Journal of Convex Analysis
Volume 19 (2012), No. 2, 575–589

A New Qualification Condition for the Maximality of the Sum of Maximal Monotone Operators in General Banach Spaces

M. Marques Alves*
Universidade Federal de Santa Catarina,
Departamento de Matemática, 88040-900 Florianópolis, Brazil
maicon@impa.br

B. F. Svaiter†
IMPA, Estrada Dona Castorina 110,
22460-320 Rio de Janeiro, Brazil
benar@impa.br, tel: 55 (21) 25295112, fax: 55 (21) 25124115

Received: November 16, 2010
Revised manuscript received: November 18, 2011

We study maximal monotonicity of the sum of maximal monotone operators in general Banach spaces. Under certain qualification condition (QC) on the domain of Fitzpatrick convex representations of the operators the sum is maximal monotone, even in nonreflexive Banach spaces.

Keywords: Maximal monotone operators, sum, nonreflexive Banach spaces, Fitzpatrick functions

2000 Mathematics Subject Classification: 47H05, 49J52, 47N10

1. Introduction

Let X be a real Banach space with dual X^* and bidual X^{**}. Whenever necessary we will identify X with its canonical injection into X^{**}. The norms in X, X^* and X^{**} will be denoted by $\|\cdot\|$ and the duality product in both $X \times X^*$ and $X^* \times X^{**}$ will be denoted by $\langle \cdot, \cdot \rangle$:

$$\langle x, x^* \rangle = x^*(x), \quad \langle x^*, x^{**} \rangle = x^{**}(x^*), \quad \forall x \in X,\, x^* \in X^*,\, x^{**} \in X^{**}.$$

A point-to-set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subset X \times X^*$$

and $T(x) = \{x^* \in X^* \mid (x, x^*) \in T\}$. The *domain* of T is the set

$$D(T) = \{x \in X \,|\, T(x) \neq \emptyset\}.$$

*Partially supported by Brazilian Ministry of Science and technology (MCT), scholarship PCI/DTI.
†Partially supported by CNPq grants 300755/2005-8, 475647/2006-8 and by PRONEX-Optimization.

ISSN 0944-6532 / $ 2.50

An operator $T : X \rightrightarrows X^*$ is *monotone* whenever

$$(x, x^*), (y, y^*) \in T \Rightarrow \langle x - y, x^* - y^* \rangle \geq 0$$

and it is *maximal monotone* if it is monotone and maximal in the family of monotone operators from X into X^*, ordered by inclusion of its graphs. Whenever $A, B \subset X$ we denote by $A + B$ the Minkowski sum of A and B.

Given two point-to-set operators $T_1, T_2 : X \rightrightarrows X^*$ the (pointwise) *sum* $T_1 + T_2 : X \rightrightarrows X^*$ is defined as

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \quad \forall x \in X$$

where, in the right hand side of the above equation we have the Minkovski sum. It is easy to verify that $T_1 + T_2$ is monotone whenever T_1 and T_2 are monotone. On the other hand, in general, maximal monotonicity is not preserved by the sum operation.

Rockafellar proposed a qualification condition (QC) on the domain of the operators to ensure maximal monotonicity of the sum in *reflexive* Banach spaces. It has been proved [14] that if X is reflexive, $T_1, T_2 : X \rightrightarrows X^*$ are maximal monotone and

$$\operatorname{int}(D(T_1)) \cap D(T_2) \neq \emptyset, \tag{1}$$

then $T_1 + T_2$ is maximal monotone.

In [2] Attouch, Riahi and Théra proved maximal monotonicity of the sum (in reflexive spaces) by replacing Rockafellar's QC (1) by the geometric and more general condition that

$$\bigcup_{\lambda > 0} \lambda \left(D(T_1) - D(T_2) \right) \tag{2}$$

is a closed linear subspace of X. Condition (2) was first introduced by Attouch and Brézis [1] in order to generalize Fenchel-Rockafellar's duality theorem. An account of several qualification conditions concerning the maximality of the sum in reflexive Banach spaces is given in [15].

