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Notes on Extended Real- and Set-Valued Functions

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An order theoretic and algebraic framework for the extended real numbers is established which includes extensions of the usual difference to expressions involving $-\infty$ and/or $+\infty$, so-called residuations. New definitions and results for directional derivatives, subdifferentials and Legendre–Fenchel conjugates for extended real-valued functions are given which admit to include the proper as well as the improper case. For set-valued functions, scalar representation theorems and a new conjugation theory are established. The common denominator is that the appropriate image spaces for set-valued functions share fundamental structures with the extended real numbers: They are order complete, residuated monoids with a multiplication by non-negative real numbers.

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1. Motivation and bibliographical comments

Without any doubts, the notion of an extended real-valued function turned out to be extremely useful in variational analysis, optimization theory and beyond. On the one hand, several operations like taking the directional derivative or the infimal convolution, even performed on real-valued or proper functions, may lead to functions which also attain the values $+\infty$ and/or $-\infty$, and it would be really awkward to exclude such cases (see already [12, p. 167]). On the other hand, the added element $+\infty$ admits the inclusion of constraints in a very elegant and concise way (compare [19, p. 23]).

Almost all textbooks and relevant papers on convex and variational analysis make use of this notion. As examples we mention [16], [11], [19], [20], [4], [12], all published before 1980.

To our opinion, the most thorough investigation of extended real-valued functions has already been made by Jean Jacques Moreau in [14], [15], [16]. It is a stunning

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and startling fact that his attempt to "algebraize" the extended reals was not exploited consequently later on. Compare [20, p. 6], and [2, p. 9], where the operation $(+\infty) + (-\infty)$ is called "risky" and "undefined and forbidden", respectively, and also the classic [19, p. 24]. Even in the more recent [21, p. 15], the authors state "there's no single, symmetric way of handling $\infty - \infty$".

Most authors try to avoid the difficulties (like in [18, p. 38]: "we won't have occasion to worry about $+\infty - \infty$ or $0 \cdot \infty$") by restricting the theory to proper functions or just ignore the problem. An extreme with respect to this "avoiding approach" is the standard volume on infinite dimensional analysis [1, p. 2] which reads "The combination $+\infty - \infty$ of symbols has no meaning. The symbols $+\infty$ and $-\infty$ are not really meant to be used for arithmetic, they are only used to avoid awkward expressions involving infima and suprema." In this note, we show that just the opposite works well.

To avoid the development (or the use) of an arithmetic for the extended reals does not only passes a chance, it may also lead to imprecise statements. An example for the latter can even be found in otherwise impressive textbooks: Theorem 2.3.1(ix) in [24] does not hold for improper functions (no such assumption made in the quoted reference) unless one uses the inf-addition on the left and the sup-addition on the right hand side (see below for definitions) of the equation $(f\Box g)^* = f^* + g^*$. The same remark applies, for example, to the first part of Theorem 1 in [12, Section 3.4, p. 178].

In this note, we give an extension of Moreau's approach to extended real-valued functions by noting that the correct algebraic framework is an order complete, residuated monoid with a multiplication with non-negative real numbers. The advantage of this complicated sounding construct is manifold: First, there is no need anymore to "explain away" the value $-\infty$ or to introduce algebraic rules for expressions like $(+\infty) + (-\infty)$ "by convention" ([21, p. 15]), or to avoid them. Secondly, new operations can be introduced which give a precise meaning to expressions like $(+\infty) - (-\infty)$, and one obtains a whole calculus for addition and residuation/difference in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. On an abstract level, some of these observations have already been made by Martínez-Legaz, Singer and Getan in [13] and [6].

Moreover, our approach will also simplify the notation avoiding symbols like $r \dot{+} -s$ (see, among others, Moreau's papers, [13, Example 2.3], [23]). Finally, it will become clear that the theory is completely symmetric, because our slightly different point of view (compared to Martínez-Legaz, Singer and others) is that there are two different ways for introducing algebraic and order structures in $\overline{\mathbb{R}}$ and more general sets as shown in the section about set-valued functions. This follows Moreau's original idea of defining convex and concave functions using different additions and image spaces.

Since we consequently work with two algebraically different copies of the extended reals, we have to say which of the two is used as an image space if we define an extended real-valued function. Thus, there are two classes of such functions. Not very surprisingly, the multiplication by -1 transfers a function of one class into one of the other, an operation which is nothing else than a duality in the sense of [23]. We show how these concepts can be used, for example, to define directional derivatives and subdifferentials of improper convex/concave functions in a coherent way.

A second new feature of our approach is that in order to obtain complete (duality) results the set of dual variables is extended by improper elements and, moreover, the definition of the Legendre-Fenchel conjugate is altered: The new definition involves an additional real variable which comes from the idea that the conjugate should be defined on the set of affine functions rather than on the set of linear functions. It does not make a difference if the function is proper, but it does if not since the improper "linear" functions are not additive. A somehow surprising result is that the conjugate of the infimal convolution of two functions turns out to be the supremal convolution of their conjugates – with respect to the new primal variable.

We mention that improper affine functions have been used in [17] in order to formulate duality results for optimization problems involving set-valued maps. We are not aware of further references, but we think there should be some.

Finally, we consider set-valued functions and give an extension of the theory formulated in [8] to improper set-valued functions using the (improper) scalar ones. In fact, the present note has been written since we wanted to have a coherent framework for proper and improper scalarizations of (closed convex) set-valued functions. The approach follows ideas of [22]: In particular, using the representation of set-valued closed convex functions by families of extended real-valued ones we give a new definition of Legendre-Fenchel conjugates for set-valued functions and conclude with a Fenchel-Moreau theorem which includes the proper as well as the improper case.

We conclude the introduction by noting that it does not take more than 5 pages and relatively elementary mathematics to introduce the two possibilities for an algebraic and order theoretic framework in $\overline{\mathrm{IR}}$, which seems affordable for classroom and textbook purposes.

2. A basic result from residuation theory

In this section, we consider a lattice ordered set with an algebraic operation which we call addition, denoted by $+$, since it corresponds to "usual" additions for the special cases we have in mind. In the following, we understand by a partially ordered groupoid a nonempty set W with a binary relation $+\colon W \times W \to W$ and a partial order $\leq$ which are compatible: $u, v, w \in W$ and $u \leq v$ imply $u + w \leq v + w$. The sum $u + M$ of $u \in W$ and $M \subseteq W$ is understood in the Minkowski sense with $u + M = \emptyset$ if $M = \emptyset$. The following theorem can be extracted, for example, from [5, Chapter XII]. See also [13, Proposition 2.6], [6, Proposition 2.1] for parts (c), (d).

Theorem 2.1. *Let $(W, +, \leq)$ be a partially ordered commutative groupoid. The following statements are equivalent:*

(a) *For each $u, v \in W$ there is $w \in W$ such that for $w' \in W$*

$$u \leq v + w' \iff w \leq w';$$

(*b*) *For each* $u, v \in W$ *the set* $\{w' \in W : u \leq v + w'\}$ *has a least element;*

(*c*) *For* $u \in W$ *and* $M \subseteq W$ *such that* $\inf M$ *exists it holds*

$$u + \inf M = \inf (u + M) ;$$

(*d*) *For each* $u, v \in W$ *there exists* $\inf \{w' \in W : u \leq v + w'\} \in W$ *and it holds*

$$u \leq v + \inf \{w' \in W : u \leq v + w'\} .$$

Proof. The equivalence of (a) and (b) is obvious. Assume (a). Then

$$\forall m \in M : \quad u + \inf M \leq u + m,$$

hence $u + \inf M$ is a lower bound of $u + M$. On the other hand, let $w \in W$ such that

$$\forall m \in M : \quad w \leq u + m.$$

By assumption, there is $\bar{w} \in W$ such that

$$w \leq u + w' \;\Leftrightarrow\; \bar{w} \leq w'.$$

Hence $\bar{w} \leq m$ for all $m \in M$ and therefore $\bar{w} \leq \inf M$. Again by assumption $w \leq u + \inf M$ which proves that $u + \inf M$ is the infimum of $u + M$.

Assuming (c) we define $w = \inf \{w' \in W : u \leq v + w'\}$. Then

$$u \leq \inf \{v + w' : w' \in W,\ u \leq v + w'\} \underset{\text{(c)}}{=} v + \inf \{w' \in W : u \leq v + w'\} = v + w.$$

Now, from (c) it follows

$$v \leq \inf \{u + w' \in W : v \leq u + w'\} = u + \inf \{w' \in W : v \leq u + w'\} ,$$

i.e. (d) holds true.

Finally, assume (d) and define $M = \{w \in W : u \leq v + w\}$. Then, by assumption, $\inf M \in W$ and

$$u \leq v + w \;\Leftrightarrow\; w \in M \;\Leftrightarrow\; \inf M \leq w$$

which proves (a).

This completes the proof of the theorem. □

We shall call a partially ordered commutative groupoid satisfying the conditions of Theorem 2.1 inf–residuated. The following theorem can be proven with parallel arguments and gives conditions for sup–residuated groupoids.

Theorem 2.2. *Let* $(W, +, \leq)$ *be a partially ordered commutative groupoid. The following statements are equivalent:*

(*a*) *For each* $u, v \in W$ *there is* $w \in W$ *such that for* $w' \in W$

$$v + w' \leq u \quad \Longleftrightarrow \quad w' \leq w;$$

(*b*) *For each* $u, v \in W$ *the set* $\{w' \in W : v + w' \leq u\}$ *has a greatest element;*

(*c*) *For* $u \in W$ *and* $M \subseteq W$ *such that* $\sup M$ *exists it holds*

$$u + \sup M = \sup (u + M);$$

(*d*) *For each* $u, v \in W$ *there exists* $\sup \{w' \in W : v + w' \leq u\} \in W$ *and it holds*

$$v + \sup \{w' \in W : v + w' \leq u\} \leq u.$$

3. An algebraic framework for extended real-valued functions

3.1. Order extension

Adding two elements $-\infty$, $+\infty$ to the set $\mathrm{I\!R}$ of real numbers we consider the set $\overline{\mathrm{I\!R}} = \mathrm{I\!R} \cup \{-\infty\} \cup \{+\infty\}$ and extend the usual order relations $\leq$, $<$ on $\mathrm{I\!R}$ to $\overline{\mathrm{I\!R}}$ by setting

$$\begin{aligned} \forall r \in \overline{\mathrm{I\!R}} &: \quad -\infty \leq r \leq +\infty, \\ \forall r \in \mathrm{I\!R} &: \quad -\infty < r < +\infty. \end{aligned}$$

With this extension of $\leq$, $\left(\overline{\mathrm{I\!R}}, \leq\right)$ becomes a partially ordered, complete lattice: Every subset has an infimum and a supremum. In particular,

$$\inf \emptyset = \sup \mathrm{I\!R} = \sup \overline{\mathrm{I\!R}} = +\infty, \tag{1}$$

$$\inf \overline{\mathrm{I\!R}} = \inf \mathrm{I\!R} = \sup \emptyset = -\infty. \tag{2}$$

Note that the commonly used conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$ are the unavoidable choice if one wants to maintain the following monotonicity property: $M \subseteq N \subseteq \overline{\mathrm{I\!R}}$ implies $\inf M \geq \inf N$ and $\sup M \leq \sup N$.

3.2. Addition

There are two ways to extend the addition from $\mathrm{I\!R}$ to $\overline{\mathrm{I\!R}}$ by means of the order relation $\leq$. We obtain two different algebraic operations in $\overline{\mathrm{I\!R}}$.

Definition 3.1. The binary operations $\dot{+} : \overline{\mathrm{I\!R}} \times \overline{\mathrm{I\!R}} \to \overline{\mathrm{I\!R}}$ and $\underset{\cdot}{+} : \overline{\mathrm{I\!R}} \times \overline{\mathrm{I\!R}} \to \overline{\mathrm{I\!R}}$ defined by

$$r \dot{+} s = \inf \{a + b : a, b \in \mathrm{I\!R},\ r \leq a,\ s \leq b\}, \tag{3}$$

$$r \underset{\cdot}{+} s = \sup \{a + b : a, b \in \mathrm{I\!R},\ a \leq r,\ b \leq s\} \tag{4}$$

for $r, s \in \overline{\mathrm{I\!R}}$ are called the inf-addition and the sup-addition in $\overline{\mathrm{I\!R}}$, respectively.

The terminology is due to [21]. Already Moreau [14] introduced the two different additions in $\overline{\mathrm{I\!R}}$. Clearly, both operations coincide with the usual addition in $\mathrm{I\!R}$. The notable differences are

$$\begin{aligned} (+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty, \\ (+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty. \end{aligned}$$

Since inf-adding $+\infty$ always gives $+\infty$ as a result, we say that $+\infty$ dominates the inf-addition. Likewise, $-\infty$ dominates the sup-addition. Both operations are compatible with the order $\leq$ on $\overline{\mathrm{IR}}$ in the usual sense. Thus, $\left(\overline{\mathrm{IR}}, +^{\centerdot}, \leq\right)$ and $\left(\overline{\mathrm{IR}}, +_{\centerdot}, \leq\right)$ are ordered commutative monoids which are complete lattices. The following result describes the relationships between inf-/sup-addition and the order relation. Compare, for example, [14], Proposition 1 and 2 with $F = M$, $G = N$ and $f(x) = g(x) = x$.

