to turn this set into a lattice.

Definition 3.7 (Exposed face lattice). Let $C \subset \mathbb{E}$ be a convex subset. The *support function* of C is $\mathbb{E} \rightarrow \mathbb{R} \cup \{\pm\infty\}$, $u \mapsto h(C, u) := \sup_{x \in C} \langle u, x \rangle$. For non-zero $u \in \mathbb{E}$

$$H(C, u) := \{x \in \mathbb{E} : \langle u, x \rangle = h(C, u)\}$$

is an affine hyperplane in \mathbb{E} unless $H(C, u) = \emptyset$ when $h(C, u) = -\infty$ with $C = \emptyset$ or $h(C, u) = \infty$, when C is unbounded in the direction of u . If $H(C, u) \neq \emptyset$, then we call it a *supporting hyperplane* of C . The *exposed face* of C by u is

$$F_\perp(C, u) := C \cap H(C, u).$$

The faces \emptyset and C are exposed faces of C by definition called *improper exposed faces*. All other exposed faces are *proper*. The set of exposed faces of C will be denoted by $\mathcal{F}_\perp(C)$ called the *exposed face lattice* of C . A face of C , which is not an exposed face is a *non-exposed face*.

It is easy to show $\mathcal{F}_\perp(C) \subset \mathcal{F}(C)$. An example of a non-exposed faces is given in Figure 1.2, left. It is well-known that the intersection of exposed faces is an exposed face, see e.g. Schneider [15], but the following details were not found in the literature.

Proposition 3.8. Let $C \subset \mathbb{E}$ be a convex set and let $U \subset \mathbb{E} \setminus \{0\}$ be a non-empty set of directions. Then $\text{ri}(\text{conv}(U)) \setminus \{0\}$ is non-empty and every vector v in this set satisfies $\bigcap_{u \in U} F_\perp(C, u) = F_\perp(C, v)$ unless the intersection is empty.

Proof. Since $U \neq \emptyset$ we have $\text{ri}(U) \neq \emptyset$ (see [14], Thm. 6.2). If we had $\text{ri}(\text{conv}(U)) = \{0\}$ then $\text{conv}(U)$ would be $\{0\}$, which was excluded in the assumptions. This proves the first assertion.

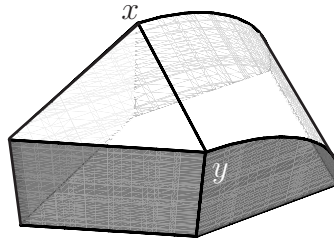


Figure 3.1: This depicted convex set K is a composition of two right prisms, one based on a triangle the other based on a quarter disk. The supremum of the extreme points x and y is the top triangle in $\mathcal{F}_\perp(K)$ and the segment $[x, y]$ in $\mathcal{F}(K)$.

Let $F := \bigcap_{u \in U} F_\perp(C, u)$ and $G := \bigcap_{u \in \text{conv}(U) \setminus \{0\}} F_\perp(C, u)$. First we show $F = G$. The non-trivial part is to prove $F \subset G$. A vector $v \in \text{conv}(U) \setminus \{0\}$ is a convex combination $v = \sum_i \lambda_i u_i$ for $u_i \in U$ and non-negative real scalars λ_i summing up to one. If $x \in F$ then $x \in F_\perp(C, u_i)$ for all i and then

$$\langle v, x \rangle = \sum_i \lambda_i \langle u_i, x \rangle = \sum_i \lambda_i \max_{s \in C} \langle u_i, s \rangle \geq \max_{s \in C} \sum_i \lambda_i \langle u_i, s \rangle = \max_{s \in C} \langle v, s \rangle,$$

so $x \in F_\perp(C, v)$. The vector v was arbitrary. So $x \in G$ and we have $F = G$ indeed.

We assume that $G \neq \emptyset$ and prove $G = F_\perp(C, v)$ for $v \in \text{ri}(\text{conv}(U)) \setminus \{0\}$. To prove the non-trivial inclusion $F_\perp(C, v) \subset G$ assume by contradiction that there is a point $y \in F_\perp(C, v) \setminus G$, i.e. there exists $u_0 \in \text{conv}(U) \setminus \{0\}$ such that

$$y \in F_\perp(C, v) \setminus F_\perp(C, u_0).$$

Since v lies in the relative interior of $\text{conv}(U)$ and u_0 lies in $\text{conv}(U)$ there exists $\lambda \in (0, 1)$ and $u_1 \in \text{conv}(U)$ such that $v = \lambda u_0 + (1 - \lambda)u_1$ (see Theorem 6.4 in [14]). We assume that $u_1 \neq 0$ by performing a small perturbation of this point along the direction $v - u_0$ if necessary. Now let $x \in G$. Then we have $x \in F_\perp(C, u_0) \cap F_\perp(C, u_1)$ so the estimation

$$\begin{aligned} \langle v, y \rangle &= \lambda \langle u_0, y \rangle + (1 - \lambda) \langle u_1, y \rangle < \lambda \max_{z \in C} \langle u_0, z \rangle + (1 - \lambda) \langle u_1, y \rangle \\ &\leq \lambda \langle u_0, x \rangle + (1 - \lambda) \langle u_1, x \rangle = \langle v, x \rangle \end{aligned}$$

gives the contradiction $y \notin F_\perp(C, v)$. □

Given a convex subset $C \subset \mathbb{E}$ the property of a subset of C to be an exposed face of C is a closure property by Prop. 3.8. Thus, by Remark 2.4 the exposed face lattice

$$(\mathcal{F}_\perp(C), \subset, \cap, \vee, \emptyset, C) \tag{11}$$

is a complete lattice ordered by inclusion and the infimum is the intersection. Although we have the inclusion of $\mathcal{F}_\perp(C) \subset \mathcal{F}(C)$ into the face lattice (4), $\mathcal{F}_\perp(C)$ is not in general a sublattice of $\mathcal{F}(C)$. Both lattices have the intersection as infimum but their suprema may be different. An example is drawn in Figure 3.1.

We prove a technical detail for the next assertion. If C is convex subset of \mathbb{E} , $x \in \mathbb{E}$ and $\{x\} \subsetneq C$ then the equality

$$\text{ri}(\text{conv}(C \setminus \{x\})) = \text{ri}(C) \tag{12}$$

holds. If $C \setminus \{x\}$ is not convex then $\text{conv}(C \setminus \{x\}) = C$ and the equality follows. If $C \setminus \{x\}$ is convex then x is an extreme point of C . Hence, unless $C = \{x\}$, we have $\text{ri}(C) \subset C \setminus \{x\} \subset C$. Therefore $C \setminus \{x\}$ lies between $\text{ri}(C)$ and the closure \overline{C} of C . Thus the relative interiors of $C \setminus \{x\}$ and C are equal by Cor. 6.3.1 in [14].

Corollary 3.9. *Let $C, D \subset \mathbb{E}$ be convex subsets. If D contains a non-zero vector, then $\text{ri}(D)$ contains a non-zero vector. If $\bigcap_{u \in D \setminus \{0\}} F_{\perp}(C, u) \neq \emptyset$ then this intersection is the exposed face $F_{\perp}(C, v)$ for any non-zero $v \in \text{ri}(D)$.*

Proof. By Prop. 3.8 we have for any vector $v \in \text{ri}(\text{conv}(D \setminus \{0\})) \setminus \{0\}$ the equality of the intersection with the face $F_{\perp}(C, v)$. With (12) applied to $x := 0$ and $C := D$ we get $\text{ri}(\text{conv}(D \setminus \{0\})) \setminus \{0\} = \text{ri}(D) \setminus \{0\}$. □

4. Normal cones

We study normal cones of a convex subset $C \subset \mathbb{E}$ of the finite-dimensional real Euclidean vector space $(\mathbb{E}, \langle \cdot, \cdot \rangle)$. There is an antitone lattice isomorphism between exposed faces and normal cones.

Definition 4.1. The *normal cone* of C at $x \in C$ is

$$N(C, x) := \{u \in \mathbb{E} : \langle u, y - x \rangle \leq 0 \text{ for all } y \in C\} \tag{13}$$

and vectors in $N(C, x)$ are called *normal vectors* of C at x .

There is a pointwise relation between exposed faces and normal cones. If $C \subset \mathbb{E}$ is a convex subset, then for arbitrary $x \in C$ and non-zero $u \in \mathbb{E}$ the equivalence of the following statements is easy to prove.

- $\langle u, x \rangle = h(C, u)$,
 - $x \in F_{\perp}(C, u)$,
 - $u \in N(C, x)$.
- (14)

The following relations are easy to prove by elementary means. If $F \subset C$ is a convex subset, $x \in \text{ri}(F)$ and $y \in C$, then we have

- (i) $N(C, x) \perp \text{lin}(F)$,
 - (ii) if $y \in F$ then $N(C, y) \supset N(C, x)$,
 - (iii) if $y \in \text{ri}(F)$ then $N(C, y) = N(C, x)$,
 - (iv) if $u, -u \in N(C, y)$ then $u \in \text{lin}(C)^{\perp}$.
- (15)

The orthogonal complement with respect to the Euclidean inner product is denoted by $^{\perp}$.

Lemma 4.2. *Let $x \in C$. Then $N(C, x) = (N(C, x) \cap \text{lin}(C)) + \text{lin}(C)^{\perp}$ holds and the following statements are equivalent.*

- the normal cone $N(C, x)$ is a vector space,
- $x \in \text{ri}(C)$,
- $N(C, x) = \text{lin}(C)^{\perp}$.

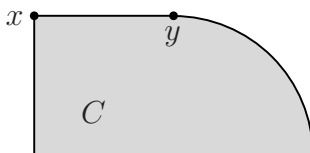


Figure 4.1: The union C of a square and a quarter disk with extreme points x, y : $\{x\}$ is an exposed face while $\{y\}$ is a non-exposed face. The face $\{y\}$ has the same normal cone as the face $[x, y]$. The normal cone of $\{y\}$ is included in the normal cone of $\{x\}$, even though $\{x\}$ and $\{y\}$ are unrelated in the partial ordering of inclusion.