In this work we will address the problem of maximal monotonicity of the sum in *general* Banach spaces. This is a topic of intense research and new results in this area were recently obtained in [17, 12, 16, 3, 7]. Our main tools are the concept of convex representation of a maximal monotone operator as well as a recent result on a sufficient condition for a convex function to represent a maximal monotone operator [10, Theorem 3.1, Corollary 3.2]. Details are given in the next section. The main results are presented in Section 6. Lemma 3.1 and Lemma 4.3 will be important results for us and we believe these are interesting results on their own.

Our notation is quite standard. We use $\operatorname{conv}(A)$ and $\operatorname{aff}(A)$ for the convex hull and the affine hull of A respectively. By $\operatorname{cl} A$ we denote the topological closure (in the strong topology) of A and $\operatorname{cl} f$ denotes the lower semicontinuous closure of $f : X \to \overline{\mathbb{R}}$. The *topological interior* of $A \subset X$ is denoted by $\operatorname{int}(A)$. Whenever $A, B \subset X$ we denote by $A + B = \{a + b \mid a \in A,\, b \in B\}$ the Minkowski sum of A and B. For any $b \in B$, we use the short notation $A + b$ for $A + \{b\}$. For $\lambda \in \mathbb{R}$

and $A \subset X$ we define $\lambda A = \{\lambda\, a \,|\, a \in A\}$. We also define $A - B = A + (-B)$ and $A - b = A - \{b\}$, where $-B = (-1)B$. By $B(a, r) = \{x \in X \,|\, \|x - a\| < r\}$ we denote the open ball of center $a \in X$ and radius $r > 0$. For $M \subset X \times X^*$ we denote by $\mathrm{Pr}_X(M)$ and $\mathrm{Pr}_{X^*}(M)$ the projections of M into X and X^*, respectively. The *indicator function* of a set A is defined as $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = \infty$ if $x \notin A$. We will use the short notation δ_a to denote the indicator function of a singleton $\{a\}$.

2. Basic results

A convex function $f : X \to \overline{\mathbb{R}}$ is *proper* whenever $f > -\infty$ and its *effective domain*

$$D(f) := \{x \in X \,|\, f(x) < \infty\}$$

is nonempty. $f : X \times X^* \to \overline{\mathbb{R}}$ is *l.s.c.* (resp. *l.s.c.* in the $s \times w^*$ topology) whenever the set

$$\mathrm{epi}(f) = \{(z, \lambda) \in (X \times X^*) \times \mathbb{R} \,|\, f(z) \le \lambda\}$$

is closed in the strong (resp. $s \times w^*$) topology of $X \times X^*$. Here s and w^* stand for strong and weak-star topology, respectively.

The *Fenchel-Legendre conjugate* of $f : X \to \overline{\mathbb{R}}$ is $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x). \tag{3}$$

Fitzpatrick proved [5] that associated to each maximal monotone operator $T : X \rightrightarrows X^*$ there exists a family $\mathcal{F}_T$ of convex, proper lower semicontinuos functions which majorizes the duality product and coincides with it on T:

$$\mathcal{F}_T = \left\{ h : X \times X^* \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ h(x, x^*) \ge \langle x, x^* \rangle, \ \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\}. \tag{4}$$

Fitzpatrick also gave an explicit formula for the smallest element of $\mathcal{F}_T$

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}}, \quad \varphi_T(x, x^*) = \sup_{(y, y^*) \in T} \langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle, \tag{5}$$

and proved that for any $h \in \mathcal{F}_T$

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle.$$

In view of the above equivalence, from now on we will call any $h \in \mathcal{F}_T$ a *convex representation* of T.

Since any maximal monotone operator is representable by a convex function, it is very natural to ask under which conditions a convex function represents a maximal monotone operator. The first work in this direction was [4], where Burachik and Svaiter determined necessary and sufficient conditions for a function to represent a maximal monotonte operator in a reflexive Banach space. This result was partially extended to non-reflexive Banach spaces by the authors in [8] and [10]. In the present work we shall use one of these extension, essentially proved in [10, Theorem 3.1, Corollary 3.2], which replaces the reflexivity assumption by a geometric condition on the domain of the function.