Proposition 3.2. *Let $M, N \subseteq \overline{\mathrm{IR}}$. Then*

$$\inf(M +^{\centerdot} N) = \inf M +^{\centerdot} \inf N, \qquad \sup(M +^{\centerdot} N) \leq \sup M +^{\centerdot} \sup N, \tag{5}$$

$$\inf M +_{\centerdot} \inf N \leq \inf(M +_{\centerdot} N), \qquad \sup M +_{\centerdot} \sup N = \sup(M +_{\centerdot} N) \tag{6}$$

where the sum of sets is understood in the Minkowski sense.

Proof. By definition,

$$\inf M +^{\centerdot} \inf N = \inf\{r + s : \ r, s \in \mathrm{IR}, \inf M \leq r, \inf N \leq s\}$$

and for all $m \in M, n \in N$ holds $\inf M +^{\centerdot} \inf N \leq m +^{\centerdot} n$, thus $\inf M +^{\centerdot} \inf N \leq \inf(M +^{\centerdot} N)$. On the other hand, if $M = \emptyset$ or $N = \emptyset$, then both of $\inf M +^{\centerdot} \inf N$ and $\inf(M +^{\centerdot} N)$ give the result $+\infty$ since the latter element dominates the inf-addition. If $M, N \neq \emptyset$ and one of them has infimum $-\infty$ then $\inf M +^{\centerdot} \inf N = \inf(M +^{\centerdot} N) = -\infty$. If $-\infty < \inf M, \inf N$ then, for each $\varepsilon > 0$ there are $m_\varepsilon \in M \cap \mathrm{IR}$, $n_\varepsilon \in N \cap \mathrm{IR}$ such that $m_\varepsilon + n_\varepsilon \leq \inf M +^{\centerdot} \inf N + \varepsilon$, hence $\inf M +^{\centerdot} \inf N \leq \inf(M +^{\centerdot} N)$ which gives, together with the first part, equality. $\square$

In particular, with $r \in \overline{\mathrm{IR}}$, $N = \{r\}$ we obtain (again, compare [14], p. 7, formulas (2.9), (2.12))

$$r +^{\centerdot} \inf M = \inf(\{r\} +^{\centerdot} M), \qquad r +_{\centerdot} \sup M = \sup(\{r\} +_{\centerdot} M). \tag{7}$$

Finally, note that the inequalities in (5) and (6) are not satisfied as equations in general. A counterexample can already be found in [15], p. 7.

3.3. Multiplication with -1

By setting

$$(-1)(+\infty) = -\infty, \qquad (-1)(-\infty) = +\infty$$

we extend the multiplication of real numbers with -1 to $\overline{\mathrm{IR}}$. As usual, we abbreviate $(-1)\,r$ to $-r$ for $r \in \overline{\mathrm{IR}}$ if no confusion arises. Obviously,

$$\forall r, s \in \overline{\mathrm{IR}}\colon\ (r \leq s \ \Leftrightarrow\ -s \leq -r)$$

and hence for each $M \subseteq \overline{\mathrm{IR}}$

$$(-1)\inf M = \sup(-1)\,M. \tag{8}$$

Thus, the multiplication with -1 is a duality of $\overline{\mathrm{IR}}$ onto itself in the sense of [23], Chapter 5. Here and in the following, we make use of $(-1)\,M = \{-r \colon r \in M\}$ if $M \neq \emptyset$ and $(-1)\,M = \emptyset$ if $M = \emptyset$.

Proposition 3.3. *For $r, s \in \overline{\mathbb{R}}$,*

$$(-1)\,(r \stackrel{\cdot}{+} s) = (-1)\,r \underset{\cdot}{+} (-1)\,s.$$

Proof. Using Definition 3.1 and (8) above we obtain

$$\begin{aligned}
(-1)\,(r \stackrel{\cdot}{+} s) &= (-1) \inf \{a + b \colon a, b \in \mathbb{R},\ r \le a,\ s \le b\} \\
&= \sup \{(-a) + (-b) : a, b \in \mathbb{R},\ r \le a,\ s \le b\} \\
&= \sup \{a' + b' \colon a', b' \in \mathbb{R},\ a' \le -r,\ b' \le -s\} \\
&= (-1)\,r \underset{\cdot}{+} (-1)\,s
\end{aligned}$$

which already proves the claim. □

3.4. Residuation and inf-/sup-difference

Proposition 3.2 together with Theorem 2.1 and 2.2 tell us that $\left(\overline{\mathbb{R}}, \stackrel{\cdot}{+}, \le\right)$ and $\left(\overline{\mathbb{R}}, \underset{\cdot}{+}, \le\right)$ are residuated semigroups with a neutral element (i.e. residuated monoids), see e.g. [5], Chap. XII. The corresponding residuation operations may serve as extensions of the difference from $\mathbb{R}$ to $\left(\overline{\mathbb{R}}, \stackrel{\cdot}{+}, \le\right)$ and $\left(\overline{\mathbb{R}}, \underset{\cdot}{+}, \le\right)$, respectively. This motivates the following definition. Residuation operations for $x \mapsto r + x$ in $\overline{\mathbb{R}}$ have not been considered by Moreau. Only Martínez-Legaz, Singer and Getan (see [13], [6]) seem to have realized the importance of residuation for the foundation of convex analysis.

Definition 3.4. Let $r, s \in \overline{\mathbb{R}}$. The element

$$r \stackrel{\cdot}{-} s = \min \left\{t \in \overline{\mathbb{R}} \colon r \le s \stackrel{\cdot}{+} t\right\}$$

is called the inf-difference of r and s. The element

$$r \underset{\cdot}{-} s = \max \left\{t \in \overline{\mathbb{R}} \colon s \underset{\cdot}{+} t \le r\right\}$$

is called the sup-difference of r and s.

One easily obtains for all $r, s \in \overline{\mathbb{R}}$

$$r \stackrel{\cdot}{-} s = \inf \{t \in \mathbb{R} \colon r \le s \stackrel{\cdot}{+} t\}, \tag{9}$$

$$r \underset{\cdot}{-} s = \sup \{t \in \mathbb{R} \colon s \underset{\cdot}{+} t \le r\} \tag{10}$$

and

$$r \stackrel{\cdot}{-} (+\infty) = -\infty, \qquad (-\infty) \stackrel{\cdot}{-} r = -\infty, \tag{11}$$

$$r \underset{\cdot}{-} (-\infty) = +\infty, \qquad (+\infty) \underset{\cdot}{-} r = +\infty. \tag{12}$$

The rules for a subtraction of least and greatest elements from each other are as shown below:

$$\begin{aligned}
(+\infty) \stackrel{\cdot}{-} (-\infty) &= +\infty, & (+\infty) \underset{\cdot}{-} (+\infty) &= +\infty, \\
(+\infty) \stackrel{\cdot}{-} (+\infty) &= -\infty, & (+\infty) \underset{\cdot}{-} (-\infty) &= +\infty, \\
(-\infty) \stackrel{\cdot}{-} (+\infty) &= -\infty, & (-\infty) \underset{\cdot}{-} (-\infty) &= +\infty, \\
(-\infty) \stackrel{\cdot}{-} (-\infty) &= -\infty, & (-\infty) \underset{\cdot}{-} (+\infty) &= -\infty.
\end{aligned}$$

Moreover, from Theorem 2.1(a) with $u = r \in \overline{\mathrm{I\!R}}$, $v = s \in \overline{\mathrm{I\!R}}$, $w = r \dot{-} s$ and $w' = 0$, and likewise with the help of Theorem 2.2, we obtain

$$r \leq s \;\Leftrightarrow\; r \dot{-} s \leq 0 \;\Leftrightarrow\; 0 \leq s \underset{\cdot}{-} r. \tag{13}$$

The following result gives relationships between inf-/sup-addition, inf-/sup-subtraction and multiplication with -1.

Proposition 3.5. *Let $r, s \in \overline{\mathrm{I\!R}}$. Then*

$$r \dot{-} s = r \underset{\cdot}{+} (-1)\, s, \tag{14}$$

$$r \underset{\cdot}{-} s = r \dot{+} (-1)\, s, \tag{15}$$

$$s \underset{\cdot}{-} r = (-1)\,(r \dot{-} s)\,, \tag{16}$$

$$s \dot{-} r = (-1) r \dot{-} (-1) s, \tag{17}$$

$$s \underset{\cdot}{-} r = (-1) r \underset{\cdot}{-} (-1) s. \tag{18}$$

Proof. If $r = -\infty$, or if $s = +\infty$, then $r \dot{-} s = r \underset{\cdot}{+} (-1)\, s = -\infty$, see (11). If $r = +\infty$ and $s < +\infty$, or if $s = -\infty$ and $r > -\infty$, then $r \dot{-} s = r \underset{\cdot}{+} (-1)\, s = +\infty$. This proves (14) since if $r, s \in \mathrm{I\!R}$, the formula is known to be true. Likewise, (15) is proven.

Next, we use (8) and (9) to obtain

$$(-1)(r \dot{-} s) = \sup \left\{ t \in \mathrm{I\!R} \colon r \leq s \dot{+} (-1) t \right\}.$$

Since it suffices to take the supremum over $t \in \mathrm{I\!R}$ we get

$$r \leq s \dot{+} (-1) t \;\Leftrightarrow\; r \underset{\cdot}{+} t \leq s$$

and (16) follows from Definition 3.4.

The last two equations are immediate from (14), (15). □

We establish a calculus for manipulating inf-/sup-differences. Since these operations are special cases of residuation mappings these rules are well-known, see for example [3, Lemma 3.2] where the sup versions can be found.

Proposition 3.6.

(a) *For each $r \in \overline{\mathrm{I\!R}}$,*

$$r \dot{-} r = \begin{cases} 0 : & r \in \mathrm{I\!R} \\ -\infty : & r \notin \mathrm{I\!R}, \end{cases} \qquad r \underset{\cdot}{-} r = \begin{cases} 0 : & r \in \mathrm{I\!R} \\ +\infty : & r \notin \mathrm{I\!R}. \end{cases}$$

(b) *For each $r, s, t \in \overline{\mathrm{I\!R}}$ with $r \leq s$,*

$$r \dot{-} t \leq s \dot{-} t, \qquad t \dot{-} s \leq t \dot{-} r,$$

and

$$r \underset{\cdot}{-} t \leq s \underset{\cdot}{-} t, \qquad t \underset{\cdot}{-} s \leq t \underset{\cdot}{-} r.$$

(c) *For each* $a, b, r, s \in \overline{\mathrm{IR}}$,

$$(a \overset{\cdot}{+} r) \overset{\cdot}{-} (b \overset{\cdot}{+} s) \leq (a \overset{\cdot}{-} b) \overset{\cdot}{+} (r \overset{\cdot}{-} s),$$

especially

$$(a \overset{\cdot}{+} r) \overset{\cdot}{-} (r \overset{\cdot}{+} b) \leq (a \overset{\cdot}{-} b).$$

Symmetrically,

$$(a \underset{\cdot}{-} b) \underset{\cdot}{+} (r \underset{\cdot}{-} s) \leq (a \underset{\cdot}{+} r) \underset{\cdot}{-} (b \underset{\cdot}{+} s),$$

especially

$$(a \underset{\cdot}{-} b) \leq (a \underset{\cdot}{+} r) \underset{\cdot}{-} (r \underset{\cdot}{+} b).$$

(d) *If* $M \subseteq \overline{\mathrm{IR}}$ *and* $a \in \overline{\mathrm{IR}}$, *then*

$$a \overset{\cdot}{-} \inf M = \sup_{m \in M} (a \overset{\cdot}{-} m),$$
$$a \underset{\cdot}{-} \sup M = \inf_{m \in M} (a \underset{\cdot}{-} m).$$

Proof. (a) is straightforward from the definitions. For (b) observe that if $r \leq s$, then $s \leq t \overset{\cdot}{+} t'$ implies $r \leq t \overset{\cdot}{+} t'$, and this in turn gives $r \overset{\cdot}{-} t \leq s \overset{\cdot}{-} t$. Moreover, if $r \leq s$, then $t \leq r \overset{\cdot}{+} t'$ implies $t \leq s \overset{\cdot}{+} t'$, and this in turn gives $t \overset{\cdot}{-} s \leq t \overset{\cdot}{-} r$. The relationships for $\underset{\cdot}{-}$ can be proven similarly. We turn to (c). Take $t_1, t_2 \in \overline{\mathrm{IR}}$ such that $a \leq b \overset{\cdot}{+} t_1$ and $r \leq s \overset{\cdot}{+} t_2$. Then $(a \overset{\cdot}{+} r) \leq (b \overset{\cdot}{+} s) \overset{\cdot}{+} (t_1 + t_2)$, hence

$$(a \overset{\cdot}{+} r) \overset{\cdot}{-} (b \overset{\cdot}{+} s) \leq t_1 \overset{\cdot}{+} t_2.$$

Taking the infimum over t_1 satisfying $a \leq b \overset{\cdot}{+} t_1$ and t_2 satisfying $r \leq s \overset{\cdot}{+} t_2$ gives the result. The second formula in (c) is immediate by setting $r = s$ in the first and applying (a). The results for $\underset{\cdot}{-}$ are proven likewise. (d) Using (14), (15) as well as (8) and (e) of Proposition 3.5 we obtain

$$a \overset{\cdot}{-} \inf M = a \underset{\cdot}{+} (-1) \inf M = a \underset{\cdot}{+} \sup -M = \sup \left(a \underset{\cdot}{+} (-M)\right) = \sup \left(a \overset{\cdot}{-} M\right).$$

The second equation follows from the first since by (15) we have $a \underset{\cdot}{-} \sup M = a \overset{\cdot}{+} \inf(-1)M$ and thus $a \underset{\cdot}{-} \sup M = \inf_{m \in M} (a \underset{\cdot}{-} m)$. $\square$

3.5. Multiplication with non-negative reals

The multiplication with non-negative reals is extended to $\overline{\mathrm{IR}}$ by

$$\forall t > 0\colon \quad t \cdot (\pm\infty) = \pm\infty$$

and $0 \cdot (\pm\infty) = 0 \in \mathrm{IR}$. The triples $\left(\overline{\mathrm{IR}}, \overset{\cdot}{+}, \cdot\right)$ and $\left(\overline{\mathrm{IR}}, \underset{\cdot}{+}, \cdot\right)$ are conlinear spaces (see appendix for a definition) consisting only of convex elements. That is, the multiplication with non-negative real numbers distributes over $\overset{\cdot}{+}$ and $\underset{\cdot}{+}$ as well as the other way around.