Proof. Let $x \in C$. The direct sum decomposition of $N(C, x)$ follows from $N(C, x) + \text{lin}(C)^\perp \subset N(C, x)$. Since $N(C, x)$ is a convex cone, it is sufficient to prove the inclusion $\text{lin}(C)^\perp \subset N(C, x)$: if $u \in \text{lin}(C)^\perp$ then $\langle u, y - x \rangle = 0$ for all $y \in C$ so $u \in N(C, x)$.

Now let us assume that $N(C, x)$ is a vector space. Then for $u \in N(C, x)$ we have $\pm u \in N(C, x)$ and by (14) we get

$$h(C, u) = \langle u, x \rangle = -\langle -u, x \rangle = -h(C, -u).$$

Thus, for the vectors $u \in \mathbb{E}$ with $h(C, u) \neq -h(C, -u)$ follows $u \notin N(C, x)$, which means $\langle u, x \rangle < h(C, u)$ by (14). These are exactly the assumption of Theorem 13.1 in [14] to prove that $x \in \text{ri}(C)$. Clearly, if $x \in \text{ri}(C)$ then $N(C, x) = \text{lin}(C)^\perp$. \square

Definition 4.3. The *normal cone* of a non-empty convex subset F of C is defined as

$$N(C, F) := N(C, x) \tag{16}$$

for any $x \in \text{ri}(F)$. This definition is consistent by (iii) in (15). The *normal cone* of the empty set is defined as the ambient space $N(C, \emptyset) := \mathbb{E}$. The *normal cone lattice* of C is the set of normal cones of all faces $\mathcal{N}(C) := \{N(C, F) \mid F \in \mathcal{F}(C)\}$. We consider the normal cone lattice as a poset ordered by set inclusion. The cones $\text{lin}(C)^\perp$ and \mathbb{E} are the *improper* normal cones, all other normal cones are *proper*.

The assignment of normal cones to faces $\mathcal{F}(C) \rightarrow \mathcal{N}(C)$, $F \mapsto N(C, F)$ is an antitone mapping between posets. This follows from (ii) in (15). But the faces of two included normal cones may be unrelated, see Figure 4.1. We work towards the antitone lattice isomorphism $\mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C)$.

Lemma 4.4. *If $F \in \mathcal{F}(C)$ is a face and $u \in \mathbb{E} \setminus \{0\}$ then $F \subset F_\perp(C, u)$ if and only if $u \in N(C, F)$. For all $u \in \mathbb{E} \setminus \{0\}$ we have $u \in N(C, F_\perp(C, u))$.*

Proof. The assertion is trivial for $F = \emptyset$. Otherwise let us assume that the inclusion $F \subset F_\perp(C, u)$ holds and consider a point $x \in \text{ri}(F_\perp(C, u))$. We have $u \in N(C, x) = N(F_\perp(C, u))$ by the relation (14) and by definition (16) of a normal cone. Since $F \subset F_\perp(C, u)$ we have $N(C, F_\perp(C, u)) \subset N(C, F)$ by the antitone normal cone assignment. Conversely, if $u \in N(C, F)$ then for $x \in \text{ri}(F)$ we have $u \in N(C, x)$. Thus $x \in F_\perp(C, u)$ by the relation (14) and (9) gives $F \subset F_\perp(C, u)$. The second assertion is the special case of $F = F_\perp(C, u)$. \square

We consider smallest upper bounds of exposed faces for arbitrary subsets of C . This is consistent by completeness (11) of the exposed face lattice $\mathcal{F}_\perp(C)$:

Definition 4.5. The *smallest exposed face* of C that contains a subset $F \subset C$ is

$$\text{sup}_\perp(F) := \bigcap \{G \in \mathcal{F}_\perp(C) \mid F \subset G\}. \tag{17}$$

Properties of the smallest exposed face are:

Lemma 4.6. *If $F \in \mathcal{F}(C)$ is a proper face, then $\text{sup}_\perp(F) = \bigcap_{u \in \text{N}(C,F) \setminus \{0\}} F_\perp(C, u)$ is a proper exposed face. We have $\text{ri}(\text{N}(C, F)) \neq \{0\}$ and for each non-zero $v \in \text{ri}(\text{N}(C, F))$ we have $\text{sup}_\perp(F) = F_\perp(C, v)$. If $F \in \mathcal{F}(C)$ is a face then $\text{N}(C, \text{sup}_\perp(F)) = \text{N}(C, F)$.*

Proof. By Lemma 4.4 if $u \in \mathbb{E}$ is non-zero, then the face F is included in $F(C, u)$ if and only if $u \in \text{N}(C, F)$.

Relative interior points of the proper face F do not belong to $\text{ri}(C)$, so by Lemma 4.2 the normal cone of F is strictly larger than $\text{lin}(C)^\perp = \text{N}(C, C)$. Choosing any $u \in \text{N}(C, F) \setminus \text{lin}(C)^\perp$ we get that F but not C is included in $F_\perp(C, u)$. So $\text{sup}_\perp(F)$ is a proper exposed face of C and the intersection expression for $\text{sup}_\perp(F)$ follows. As $F \neq \emptyset$, any non-zero vector $v \in \text{ri}(\text{N}(C, F))$ gives $\text{sup}_\perp(F) = F_\perp(C, v)$ by Cor. 3.9.

Since $F \subset \text{sup}_\perp(F)$, the inclusion $\text{N}(C, \text{sup}_\perp(F)) \subset \text{N}(C, F)$ follows from antitone assignment of normal cones. For every non-zero vector $u \in \text{N}(C, F)$ we have $F \subset F_\perp(C, u)$. Hence $\text{sup}_\perp(F) \subset F_\perp(C, u)$ and so $u \in \text{N}(C, \text{sup}_\perp(F))$. \square

We arrive at the main results of this section.

Proposition 4.7. *Assume that C has not exactly one point. Then the assignment of normal cones to exposed faces $\text{N}(C) : \mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C)$, $F \mapsto \text{N}(C, F)$ is an antitone lattice isomorphism.*

Proof. The two lattices $\mathcal{F}_\perp(C)$ and $\mathcal{N}(C)$ are partially ordered by set inclusion. They are linked by the antitone mapping of posets

$$\text{N}(C)|_{\mathcal{F}_\perp(C)} : \mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C), \quad F \mapsto \text{N}(C, F).$$

This mapping is surjective because a face F of C has the same normal cone as the smallest exposed face that contains F , see Lemma 4.6.

We can show that $\text{N}(C)|_{\mathcal{F}_\perp(C)}$ has an antitone inverse. Then Remark 2.2 implies that $\text{N}(C)|_{\mathcal{F}_\perp(C)}$ is an (antitone) lattice isomorphism. Let us prove that this map is injective and consider two proper exposed faces F, G of C with the same normal cone N . Then there exists by Lemma 4.2 a non-zero vector $u \in N$, so there is a non-zero $v \in \text{ri}(N)$. As $F, G \neq \emptyset$, Lemma 4.6 proves that $F = F_\perp(C, v) = G$. By Lemma 4.2 only the improper face C has the smallest possible normal cone $\text{lin}(C)^\perp$. It remains to show that $\text{N}(C, F) = \mathbb{E}$ implies $F = \emptyset$ for an exposed face F of C . If $\text{N}(C, F) = \mathbb{E}$ holds for a non-empty face F then Lemma 4.2 shows that $F = C$ and $\text{lin}(C) = \mathbb{E}^\perp = \{0\}$. Thus, C has exactly one point but this case was excluded in the assumptions.

We show that the inverse $\mathcal{N}(C) \rightarrow \mathcal{F}_\perp(C)$ is antitone. For proper exposed faces F, G of C the inclusion $N(G) \subset N(F)$ implies $\text{sup}_\perp(F) \subset \text{sup}_\perp(G)$ by Lemma 4.6. As F, G are exposed we have $F = \text{sup}_\perp(F)$ and $G = \text{sup}_\perp(G)$, hence $F \subset G$. The greatest element \mathbb{E} of $\mathcal{N}(C)$ maps to the smallest element \emptyset of $\mathcal{F}_\perp(C)$ and the smallest element $\text{lin}(C)^\perp$ of $\mathcal{N}(C)$ maps to the greatest element C of $\mathcal{F}_\perp(C)$. \square

By definition of the normal cone of a face and by antitone assignment of normal cones the isomorphism $\mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C)$ in Prop. 4.7 is for proper exposed faces $F \in \mathcal{F}_\perp(C)$

$$\begin{aligned} F &\mapsto \bigcap_{x \in F} N(C, x) = N(C, y) \quad \text{for any } y \in \text{ri}(F), \\ N &\mapsto \bigcap_{u \in N \setminus \{0\}} F_\perp(C, u) = F_\perp(C, v) \quad \text{for any } v \in \text{ri}(N) \setminus \{0\}. \end{aligned} \tag{18}$$

The second mapping defined for proper normal cones $N \in \mathcal{N}(C)$ describes the inverse $\mathcal{N}(C) \rightarrow \mathcal{F}_\perp(C)$ by Lemma 4.6. Now we shows that intersections of normal cones are normal cones, so by Remark 2.4 the normal cone lattice is a complete lattice with intersection as the infimum

$$(\mathcal{N}(C), \subset, \cap, \vee, \text{lin}(C)^\perp, \mathbb{E}). \tag{19}$$

Proposition 4.8. *If $\{N_\alpha\}_{\alpha \in I} \subset \mathcal{N}(C)$ is a non-empty family of normal cones, then $\bigwedge_{\alpha \in I} N_\alpha = \bigcap_{\alpha \in I} N_\alpha$ and this intersection is a face of $N_{\tilde{\alpha}}$ for every $\tilde{\alpha} \in I$ with $N_{\tilde{\alpha}} \neq \mathbb{E}$.*

Proof. As \mathbb{E} is the greatest element of $\mathcal{N}(C)$ we assume $N_\alpha \neq \mathbb{E}$ for all $\alpha \in I$ and we assume that C has not exactly one point, without restricting generality. As $N(C, \emptyset) = \mathbb{E}$ we choose throughout for $\alpha \in I$ a family of (non-empty) faces F_α with $N(C, F_\alpha) = N_\alpha$. Let $x_\alpha \in \text{ri}(F_\alpha)$ for $\alpha \in I$ and let $z \in \text{ri}(\text{conv}\{x_\alpha \mid \alpha \in I\})$. So Lemma 3.6 shows $F(C, z) = \bigvee_{\alpha \in I} F_\alpha$. By Prop. 4.7 we have $K := \bigwedge_{\alpha \in I} N(C, F_\alpha) = N(C, \bigvee_{\alpha \in I} F_\alpha) = N(C, z)$.