Theorem 2.1. *Let $h : X \times X^* \to \overline{\mathbb{R}}$ be a proper convex function satisfying*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*.$$

If

$$\bigcup_{\lambda > 0} \lambda \left(\mathrm{Pr}_X(D(h)) - x_0 \right)$$

is a closed subspace (for some $x_0 \in X$) then

$$T := \{(x, x^*) \in X \times X^* \,|\, h^*(x^*, x) = \langle x, x^* \rangle\}$$

is maximal monotone. Moreover, if h is l.s.c. in the strong×weak-$*$ topology of $X \times X^*$, then

$$T := \{(x, x^*) \in X \times X^* \,|\, h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone.

Proof. Apply [10, Theorem 3.1, Corollary 3.2] to $\tilde{h} : X \times X^* \to \overline{\mathbb{R}}$, $\tilde{h}(x, x^*) = h(x + x_0, x^*) - \langle x_0, x^* \rangle$. $\square$

Next Lemma was proved inside of [12, Lemma 3.5]. For the convenience of the reader we present a proof.

Lemma 2.2. *Let $h_1, h_2 : X \times X^* \to \overline{\mathbb{R}}$ be proper l.s.c. convex functions. Define*

$$\begin{aligned} &h : X \times X^* \to \overline{\mathbb{R}} \\ &h(x, x^*) = \left(h_1(x, \cdot) \square h_2(x, \cdot)\right)(x^*) = \inf_{y^* \in X^*} h_1(x, y^*) + h_2(x, x^* - y^*). \end{aligned} \tag{6}$$

If

$$\bigcup_{\lambda > 0} \lambda(\mathrm{Pr}_X(D(h_1)) - \mathrm{Pr}_X(D(h_2))) \tag{7}$$

is a closed subspace then

$$h^*(x^*, x^{**}) = \min_{u^* \in X^*} h_1^*(u^*, x^{**}) + h_2^*(x^* - u^*, x^{**}). \tag{8}$$

Proof. Let $(x^*, x^{**}) \in X^* \times X^{**}$. Using the definition of h we have

$$\begin{aligned} &h^*(x^*, x^{**}) \\ = &\sup_{(z, z^*) \in X \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h(z, z^*) \\ = &\sup_{(z, z^*, y^*) \in X \times X^* \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h_1(z, y^*) - h_2(z, z^* - y^*) \\ = &\sup_{(z, y^*, w^*) \in X \times X^* \times X^*} \langle z, x^* \rangle + \langle y^*, x^{**} \rangle + \langle w^*, x^{**} \rangle - h_1(z, y^*) - h_2(z, w^*), \end{aligned}$$

where we used the substitution $z^* = w^* + y^*$ in the last term. Thus, defining $H_1, H_2 : X \times X^* \times X^* \to \overline{\mathbb{R}}$,

$$H_1(x, y^*, z^*) = h_1(x, y^*), \qquad H_2(x, y^*, z^*) = h_2(x, z^*) \tag{9}$$

we have

$$h^*(x^*, x^{**}) = (H_1 + H_2)^*(x^*, x^{**}, x^{**}). \tag{10}$$

Since

$$D(H_1) - D(H_2) = \{x \in X \mid x \in Pr_X(D(h_1)) - Pr_X(D(h_2))\} \times X^* \times X^*,$$

using (7), the Attouch-Brézis's extension [1, Theorem 1.1] of Fenchel-Rockafellar duality theorem and (9) we conclude that the conjugate of the sum at the right hand side of (10) is the *exact* inf-convolution of the conjugates. Therefore,

$$h^*(x^*, x^{**}) = \min_{(u^*, y^{**}, z^{**})} H_1^*(u^*, y^{**}, z^{**}) + H_2^*(x^* - u^*, x^{**} - y^{**}, x^{**} - z^{**}).$$

Direct use of (9) shows that, for any $(x, y^*, z^*) \in X \times X^* \times X^*$,

$$H_1^*(u^*, y^{**}, z^{**}) = h_1^*(u^*, y^{**}) + \delta_0(z^{**}), \qquad H_2^*(u^*, y^{**}, z^{**}) = h_2^*(u^*, z^{**}) + \delta_0(y^{**})$$

To end the proof, combine the two above equations. □

3. Topological preliminaries

The *algebraic interior* of $A \subset X$, denoted by $\mathrm{ai}(A)$, is the set of points $a \in X$ such that

$$\begin{array}{l} \forall x \in X,\ \exists \delta > 0 \\ \ a + \lambda x \in A \quad \forall \lambda \in [0, \delta]. \end{array}$$

A set A is said to be a F_σ set if A is an enumerable union of closed sets. The next lemma is concerned with algebraic and topological interiority notions for F_σ sets.