The order relation $\leq$ as defined above is compatible with this algebraic structure in the usual sense. We write $\mathrm{IR}^{\vartriangle} = \left(\overline{\mathrm{IR}}, \overset{\cdot}{+}, \cdot, \leq\right)$, $\mathrm{IR}^{\triangledown} = \left(\overline{\mathrm{IR}}, \underset{\cdot}{+}, \cdot, \leq\right)$, and drop the $\cdot$ for multiplication if no confusion arises.

Proposition 3.7. *For all $a, b \in \overline{\mathrm{IR}}$ and $t \geq 0$ it holds*

$$t(a -^{\cdot} b) = ta -^{\cdot} tb, \qquad t(a -_{\cdot} b) = ta -_{\cdot} tb.$$

Proof. The relationships are trivial for $t = 0$. If $t > 0$ then

$$\begin{aligned} t(a -^{\cdot} b) &= \inf\{ts \in \mathrm{IR} \colon a \leq b +^{\cdot} s\} \\ &= \inf\{s' \in \mathrm{IR} \colon ta \leq tb +^{\cdot} s'\} = ta -^{\cdot} tb. \end{aligned}$$

The result for $-_{\cdot}$ follows similarly. □

4. Extended real-valued functions

From the above, it should be clear that there are two types of extended real-valued functions, those mapping into $\mathrm{IR}^{\vartriangle}$ and those mapping into $\mathrm{IR}^{\triangledown}$. The point-wise multiplication with -1 transfers a function of one class into a function of the other. This point of view differs slightly from [6] where (only one copy of) $\overline{\mathrm{IR}}$ is considered with two additions and two corresponding residuation operations. This might appear to be just a tiny shift of weight, but it becomes important when it comes to set-valued functions: The replacements of $\mathrm{IR}^{\vartriangle}$ and $\mathrm{IR}^{\triangledown}$ will have rather different looking elements. In the following, let X be a linear space.

Definition 4.1. Let $f \colon X \to \overline{\mathrm{IR}}$. The epigraph and the hypographs of f are the sets

$$\begin{aligned} \operatorname{epi} f &= \{(x, r) \in X \times \mathrm{IR} \colon f(x) \leq r\}, \\ \operatorname{hypo} f &= \{(x, r) \in X \times \mathrm{IR} \colon r \leq f(x)\}, \end{aligned}$$

respectively. The effective domain of a function $g \colon X \to \mathrm{IR}^{\vartriangle}$ is the set

$$\operatorname{dom} g = \{x \in X \colon g(x) < +\infty\}$$

whereas the effective domain of a function $h \colon X \to \mathrm{IR}^{\triangledown}$ is the set

$$\operatorname{dom} h = \{x \in X \colon -\infty < h(x)\}.$$

The concept of the domain depends on the image space, so if one knows the latter, one also knows which definition to use. Therefore, we do not introduce different symbols. Note also that the collection of functions into $\mathrm{IR}^{\vartriangle}$ ($\mathrm{IR}^{\triangledown}$) is a conlinear space under point-wise addition $+^{\cdot}$ ($+_{\cdot}$) and multiplication with non-negative reals, but neither collection is a linear space. Mixing up the image spaces may lead to strange effects as the next example shows.

Example 4.2. Consider the functions $f, g \colon \mathrm{IR} \to \mathrm{IR}^{\vartriangle}$ defined by

$$f(x) = \begin{cases} +\infty : & x < 2, \\ -\infty : & x \geq 2, \end{cases} \qquad g(x) = \begin{cases} -\infty : & |x| \leq 1, \\ +\infty : & |x| > 1. \end{cases}$$

Both functions have a convex epigraph. So has the function $x \mapsto f(x) +^{\cdot} g(x)$. However, the function $x \mapsto f(x) +_{\cdot} g(x)$ neither has a convex epigraph nor a convex hypograph.

The previous example also shows that convexity and $\dotplus$ are linked as are concavity and $\underset{\cdot}{+}$. This justifies the following definition (already in [14, p. 19]).

Definition 4.3. A function $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ is called convex if

$$\forall t \in (0,1)\,,\ \forall x_1, x_2 \in X\colon \quad g\left(tx_1 + (1-t)\,x_2\right) \le tg\left(x_1\right) \dotplus (1-t)\,g\left(x_2\right).$$

A function $h\colon X \to \mathrm{I\!R}^{\triangledown}$ is called concave if

$$\forall t \in (0,1)\,,\ \forall x_1, x_2 \in X\colon \quad th\left(x_1\right) \underset{\cdot}{+} (1-t)\,h\left(x_2\right) \le h\left(tx_1 + (1-t)\,x_2\right).$$

Again, the collection of all convex functions into $\mathrm{I\!R}^{\vartriangle}$ (concave functions into $\mathrm{I\!R}^{\triangledown}$) is a conlinear space under point-wise addition $\dotplus$ ($\underset{\cdot}{+}$) and multiplication with non-negative reals. Apparently, it does not make much sense to consider convex functions into $\mathrm{I\!R}^{\triangledown}$, nor concave into $\mathrm{I\!R}^{\vartriangle}$.

A function $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ is convex if and only if $-g\colon X \to \mathrm{I\!R}^{\triangledown}$ is concave. A function g mapping into $\mathrm{I\!R}^{\vartriangle}$ is convex if and only if its epigraph is a convex subset of the linear space $X \times \mathrm{I\!R}$, and a function h mapping into $\mathrm{I\!R}^{\triangledown}$ is concave if and only if its hypograph is convex.

Definition 4.4. A function $f\colon X \to \overline{\mathrm{I\!R}}$ is called positively homogeneous if

$$\forall t > 0,\ \forall x \in X\colon \quad f\left(tx\right) = tf\left(x\right).$$

A positively homogeneous convex function into $\mathrm{I\!R}^{\vartriangle}$ is called sublinear, and a positively homogeneous concave function into $\mathrm{I\!R}^{\triangledown}$ is called superlinear.

Notice that we do not include the case $t = 0$ in the definition of positive homogeneity. Thus,

$$g(x) = \begin{cases} -\infty : & x \le 0, \\ +\infty : & x > 0 \end{cases}$$

is a positively homogeneous function, while $0 \cdot g(x) \neq g(0 \cdot x)$ holds for all $x \in X$.

Example 4.5 (improper affine functions). Let X be a topological linear space and X^* its topological dual. We write $x^*\left(x\right)$ for the value of an element $x^* \in X^*$ at $x \in X$. Let $r \in \mathrm{I\!R}$ and set $x^*_r(x) = x^*(x) - r$ for $x \in X$. Each $x^* \in X^*$ generates a closed improper function $\hat{x}^*_r\colon X \to \mathrm{I\!R}^{\vartriangle}$ given by

$$\hat{x}^*_r\left(x\right) = \begin{cases} -\infty : & x^*_r\left(x\right) \le 0 \\ +\infty : & x^*_r\left(x\right) > 0 \end{cases}$$

which we call the inf-extension of the affine function $x \mapsto x^*_r\left(x\right)$. Analogously, the improper sup-extension of $x \mapsto x^*_r\left(x\right)$ (with a closed hypograph) mapping into $\mathrm{I\!R}^{\triangledown}$ can be obtained by reversing the roles of $-\infty$ and $+\infty$. The functions $\hat{x}^*_0$ are positively homogeneous, subadditive and superadditive, but not additive, i.e. in general $\hat{x}^*_0\left(x_1 + x_2\right) \neq \hat{x}^*_0\left(x_1\right) \dotplus \hat{x}^*_0\left(x_2\right)$ for $x_1, x_2 \in X$. Below, this will force us to define Legendre–Fenchel conjugates acting on the set of affine rather than linear functions.

In the sequel, we shall write x^* for x_0^* and $\hat{x}^*$ for $\hat{x}_0^*$. Define

$$\hat{X}^* = \{\hat{x}^* \colon x^* \in X^*\}$$

and $X^\vartriangle = X^* \cup \hat{X}^*$. The set $X^\vartriangle$ is called the (topological) inf-dual of X (correspondingly, the sup-dual $X^\triangledown$ of X can be defined using the sup-extensions of continuous linear functions). On $X^\vartriangle$, an addition can be introduced by

$$\xi + \eta = \eta + \xi = \begin{cases} \text{inf-extension of } x^* + y^* : & \xi = \hat{x}^*, \ \eta = \hat{y}^*, \\ \xi : & \xi = \hat{x}^*, \ \eta = y^*, \\ x^* + y^* : & \xi = x^*, \ \eta = y^* \end{cases}$$

for $\xi, \eta \in X^\vartriangle$ with $x^*, y^* \in X^*$. Taking the multiplication by non-negative numbers pointwise, $(X^\vartriangle, +, \cdot)$ is a conlinear space with neutral element $0 \in X^*$.

The following representation formulas for (proper and improper) affine functions will be used later on.

Proposition 4.6. *Let $\xi \in X^\vartriangle$, $r \in \mathrm{I\!R}$. Then*

$$\forall x_1, x_2 \in X\colon \quad \xi_r(x_1 + x_2) = \sup_{r_1 + r_2 = r} \left[\xi_{r_1}(x_1) \mathbin{\underset{\cdot}{+}} \xi_{r_2}(x_2)\right], \tag{19}$$

$$\forall x_1, x_2 \in X\colon \quad \xi_r(x_1 - x_2) = \sup_{r_1 + r_2 = r} \left[\xi_{r_1}(x_1) \mathbin{\dot{-}} \xi_{-r_2}(x_2)\right]. \tag{20}$$

Proof. Exemplarily, we prove (19). The formula is obvious for $\xi = x^* \in X^*$ and for $\xi = \hat{0}^*$. Let $\xi = \hat{x}^* \in \hat{X}^* \backslash \{\hat{0}^*\}$. Then, by definition of $\hat{x}^*$ and since $-\infty$ dominates the sup-addition $\sup \left\{\hat{x}^*_{r_1}(x) \mathbin{\underset{\cdot}{+}} \hat{x}^*_{r_2}(y) : r_1 + r_2 = r\right\} = +\infty$ if and only if there are $r_1, r_2 \in \mathrm{I\!R}$ such that $r_1 + r_2 = r$ and $x^*(x) - r > 0$, $x^*(y) - r > 0$. But this is equivalent to $x^*(x+y) - r > 0$, and this in turn to $\hat{x}^*_r(x+y) = +\infty$. Otherwise, $\hat{x}^*_r(x+y) = \sup_{r_1+r_2=r}\left(\hat{x}^*_{r_1}(x) \mathbin{\underset{\cdot}{+}} \hat{x}^*_{r_2}(y)\right) = -\infty$. $\square$

Replacing sup by inf, $\mathbin{\underset{\cdot}{+}}$ by $\mathbin{\dot{+}}$ and $\mathbin{\dot{-}}$ by $\mathbin{\underset{\cdot}{-}}$ one may obtain another pair of representation formulas.