The assignment of a normal cone is antitone, so for all $\tilde{\alpha} \in I$ we have $K \subset N(C, F_{\tilde{\alpha}})$. This proves one inclusion, it remains to show $\bigcap_{\alpha \in I} N(C, F_\alpha) \subset K$. We write z as a convex combination for $n \in \mathbb{N}$, $\lambda_i > 0$ and $\alpha(i) \in I$ for $i = 1, \dots, n$ in the form $z = \sum_{i=1}^n \lambda_i x_{\alpha(i)}$. Hence, if $u \in \bigcap_{\alpha \in I} N(C, F_\alpha)$, then for all $x \in C$ we have the inequality $\langle u, x - z \rangle = \sum_{i=1}^n \lambda_i \langle u, x - x_{\alpha(i)} \rangle \leq 0$. This proves $u \in N(C, z)$.

For $\tilde{\alpha} \in I$ let us prove that K is a face of $N_{\tilde{\alpha}}$. Let $u, v, w \in N_{\tilde{\alpha}}$, $v \in K$ and $v \in]u, w[$. If $u = 0$ then $w = \lambda v$ for some $\lambda > 0$, then $u, w \in K$ because K is a convex cone including v . If $u, w \neq 0$ and $v = 0$ then $u, w \in \text{lin}(C)^\perp$. By Lemma 4.2 the vector space $\text{lin}(C)^\perp$ belongs to every normal cone of C , so $u, w \in K$. Let us assume $u, v, w \neq 0$. For every $\alpha \in I$ holds $v \in N_\alpha = N(C, F_\alpha)$ so $F_\alpha \subset F_\perp(C, v)$ by Lemma 4.4. Now Prop. 3.8 shows $F_\perp(C, v) = F_\perp(C, u) \cap F_\perp(C, w)$, so we have

$$F_\alpha \subset F_\perp(C, v) = F_\perp(C, u) \cap F_\perp(C, w) \subset F_\perp(C, u).$$

This gives $N(C, F_\perp(C, u)) \subset N(C, F_\alpha)$ and Lemma 4.4 completes the proof with $u \in N(C, F_\perp(C, u))$. The proof of $w \in N(C, F_\alpha)$ is a complete analogue. \square

5. Cylinders

This section explains a lifting construction for projections of convex sets. Lifting is an isomorphism for face lattices, we characterize lifted faces. As an application, the projections of state spaces introduced in Section 1.3 are decomposed by Weis [16] using a lifting. Throughout this section let C be a convex subset of a finite-dimensional real Euclidean vector space $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ and let V be a linear subspace of \mathbb{E} .

If $\emptyset \neq \mathbb{A} \subset \mathbb{E}$ is an affine subspace, then there is a unique affine mapping $\pi_{\mathbb{A}} : \mathbb{E} \rightarrow \mathbb{A}$, called the *orthogonal projection* to \mathbb{A} , such that for all $x \in \mathbb{E}$ we have

$$(x - \pi_{\mathbb{A}}(x)) \perp \text{lin}(\mathbb{A}). \tag{20}$$

We study the orthogonal projection $\pi_V : \mathbb{E} \rightarrow V$ to V . This, thought of as acting on sets, may be written for $M \subset \mathbb{E}$ in the form

$$\pi_V(M) = (M + V^\perp) \cap V. \tag{21}$$

In addition to the projection $\pi_V(C)$ we will study the cylinder $C + V^\perp$, which connects the projection $\pi_V(C)$ to C .

There is a basic tool for the study of cylinders, which is reminiscent of the modular law for lattices (3).

Lemma 5.1. *Let $X, Y, Z \subset \mathbb{E}$ such $Z \pm X \subset Z$. Then $X + (Y \cap Z) = (X + Y) \cap Z$.*

Proof. The inclusion $(X + Y) \cap Z \subset X + (Y \cap Z)$ is proved by taking vectors $x \in X$ and $y \in Y$ such that $x + y \in Z$. Then $y = (x + y) - x \in Z$. For the converse $X + (Y \cap Z) \subset (X + Y) \cap Z$ we choose vectors $x \in X$ and $t \in Y \cap Z$. Then $t + x \in Z$. □

A special case of Lemma 5.1 is the *modular law for affine spaces*. Let $\mathbb{A} \subset \mathbb{E}$ be an affine subspace with translation vector space $\text{lin}(\mathbb{A})$. If $X \subset \text{lin}(\mathbb{A})$ then for arbitrary $Y \subset \mathbb{E}$ we have

$$X + (Y \cap \mathbb{A}) = (X + Y) \cap \mathbb{A}. \tag{22}$$

We will use this modular law as indicated in Figure 5.1.

Definition 5.2. We define the *lift* from V to C (or along V^\perp to C) as the mapping $L_V^C : 2^{\mathbb{E}} \rightarrow 2^C$, $M \mapsto (M + V^\perp) \cap C$. Here $2^{\mathbb{E}}$ denotes the power set of \mathbb{E} and 2^C the power set of C .

Lemma 5.3. *The projection $\pi_V : 2^{\mathbb{E}} \rightarrow 2^V$ is isotone with respect to set inclusion and we have*

$$L_V^C = L_V^C \circ L_V^C = L_V^C \circ \pi_V.$$

If \mathcal{M} is a family of subsets of $\pi_V(C)$, then π_V is left inverse to $L_V^C|_{\mathcal{M}}$. In particular

$$L_V^C|_{\mathcal{M}} : \mathcal{M} \rightarrow \{L_V^C(M) : M \in \mathcal{M}\}$$

is a bijection. The mapping $L_V^C|_{\mathcal{M}}$ is an isotone isomorphism of posets (partially ordered by set inclusion).

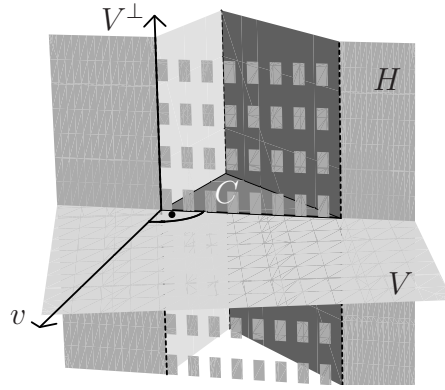


Figure 5.1: We start with a plane V and an arbitrary subset C in \mathbb{R}^3 . For simplicity in the drawing we choose C a triangle in V . A non-zero vector $v \in V$ defines the supporting hyperplane $H = H(C, v)$ with $v \perp H$. We have $V^\perp \subset \{v\}^\perp = \text{lin}(H)$. So by the modular law for affine spaces $V^\perp + (C \cap H) = (V^\perp + C) \cap H$ holds. This set is drawn tiled.

Proof. Trivial. □

Lemma 5.4 (Lifted faces). *If F is a face of $\pi_V(C)$ then the lift $L_V^C(F)$ is a face of C . The exposed face for non-zero $v \in V$ transforms according to $L_V^C(F_\perp(\pi_V(C), v)) = F_\perp(C, v)$.*

Proof. For a face F of $\pi_V(C)$ we show that $L_V^C(F)$ is a face of C . To this aim we choose $x, y, z \in C$ such that $y \in]x, z[$ and $y \in L_V^C(F)$. We have to prove $x, z \in L_V^C(F)$. By (8) the projection π_V commutes with reduction to the relative interior of a convex set, so we have $\pi_V(y) \in]\pi_V(x), \pi_V(z)[$. Since $y \in L_V^C(F)$ we have $\pi_V(y) \in F$. Since F is a face we obtain $\pi_V(x), \pi_V(z) \in F$. Then

$$x \in L_V^C \circ \pi_V(x) = (\pi_V(x) + V^\perp) \cap C \subset (F + V^\perp) \cap C = L_V^C(F).$$

Analogously we have $z \in L_V^C(F)$, so $L_V^C(F)$ is a face of C .