Lemma 3.1. *Let X be a general Banach space and $A \subset X$ be a F_σ set. If* $\mathrm{ai}(A)$ *is nonempty, then* $\mathrm{int}(A)$ *is nonempty.*

Proof. Take $a \in \mathrm{ai}(A)$. Since A is a F_σ set, there exists a countable family $\{C_n\}$ of closed sets such that

$$A = \bigcup_{n \in \mathbb{N}} C_n.$$

We claim that

$$X = \bigcup_{n, m \in \mathbb{N}} m(C_n - a). \tag{11}$$

For proving this claim, take $x \in X$. Since $a \in \mathrm{ai}(A)$, there exists $m > 0$ such that

$$y := a + \frac{1}{m} x \in A.$$

Therefore, there exists n such that $y \in C_n$ and so

$$x = m(y - a) \in m(C_n - a).$$

As x is a generic point of X, we proved that

$$X \subset \bigcup_{n,m \in \mathbb{N}} m(C_n - a),$$

which readily implies our claim.

To end the proof of the lemma, use (11) and Baire's Lemma. $\square$

We define the *relative algebraic interior* of $A \subset X$, denoted by $\operatorname{rai}(A)$, as the set of points $a \in A$ such that

$$\begin{aligned} &\forall x \in \operatorname{aff}(A) - a, \ \exists \delta > 0 \\ &a + \lambda x \in A \quad \forall \lambda \in [0, \delta]. \end{aligned}$$

The *relative interior* of A, denoted by $\operatorname{ri}(A)$, is the interior of A in the relative topology of $\operatorname{cl} \operatorname{aff}(A)$, that is, $a \in \operatorname{ri}(A)$ if $a \in A$ and there exists $r > 0$,

$$B(a, r) \cap \operatorname{cl} \operatorname{aff}(A) \subset A.$$

Note that if $\operatorname{aff}(A)$ is not closed, then $\operatorname{ri}(A)$ is empty.

Next we generalize Lemma 3.1.

Corollary 3.2. *Let X be a general Banach space and $A \subset X$ be a F_σ set with $\operatorname{aff}(A)$ closed. If $\operatorname{rai}(A)$ is nonempty, then $\operatorname{ri}(A)$ is nonempty.*

Proof. Take $x_0 \in \operatorname{rai}(A)$ and define

$$V = \operatorname{aff}(A) - x_0, \qquad A_0 = A - x_0.$$

Note that V is a closed subspace. Endowing V with the (restriction of the) norm of X we get a Banach space. Moreover $A_0 \subset V$ is a F_σ set in V and has a nonempty algebraic interior (when regarded as a subset of V). Therefore, using Lemma 3.1, we conclude that A_0 has a nonempty interior in V, which is equivalent to A having a nonempty relative interior. $\square$

Proposition 3.3. *Let X be a general Banach space. If $A \subset X$ is convex, then, for any $x \in \operatorname{ri}(A)$, $y \in \operatorname{cl} A$ and $\theta \in (0, 1]$, we have*

$$\theta x + (1 - \theta) y \in \operatorname{ri}(A).$$

Proof. It suffices to prove the proposition in the case that $\operatorname{aff}(A) = X$ and $x \in \operatorname{int}(A)$. In this case, there exists $r > 0$ such that $B(x, r) \subset A$. Take $\theta \in (0, 1]$ and let

$$x_\theta = \theta x + (1 - \theta) y.$$

Since $y \in \operatorname{cl} A$, there exists $y' \in A$ such that $\|y - y'\| < \theta r$. Using the convexity of A we have

$$\theta B(x, r) + (1-\theta)y' \subset A.$$

Since

$$\theta B(x, r) + (1-\theta)y' = (\theta x + (1-\theta)y') + B(0, \theta r)$$

and

$$\|x_\theta - (\theta x + (1-\theta)y')\| = (1-\theta)\|y - y'\| < \theta r$$

we conclude that $x_\theta \in \theta B(x, r) + (1-\theta)y' \subset A$, i.e., $x_\theta \in \operatorname{int}(A)$. For the general case recall that $\operatorname{ri}(A) \neq \emptyset$ implies $\operatorname{aff}(A)$ closed and so that $\operatorname{cl} A \subset \operatorname{aff}(A)$. □