5. Applications

5.1. Directional derivatives of convex functions

In this section, we shall show that the residuations $\mathbin{\dot{-}}$ and $\mathbin{\underset{\cdot}{-}}$ may serve as substitutes for the usual difference in the definition of the directional derivative of extended real-valued convex and concave functions. Besides the obvious result (may be useful or not) of having a coherent definition of directional derivatives also for improper functions the constructions below indicate the deep connection between the order relation in the image space, crucial for the definition of residuations, and directional derivatives.

Definition 5.1. The directional derivative of a function $g\colon X \to \mathrm{I\!R}^\vartriangle$ at $x_0 \in X$ in direction $x \in X$ is given by

$$g'(x_0, x) = \lim_{t \downarrow 0} \frac{1}{t}\left[\left(g(x_0 + tx) \mathbin{\dot{-}} g(x_0)\right)\right].$$

The directional derivative of a function $h\colon X \to \mathrm{I\!R}^{\triangledown}$ at $x_0 \in X$ in direction $x \in X$ is given by

$$h'(x_0, x) = \lim_{t \downarrow 0} \frac{1}{t} \left[\left(h(x_0 + tx) \underset{\cdot}{-} h(x_0) \right) \right].$$

If $g(x_0) = +\infty$ then $g'(x_0, x) = -\infty$ since $r \dot{-} (+\infty) = -\infty$ for all $r \in \overline{\mathrm{I\!R}}$. If g is convex and $g(x_0) = -\infty$ then $g'(x_0, x) = -\infty$ iff there is $t > 0$ such that $g(x_0 + tx) = -\infty$ and $g'(x_0, x) = +\infty$ otherwise. If $g(x_0) \in \mathrm{I\!R}$ the directional derivative coincides with the classical one (see e.g. [12], p. 193). Similar remarks apply to $h'(x_0, x)$.

Remark 5.2. We have $g'(x_0, x) = (-1)(-g)'(x_0, x)$ and similar for h. This can be seen with the help of Proposition 3.5.

Proposition 5.3. *If $g\colon X \to \mathrm{I\!R}^{\triangle}$ is convex, then $x \mapsto g'(x_0, x)$ is a sublinear function from X into $\mathrm{I\!R}^{\triangle}$. In this case,*

$$g'(x_0, x) = \inf_{t>0} \frac{1}{t} \left[\left(g(x_0 + tx) \dot{-} g(x_0) \right) \right]. \tag{21}$$

Likewise, if $h\colon X \to \mathrm{I\!R}^{\triangledown}$ is concave, then $x \mapsto h'(x_0, x)$ is a superlinear function from X into $\mathrm{I\!R}^{\triangledown}$ and

$$h'(x_0, x) = \sup_{t>0} \frac{1}{t} \left[\left(h(x_0 + tx) \underset{\cdot}{-} h(x_0) \right) \right]. \tag{22}$$

Proof. The basic fact is the monotonicity of the difference quotient

$$t \mapsto \frac{1}{t} \left[\left(g(x_0 + tx) \dot{-} g(x_0) \right) \right].$$

Indeed, taking $t > 0$ and $\tau \in (0, 1)$ we obtain with the help of Proposition 3.6(c) and Proposition 3.7

$$\begin{aligned} \frac{1}{\tau t} \left[g(x_0 + \tau t x) \dot{-} g(x_0) \right] &\leq \frac{\tau}{\tau t} \left[g(x_0 + tx) \dot{-} g(x_0) \right] \dot{+} \frac{1 - \tau}{\tau t} \left[(g(x_0) \dot{-} g(x_0)) \right] \\ &\leq \frac{1}{t} \left[g(x_0 + tx) \dot{-} g(x_0) \right]. \end{aligned}$$

Therefore, for $0 < s = \tau t < t$

$$\frac{1}{s} \left[\left(g(x_0 + sx) \dot{-} g(x_0) \right) \right] \leq \frac{1}{t} \left[\left(g(x_0 + tx) \dot{-} g(x_0) \right) \right]$$

and hence

$$g'(x_0, x) = \inf_{t>0} \frac{1}{t} \left[\left(g(x_0 + tx) \dot{-} g(x_0) \right) \right]$$

holds true. For $s > 0$,

$$g'(x_0, sx) = s \cdot \inf_{st>0} \frac{1}{st} \left[\left(g(x_0 + stx) \dot{-} g(x_0) \right) \right] = s g'(x_0, x),$$

thus the directional derivative is positively homogeneous.

Finally, consider the function $g_t \colon X \to \mathrm{I\!R}^\vartriangle$ defined by

$$g_t(x) = \frac{1}{t}\left(g(x_0+tx) \dot{-} g(x_0)\right) \quad \text{for } x \in X.$$

The epigraph of each function g_t is convex and by the monotonicity of the difference quotient,

$$\operatorname{epi} g'(x_0,\cdot) = \bigcup_{t>0} \operatorname{epi} g_t$$

is convex. Thus, the directional derivative is a sublinear function.

The concave case is immediate, considering that h is concave iff $-h$ is convex. $\square$

The first part of the previous proposition is an extension of results like theorem 2.1.13 in [24]. In the "proper" theory, it is well-established that the linear minorants of $x \mapsto g'(x_0, x)$ are precisely the elements of the subdifferential of g at x_0. Admitting improper affine functions (see Example 4.5) it is now possible to formulate an "improper" supplement for the "proper" theory. Recall $X^\vartriangle = X^* \cup \hat{X}^*$ (see Example 4.5).

Definition 5.4. An element $\xi \in X^\vartriangle$ is called an extended subgradient of $g \colon X \to \mathrm{I\!R}^\vartriangle$ at $x_0 \in X$ iff

$$\forall x \in X\colon \quad \xi(x - x_0) \le g(x) \dot{-} g(x_0) \tag{23}$$

The set of all subgradients of g at x_0 is denoted by $\partial^{ex} g(x_0)$, the set of improper subgradients of g at x_0 is denoted by $\partial^{ip} g(x_0)$, and the (classical) subdifferential is $\partial g(x_0) = \partial^{ex} g(x_0) \backslash \partial^{ip} g(x_0)$.

Obviously, $-\infty \in \partial^{ex} g(x_0)$ for all g and all $x_0 \in X$.

Proposition 5.5. *Let $g \colon X \to \mathrm{I\!R}^\vartriangle$ be a convex function. The following two statements are equivalent for $\xi \in X^\vartriangle$, $x_0 \in X$:*

(a) ξ is a subgradient of g at x_0;
(b) $\forall x \in X:\ \xi(x) \le g'(x_0, x)$.

Proof. First, assume (a) and choose $x = x_0 + ty$ with $t > 0$, $y \in X$. We get

$$\forall y \in X,\ \forall t > 0\colon \quad \xi(ty) \le g(x_0 + ty) \dot{-} g(x_0).$$

Since $\frac{1}{t}\xi(ty) = \xi(y)$ for all $t > 0$, $y \in X$ (no matter if ξ is proper or improper) we may conclude

$$\forall y \in X,\ \forall t > 0\colon \quad \xi(y) \le \frac{1}{t}\left[g(x_0 + ty) \dot{-} g(x_0)\right].$$

Formula (21) produces (b). Next, assume (b). Using (21) and choosing $t = 1$ gives (23). $\square$

5.2. Legendre–Fenchel conjugation

The following concepts and results supplement the conjugation theory for functions with a proper closure. Throughout this section, we assume that X is a separated locally convex space with topological dual X^*. It is well-known that if g is convex and improper, then $(\operatorname{cl} g)(x) \in \{\pm\infty\}$ for all $x \in X$ (see [24, Proposition 2.2.5]). In this case, $\operatorname{dom}(\operatorname{cl} g) = \operatorname{cl}(\operatorname{dom} g)$.

Theorem 5.6. *Let $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ be improper closed convex. Then g is the pointwise supremum of its improper closed minorants, and if $g \not\equiv -\infty$, then there are $x^* \in X^* \backslash \{0\}$ and $r \in \mathrm{I\!R}$ such that*

$$\forall x \in X\colon \quad \hat{x}^*_r(x) \leq g(x).$$

Proof. If g is $g \equiv +\infty$ or $\equiv -\infty$, then $g(x) = \hat{0}^*_{-1}$ or $g(x) = \hat{0}^*$ for all $x \in X$, respectively. Thus, let us assume $\operatorname{dom} g \not\equiv \emptyset$ and $g \not\equiv -\infty$. Since $\operatorname{dom} g$ is a closed convex subset of X it is the intersection of all closed half spaces including it. Each such half space has the form

$$\{x \in X\colon x^*(x) - r \leq 0\} \supseteq \operatorname{dom} g \tag{24}$$

for some $x^* \in X^*$, $r \in \mathrm{I\!R}$. The function $\hat{x}^*_r : X \to \overline{\mathrm{I\!R}}$ certainly is a closed improper affine minorant of g. Since $\operatorname{dom} g$ is the intersection of all $\operatorname{dom} x^*_r$ with x^*, r satisfying (24) the result follows. $\square$

The next result characterizes improper affine minorants of improper $\mathrm{I\!R}^{\vartriangle}$-valued functions.

Theorem 5.7. *Let $g\colon X \to \mathrm{I\!R}^{\vartriangle}$. The following statements are equivalent for $\xi \in X^{\vartriangle}$ and $r \in \mathrm{I\!R}$:*

(*a*) $\forall x \in X : \xi_r(x) \leq g(x)$,
(*b*) $\sup_{x\in X} \{\xi_r(x) \dot{-} g(x)\} \leq 0$,
(*c*) $\sup_{x\in X} \left\{\hat{\xi}_r(x) \underset{\cdot}{+} (-1)\, g(x)\right\} \leq 0$,
(*d*) $\inf_{x\in X} \{g(x) \underset{\cdot}{-} \xi_r(x)\} \geq 0$,
(*e*) $\inf_{x\in X} \{g(x) \dot{+} (-1)\, \xi_r(x)\} \geq 0$.

Moreover, if $\xi = \hat{x}^*$ with $x^* \in X^*$ then (a) through (e) are equivalent to

(*f*) $\operatorname{dom} \hat{x}^*_r \supseteq \operatorname{dom} g$,

and the suprema in (b), (c) are $-\infty$ whereas the infima in (d), (e) are $+\infty$.

Proof. The equivalence of (*a*) and (*b*) follows from (13). The equivalence of (*b*) and (*c*) is immediate from (14). Likewise, (*d*) and (*e*) are equivalent because of (15). The equivalence of (*b*) and (*d*) follows from (14) and (8).

If $\xi = \hat{x}^*$ then the difference $\hat{x}^*_r(x) \dot{-} g(x)$ is $+\infty$ if and only if $x \in \operatorname{dom} g \backslash \operatorname{dom} \hat{x}^*_r$, so (*b*) and (*f*) are equivalent. $\square$

The previous theorem together with the effect described in Example 4.5 gives rise to the following definition.

Definition 5.8. The Legendre-Fenchel conjugate of $g\colon X \to \mathbb{R}^{\vartriangle}$ is the function $g^*\colon X^{\vartriangle} \times \mathbb{R} \to \mathbb{R}^{\triangledown}$ which is given by

$$g^*(\xi, r) = \sup_{x \in X} \{\xi_r(x) \dot{-} g(x)\}.$$

for $\xi \in X^{\vartriangle}$, $r \in \mathbb{R}$.

Take $x^* \in X^*$ and $r \in \mathbb{R}$. Of course,

$$g^*(x^*, r) = \sup_{x \in X} \{x^*_r(x) \dot{-} g(x)\} = \sup_{x \in X} \{x^*(x) \dot{-} g(x)\} - r = g^*(x^*) - r$$

where $g^*(x^*)$ is the classical Legendre-Fenchel conjugate of g at $x^* \in X^*$. Moreover (compare the previous theorem),

$$g^*(\hat{x}^*, r) = \begin{cases} -\infty: & \operatorname{dom} \hat{x}^*_r(\cdot) \supseteq \operatorname{dom} g, \\ +\infty: & \text{otherwise.} \end{cases} \tag{25}$$

Therefore, ξ_r with $\xi \in X^{\vartriangle}$, $r \in \mathbb{R}$ is a (proper or improper) closed affine minorant of g if and only if $g^*(\xi, r) \le 0$.