The support functions of C and $\pi_V(C)$ are equal on V because for all $x \in \mathbb{E}$ and $v \in V$ we have $\langle v, x \rangle = \langle v, \pi_V(x) \rangle$. If $v \in V$ is a non-zero vector then the hyperplanes $H(C, v)$ and $H(\pi_V(C), v)$ are equal. Since $v \in V$ we have $V^\perp \subset \{v\}^\perp = \text{lin}(H(\pi_V(C), v))$ and we can apply the modular law for affine spaces (22) as follows

$$\begin{aligned} V^\perp + F_\perp(\pi_V(C), v) &= V^\perp + [\pi_V(C) \cap H(\pi_V(C), v)] \\ &= [V^\perp + \pi_V(C)] \cap H(\pi_V(C), v) = (V^\perp + C) \cap H(C, v). \end{aligned}$$

This gives

$$\begin{aligned} L_V^C(F_\perp(\pi_V(C), v)) &= (F_\perp(\pi_V(C), v) + V^\perp) \cap C = (V^\perp + C) \cap H(C, v) \cap C \\ &= H(C, v) \cap C = F_\perp(C, v) \end{aligned}$$

finally. □

Definition 5.5. With respect to C and V , the face $L_V^C(F) \in \mathcal{F}(C)$ is called the *lifted face* of $F \in \mathcal{F}(\pi_V(C))$. The *lifted face lattice* is

$$\mathcal{F}_V^C := \{L_V^C(F) : F \in \mathcal{F}(\pi_V(C))\}.$$

The *lifted exposed face lattice* is

$$\mathcal{F}_{V,\perp}^C := \{L_V^C(F) : F \in \mathcal{F}_\perp(\pi_V(C))\} \tag{23}$$

where $\mathcal{F}(\pi_V(C))$ is the face lattice of $\pi_V(C)$ and $\mathcal{F}_\perp(\pi_V(C))$ is the exposed face lattice of $\pi_V(C)$. We consider \mathcal{F}_V^C and $\mathcal{F}_{V,\perp}^C$ partially ordered by set inclusion.

Proposition 5.6 (Lifted face lattices). *The lifts from V to C restricted to the face lattices of $\pi_V(C)$ are*

$$\begin{aligned} L_V^C|_{\mathcal{F}(\pi_V(C))} : \mathcal{F}(\pi_V(C)) &\rightarrow \mathcal{F}_V^C \subset \mathcal{F}(C), \\ L_V^C|_{\mathcal{F}_\perp(\pi_V(C))} : \mathcal{F}_\perp(\pi_V(C)) &\rightarrow \mathcal{F}_{V,\perp}^C \subset \mathcal{F}_\perp(C). \end{aligned}$$

These mappings are isotone lattice isomorphisms. The infimum in the lifted face lattices is the intersection.

Proof. The mapping L_V^C restricted to $\mathcal{F}(\pi_V(C))$ resp. to $\mathcal{F}_\perp(\pi_V(C))$ is a bijection to \mathcal{F}_V^C resp. to $\mathcal{F}_{V,\perp}^C$ by Lemma 5.3. The ranges are included in the face lattice of C resp. in the exposed face lattice of C by Lemma 5.4.

The mappings L_V^C and π_V (on the considered domains) are inverse to each other and they are isotone with respect to set inclusion by Lemma 5.3. Hence the lift is a lattice isomorphism in each case by Remark 2.2.

Finally, by direct sum decomposition of $\mathbb{E} = V + V^\perp$ we have for a non-empty family $\{F_\alpha\}_{\alpha \in I}$ of faces of $\pi_V(C)$

$$L_V^C(\bigcap_{\alpha \in I} F_\alpha) = (\bigcap_{\alpha \in I} F_\alpha + V^\perp) \cap C = \bigcap_{\alpha \in I} (F_\alpha + V^\perp) \cap C = \bigcap_{\alpha \in I} L_V^C(F_\alpha),$$

the infimum in the lifted face lattices is the intersection. □

We notice that the lifted exposed face lattice $\mathcal{F}_{V,\perp}^C$ is not a sublattice of the face lattice $\mathcal{F}(C)$ because the supremum of lifted faces in $\mathcal{F}(C)$ is not necessarily a lifted face. An example is a triangle projected to the linear span of one of its sides, say c . Then the corners A and B of c belong to $\mathcal{F}_{V,\perp}^C$, but c does not. We characterize the lifted face lattice:

Proposition 5.7 (Lift invariant faces). *A face $F \in \mathcal{F}(C)$ belongs to the lifted face lattice \mathcal{F}_V^C if and only if $L_V^C(F) = F$.*

Proof. Let us choose a face $F \in \mathcal{F}(C)$. If F belongs to \mathcal{F}_V^C then there is a face $G \in \mathcal{F}(\pi_V(C))$ such that $F = L_V^C(G)$. With Lemma 5.3 we obtain

$$L_V^C(F) = L_V^C \circ L_V^C(G) = L_V^C(G) = F.$$

For the converse we assume that $F = L_V^C(F)$. If $\pi_V(F)$ is a face of $\pi_V(C)$ then we have $F = L_V^C \circ \pi(F)$ and so F is a lifted face. It remains to prove $\pi_V(F) \in \mathcal{F}(\pi_V(C))$.

To this end let $x, y, z \in \pi_V(C)$ such that $y \in]x, z[$ and $y \in \pi_V(F)$. We must show $x, z \in \pi_V(F)$. We choose $\tilde{x} \in L_V^C(x)$ and $\tilde{z} \in L_V^C(z)$. Then $[\tilde{x}, \tilde{z}] \xrightarrow{\pi_V} [x, z]$ is a bijection so there exists $\tilde{y} \in]\tilde{x}, \tilde{z}[\cap L_V^C(y)$. Since $y \in \pi_V(F)$ we have $\tilde{y} \in L_V^C \circ \pi_V(F) = L_V^C(F) = F$ and this proves $\tilde{x}, \tilde{z} \in F$ because F is a face of C . Then $x = \pi_V(\tilde{x})$ and $z = \pi_V(\tilde{z})$ belong to $\pi_V(F)$ and we have proved that $\pi_V(F)$ is a face of $\pi_V(C)$. \square

There is a canonical space to project onto.

Corollary 5.8. *Let U be the orthogonal projection of V onto the vector space of C , i.e. $U := \pi_{\text{lin}(C)}(V)$. Then for all $F \subset C$ we have $L_V^C(F) = L_U^C(F)$. In particular $\mathcal{F}_V^C = \mathcal{F}_U^C$ holds.*

Proof. We put $W := \text{lin}(C)$. By straight forward calculation we have for any $F \subset C$

$$L_U^C(F) = ((V^\perp \cap W) + (F + W^\perp)) \cap \text{aff}(C) \cap C.$$

By the modular law (22) applied to the first two intersection sets this simplifies to $L_V^C(F)$. The second statement follows now from Prop. 5.7. \square

Finally we write down the normal cones.

Lemma 5.9 (Normal cones). *Let $a \in C + V^\perp$. Then $N(\pi_V(C), \pi_V(a)) = N(C + V^\perp, a) + V^\perp$. If a belongs to C then $N(C + V^\perp, a) = N(C, a) \cap V$.*

Proof. Let $a \in C + V^\perp$. We use the relation (14) to prove the first identity. We decompose a vector $u \in \mathbb{E}$ in the form $u = v + w \in \mathbb{E}$ for $v \in V$ and $w \in V^\perp$. If $u \in N(\pi_V(C), \pi_V(a))$ then

$$h(C + V^\perp, v) = h(\pi_V(C), v) = h(\pi_V(C), u) = \langle u, \pi_V(a) \rangle = \langle v, \pi_V(a) \rangle = \langle v, a \rangle,$$

so $v \in N(C + V^\perp, a)$ and $u \in N(C + V^\perp, a) + V^\perp$. Conversely, if $v \in N(C + V^\perp, a)$ then

$$\langle u, \pi_V(a) \rangle = \langle v, \pi_V(a) \rangle = \langle v, a \rangle = h(C + V^\perp, v) = h(\pi_V(C), v) = h(\pi_V(C), u),$$

so $u \in N(\pi_V(C), \pi_V(a))$.

The second equation is as follows. If $u \in N(C + V^\perp, a)$, then $u \in N(C, a)$ because there are less conditions on normal cones for the smaller set C . For all $w \in V^\perp$ we have $\langle u, \pm w \rangle \leq 0$ so $u \in V$. Conversely, if $u \in N(C, a) \cap V$, then for all $x \in C$ and for all $w \in V^\perp$ we have $\langle u, x + w - a \rangle = \langle u, x - a \rangle \leq 0$ and this proves $u \in N(C + V^\perp, a)$. \square

6. Sharp relations

Let $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real Euclidean vector space and $C \subset \mathbb{E}$ a convex subset. There is a relation (14) between exposed faces and normal cones, this is for $x \in C$ and $u \in \mathbb{E} \setminus \{0\}$

$$x \in F_\perp(C, u) \iff u \in N(C, x).$$

We define two alterations:

Definition 6.1. A vector $u \in \mathbb{E} \setminus \{0\}$ is *sharp normal* for C if

$$x \in \text{ri}(F_{\perp}(C, u)) \implies u \in \text{ri}(N(C, x)). \tag{24}$$

A point $x \in C$ is *sharp exposed* in C if

$$u \in \text{ri}(N(C, x)) \setminus \{0\} \implies x \in \text{ri}(F_{\perp}(C, u)). \tag{25}$$

A connection of sharp normal vectors to normal cones will be shown in the following section. In this section we show that the above definitions do not depend on the ambient space (through the normal cones), the argument for sharp exposed points connects these to exposed faces. We show that sharp normal vectors are preserved under orthogonal projection of a convex set and sharp exposed points are preserved under intersection. An example where both (24) and (25) hold is a state space:

Example 6.2. For $n \in \mathbb{N}$ let $\text{Mat}(\mathbb{C}, n)$ be the set of complex $n \times n$ matrices acting as linear operators on the complex Hilbert space \mathbb{C}^n with the standard inner product, 0_n resp. $\mathbb{1}_n$ denoting the zero resp. the multiplicative identity. We consider the Euclidean space of self-adjoint matrices endowed with the Hilbert-Schmidt inner product $(a, b) \mapsto \text{tr}(ab)$ for $a, b \in \text{Mat}(\mathbb{C}, n)$ self-adjoint. Here tr denotes the standard trace. By $a \geq 0$ we mean that $a \in \text{Mat}(\mathbb{C}, n)$ is positive semidefinite, i.e. self-adjoint and having non-negative eigenvalues. The *state space* of $\text{Mat}(\mathbb{C}, n)$ is the convex body

$$\mathbb{S}(n) := \{\rho \in \text{Mat}(\mathbb{C}, n) \mid \rho \geq 0 \text{ and } \text{tr}(\rho) = 1\}. \tag{26}$$

The Pauli σ -matrices $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ together with $\mathbb{1}_2$ are an orthogonal basis for the self-adjoint part of $\text{Mat}(\mathbb{C}, 2)$. The *Bloch ball* is

$$\mathbb{S}(2) = \left\{ \frac{1}{2}(\mathbb{1}_2 + b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3) \mid (b_1, b_2, b_3) \in \mathbb{B}^3 \right\}.$$

For $m, n \in \mathbb{N}$ the state space of the direct sum algebra $\mathcal{A} := \text{Mat}(\mathbb{C}, m) \oplus \text{Mat}(\mathbb{C}, n)$ is the convex hull of the individual state spaces

$$\mathbb{S}(\mathcal{A}) := \mathbb{S}(m+n) \cap \mathcal{A} = \text{conv}(\mathbb{S}(m) \oplus 0_n, 0_m \oplus \mathbb{S}(n)).$$

With n direct summands we have e.g. the $n - 1$ dimensional simplex $\mathbb{S}(\mathbb{C}^n)$.