Corollary 3.4. *Let X be a general Banach space and $A \subset X$ be a F_σ set with* $\operatorname{aff}(A)$ *closed. If A is convex then* $\operatorname{rai}(A) = \operatorname{ri}(A)$.

Proof. The inclusion $\operatorname{ri}(A) \subset \operatorname{rai}(A)$ holds trivially. To complete the proof, assume that $\operatorname{rai}(A)$ is nonempty and take $z \in \operatorname{rai}(A)$. Using Corollary 3.2 we conclude that $\operatorname{ri}(A) \neq \emptyset$. Take $x \in \operatorname{ri}(A)$. Then, in particular, $x \in A$ and since $z \in \operatorname{rai}(A)$, there exists $\lambda > 0$ such that $y := z + \lambda(z - x) \in A$. Since

$$z = \frac{1}{1+\lambda} y + \frac{\lambda}{1+\lambda} x,$$

with $y \in A$ and $x \in \operatorname{ri}(A)$, using Proposition 3.3 we conclude that $z \in \operatorname{ri}(A)$. □

4. On the domain of maximal monotone operators

Recall that the domain of $T : X \rightrightarrows X^*$ is defined by

$$D(T) = \{x \in X \mid T(x) \neq \emptyset\}.$$

In the next proposition we analyze the relationship between the domain of T and the effective domain of its convex representations.

Proposition 4.1. *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then, for any $h \in \mathcal{F}_T$,*

$$D(T) \subset \operatorname{Pr}_X(D(h)) \subset \operatorname{cl} \operatorname{conv} D(T)\,.$$

Proof. Let $h \in \mathcal{F}_T$. Using (4) we conclude that $D(T) \subset \operatorname{Pr}_X(D(h)))$. To prove the second inclusion, suppose that $x_0 \notin \operatorname{cl} \operatorname{conv} D(T)$. Using the geometric version of Hahn-Banach Lemma, we conclude that there exists $x_0^* \in X^*$ and $\beta \in \mathbb{R}$ such that

$$\langle y, x_0^* \rangle < \beta < \langle x_0, x_0^* \rangle, \quad \forall y \in D(T).$$

Take $x^* \in X^*$. Since $x_0 \notin D(T)$, for any $\lambda > 0$, we have $\lambda x_0^* + x^* \notin T(x_0)$. Therefore there exists $(y, y^*) \in T$ such that

$$\langle x_0 - y, (\lambda x_0^* + x^*) - y^* \rangle < 0.$$

Using the two above inequalities and defining $\varepsilon = \langle x_0, x_0^* \rangle - \beta > 0$ we have

$$\lambda\varepsilon < \langle x_0 - y, y^* - x^* \rangle = \langle x_0, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle - \langle x_0, x^* \rangle.$$

Using the above equation and (5) we conclude that

$$\varphi_T(x_0, x^*) \geq \lambda\varepsilon + \langle x_0, x^* \rangle.$$

As $\lambda > 0$ is arbitrary and $\varphi_T \leq h$ we have $h(x_0, x^*) = \infty$. Since x^* is an arbitrary element of X^*, we conclude that $x_0 \notin \mathrm{Pr}_X(D(h))$. □

Corollary 4.2. *Let $T : X \rightrightarrows X^*$ be maximal monotone. If $\mathrm{aff}(D(T))$ is closed, then, for any $h \in \mathcal{F}_T$,*

$$\begin{aligned} \mathrm{aff}(\mathrm{Pr}_X(D(h))) &= \mathrm{aff}(D(T))\,, \\ \mathrm{rai}(D(T)) &\subset \mathrm{rai}(\mathrm{Pr}_X(D(h)))\,, \\ \mathrm{ri}(D(T)) &\subset \mathrm{ri}(\mathrm{Pr}_X(D(h))). \end{aligned}$$