Proposition 5.9. *Let $g\colon X \to \mathbb{R}^{\vartriangle}$ be a function, $(\xi, r) \in X^{\vartriangle} \times \mathbb{R}$. The following equivalent statements are true:*

(a) *The Young-Fenchel inequality*

$$\forall x \in X\colon \quad \xi_r(x) \dot{-} g(x) \le g^*(\xi; r).$$

(b)

$$\forall x \in X\colon \quad \xi_r(x) \le g(x) \dot{+} g^*(\xi; r).$$

(c)

$$\forall x \in X\colon \quad \xi_r(x) \dot{-} g^*(\xi; r) \le g(x).$$

Proof. The first equation is immediate from the definition on the conjugate. For (b) and (c), recall $r \dot{-} s \le t$ is equivalent to $r \le s \dot{+} t$ (see Theorem 2.1(a)). This gives the equivalence of (a) with (b) and (c), respectively. □

Many of the known rules for the manipulation of conjugates apply also to g^* at improper elements. There are, however, some differences. We shall indicate one of them, a rule for conjugates of an infimal convolution which is defined for $f, g\colon X \to \mathbb{R}^{\vartriangle}$ (see [14]) as

$$(f \Box g)(x) = \inf \{f(x_1) \dot{+} g(x_2) : x_1 + x_2 = x\}.$$

Theorem 5.10. *Let $f, g\colon X \to \mathbb{R}^{\vartriangle}$ and $\xi \in X^{\vartriangle}$, $r \in \mathbb{R}$. Then*

$$(f \Box g)^*(\xi, r) = \sup \{f^*(\xi, r_1) \underset{\cdot}{+} g^*(\xi, r_2) : r_1 + r_2 = r\}. \tag{26}$$

If, in particular, $\xi = x^* \in X^*$ then

$$(f \Box g)^*(x^*, r) = f^*(x^*, 0) \underset{\cdot}{+} g^*(x^*, 0) \underset{\cdot}{+} (-r). \tag{27}$$

Proof. From the first formula in (d) of Proposition 3.6 we obtain

$$(f\Box g)^*(\xi, r) = \sup_{x,y\in X} \left(\xi_r(x+y) \dot{-} (f(x) \dot{+} g(y))\right).$$

From (19) and (7), (14) we may conclude

$$\begin{aligned}(f\Box g)^*(\xi, r) &= \sup_{x,y\in X} \left(\sup_{r_1+r_2=r} (\xi_{r_1}(x) \underset{\cdot}{+} \xi_{r_2}(y)) \underset{\cdot}{+} (-1)\,(f(x) \dot{+} g(y))\right) \\ &= \sup_{\substack{x,y\in X \\ r_1+r_2=x_0}} \Big(\big(\xi_{r_1}(x) \underset{\cdot}{+} \xi_{r_2}(y)\big) \underset{\cdot}{+} (-1)\big(f(x) \dot{+} g(y)\big)\Big)\end{aligned}$$

Using Proposition 3.3 we get

$$\begin{aligned}(f\Box g)^*(\xi, r) &= \sup_{\substack{x,y\in X \\ r_1+r_2=r}} \Big(\big(\xi_{r_1}(x) \underset{\cdot}{+} \xi_{r_2}(y)\big) \underset{\cdot}{+} \big((-1)f(x) \underset{\cdot}{+} (-1)g(y)\big)\Big) \\ &= \sup_{\substack{x,y\in X, \\ r_1+r_2=r}} \Big(\big(\xi_{r_1}(x) \underset{\cdot}{+} (-1)f(x)\big) \underset{\cdot}{+} \big(\xi_{r_2}(y) \underset{\cdot}{+} (-1)g(y)\big)\Big).\end{aligned}$$

Again (14) yields

$$(f\Box g)^*(\xi, r) = \sup_{\substack{x,y\in X, \\ r_1+r_2=r}} \left((\xi_{r_1}(x) \dot{-} f(x)) \underset{\cdot}{+} (\xi_{r_2}(y) \dot{-} g(y))\right).$$

Taking the supremum with respect to x while applying the second part of (7), and then doing the same with the supremum with respect to y we arrive at

$$\begin{aligned}(f\Box g)^*(\xi, r) &= \sup_{\substack{y\in X, \\ r_1+r_2=r}} \left(f^*(\xi, r_1) \underset{\cdot}{+} (\xi_{r_2}(y) \dot{-} g(y))\right) \\ &= \sup_{r_1+r_2=r} \left(f^*(\xi, r_1) \underset{\cdot}{+} g^*(\xi, r_2)\right).\end{aligned}$$

This completes the proof. □

Remark 5.11. Taking $r = 0$ in (27) and observing that $g^*(x^*, 0) = g^*(x^*)$ is the classical Legendre-Fenchel conjugate we arrive at the correct version of Theorem 2.3.1(ix) in [24]. See already Moreau's paper [16], Paragraph 6.h.

Remark 5.12. Surprisingly, the right hand side of (26) turns out to be the supremal convolution of the conjugates of f and g – with respect to the real variable r. One may observe once again that conjugation via the formula given in Definition 5.8 changes the image space: $\underset{\cdot}{+}$ has to be used on the right hand side of (26) whereas $\dot{+}$ appears on the left hand side.

Definition 5.13. The Legendre-Fenchel biconjugate of $g\colon X \to \mathrm{I\!R}^\vartriangle$ is the function $g^{**}\colon X \to \mathrm{I\!R}^\vartriangle$ which is given by

$$g^{**}(x) = \sup_{(\xi,r)\in X^\triangledown\times\mathrm{I\!R}} \left\{\xi_r(x) \dot{-} g^*(\xi, r)\right\}$$

for $x \in X$.

Theorem 5.14. *Let $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ be closed convex. Then $g = g^{**}$.*

Proof. Note that for $x^* \in X^*$ and $r \in \mathrm{I\!R}$ we have

$$\hat{x}^*_r(x) \dot{-} g^*(\hat{x}^*, r) = \begin{cases} +\infty: & x \notin \operatorname{dom} \hat{x}^*_r \text{ and } \operatorname{dom} g \subseteq \operatorname{dom} \hat{x}^*_r \\ -\infty: & \text{otherwise} \end{cases} \tag{28}$$

and

$$x^*_r(x) \dot{-} g^*(x^*, r) = x^*(x) \dot{-} g^*(x^*, 0). \tag{29}$$

If $\operatorname{epi} g = \emptyset$, then $g^{**}(x) = g(x) = +\infty$ for all $x \in X$. If $\operatorname{epi} g \neq \emptyset$, then g is the pointwise supremum of its affine minorants. Especially, if g is improper, then it is the pointwise supremum of its improper affine minorants, thus by (28) $g^{**}(x) = g(x)$ holds for all $x \in X$. If g is proper, then the well-known biconjugation theorem (see e.g. [24, Theorem 2.3.3]) combined with (29) delivers the desired result. □

Remark 5.15. The closed convex hull of $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ is defined by

$$\forall x \in X\colon \quad (\operatorname{cl}\operatorname{co} g)(x) = \inf\{t \in \mathrm{I\!R}\colon (x,t) \in \operatorname{cl}\operatorname{co}(\operatorname{epi} g)\}.$$

Since the conjugate of an arbitrary function coincides with the conjugate of its closed convex hull, the biconjugate of the function yields precisely the closed convex hull.

The well-known relationship between the subdifferential and the Fenchel conjugate can be extended to the improper case as follows.

Proposition 5.16. *Let $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ be a convex function and $x_0 \in \operatorname{dom} g$. Then, the following statements are equivalent for $\xi = \hat{x}^* \in \hat{X}^*$, $r = x^*(x_0)$:*

(*a*) $\xi \in \partial^{ex} g(x_0)$,
(*b*) $\operatorname{dom} \xi_r \supseteq \operatorname{dom} g$,
(*c*) $\forall x \in X: (\xi_r(x) \dot{-} g(x)) \dot{+} g(x_0) \leq \xi_r(0) = -\infty$.
(*d*) $g^*(\xi, r) = -\infty$.

Proof. It holds $\xi_{x^*(x_0)}(x) = \xi(x - x_0)$ for all $x \in X$. Moreover, $\xi_{x^*(x_0)}(x) = -\infty$ if $x^*(x) \leq x^*(x_0)$ and $\xi_{x^*(x_0)}(x) = +\infty$ otherwise. $(a) \Leftrightarrow (b)$ can be checked directly. $(a) \Leftrightarrow (d)$ is formula (25). $(d) \Rightarrow (c)$ is clear from the definition of g^*, while (c) and "not (b)" produce a contradiction. □

Proposition 5.17. *Let $g\colon X \to \mathrm{I\!R}^{\vartriangle}$ be a convex function and $x_0 \in X$. If $x_0 \notin \operatorname{dom} g$, then $\partial^{ex} g(x_0) = \{-\infty\}$. If $x_0 \in \operatorname{dom} g$, then*

$$\partial^{ex} g(x_0) = \{\xi \in X^{\vartriangle}\colon g^*(\xi, x^*(x_0)) \dot{+} g(x_0) \leq \xi(0)\}.$$

Proof. Assume $x_0 \in \operatorname{dom} g$. First, if $\xi = \hat{x}^* \in \hat{X}^*$ then $\xi(0) = -\infty$. If $\xi \in \partial^{ex} g(x_0)$ then Proposition 5.16(d) gives $g^*(\xi, x^*(x_0)) = -\infty$, hence $g^*(\xi, x^*(x_0)) \dot{+} g(x_0) \leq \xi(0)$. If ξ satisfies the latter inequality, then, by definition of g^*, also the one in Proposition 5.16(c), hence $\xi \in \partial^{ex} g(x_0)$. Secondly, let $\xi = x^* \in X^*$. Then g is proper (otherwise $\partial g(x_0) = \partial^{ex} g(x_0) \cap X^* = \emptyset$ and $g^*(\xi, x^*(x_0)) \equiv +\infty$), and the result is well-known since $g^*(\xi, x^*(x_0)) = g^*(x^*) - x^*(x_0)$ in this case. □

We close this subsection by noting that the theory in Sections 5.1, 5.2 has a symmetric counterpart for functions mapping into $\mathrm{I\!R}^{\triangledown}$ which requires modified definitions. For example, Moreau [14, p. 10], already introduced the sup-convolution for $\mathrm{I\!R}^{\triangledown}$-valued functions.

5.3. Set-valued convex and concave functions

5.3.1. Image spaces

Let Z be a topological linear space and $\mathcal{P}(Z)$ the collection of all subsets of Z including $\emptyset$. Let $C \subseteq Z$ be a convex cone including $0 \in Z$. We shall write $z_1 \leq_C z_2$ for $z_2 - z_1 \in C$, and this defines a reflexive, transitive relation. The relation $\leq_C$ can be extended in two different ways to $\mathcal{P}(Z)$, see for example [7], [8] for details and references. A basic idea in these references is to use the equivalence classes in $\mathcal{P}(Z)$ of the extension of $\leq_C$ to construct appropriate image spaces for set-valued functions. It turned out (see [8]) that the following sets are appropriate choices as image spaces for set-valued closed convex (concave) functions:

$$\mathcal{Q}_C^t(Z) = \{A \subseteq Z \colon A = \operatorname{cl}\operatorname{co}(A + C)\} \quad \text{and}$$
$$\mathcal{Q}_t^C(Z) = \{A \subseteq Z \colon A = \operatorname{cl}\operatorname{co}(A - C)\}.$$

We redefine the addition for elements A, B of $\mathcal{Q}_C^t(Z)$ and $\mathcal{Q}_t^C(Z)$ and the multiplication with $0 \in \mathrm{I\!R}$ by

$$A \oplus B = \operatorname{cl}(A + B) \tag{30}$$

and $0 \cdot A = \operatorname{cl} C$ in $\mathcal{Q}_C^t(Z)$ and $0 \cdot A = -\operatorname{cl} C$ in $\mathcal{Q}_t^C(Z)$, respectively. Then, $(\mathcal{Q}_C^t(Z), \oplus, \cdot, \supseteq)$ and $\left(\mathcal{Q}_t^C(Z), \oplus, \cdot, \subseteq\right)$ are partially ordered conlinear spaces. Again, the multiplication with -1, transforming $\mathcal{Q}_C^t(Z)$ into $\mathcal{Q}_t^C(Z)$ and vice versa, is a duality in the sense of [23]. We shall abbreviate $\mathcal{Q}^{\vartriangle} = (\mathcal{Q}_C^t(Z), \oplus, \cdot, \supseteq)$ and $\mathcal{Q}^{\triangledown} = \left(\mathcal{Q}_t^C(Z), \oplus, \cdot, \subseteq\right)$. The advantage of using these image spaces compared to other, more common approaches in vector and set optimization is that they are partially ordered lattices with formulas for inf and sup as given below. Moreover, inf and sup in these spaces are not "utopia elements", but they are strongly related to known extremality concepts based on minimal/maximal points with respect to $\leq_C$ or set relations. Compare [7], [9] and, most notably, the discussion in [10].

Proposition 5.18.

(*a*) *Let $\mathcal{A} \subseteq \mathcal{Q}^{\vartriangle}$ and $B \in \mathcal{Q}^{\vartriangle}$. Then*

$$\inf \mathcal{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$$

and

$$\inf \mathcal{A} \oplus B = \inf (\mathcal{A} \oplus \{B\}), \qquad \sup \mathcal{A} \oplus B \subseteq \sup (\mathcal{A} \oplus \{B\}).$$

(*b*) *Let $\mathcal{A} \subseteq Q^{\triangledown}$ and $B \in Q^{\triangledown}$. Then*

$$\sup \mathcal{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} A, \qquad \inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$$

and

$$\sup \mathcal{A} \oplus B = \sup\left(\mathcal{A} \oplus \{B\}\right), \qquad \inf \mathcal{A} \oplus B \subseteq \inf\left(\mathcal{A} \oplus \{B\}\right).$$

The addition for sets of sets is defined as $\mathcal{A} \oplus \{B\} = \{A \oplus B \colon A \in \mathcal{A}\}$ if $\mathcal{A}$ is non-empty and $\mathcal{A} \oplus B = \emptyset$ otherwise.