An element $p \in \mathcal{A}$ is an *orthogonal projection* if $p^2 = p = p^*$. The set of orthogonal projections of \mathcal{A} are partially ordered by: $p \leq q$ if and only if $pq = p$ for p, q orthogonal projections. The *support projection* $s(\rho)$ of $\rho \in \mathbb{S}(\mathcal{A})$ is the sum of the spectral projections of ρ belonging to non-zero eigenvalues. The *maximal projection* $p_+(u)$ of a vector u in the space \mathcal{A}_{sa} of self-adjoint matrices is the spectral projection of u for the largest eigenvalue of u . For non-zero $u \in \mathcal{A}_{\text{sa}}$ we have the exposed faces (see Weis [16], Section 2.3)

$$\begin{aligned} F_{\perp}(\mathbb{S}(\mathcal{A}), u) &= \{\rho \in \mathbb{S}(\mathcal{A}) \mid s(\rho) \leq p_+(u)\}, \\ \text{ri}(F_{\perp}(\mathbb{S}(\mathcal{A}), u)) &= \{\rho \in \mathbb{S}(\mathcal{A}) \mid s(\rho) = p_+(u)\} \end{aligned} \tag{27}$$

and for $\rho \in \mathbb{S}(\mathcal{A})$ we have the normal cones

$$\begin{aligned} \mathbb{N}(\mathbb{S}(\mathcal{A}), \rho) &= \{u \in \mathcal{A}_{\text{sa}} \mid s(\rho) \leq p_+(u)\}, \\ \text{ri}(\mathbb{N}(\mathbb{S}(\mathcal{A}), \rho)) &= \{u \in \mathcal{A}_{\text{sa}} \mid s(\rho) = p_+(u)\}. \end{aligned} \tag{28}$$

Much more general the facial structure of the state space of C^* -algebra is treated by Alfsen and Schultz [1]. It is immediate from (27) and (28) that every non-zero vector $u \in \mathcal{A}_{\text{sa}}$ is sharp normal for $\mathbb{S}(\mathcal{A})$ and every element $\rho \in \mathbb{S}(\mathcal{A})$ is sharp exposed in $\mathbb{S}(\mathcal{A})$. We will extend this example in Example 7.8.

The definitions (24) and (25) depend *a priori* on the ambient space \mathbb{E} through the normal cone. For sharp normal vectors we show independence in the following lemma. To keep notation clear we use orthogonal projections π_V onto a vector space $V \subset \mathbb{E}$ and not onto an affine space.

Lemma 6.3. *Let $C \subset V$. Then every non-zero $v \in V^\perp$ is sharp normal for C in the ambient space \mathbb{E} . A vector $v \in \mathbb{E} \setminus V^\perp$ is sharp normal for C in the ambient space \mathbb{E} if and only if the vector $\pi_V(v)$ is sharp normal for C in the ambient space V .*

Proof. For $v \in V^\perp \subset \text{lin}(C)^\perp$ we have $F_\perp(C, v) = C$ (notice that $h(C, v) = 0$ unless $C = \emptyset$). Then for every $x \in \text{ri}(C)$ the normal cone $\mathbb{N}(C, x) = \text{lin}(C)^\perp$ is a vector space by Lemma 4.2, so $v \in \text{ri}(\mathbb{N}(C, x))$ and v is sharp normal for C .

If $v \in \mathbb{E} \setminus V^\perp$ then we have $F_\perp(C, v) = F_\perp(C, \pi_V(v))$. For a point $x \in \text{ri}(F_\perp(C, v))$ we distinguish between the normal cone $\mathbb{N}_\mathbb{E}(C, x)$ in the ambient space \mathbb{E} and the normal cone $\mathbb{N}_V(C, x) \subset V$ in the ambient space V . These satisfy $\mathbb{N}_\mathbb{E}(C, x) = \mathbb{N}_V(C, x) + V^\perp$. By the sum formula (6) for the relative interior we have

$$\text{ri}(\mathbb{N}_\mathbb{E}(C, x)) = \text{ri}(\mathbb{N}_V(C, x)) + V^\perp.$$

Then we get $v \in \text{ri}(\mathbb{N}_\mathbb{E}(C, x))$ if and only if $\pi_V(v) \in \text{ri}(\mathbb{N}_V(C, x))$, i.e. v is sharp normal for C in \mathbb{E} if and only if $\pi_V(v)$ is sharp normal for C in V . \square

Sharp normal vectors are preserved under projection.

Proposition 6.4. *If a non-zero vector $v \in V$ is sharp normal for C , then v is sharp normal for $\pi_V(C)$.*

Proof. We choose $x \in \text{ri}(F_\perp(\pi_V(C), v))$ and we have to show that $v \in \text{ri}(\mathbb{N}(\pi_V(C), x))$. By Lemma 5.4 we have

$$F_\perp(\pi_V(C), v) = \pi_V(F_\perp(C, v))$$

so by (8) we can choose a point $a \in \text{ri}(F_\perp(C, v))$ such that $x = \pi_V(a)$. By assumption the vector v is sharp normal for C so $v \in \text{ri}(\mathbb{N}(C, a))$. By the formula for normal cones of a projected set in Lemma 5.9 we have

$$\mathbb{N}(\pi_V(C), x) = (\mathbb{N}(C, a) \cap V) + V^\perp.$$

Since $v \in \text{ri}(N(C, a))$ we have $v \in \text{ri}(N(C, a) \cap V)$ by the intersection formula (7) for relative interiors. The sum formula (6) for the relative interior shows $v \in \text{ri}(N(\pi_V(C), x))$, i.e. v is sharp normal for $\pi_V(C)$ in \mathbb{E} . \square

We shortly discuss sharp exposed points and connect these to exposed faces. The following lemma shows also that the definition (25) of sharp exposed is independent of the ambient space because exposed faces are independent of the ambient space.

Lemma 6.5. *A non-empty face F of C is exposed if and only if there is a point in $\text{ri}(F)$, which is sharp exposed in C . If there is a point in $\text{ri}(F)$, which is sharp exposed in C , then all points in $\text{ri}(F)$ are sharp exposed in C .*

Proof. Let F be a non-empty exposed face of C . If $x \in \text{ri}(F)$ then we have $N(C, F) = N(C, x)$ by definition of the normal cone of F . We want to show that x is sharp exposed in C . If $N(C, x) = \{0\}$ then there is nothing to prove. Otherwise by Lemma 4.6 for all non-zero $u \in \text{ri}(N(C, F))$ we have $F = F_\perp(C, u)$. In other words for each $u \in \text{ri}(N(C, x)) \setminus \{0\}$ we have $x \in \text{ri}(F_\perp(C, u))$, i.e. x is sharp exposed in C .

Conversely let $F \neq \emptyset$ be a face of C , not necessarily exposed. Since C is exposed we can assume $F \neq C$, so $N(C, F) \neq \{0\}$ by Lemma 4.2. Let us choose a point $x \in \text{ri}(F)$ and consider a non-zero vector $u \in \text{ri}(N(C, F)) = \text{ri}(N(C, x))$. If we assume that x is sharp exposed in C , then we have $x \in \text{ri}(F_\perp(C, u))$. Therefore $F = F_\perp(C, u)$ is an exposed face by the decomposition (10). \square

Exposed faces are preserved under intersection.

Lemma 6.6. *Let $\mathbb{A} \subset \mathbb{E}$ be an affine subspace and let $x \in C \cap \mathbb{A}$. If $F(C, x)$ is an exposed face of C , then $F(C \cap \mathbb{A}, x)$ is an exposed face of $C \cap \mathbb{A}$.*

Proof. If $x \in \text{ri}(C)$ then $x \in \text{ri}(C \cap \mathbb{A})$ by the intersection formula (7) for relative interiors. So $F(C \cap \mathbb{A}, x) = C \cap \mathbb{A}$ is exposed. Otherwise there is a non-zero $u \in \mathbb{E}$ such that $x \in \text{ri}(F_\perp(C, u))$. As $x \in \mathbb{A}$ we have $h(C, u) = \langle u, x \rangle = h(C \cap \mathbb{A}, u)$, so we obtain $F_\perp(C, u) \cap \mathbb{A} = F_\perp(C \cap \mathbb{A}, u)$. By the intersection formula (7) for relative interiors this gives $x \in \text{ri}(F_\perp(C \cap \mathbb{A}, u))$ and completes the proof. \square

7. Touching cones

Let C be a convex subset of a finite-dimensional real Euclidean vector space $(\mathbb{E}, \langle \cdot, \cdot \rangle)$. We connect sharp normal vectors for C to Schneider’s [15] concept of touching cone. Touching cones form a complete lattice with infimum the intersection. They include all normal cones, which are preserved under projection. Touching cones can detect the exposed faces which are intersections of coatoms.

Definition 7.1. If $v \in \mathbb{E}$ is a non-zero vector and if the exposed face $F_\perp(C, v)$ is non-empty, then the *touching cone* of C for u is defined by $T(C, u) := F(N(C, F_\perp(C, u)), u)$. This is the face of the normal cone $N(C, F_\perp(C, u))$, which has u in the relative interior. The normal cones $\text{lin}(C)^\perp$ and \mathbb{E} are touching cones by definition, called *improper*. All other touching cones are *proper*. The set of touching cones of C , called *touching cone lattice* is denoted by $\mathcal{T}(C)$.

Perhaps the analogy with the face-function (as studied by Klee and Martin [8] and others) should be pointed out here. The face-function associates with each $x \in C$ the smallest face $F(C, x)$ of C containing x . Analogously (or dually) Definition 7.1 associates with each vector $u \neq 0$ the smallest touching cone of C containing it.

Lemma 7.2. *If T is a touching cone of C , then $T = (T \cap \text{lin}(C)) + \text{lin}(C)^\perp$. Every normal cone of C is a touching cone of C . If T is a touching cone of C but $T \neq \mathbb{E}$, then*

- (a) *if $u \in \text{ri}(T) \setminus \{0\}$, then $F_\perp(C, u) = \bigcap_{v \in T \setminus \{0\}} F_\perp(C, v)$ is non-empty,*
- (b) *if $u \in \text{ri}(T) \setminus \{0\}$, then $T = T(C, u)$,*
- (c) *if $0 \in \text{ri}(T)$, then $T = \text{lin}(C)^\perp$.*

Proof. The first assertion is clear for $T = \text{lin}(C)^\perp$ or $T = \mathbb{E}$. The normal cone N of $x \in C$ is a direct sum of $N \cap \text{lin}(C)$ and of $\text{lin}(C)^\perp$ by Lemma 4.2, so this holds also for all its faces including T .

Let us prove that every proper normal cone N of C belongs to $\mathcal{T}(C)$. By the antitone lattice isomorphism $\mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C)$ in Prop. 4.7 there is a proper exposed face F , such that $N = N(C, F)$. By Lemma 4.6 there exists $u \in \text{ri}(N(C, F)) \setminus \{0\}$ such that $F = F_\perp(C, u)$. Now $u \in \text{ri}(N(C, F)) = \text{ri}(N(C, F_\perp(C, u)))$ gives $T(C, u) = N(C, F)$ by definition of a touching cone.

(a)–(c) are trivial if $T = \{0\}$. Otherwise the touching cone T arises from a non-zero vector $w \in \mathbb{E}$ as $T = T(C, w)$ such that $F_\perp(C, w) \neq \emptyset$ (also in the case $T = \text{lin}(C)^\perp \neq \{0\}$).

To show (a) we notice $T \subset N(C, F_\perp(C, w))$, so the intersection $\bigcap_{v \in T \setminus \{0\}} F_\perp(C, v)$ is non-empty by Lemma 4.6. For any $u \in \text{ri}(T) \setminus \{0\}$ this intersection equals $F_\perp(C, u)$ by Cor. 3.9.

To prove (b) we recall $w \in \text{ri}(T)$ by definition of a touching cone. If a non-zero $u \in \text{ri}(T)$ is chosen then by (a) we have $F_\perp(C, u) = F_\perp(C, w)$ and the two vectors u, w belong to the relative interior $\text{ri}(T)$ of the same face T of $N(C, F_\perp(C, u))$, so $T(C, u) = T(C, w) = T$ by the partition (10) of a convex set into relative interiors of faces.

For (c) we recall that a convex cone with zero in the relative interior is a linear space. Since $w \in \text{ri}(T)$ the opposite vector $-w$ belongs also to $\text{ri}(T)$ and from (a) follows $F_\perp(C, w) = F_\perp(C, -w)$ so $F_\perp(C, w) = C$. The normal cone of C is $N(C, C) = \text{lin}(C)^\perp$ by Lemma 4.2 hence $T = T(C, w) = \text{lin}(C)^\perp$. □

Remark 7.3. If K is a convex body and $u \in \mathbb{E}$ a non-zero vector, then $F_\perp(K, u)$ is a non-empty exposed face and the touching cone $T := T(C, u)$ with $u \in \text{ri}(T)$ is defined. So $\mathbb{E} \setminus \{0\}$ is covered by the relative interiors of touching cones $\neq \mathbb{E}$. Lemma 7.2(b) and (c) make sure that this cover is disjoint. We notice that this partition follows also from Thm. 8.3

Next we show beyond $\mathcal{T}(C) \supset \mathcal{N}(C)$ that the touching cone lattice consists of all non-empty faces of normal cones. The infimum in $\mathcal{T}(C)$ is the intersection and

$\mathcal{T}(C)$ is a complete lattice

$$(\mathcal{T}(C), \subset, \cap, \vee, \text{lin}(C)^\perp, \mathbb{E}). \tag{29}$$

Theorem 7.4. *The touching cones of C are exactly the non-empty faces of the normal cones of C , i.e. $\mathcal{T}(C) = \{T \mid T \neq \emptyset \text{ is a face of } N, N \in \mathcal{N}(C)\}$. The touching cone lattice is a complete lattice ordered by inclusion. If $\{T_\alpha\}_{\alpha \in I} \subset \mathcal{T}(C)$ is a non-empty family of touching cones, then $\bigwedge_{\alpha \in I} T_\alpha = \bigcap_{\alpha \in I} T_\alpha$ and this intersection is a face of $T_{\tilde{\alpha}}$ for every $\tilde{\alpha} \in I$ with $T_{\tilde{\alpha}} \neq \mathbb{E}$.*

Proof. By definition, every touching cone is a non-empty face of a normal cone. For the converse we need not treat the improper cones $\text{lin}(C)^\perp$ and \mathbb{E} , they have only one non-empty face, which is already included to the touching cones. Let N be a proper normal cone of C . By the partition (10) of N into relative interiors of its faces, it is sufficient to show for any non-zero vector $v \in N$ that $T(C, v) = F(N, v)$, i.e. the touching cone of v is the face of N with v in the relative interior.

There is a proper exposed face F with normal cone $N(C, F) = N$ by Prop. 4.7. Since $v \in N(C, F)$ we have $F \subset F_\perp(C, v)$ as proved in Lemma 4.4. By the antitone assignment of normal cones we get $N(C, F_\perp(C, v)) \subset N$ and this statement includes by Prop. 4.8 that $N(C, F_\perp(C, v))$ is a face of N . By definition of a touching cone, $T(C, v)$ is a face of the normal cone $N(C, F_\perp(C, v))$, so it is a face of N . As v belongs to the relative interior of $T(C, v)$, we conclude that $T(C, v) = F(N, v)$.

In order to prove that $\mathcal{T}(C)$ is a complete lattice with intersection as infimum, we can show by Remark 2.4 for a non-empty family $\{T_\alpha\}_{\alpha \in I}$ that the intersection $\bigcap_{\alpha \in I} T_\alpha$ is a touching cone of C . Since $\text{lin}(C)^\perp$ is the smallest element of $\mathcal{T}(C)$ by Lemma 7.2 and since \mathbb{E} is the greatest element of $\mathcal{T}(C)$ we assume that all T_α are proper touching cones. Then for every $\alpha \in I$ there is a non-zero $u_\alpha \in \mathbb{E}$ such that $T_\alpha = T(C, u_\alpha)$. We put $N_\alpha := N(C, F_\perp(C, u_\alpha))$ so $T(C, u_\alpha)$ is a face of N_α . The normal cone $N := \bigcap_{\tilde{\alpha} \in I} N_{\tilde{\alpha}}$ is a face of N_α by Prop. 4.8, so the intersection $N \cap T(C, u_\alpha)$ is a face of N_α and also of N . But then

$$\bigcap_{\tilde{\alpha} \in I} T_{\tilde{\alpha}} = \bigcap_{\tilde{\alpha} \in I} (N \cap T(C, u_{\tilde{\alpha}}))$$

is a face of N , which is a touching cone by the first part of this theorem. Since the normal cone N is a face of N_α the intersection $\bigcap_{\tilde{\alpha} \in I} T_{\tilde{\alpha}}$ is a face of N_α . \square

We prove an independence of touching cones.

Corollary 7.5. *The lattice orderings of $\mathcal{N}(C)$ and $\mathcal{T}(C)$ and the embedding $\mathcal{N}(C) \rightarrow \mathcal{T}(C)$ are independent of the ambient space \mathbb{E} .*

Proof. A normal cone $N \in \mathcal{N}(C)$ has the direct sum form $N = (N \cap \text{lin}(C)) + \text{lin}(C)^\perp$ by Lemma 4.2. Thus the normal cone lattice $\mathcal{N}(C)$ can be reconstructed from

$$\tilde{\mathcal{N}} := \{N \cap \text{lin}(C) \mid N \in \mathcal{N}(C)\}$$

by adding the direct summand $\text{lin}(C)^\perp$. This defines a lattice isomorphism $\tilde{\mathcal{N}} \rightarrow \mathcal{N}(C)$ and the lattice $\tilde{\mathcal{N}}$ is independent of the ambient space \mathbb{E} because $\tilde{\mathcal{N}}$ is the

normal cone lattice of C in the ambient space $\text{lin}(C)$. By Thm. 7.4 the touching cone lattice $\mathcal{T}(C)$ consists of all non-empty faces T of $\mathcal{N}(C)$, so $T = (T \cap \text{lin}(C)) + \text{lin}(C)^\perp$ holds. The same argument as above shows independence of the lattice $\mathcal{T}(C)$ from the ambient space \mathbb{E} . The question which touching cones are normal cones is solved by the embedding $\mathcal{N}(C) \rightarrow \mathcal{T}(C)$, which is also induced from the ambient space $\text{lin}(C)$. \square

Sharp normal vectors characterize the normal cones among all touching cones.