Proof. Exemplarily, we add the proof of infimum-additivity in $Q^\vartriangle$. Indeed, for $\mathcal{A} \subseteq \mathcal{Q}^\vartriangle$ and $B \in \mathcal{Q}^\vartriangle$ we have

$$\begin{aligned}\inf \mathcal{A} \oplus B &= \left(\operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} A\right) \oplus B = \left(\operatorname{co} \bigcup_{A \in \mathcal{A}} A\right) \oplus B \\ &= \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathcal{A}} (A \oplus B) = \inf\left(\mathcal{A} \oplus \{B\}\right).\end{aligned}$$

Since the roles of inf and sup are exchanged in $Q^\triangledown$, the supremum-additivity in $Q^\triangledown$ follows directly. The other results are essentially a consequence of the definitions of inf, sup and $\oplus$. $\square$

In view of the Theorems 2.1, 2.2, the previous proposition tells us that $\mathcal{Q}^\vartriangle$ and $\mathcal{Q}^\triangledown$ are order complete residuated lattices. Note that $\emptyset$ is the greatest element in $\mathcal{Q}^\vartriangle$ and the least in $\mathcal{Q}^\triangledown$. In both cases, it dominates the addition which is in complete analogy with $+\infty$ dominating the inf-addition in $\mathbb{R}^\vartriangle$ and $-\infty$ the sup-addition in $\mathbb{R}^\triangledown$. The residuation can be used to define a difference for sets.

Definition 5.19. Let $A, B \in Q^\vartriangle$. The set

$$A \dot{-} B = \inf\left\{M \in Q^\vartriangle \colon A \supseteq B \oplus M\right\} = \left\{z \in Z \colon B + \{z\} \subseteq A\right\} \tag{31}$$

is called the inf-difference of A and B. Likewise, for $A, B \in Q^\triangledown$ the set

$$A \underset{\cdot}{-} B = \sup\left\{M \in Q^\triangledown \colon B \oplus M \subseteq A\right\} = \left\{z \in Z \colon B + \{z\} \subseteq A\right\} \tag{32}$$

is called the sup-difference of A and B.

The reader may wonder why we use the same expression for the difference in $\mathcal{Q}^\vartriangle$ and $\mathcal{Q}^\triangledown$ which is not the case in $\mathbb{R}^\vartriangle$ and $\mathbb{R}^\triangledown$. Of course, the reason is that we use different order relations, namely $\supseteq$ in $\mathcal{Q}^\vartriangle$ and $\subseteq$ in $\mathcal{Q}^\triangledown$ and therefore, the infimum in $\mathcal{Q}^\vartriangle$ is a union as it is the supremum in $\mathcal{Q}^\triangledown$. We had the same effect if we would use $\leq$ in $\mathbb{R}^\vartriangle$ and $\geq$ in $\mathbb{R}^\triangledown$. However, note that $A \dot{-} B \in \mathcal{Q}^\vartriangle$ for $A, B \in \mathcal{Q}^\vartriangle$ while $A \underset{\cdot}{-} B \in \mathcal{Q}^\triangledown$ for $A, B \in \mathcal{Q}^\triangledown$, so $A \dot{-} B$ and $A \underset{\cdot}{-} B$ look very different in general.

From now on, let X and Z be separated, locally convex spaces with topological duals X^* and Z^*, respectively. The (negative) dual cone of C is the set

$$C^- = \left\{z^* \in Z^* \colon \forall z \in C \colon z^*(z) \leq 0\right\}.$$

It is well-known (and a consequence of a separation argument) that sets $A \in \mathcal{Q}^{\vartriangle}$, $B \in \mathcal{Q}^{\triangledown}$ can be described dually as

$$A = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \colon \inf_{a \in A} -z^*(a) \le -z^*(z) \right\},$$

$$B = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \colon -z^*(z) \le \sup_{b \in B} -z^*(b) \right\},$$

respectively. The above representation of elements in $\mathcal{Q}^{\vartriangle}$ and $\mathcal{Q}^{\triangledown}$ can be used to characterize the set differences from Definition 5.19 in terms of support functions. For $D \subseteq Z$, define the extended real-valued functions $\sigma^{\vartriangle}_D \colon Z^* \to \mathrm{I\!R}^{\vartriangle}$, $\sigma^{\triangledown}_D \colon Z^* \to \mathrm{I\!R}^{\triangledown}$ by

$$\sigma^{\vartriangle}_D(z^*) = \inf_{z \in D} -z^*(z), \qquad \sigma^{\triangledown}_D(z^*) = \sup_{z \in D} -z^*(z).$$

Proposition 5.20.

(a) *For all $A, B \in \mathcal{Q}^{\vartriangle}$,*

$$A \mathbin{\dot{-}} B = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \colon \sigma^{\vartriangle}_A(z^*) \mathbin{\dot{-}} \sigma^{\vartriangle}_B(z^*) \le -z^*(z) \right\}.$$

In particular, if $A = \{z \in Z \colon \sigma^{\vartriangle}_A(z^*) \le -z^*(z)\}$ for $z^* \in C^- \setminus \{0\}$, then

$$A \mathbin{\dot{-}} B = \left\{ z \in Z \colon \sigma^{\vartriangle}_A(z^*) \mathbin{\dot{-}} \sigma^{\vartriangle}_B(z^*) \le -z^*(z) \right\}.$$

(b) *For all $A, B \in \mathcal{Q}^{\triangledown}$,*

$$A \mathbin{\underset{\cdot}{-}} B = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{ z \in Z \colon -z^*(z) \le \sigma^{\triangledown}_A(z^*) \mathbin{\underset{\cdot}{-}} \sigma^{\triangledown}_B(z^*) \right\}.$$

In particular, if $A = \{z \in Z \colon -z^*(z) \le \sigma^{\triangledown}_A(z^*)\}$ for $z^* \in C^- \setminus \{0\}$, then

$$A \mathbin{\underset{\cdot}{-}} B = \left\{ z \in Z \colon -z^*(z) \le \sigma^{\triangledown}_A(z^*) \mathbin{\underset{\cdot}{-}} \sigma^{\triangledown}_B(z^*) \right\}.$$

Proof. (a) By definition of $A \mathbin{\dot{-}} B$, $\sigma^{\vartriangle}_A(z^*) \le \sigma^{\vartriangle}_{B+z}(z^*)$ whenever $z \in A \mathbin{\dot{-}} B$, and $\sigma^{\vartriangle}_{B+z}(z^*) = \sigma^{\vartriangle}_B(z^*) \mathbin{\dot{+}} z^*(-z)$ for all $z^* \in C^- \setminus \{0\}$. Thus, $\sigma^{\vartriangle}_A(z^*) \mathbin{\dot{-}} \sigma^{\vartriangle}_B(z^*) \le -z^*(z)$.

On the other hand, take $z \in Z$ such that $\sigma^{\vartriangle}_A(z^*) \mathbin{\dot{-}} \sigma^{\vartriangle}_B(z^*) \le -z^*(z)$ for all $z^* \in C^- \setminus \{0\}$ and assume $z \notin A \mathbin{\dot{-}} B$. Then there is $b \in B$ such that $z + b \notin A$. A separation argument produces $z^* \in C^- \setminus \{0\}$ and $t \in \mathrm{I\!R}$ such that $-z^*(b+z) < t \le \sigma^{\vartriangle}_A(z^*)$. Since $\sigma^{\vartriangle}_B(z^*) \le -z^*(b)$ we may conclude $-z^*(z) < \sigma^{\vartriangle}_A(z^*) \mathbin{\dot{-}} \sigma^{\vartriangle}_B(z^*)$, a contradiction.

If, additionally, $A = \{z \in Z \colon \sigma^{\vartriangle}_A(z^*) \le -z^*(z)\}$ with $z^* \in C^- \setminus \{0\}$ and $\sigma^{\vartriangle}_A(z^*) \mathbin{\dot{-}} \sigma^{\vartriangle}_B(z^*) \le -z^*(z)$ for $z \in Z$, then $\sigma^{\vartriangle}_A(z^*) \le -z^*(b+z)$ for all $b \in B$, thus $B + z \subseteq A$.

(b) is proven with parallel arguments. $\square$

From Proposition 5.20 we immediately obtain for all $A, B \subseteq \mathcal{Q}^{\vartriangle}$ and $z^* \in C^- \setminus \{0\}$

$$\sigma^{\vartriangle}_{A \dot{-} B}(z^*) \geq \sigma^{\vartriangle}_{A}(z^*) \dot{-} \sigma^{\vartriangle}_{B}(z^*)$$

with equality if $A = \{z \in Z \colon \sigma^{\vartriangle}_{A}(z^*) \leq -z^*(z)\}$. A parallel relationship holds for elements of $\mathcal{Q}^{\triangledown}$ and $\underset{\cdot}{-}$.

Remark 5.21. For $z^* \in Z^*$ we set $H(z^*) = \{z \in Z \colon z^*(z) \leq 0\}$. The additional assumption in Proposition 5.20(a) is equivalent to $A \oplus H(z^*) = A$ since

$$A \oplus H(z^*) = \{z \in Z \colon \varphi^{\vartriangle}_{z^*}(A) \leq -z^*(z)\}.$$

5.3.2. Set-valued functions and their scalar representation

The graph of a function $f \colon X \to \mathcal{P}(Z)$ is the set

$$\operatorname{gr} f = \{(x, z) \in X \times Z \colon z \in f(x)\},$$

and the effective domain of f is $\operatorname{dom} f = \{x \in X \colon f(x) \neq \emptyset\}$.

Definition 5.22. A function $g \colon X \to \mathcal{Q}^{\vartriangle}$ is called convex (positively homogenous, closed) iff $\operatorname{gr} g \subseteq X \times Z$ is convex (a cone, closed). A positively homogenous convex function into $\mathcal{Q}^{\vartriangle}$ is called sublinear. A function $h \colon X \to \mathcal{Q}^{\triangledown}$ is called concave (positively homogenous, closed) iff $\operatorname{gr} h \subseteq X \times Z$ is a convex (a cone, closed). A positively homogenous concave function into $\mathcal{Q}^{\triangledown}$ is called superlinear.

The collection of all convex functions into $\mathcal{Q}^{\vartriangle}$ ($\mathcal{Q}^{\triangledown}$) is a conlinear space under pointwise addition and multiplication with non-negative reals. As in the scalar case, it does not make much sense to consider convex functions into $\mathcal{Q}^{\triangledown}$ or concave into $\mathcal{Q}^{\vartriangle}$. A function $g \colon X \to \mathcal{Q}^{\vartriangle}$ is convex if and only if $-g \colon X \to \mathcal{Q}^{\triangledown}$ is concave.

The next goal is to represent $\mathcal{Q}^{\vartriangle}$-valued functions by families of $\mathrm{I\!R}^{\vartriangle}$-valued ones and, likewise, $\mathcal{Q}^{\triangledown}$-valued functions by families of $\mathrm{I\!R}^{\triangledown}$-valued ones. Let $z^* \in C^-$ and $g \colon X \to \mathcal{Q}^{\vartriangle}$, $h \colon X \to \mathcal{Q}^{\triangledown}$. Define $\varphi^{\vartriangle}_{g,z^*} \colon X \to \mathrm{I\!R}^{\vartriangle}$ and $\varphi^{\triangledown}_{h,z^*} \colon X \to \mathrm{I\!R}^{\triangledown}$, respectively, by

$$\varphi^{\vartriangle}_{g,z^*}(x) = \inf_{z \in g(x)} -z^*(z) \quad \text{and} \quad \varphi^{\triangledown}_{h,z^*}(x) = \sup_{z \in h(x)} -z^*(z) \tag{33}$$

for $x \in X$. Note that g is convex if and only if $\varphi^{\vartriangle}_{g,z^*}$ is convex for all $z^* \in C^-$, and h is concave if and only if $\varphi^{\triangledown}_{h,z^*}$ is concave for all $z^* \in C^-$, see [22, Lemma 3.2.3.]. If $z^* = 0$ then

$$\varphi^{\vartriangle}_{g,z^*}(x) = \begin{cases} 0 : & x \in \operatorname{dom} g \\ +\infty : & \text{otherwise.} \end{cases}$$

Moreover, $-\varphi^{\triangledown}_{-g,z^*}(x) = \varphi^{\vartriangle}_{g,z^*}(x)$ for all $x \in X$. From the dual description of elements of $\mathcal{Q}^{\vartriangle}$, $\mathcal{Q}^{\triangledown}$ the following formulas are immediate for functions $g \colon X \to \mathcal{Q}^{\vartriangle}$, $h \colon X \to \mathcal{Q}^{\triangledown}$:

$$\forall x \in X \colon \quad g(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z \colon \varphi^{\vartriangle}_{g,z^*}(x) \leq -z^*(z)\right\} \tag{34}$$

$$\forall x \in X \colon \quad h(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z \colon -z^*(z) \leq \varphi^{\triangledown}_{h,z^*}(x)\right\}. \tag{35}$$

These formulas tell us that $\mathcal{Q}^{\vartriangle}$- and $\mathcal{Q}^{\triangledown}$-valued functions can be represented by families of extended real-valued functions. In general, the scalarizations may behave "very badly".

Example 5.23. Let the set $\mathbb{R}^2$ be ordered by the cone $C = \mathbb{R}^2_+$, $z^* = (0, -1)$ and $g : \mathbb{R} \to \mathbb{R}^2$ be defined as $g(x) = \left\{(\frac{1}{x}, 0)\right\} + C$ if $x > 0$, and $g(x) = \emptyset$ otherwise. Then, $\varphi^{\vartriangle}_{g,z^*}(0) = +\infty$, while $\varphi^{\vartriangle}_{g,z^*}(x) = 0$ holds for all $x > 0$ and thus $\operatorname{cl} \varphi^{\vartriangle}_{g,z^*}(0) = 0$.

The above example shows that the scalarizations $\varphi^{\vartriangle}_{g,z^*}$ of a closed function need not be closed. However, one can restrict the scalarizations to closed ones as already shown in [22, Proposition 3.3.5].

Proposition 5.24. *Let $g\colon X \to \mathcal{Q}^{\vartriangle}$ and $h\colon X \to \mathcal{Q}^{\triangledown}$ be a closed convex and a closed concave function, respectively. Then*

$$\forall x \in X\colon \quad g\left(x\right) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z\colon \left(\operatorname{cl} \varphi^{\vartriangle}_{g,z^*}\right)(x) \le -z^*\left(z\right)\right\} \tag{36}$$

$$\forall x \in X\colon \quad h\left(x\right) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z\colon -z^*\left(z\right) \le \left(\operatorname{cl} \varphi^{\triangledown}_{h,z^*}\right)(x)\right\}. \tag{37}$$

Proof. A function $h\colon X \to Q^{\triangledown}$ is closed and concave if and only if $-h\colon X \to Q^{\vartriangle}$ is closed and convex. Thus, it is sufficient to prove the statement for convex functions. Let $g\colon X \to Q^{\vartriangle}$ be closed and convex. If $\operatorname{gr} g = \emptyset$, then there is nothing to prove. Let $(x_0, z_0) \notin \operatorname{gr} g$. Then by a separation argument in $X \times Z$ there exists $(x^*, z^*) \in (X^* \times Z^*) \setminus \{(0, 0)\}$ and $t \in \mathbb{R}$ such that

$$-x^*(x_0) - z^*(z_0) < t < \inf_{(x,z) \in \operatorname{gr} g} \left(-x^*(x) - z^*(z)\right).$$

Obviously, $z^* \in C^-$ and

$$\forall x \in X\colon \quad x^*(x) - (x^*(x_0) + z^*(z_0)) < x^*(x) + t < \inf_{z \in g(x)} \left(-z^*(z)\right) = \varphi^{\vartriangle}_{g,z^*}(x).$$

Thus, $x \mapsto x^*(x) + t$ is an affine minorant of $\operatorname{cl} \varphi^{\vartriangle}_{g,z^*}$ and

$$z_0 \notin \left\{z \in Z : \ \varphi^{\vartriangle}_{g,z^*}(x_0) \le -z^*(z)\right\}.$$

This proves $\supseteq$ in (36), and the converse inclusion is trivial. $\square$

A function $g\colon X \to \mathcal{Q}^{\vartriangle}$ is called proper (C-proper) if $\operatorname{dom} f \neq \emptyset$ and $f\left(x\right) \neq Z$ ($f\left(x\right) \neq f\left(x\right) - C$) holds for all $\in X$. A function $g\colon X \to \mathcal{Q}^{\vartriangle}$ is proper (C-proper) if and only if there exists at least one $z^* \in C^- \setminus \{0\}$ ($z^* \in C^- \setminus -C^-$) such that $\varphi^{\vartriangle}_{g,z^*} : X \to \mathbb{R}^{\vartriangle}$ is proper. However, even if g is C-proper, not all scalarizations are proper in general.

Example 5.25. Let $Z = \mathbb{R}^2$, $C = \operatorname{cl} \operatorname{cone}\{c\}$ with $c = (0, 1)^T$. Take $z^*_0 = (0, -1)^T \in \mathbb{R}^2$ and define a function $g\colon \mathbb{R} \to \mathcal{Q}^{\vartriangle}$ by

$$g(x) = \begin{cases} H(z^*_0) = \left\{z \in \mathbb{R}^2\colon \left(z^*_0\right)^T z \le 0\right\} : & x > 0 \\ C : & x = 0 \\ \emptyset : & x < 0 \end{cases}$$

Then g is convex C-proper, and $\varphi^{\vartriangle}_{g,z^*}\colon X \to \mathrm{I\!R}^{\vartriangle}$ is proper if and only if z^* is collinear with z_0^*. Moreover, the function g is not completely characterized by its proper scalarizations since $g(0) = C \subsetneq H(z_0^*)$ holds.

The previous example shows that while analyzing set-valued functions, improper scalar functions appear naturally. Providing a calculus on the space of improper functions allows a unified approach to the theory of set-valued functions via scalarization.

5.3.3. Conaffine proper and improper functions

In the remaining two sections we focus on the convex case mentioning that the corresponding constructions for the concave one are easily obtained.

Definition 5.26. Let $x^* \in X^*$, $z^* \in C^-$. The function $S^{\vartriangle}_{(x^*,z^*)}\colon X \to \mathcal{Q}^{\vartriangle}$ given by

$$S^{\vartriangle}_{(x^*,z^*)}(x) = \{z \in Z \colon x^*(x) + z^*(z) \le 0\}$$

for $x \in X$ is called an lower conlinear function.

Obviously, $S^{\triangledown}_{(x^*,z^*)}(x) = -S^{\vartriangle}_{(x^*,z^*)}(x)$ for $x \in X$ is the corresponding upper conlinear function to be used in the concave case. For each $z^* \neq 0$, the functions $S^{\vartriangle}_{(x^*,z^*)}$, $S^{\triangledown}_{(x^*,z^*)}$ are "finite-valued" in the sense that they attain neither the value Z nor $\emptyset$.

It is easy to find situations in which $S^{\vartriangle}_{(x^*,z^*)}$ is C-proper, but its scalarization with z_0^* is improper (in which case z_0^* is not collinear with z^*). Of course, the scalarization with z^* is linear if $z^* \neq 0$. If $z^* = 0$, then the scalarization of a lower conlinear function is $\hat{x}^*$.

For each $z^* \in Z^*\backslash\{0\}$, there is a one-to-one correspondence between functions $x^* \in X^*$ and $S^{\vartriangle}_{(x^*,z^*)}\colon X \to Q^{\vartriangle}$. The situation is different for $z^* = 0$, see Remark 5.28 below.

In the same way as we extended the definition of affine functions from real-valued to extended real-valued ones we extend the definition of conlinear functions.

Definition 5.27. Let $\xi \in X^{\vartriangle}$ and $r \in \mathrm{I\!R}$. We define the functions $S^{\vartriangle}_{(\xi,r,z^*)}\colon X \to Q^{\vartriangle}$ by

$$S^{\vartriangle}_{(\xi,r,z^*)}(x) = \{z \in Z \colon \xi_r(x) \le -z^*(z)\}$$

for $x \in X$. Each such function is called a (closed) conaffine function. If $r = 0$ it is called a (closed) conlinear function.

Of course, a $Q^{\vartriangle}$-valued conaffine function is proper if and only if $z^* \neq 0$ and $\xi \in X^*$.

Remark 5.28. Let $z^* \in Z^*$, $\xi = \hat{x}^* \in \hat{X}^*$ and $r \in \mathrm{I\!R}$. Then

$$\begin{aligned}\forall x \in X\colon\quad S^{\vartriangle}_{(\hat{x}^*,r,z^*)}(x) &= \{z \in Z \colon\ \hat{x}^*_r(x) \le -z^*(z)\}\\ &= \begin{cases} Z\colon & x^*_r(x) \le 0 \ (\Leftrightarrow\ x \in \operatorname{dom}\hat{x}^*_r) \\ \emptyset\colon & x^*_r(x) > 0 \ (\Leftrightarrow\ x \notin \operatorname{dom}\hat{x}^*_r)\end{cases} = S^{\vartriangle}_{(x^*,r,0)}(x).\end{aligned}$$

This means, one can replace $\hat{x}^*$ by x^* and z^* by $0 \in Z^*$, and in this way the case $z^* = 0$ "includes" all improper cases. In particular,

$$\forall x \in X\colon \quad S^{\vartriangle}_{(x^*,0)}(x) = \{z \in Z\colon x^*(x) \le 0\}$$
$$= \begin{cases} Z: & x^*(x) \le 0 \ (\Leftrightarrow \ x \in \operatorname{dom} \hat{x}^*) \\ \emptyset: & x^*(x) > 0 \ (\Leftrightarrow \ x \notin \operatorname{dom} \hat{x}^*) \end{cases}$$

Consequently, we obtain a one-to-one correspondence between the set of improper affine scalar functions $\hat{x}^*_r\colon X \to \mathrm{I\!R}^{\vartriangle}$ and the set of conaffine functions $S^{\vartriangle}_{(\hat{x}^*,r,0)}\colon X \to Q^{\vartriangle}$. Note that many of those scalar and set-valued functions coincide since $\hat{x}^* \equiv t\hat{x}^*$ for $t > 0$.

Finally, we turn to scalarizations of proper and improper conaffine functions.

Proposition 5.29. *Let $z^*, z_0^* \in Z^* \backslash \{0\}$, $x^* \in X^*$, $r \in \mathrm{I\!R}$. Then*

$$\varphi^{\vartriangle}_{\left(S^{\vartriangle}_{(x^*,r,z_0^*)},z^*\right)}(x) = \begin{cases} -\infty: & \forall t > 0\colon z^* \ne tz_0^* \\ tx^*(x - x_0): & z^* = tz_0^*, \ t > 0 \end{cases} \tag{38}$$

$$\varphi^{\vartriangle}_{S^{\vartriangle}_{(\hat{x}^*,r,0)},z^*}(x) = \varphi^{\vartriangle}_{S^{\vartriangle}_{(x^*,r,0)},z^*}(x) = \hat{x}^*_r(x). \tag{39}$$

Proof. Obvious from the definition in (33). □

5.3.4. Legendre–Fenchel conjugates of set-valued functions

Let $\xi \in X^{\vartriangle}$, $r \in \mathrm{I\!R}$, $z^* \in C^- \backslash \{0\}$. The function $S^{\vartriangle}_{(\xi,r,z^*)}\colon X \to Q^{\vartriangle}$ is a closed conaffine minorant of $g\colon X \to Q^{\vartriangle}$ if and only if

$$\forall x \in X\colon \quad S^{\vartriangle}_{(\xi,r,z^*)}(x) \supseteq g(x). \tag{40}$$

The following result runs parallel to Theorem 5.6. It should be clear how to formulate the counterpart for concave functions.

Theorem 5.30. *Let $g\colon X \to \mathcal{Q}^{\vartriangle}$ be improper closed convex. Then g is the pointwise supremum of its improper closed conaffine minorants, that is*

$$\forall x \in X\colon$$
$$g(x) = \bigcap \left\{ S^{\vartriangle}_{(\hat{x}^*,r,z^*)}(x) \colon (\hat{x}^*, r, z^*) \in \hat{X}^* \times \mathrm{I\!R} \times C^- \backslash \{0\} : (40) \textit{ is satisfied} \right\}. \tag{41}$$

In particular, if $g \not\equiv Z$, then there are $x^* \in X^* \backslash \{0\}$, $z^* \in C^- \backslash \{0\}$ and $r \in \mathrm{I\!R}$ such that

$$\forall x \in X\colon \quad S^{\vartriangle}_{(\hat{x}^*,r,z^*)}(x) \supseteq g(x). \tag{42}$$

Proof. The theorem is trivial if $g \equiv Z$ and $g \equiv \emptyset$. Let us assume that g is different from these two functions. In this case, $\operatorname{dom} g \ne \emptyset$ is a closed convex set and $g = Z$ on $\operatorname{dom} g$ (see Proposition 5 in [8]). Hence $\operatorname{dom} g$ is the intersection of non-trivial closed half spaces including it. Each such half space is generated by some $x^* \in X^* \backslash \{0\}$, $r \in \mathrm{I\!R}$, and it is easily seen with the help of Remark 5.28 that the corresponding $S^{\vartriangle}_{(\hat{x}^*,r,z^*)}$ satisfies (42) with an arbitrary $z^* \in C^- \backslash \{0\}$. □

The "proper/C-proper" version of this Theorem is Theorem 1 in [8]: $g\colon X\to Q^{\vartriangle}$ is closed, convex and proper, or identically Z or $\emptyset$, if and only if g is the pointwise supremum of its proper conaffine minorants (that is, with $z^*\in C^-\backslash\{0\}$). Therefore, a closed convex function with values in $\mathcal{Q}^{\vartriangle}$ is the pointwise supremum of its conaffine closed minorants. The next result is the set-valued counterpart of Theorem 5.7.