Proposition 7.6. *A proper touching cone T of C is a normal cone of C if and only if there is a vector in $\text{ri}(T) \setminus \{0\}$, which is sharp normal for C . If there is a vector in $\text{ri}(T) \setminus \{0\}$, which is sharp normal for C , then all vectors in $\text{ri}(T) \setminus \{0\}$ are sharp normal for C .*

Proof. Let K be a proper touching cone of C and let us assume that $u \in \text{ri}(K) \setminus \{0\}$ is sharp normal for C . Then there exists $x \in \text{ri}(F_\perp(C, u))$ and we have $u \in \text{ri}(N(C, x))$. By definition of the normal cone of a face we have $N(C, x) = N(C, F_\perp(C, u))$ hence $u \in \text{ri}(N(C, F_\perp(C, u)))$ and this gives us $T(C, u) = N(C, F_\perp(C, u))$. Since $u \in \text{ri}(K)$ we have $K = T(C, u)$ by Lemma 7.2(b). Hence K is the normal cone of the non-empty face $F_\perp(C, u)$.

Conversely let us assume that the touching cone K is the normal cone of a non-empty face of C . Then by Prop. 4.7 we have $K = N(C, F)$ for some non-empty exposed face F of C . Now Lemma 4.6 shows for any non-zero $u \in \text{ri}(K)$ that $F = F_\perp(C, u)$ holds. Then for any $x \in \text{ri}(F_\perp(C, u))$ we have

$$N(C, x) = N(C, F) = K$$

and this shows that $u \in \text{ri}(N(C, x))$. We have proved that u is sharp normal for C . If K is proper, then existence of a non-zero vector u in $\text{ri}(K)$ is assured. \square

Projection properties of sharp normal vectors apply to touching cones. We denote π_V the orthogonal projection onto a vector space $V \subset \mathbb{E}$.

Corollary 7.7. *Let $v \in V \setminus \{0\}$. If the touching cone $T(\pi_V(C), v)$ exists and is not a normal cone, then $T(C, v)$ exists and is not a normal cone. In particular, if $T(C) = \mathcal{N}(C)$ then $T(\pi_V(C)) = \mathcal{N}(\pi_V(C))$.*

Proof. If $T(\pi_V(C), v)$ exists, then $F_\perp(\pi_V(C), v) \neq \emptyset$ and by Lemma 5.4 we have $F_\perp(\pi_V(C), v) = \pi_V(F_\perp(C, v))$. So $F_\perp(C, v) \neq \emptyset$ and $T(C, v)$ exists. If in addition $T(C, v)$ is a normal cone of C , then v is sharp normal for C by Prop. 7.6 as $v \in \text{ri}(T(C, v))$. Then by Prop. 6.4 v is sharp normal for $\pi_V(C)$ and this implies that $T(\pi_V(C), v)$ is a normal cone of $\pi_V(C)$. \square

Example 7.8. We return to Example 6.2 and denote by $K := \mathbb{S}(\mathcal{A})$ the state space of the algebra $\mathcal{A} := \text{Mat}(\mathbb{C}, 2) \oplus \mathbb{C}$. We have seen that every non-zero $u \in \mathcal{A}_{\text{sa}}$ is sharp normal for K and every $\rho \in K$ is sharp exposed in K . This implies $\mathcal{T}(K) = \mathcal{N}(K)$ and $\mathcal{F}(K) = \mathcal{F}_\perp(K)$ by Prop. 7.6 and Lemma 6.5. Now we consider

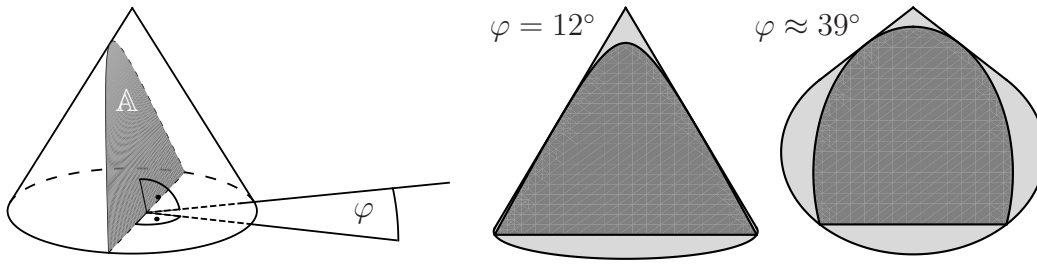


Figure 7.1: The cone of revolution C of an equilateral triangle (left). An affine plane \mathbb{A} through the center of gravity of C is specified by the angle $\varphi < 90^\circ$. The intersection $C \cap \mathbb{A}$ is hyperbolic for $\varphi = 12^\circ$ (middle) and elliptic for $\varphi \approx 39^\circ$ (right), drawn dark. The union with the bright region surrounding it is the projection $\pi_{\mathbb{A}}(C)$.

a family of two-dimensional projections and intersections of K produced by a three-dimensional affine space of self-adjoint matrices without σ_3 -contribution in the first summand

$$\tilde{\mathbb{A}} := \{a \in \text{Mat}(\mathbb{C}, 2) \oplus \mathbb{C} \mid a^* = a, \text{tr}[a(\sigma_3 \oplus 0)] = 0 \text{ and } \text{tr}(a) = 1\}.$$

If $\pi_{\tilde{\mathbb{A}}}$ denotes orthogonal projection to $\tilde{\mathbb{A}}$, then (see [10] Section 2)

$$C := \pi_{\tilde{\mathbb{A}}}(K) = K \cap \tilde{\mathbb{A}} = \text{conv} [\{\rho \in \mathbb{S}(2) \mid \text{tr}(\rho\sigma_3) = 0\} \oplus 0, 0_2 \oplus 1]$$

is the three-dimensional cone depicted in Figure 7.1, left. By Cor. 7.7 we have $\mathcal{T}(C) = \mathcal{N}(C)$ because C is the projection of K to $\tilde{\mathbb{A}}$. By Lemma 6.6 we have $\mathcal{F}(C) = \mathcal{F}_\perp(C)$ because C is the intersection of K with $\tilde{\mathbb{A}}$.

Let $\mathbb{A} \subset \tilde{\mathbb{A}}$ be the two-dimensional affine subspace containing $\frac{1}{3}\mathbb{1}_3$ and having the angle φ with the direction $-\mathbb{1}_2 \oplus 2$. Two example are shown in Figure 7.1, right. The projection shapes $\pi_{\mathbb{A}}(C)$ have every touching cone a normal cone. So, according to Thm. 7.10 and Remark 1.1 a face of $\pi_{\mathbb{A}}(C)$ is non-exposed if and only if it is the endpoint of a unique one-dimensional face. The examples with $\varphi = 12^\circ$ and $\varphi \approx 39^\circ$ have two non-exposed faces: the tangent points of boundary segments to the elliptic boundary arcs. The intersections $C \cap \mathbb{A}$ have all faces exposed. In the depicted examples exist touching cones, which are not normal cones. It is instructive to realize that projection and intersection for the same affine space \mathbb{A} are polars of each other up to the sign (see e.g. Weis [16] Section 2.4).

An easy corollary of Minkowski’s and Carathéodory’s theorem characterizes normal cones and exposed faces in terms of touching cones.

Theorem 7.9. *Let N be a proper normal cone of C such that every touching cone included in N is a normal cone. Then N can be written as a supremum of atoms of $\mathcal{N}(C)$. A number of $\dim(N) - \dim(\text{lin}(C)^\perp)$ atoms suffice in the supremum.*

Proof. By Cor. 7.5 we assume that C has non-empty interior $\text{int}(C) \neq \emptyset$, so $\{0\}$ is the smallest element in $\mathcal{N}(C)$. Let N be a proper normal cone of C . Then N does

not contain a line, for otherwise by (iv) in (15) we had $\text{int}(C) = \emptyset$. Therefore there is an affine hyperplane $H \subset \mathbb{E}$ such that $K := N \cap H$ is a convex body and N is the positive hull $N = \text{pos}(K)$. Let $u \in \text{ri}(K)$. By Minkowski's theorem we write u as a convex combination $u = \sum_{i=1}^d \lambda_i u_i$ for (non-zero) extreme points u_i of K . By Carathéodory's theorem we choose $d = \dim(K) + 1 = \dim(N)$.

We show that N is a supremum of the d normal cones $r_i := \{\lambda u_i \mid \lambda \geq 0\}$, $i = 1, \dots, d$. By Lemma 3.4 u belongs to $\text{ri}(N)$ and the ray r_i is a face of N so r_i is a touching cone by Thm. 7.4. By assumption the touching cone r_i is a normal cone so it is an atom in $\mathcal{N}(C)$. If the supremum $\tilde{N} := r_1 \vee \dots \vee r_d$ is strictly included into N , then \tilde{N} must be a proper face of N by Prop. 4.8, so $\tilde{N} \subset \text{rb}(N)$ by the partition (10) of N into relative interiors of its faces. This contradicts $u \in \text{ri}(N)$. \square

The isomorphism $\mathbf{N}(C) : \mathcal{F}_\perp(C) \rightarrow \mathcal{N}(C)$ in Prop. 4.7 gives an equivalent form of this theorem (which is trivial if C is a single point).

Theorem 7.10. *Let F be a proper exposed face of C such that every touching cone included in the normal cone $\mathbf{N}(C, F)$ is a normal cone. Then F can be written as an intersection of coatoms of $\mathcal{F}_\perp(C)$. A number of $\dim(\mathbf{N}(C, F)) - \dim(\text{lin}(C)^\perp)$ coatoms suffice in the intersection.*

One may check the Thm. 7.9 and Thm. 7.10 on Figure 1.5, 1.3 and 1.1. The theorems have no converse by example in Figure 1.4. The bound on coatoms is saturated by a corner of a cube, it is not saturated for the apex of the cone in Figure 7.1, left.