Theorem 5.31. *Let $g\colon X\to\mathcal{Q}^{\vartriangle}$ and $\xi\in X^{\vartriangle}$, $z^*\in C^-\backslash\{0\}$. The following statements are equivalent:*

(a) $\quad\forall x\in X\colon S^{\vartriangle}_{(\xi,r,z^*)}(x)\supseteq g(x)$, *i.e.* $S^{\vartriangle}_{(\xi,r,z^*)}$ *is a closed conaffine minorant of* g,

(b) $\quad\forall x\in X\colon \xi_r(x)\le\varphi^{\vartriangle}_{g,z^*}(x)$, *i.e.* ξ_r *is an affine minorant of* $\varphi^{\vartriangle}_{g,z^*}$,

(c) $\quad\left(\varphi^{\vartriangle}_{g,z^*}\right)^*(\xi,r)\le 0$,

(d) $\quad\sup\left\{S^{\vartriangle}_{(\xi,r,z^*)}(x)\mathbin{\dot-}g(x)\colon x\in X\right\}\supseteq H(z^*)$.

Moreover, if $\xi=\hat{x}^*$ with $x^*\in X^*$, then (a) through (d) are equivalent to

(e) $\quad\operatorname{dom}\hat{x}^*_r\supseteq\operatorname{dom}g$,

and the supremum in (d) is Z.

Proof. $(a)\Rightarrow(b)$: By assumption,

$$\forall x\in X\colon\quad \varphi^{\vartriangle}_{S^{\vartriangle}_{(\xi,r,z^*)},z^*}(x)\le\varphi^{\vartriangle}_{g,z^*}(x). \tag{43}$$

If $z^*\neq 0$ then, by definition of $S^{\vartriangle}_{(\xi,r,z^*)}$,

$$\varphi^{\vartriangle}_{S^{\vartriangle}_{(\xi,r,z^*)},z^*}(x)=\inf\{-z^*(z)\colon\xi_r(x)\le -z^*(z)\}=\xi_r(x)$$

for each $x\in X$. Now, (b) follows from (43). If $z^*=0$ then $\varphi^{\vartriangle}_{g,0}(x)=I_{\operatorname{dom}g}(x)$ and

$$\varphi^{\vartriangle}_{S^{\vartriangle}_{(\xi,r,0)},0}(x)=\begin{cases}+\infty\colon & \xi_r(x)>0\\ 0\colon & \xi_r(x)\le 0\end{cases}$$

With this, (b) is immediate from (43).

$(b)\Rightarrow(c)$: We have

$$\left(\varphi^{\vartriangle}_{g,z^*}\right)^*(\xi,r)=\sup_{x\in X}\left\{\xi_r(x)\mathbin{\dot-}\varphi^{\vartriangle}_{g,z^*}(x)\right\}\le\sup_{x\in X}\{\xi_r(x)\mathbin{\dot-}\xi_r(x)\}\le 0$$

where the first inequality follows with the help of Proposition 3.6(b), second formula, and the second inequality with the help of Proposition 3.6(a).

$(c)\Rightarrow(d)$: If $z^*\neq 0$ we obtain from Proposition 5.20(a) with $A=S^{\vartriangle}_{(\xi,r,z^*)}(x)$, $\sigma^{\vartriangle}_A(z^*)=\xi_r(x)$ and $B=g(x)$

$$S^{\vartriangle}_{(\xi,r,z^*)}(x)\mathbin{\dot-}g(x)=\left\{z\in Z\colon\xi_r(x)\mathbin{\dot-}\varphi^{\vartriangle}_{g,z^*}(x)\le -z^*(z)\right\}.$$

Consequently,

$$\begin{aligned}\bigcap_{x\in X}\left[S^{\vartriangle}_{(\xi,r,z^*)}(x)\mathbin{\dot-}g(x)\right] &= \left\{z\in Z\colon\sup_{x\in X}\left[\xi_r(x)\mathbin{\dot-}\varphi^{\vartriangle}_{g,z^*}(x)\right]\le -z^*(z)\right\}\\ &= \left\{z\in Z\colon\left(\varphi^{\vartriangle}_{g,z^*}\right)^*(\xi,r)\le -z^*(z)\right\}.\end{aligned}$$

The desired implication follows since $H(z^*) = \{z \in Z \colon 0 \leq -z^*(z)\}$.

$(d) \Rightarrow (a)$: By assumption, $H(z^*) \subseteq \left\{z \in Z \colon g(x) + \{z\} \subseteq S^{\vartriangle}_{(\xi,r,z^*)}(x)\right\}$ for all $x \in X$. Since $0 \in H(z^*)$, this implies $g(x) \subseteq S^{\vartriangle}_{(\xi,r,z^*)}(x)$ for all $x \in X$ which is (a).

Finally, if $\xi = \hat{x}^*$, the equivalence of (a) and (e) follows from Remark 5.28. □

The previous theorem motivates the following definition of set-valued conjugates for functions $g \colon X \to Q^{\vartriangle}$. In contrast to [8], we define $Q^{\vartriangle}$-valued conjugates via a supremum instead negative conjugates via an infimum. The definition below is also slightly different from the one given in [22], but uses the same basic idea, namely the set difference.

Definition 5.32. The Legendre-Fenchel conjugate $g^* \colon X^{\vartriangle} \times \mathbb{R} \times C^- \to Q^{\vartriangle}$ of a function $g \colon X \to Q^{\vartriangle}$ is the function $g^* \colon X^{\vartriangle} \times \mathbb{R} \times C^- \to Q^{\vartriangle}$ defined by

$$g^*(\xi, r, z^*) = \bigcap_{x \in X} \left(S^{\vartriangle}_{(\xi,r,z^*)}(x) \dot{-} g(x)\right).$$

By Remark 5.28 it holds

$$\forall (x^*, r, z^*) \in X^* \times \mathbb{R} \times C^- : \quad g^*(\hat{x}^*, r, z^*) = g^*(x^*, r, 0^*). \tag{44}$$

The definition of the conjugate together with Theorem 5.31 produces the following scalarization formula for conjugates

$$g^*(\xi, r, z^*) = \left\{z \in Z \colon \left(\varphi^{\vartriangle}_{g,z^*}\right)^*(\xi, r) \leq -z^*(z)\right\} \tag{45}$$

for all $\xi \in X^{\vartriangle}$, $r \in \mathbb{R}$, $z^* \in C^-$.

Most rules for manipulating conjugates carry over from the scalar case. In particular, if $r = 0$, $\xi = x^* \in X^*$ and $z^* \in C^- \backslash \{0\}$, then all classic duality results from the scalar theory can be proven for the set-valued case as well, compare [8], [22], [9].

We will close this section illustrating the previous statement using the biconjugation theorem as an example. If $g \colon X \to Q^{\vartriangle}$ is a function, its biconjugate is defined to be

$$\forall x \in X \colon \quad g^{**}(x) = \bigcap_{\xi \in X^{\vartriangle},\, r \in \mathbb{R},\, z^* \in C^-} \left(S^{\vartriangle}_{(\xi,r,z^*)}(x) \dot{-} g^*(\xi, r, z^*)\right). \tag{46}$$

The function $g^{**} \colon X \to Q^{\vartriangle}$ maps indeed into $Q^{\vartriangle}$ and is closed and convex. By equation (44), it holds

$$\forall x \in X : \quad g^{**}(x) = \bigcap_{\substack{(\xi,r) \in X^{\vartriangle} \times \mathbb{R}, \\ z^* \in C^- \backslash \{0\}}} \left(S^{\vartriangle}_{(\xi,r,z^*)}(x) \dot{-} g^*(\xi, r, z^*)\right). \tag{47}$$

Theorem 5.33. *Let $g \colon X \to Q^{\vartriangle}$ be closed and convex. Then $g^{**} = g$.*

Proof. According to Proposition 5.24 we have

$$\forall x \in X\colon \quad g(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z\colon \left(\operatorname{cl} \varphi^{\vartriangle}_{g,z^*}\right)(x) \leq -z^*(z)\right\},$$

and $\varphi^{\vartriangle}_{g,z^*}$ is convex for all $z^* \in C^- \setminus \{0\}$. Applying Theorem 5.14 to $\operatorname{cl} \varphi^{\vartriangle}_{g,z^*}$ we obtain

$$\begin{aligned} \forall x \in X\colon \quad g(x) &= \bigcap_{z^* \in C^- \setminus \{0\}} \left\{z \in Z\colon \left(\varphi^{\vartriangle}_{g,z^*}\right)^{**}(x) \leq -z^*(z)\right\} \\ &= \bigcap_{\substack{(\xi,r) \in X^{\vartriangle} \times \mathbb{R} \\ z^* \in C^- \setminus \{0\}}} \left\{z \in Z\colon \left(\xi_r(x) \dot{-} \left(\varphi^{\vartriangle}_{g,z^*}\right)^*(\xi,r)\right) \leq -z^*(z)\right\}. \end{aligned}$$

On the other hand, by formula (45) for all $x \in X$ and $(\xi, r, z^*) \in X^{\vartriangle} \times \mathbb{R} \times C^- \setminus \{0\}$ it holds

$$\begin{aligned} S^{\vartriangle}_{(\xi,r,z^*)}(x) \dot{-} g^*(\xi,r) &= \left\{z \in Z\colon S^{\vartriangle}_{(\xi,r,z^*)}(x) \supseteq g^*(\xi,r) + z\right\} \\ &= \left\{z \in Z\colon \xi_r(x) \leq \left(\varphi^{\vartriangle}_{g,z^*}\right)^*(\xi,r) \dot{+} (-z^*(z))\right\}. \end{aligned}$$

Thus, in view of formula (47) the statement is proven. □

Remark 5.34. Defining the closed convex hull $\operatorname{cl}\operatorname{co} g$ of an arbitrary function $g\colon X \to Q^{\vartriangle}$ by $\operatorname{gr}(\operatorname{cl}\operatorname{co} g) = \operatorname{cl}\operatorname{co}(\operatorname{gr} g)$ we can draw the same conclusion as in the scalar case since, as in the scalar case, the conjugate of g coincides with the one of $\operatorname{cl}\operatorname{co} g$ (see e.g. (45)), namely $g^{**} = \operatorname{cl}\operatorname{co} g$.

6. Appendix

6.1. Power sets of linear spaces

Let Z be a linear space and $\mathcal{P}(Z)$ the set of all subsets of Z including the empty set. The usual Minkowski addition of two sets $A, B \subseteq Z$

$$A + B = \{a + b\colon a \in A,\, b \in B\}$$

is extended to $\mathcal{P}(Z)$ by setting $A + B = \emptyset$ if $A = \emptyset$ or $B = \emptyset$, or both. If $B = \{z\}$ is a singleton, we abbreviate $A + B = A + \{z\} = A + z$.

The point-wise multiplication of a set $A \subseteq Z$ by a non-negative $t \in \mathbb{R}$

$$t \cdot A = \{ta\colon a \in A\}$$

is extended to $\mathcal{P}(Z)$ by setting $t \cdot \emptyset = \emptyset$ for all $t > 0$, and finally $0 \cdot \emptyset = \{0\}$ where the first 0 is in $\mathbb{R}$, the second in Z. Finally, we write $A - B$ for $A + (-B)$.

6.2. Conlinear spaces

The following definition is taken from [7] where references and more material about structural properties of conlinear spaces can be found.

Definition 6.1. A nonempty set W together with two algebraic operations $+\colon W \times W \to W$ and $\cdot\colon \mathrm{I\!R}_+ \times W \to W$ is called a conlinear space provided that

(C1) $(W, +)$ is a commutative monoid with neutral element θ,
(C2) (i) $\forall w_1, w_2 \in W$, $\forall r \in \mathrm{I\!R}_+\colon\ r \cdot (w_1 + w_2) = r \cdot w_1 + r \cdot w_2$,
(ii) $\forall w \in W$, $\forall r, s \in \mathrm{I\!R}_+\colon\ s \cdot (r \cdot w) = (rs) \cdot w$,
(iii) $\forall w \in W\colon\ 1 \cdot w = w$,
(iv) $0 \cdot \theta = \theta$.

An element $w \in W$ is called a convex element of the conlinear space W if

$$\forall s, t \geq 0\colon\ (s + t) \cdot w = s \cdot w + t \cdot w.$$

A conlinear space $(W, +, \cdot)$ together with a partial order $\preceq$ on W (a reflexive, antisymmetric, transitive relation) is called ordered conlinear space provided that

(iv) $w, w_1, w_2 \in W$, $w_1 \preceq w_2$ imply $w_1 + w \preceq w_2 + w$,
(v) $w_1, w_2 \in W$, $w_1 \preceq w_2$, $r \in \mathrm{I\!R}_+$ imply $r \cdot w_1 \preceq r \cdot w_2$.

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