8. Polar convex bodies

This section is restricted to a convex body $K \subset \mathbb{E}$ in a finite-dimensional real Euclidean vector space $(\mathbb{E}, \langle \cdot, \cdot \rangle)$. Unless specified other we assume that K has non-empty interior $\text{int}(K) \neq \emptyset$ containing the origin $0 \in \text{int}(K)$ and second we assume that K has at least two points. Conjugate faces induce an isotone lattice isomorphism between the faces of the polar convex body K° and the touching cones of K . This implies an equivalent theorem to Thm. 7.9, which can be proved directly using only Minkowski's and Carathéodory's theorem. The antitone lattice isomorphism $\mathcal{F}_\perp(K) \rightarrow \mathcal{N}(K)$ (see Prop. 4.7) gives a fourth equivalent form of Thm. 7.9.

Definition 8.1. The *polar body* of K is $K^\circ := \{x \in \mathbb{E} \mid \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$. If F is a subset of K , then the *conjugate face* of F is $\widehat{F} := \{x \in K^\circ \mid \langle x, y \rangle = 1 \text{ for all } y \in F\}$.

The polar body K° is a convex body with $0 \in \text{int}(K^\circ)$ and such that $K^{\circ\circ} = K$, see Schneider [15], Section 1.6. An example of a convex body with its polar body is depicted in Figure 1.2, right. We recall that \emptyset and K are exposed faces of K so as to make $\mathcal{F}_\perp(K)$ a lattice (this deviates from definitions by Rockafellar or Schneider [14, 15]). By Schneider, Thm. 2.1.4, a subset $F \subset K$ is included in a proper exposed face of K if and only if the conjugate face \widehat{F} is a proper exposed face of

K° . Further, if these conditions hold, then $(\widehat{F})^\wedge = \text{sup}_\perp(F)$ is the smallest exposed face of K containing F . Obviously $\widehat{\emptyset} = K^\circ$ and $\widehat{K} = \emptyset$. So $\mathcal{F}_\perp(K) \rightarrow \mathcal{F}_\perp(K^\circ)$, $F \mapsto (\widehat{F})^\wedge$ is the identity and we get an antitone lattice isomorphism:

$$\mathcal{F}_\perp(K) \rightarrow \mathcal{F}_\perp(K^\circ), \quad F \mapsto \widehat{F}. \tag{30}$$

An example is shown in Figure 1.3. The following remark may help our intuition.

Remark 8.2. The *polar* of an affine space \mathbb{A} in \mathbb{E} with respect to the unit sphere $\{x \in \mathbb{E} \mid \langle x, x \rangle = 1\}$ is the affine space

$$\mathbb{A}^{\text{polar}} := \{x \in \mathbb{E} \mid \langle x, y \rangle = 1 \text{ for all } y \in \mathbb{A}\}.$$

The polar is well-known in projective geometry (see e.g. Coxeter or Fischer [4, 5]), it defines an antitone lattice isomorphism on the set of affine subspaces of $\mathbb{A} \subset \mathbb{E}$ with $0 \notin \mathbb{A}$ with \mathbb{E} joined. The polar is an involution, i.e. $\mathbb{A}^{\text{polarpolar}} = \mathbb{A}$ such that $\dim(\mathbb{A}) + \dim(\mathbb{A}^{\text{polar}}) = \dim(\mathbb{E}) - 1$. E.g. $\emptyset^{\text{polar}} = \mathbb{E}$ and $\mathbb{E}^{\text{polar}} = \emptyset$. In fact it restricts a *correlation* of a projective space.

The conjugate face of an arbitrary subset $F \subset K$ is $\widehat{F} = \text{aff}(F)^{\text{polar}} \cap K^\circ$. It is possible, e.g. for a disk, that $\text{aff}(\widehat{F}) \subsetneq \text{aff}(F)^{\text{polar}}$. Equality holds for all polytopes K and their faces F , see Grünbaum [6], Section 3.4.

The next observation is that the normal cone of every non-empty exposed face F of K is the positive hull of the conjugate face $N(K, F) = \text{pos}(\widehat{F})$ (we have $\text{pos}(\emptyset) = \{0\}$). This statement is proved in a more general form by Schneider [15], Lemma 2.2.3. We include the empty face with $\widehat{\emptyset} = K^\circ$ and with normal cone $N(K, \emptyset) = \text{pos}(K^\circ) = \mathbb{E}$. Combining this with the two antitone lattice isomorphisms $\mathcal{F}_\perp(K) \rightarrow \mathcal{F}_\perp(K^\circ)$ in (30) and $\mathcal{F}_\perp(K) \rightarrow \mathcal{N}(K)$ in Prop. 4.7 we get an isotone lattice isomorphism

$$\mathcal{F}_\perp(K^\circ) \rightarrow \mathcal{N}(K), \quad F \mapsto \text{pos}(F) \tag{31}$$

from the commuting diagram

$$\begin{array}{ccc} \mathcal{F}_\perp(K) & \longrightarrow & \mathcal{F}_\perp(K^\circ) \\ \downarrow & \swarrow \text{pos} & \\ \mathcal{N}(K) & & \end{array}$$

Every proper exposed face F of K° has a supporting hyperplane H of K° with $F = K^\circ \cap H$. We get $F = \text{pos}(F) \cap H$ and since $\emptyset \neq \text{int}(K^\circ)$ we have also $F = \text{pos}(F) \cap (K^\circ)$. So the inverse to (31) is

$$\mathcal{N}(K) \rightarrow \mathcal{F}_\perp(K^\circ), \quad \begin{cases} N \mapsto \text{rb}(K^\circ) \cap N & \text{if } N \neq \mathbb{E}, \\ \mathbb{E} \mapsto K^\circ. \end{cases}$$

By examples in Figure 1.3 the antitone isomorphism $\mathcal{F}_\perp(K) \rightarrow \mathcal{N}(K)$ does not extend to $\mathcal{F}(K) \rightarrow \mathcal{T}(K)$ but we prove extension of $\text{pos} : \mathcal{F}_\perp(K^\circ) \rightarrow \mathcal{N}(K)$.

Theorem 8.3. *Let K be a convex body containing at least two points and with $0 \in \text{int}(K)$. If K° denotes the polar body, then the positive hull operator pos defines an isotone lattice isomorphism $\mathcal{F}(K^\circ) \rightarrow \mathcal{T}(K)$.*

Proof. We consider a proper exposed face F of K° . For $N := \text{pos}(F)$ we have a bijection $\text{pos} : \mathcal{F}(F) \rightarrow \mathcal{F}(N) \setminus \{\emptyset\}$ by Lemma 3.4, which may be written in the form

$$\begin{aligned} \text{pos}(\tilde{F}) \cap F &= \tilde{F} \quad \text{for all } \tilde{F} \in \mathcal{F}(F), \\ \text{pos}(G \cap F) &= G \quad \text{for all } G \in \mathcal{F}(N) \setminus \{\emptyset\}. \end{aligned} \tag{32}$$

By (31) and the paragraph following it, we have $F = \text{rb}(K^\circ) \cap N$ so we replace F by $\text{rb}(K^\circ)$ in (32) except $\mathcal{F}(F)$, which we leave unchanged. This gives us the bijection

$$\text{pos} : \left\{ \begin{array}{l} \text{faces of proper} \\ \text{exposed faces of } K^\circ \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{non-empty faces of proper} \\ \text{normal cones of } K \end{array} \right\}.$$

The domain is clearly $\mathcal{F}(K^\circ) \setminus \{K^\circ\}$ and the target is $\mathcal{T}(K) \setminus \{\mathbb{E}\}$ by Thm. 7.4. Since K has more than two points we have $\mathbb{E} \neq \{0\}$ so $\text{pos}(K^\circ) = \mathbb{E}$ extends this map to an isotone lattice isomorphism $\mathcal{F}(K^\circ) \rightarrow \mathcal{T}(K)$. \square

Theorem 8.3 partitions $\mathbb{E} \setminus \{0\}$ into relative interiors of touching cones, see Rem. 7.3. We translate Thm. 7.9 by interchanging exposed faces with normal cones and touching cones with faces, using (31) and Thm. 8.3. Through affine embeddings we can drop the condition $0 \in \text{ri}(K)$ in the sequel, the condition that K has at least two points is not needed.

Theorem 8.4. *Let K be a convex body and let F be a proper exposed face of K such that every face included in F belongs to $\mathcal{F}_\perp(K)$. Then F can be written as a supremum of at most $\dim(F) + 1$ atoms of $\mathcal{F}_\perp(K)$.*

Thm. 8.4 follows directly from Minkowski’s and Carathéodory’s theorem. It is wrong if K is not closed (e.g. a closed triangle with an extreme point missing) or unbounded (e.g. the strip $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, y \leq 1\}$).

The antitone lattice isomorphism $N(K) : \mathcal{F}_\perp(K) \rightarrow \mathcal{N}(K)$ in Prop. 4.7 gives us an equivalent form of Thm. 8.4. We denote by $F_\perp(K, N)$ the unique exposed face F of K with $N(K, F) = N$ and we use intersection for the infimum in $\mathcal{N}(K)$ by Prop. 4.8.

Theorem 8.5. *Let K be a convex body and let N be a proper normal cone of K such that every face included in $F_\perp(K, N)$ belongs to $\mathcal{F}_\perp(K)$. Then N can be written as an intersection of at most $\dim(F_\perp(K, N)) + 1$ coatoms of $\mathcal{N}(K)$.*

The bound on coatoms is saturated by the normal vector of a square face of the cube.

Acknowledgements. Thanks to Andreas Knauf for discussions and encouragement in the early stage of this work, to Hermann Schulz-Baldes for suggestions to improve the text,

and to Rolf Schneider for drawing my attention to the Aleksandrov-Fenchel inequality and for correcting errors. Thanks to the anonymous referee for useful comments.

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