Proof. Using Proposition 4.1 and the fact that $\mathrm{aff}(D(T))$ is closed we have

$$\mathrm{Pr}_X(D(h)) \subset \mathrm{cl\,conv}\, D(T) \subset \mathrm{cl\,aff}(D(T)) = \mathrm{aff}(D(T)).$$

Therefore $\mathrm{aff}(\mathrm{Pr}_X(D(h))) \subset \mathrm{aff}(D(T))$. By Proposition 4.1 $D(T) \subset \mathrm{Pr}_X(D(h))$ and so the reverse inclusion also holds true. The two inclusions follow directly from the equality just proved and the assumption of $\mathrm{aff}(D(T))$ being closed. □

To further elucidate the structure of the domain of maximal monotone operators which domain have a closed affine hull we need an auxiliary result, proved by Rockafellar in the demonstration of [13, Theorem 1]. For the sake of completeness, we will provide a proof of this result.

Lemma 4.3. *The domain of a maximal monotone operator in a real Banach space is a F_σ set, that is, the union of a countable family of closed sets.*

Proof. Let X be a real Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Define, for $\lambda \geq 0$

$$C_\lambda = \{x \in X \mid \exists\, x^* \in T(x),\ \|x^*\| \leq \lambda\}.$$

We claim that C_λ is closed for any $\lambda < \infty$. For proving this claim, suppose that

$$x_n \in C_\lambda,\ n = 1, 2, \ldots, \qquad x_n \to x \ \text{ as } n \to \infty.$$

For each n there exists $x_n^* \in T(x_n)$, $\|x_n^*\| \leq \lambda$. Using Banach-Alaoglu Theorem we conclude that the sequence $\{x_n^*\}$ has a weak-$*$ cluster point x^* such that $\|x^*\| \leq \lambda$. To prove that $x^* \in T(x)$ take an arbitrary $(y, y^*) \in T$. For any $\varepsilon > 0$ there exists n such that

$$\|x - x_n\| < \varepsilon, \qquad |\langle x - y, x_n^* - x^* \rangle| < \varepsilon.$$

Hence, using the above inequalities, the monotonicity of T and the inclusions $x_n^* \in T(x_n)$, $y^* \in T(y)$ we have

$$\begin{aligned} \langle x - y, x^* - y^* \rangle &= \langle x - y, x^* - x_n^* \rangle + \langle x - y, x_n^* - y^* \rangle \\ &\geq -\varepsilon + \langle x - y, x_n^* - y^* \rangle \\ &\geq -\varepsilon + \langle x - x_n, x_n^* - y^* \rangle + \langle x_n - y, x_n^* - y^* \rangle \\ &\geq -\varepsilon - \varepsilon\|x_n^* - y^*\| \\ &\geq -\varepsilon(1 + \lambda + \|y^*\|). \end{aligned}$$

Since the above inequality holds for any $\varepsilon > 0$, we have $\langle x - y, x^* - y^* \rangle \geq 0$, which, combined with the maximal monotonicity of T yields $x^* \in T(x)$. As $\|x^*\| \leq \lambda$ we conclude that $x \in C_\lambda$.

To end the proof of the lemma, note that $D(T) = \bigcup_{n \in \mathbb{N}} C_n$. □

In the next theorem we use previous results to analyze algebraic and topological properties of the domain of maximal monotone operators.

Theorem 4.4. *Let $T : X \rightrightarrows X^*$ be maximal monotone with* $\mathrm{aff}(D(T))$ *closed. Then*

$$\mathrm{ai}(D(T)) = \mathrm{int}(D(T)), \tag{12}$$

and both sets in (12) *are convex. Moreover, if the sets in* (12) *are nonempty, then*

$$\mathrm{cl}\, D(T) = \mathrm{cl}\, \mathrm{int}(D(T)) = \mathrm{cl}\, \mathrm{ai}(D(T)).$$

Proof. Trivially $\mathrm{int}(D(T)) \subset \mathrm{ai}(D(T))$. Thus, if $\mathrm{ai}(D(T))$ is an empty set, all statements of the theorem hold.

Suppose that

$$\mathrm{ai}(D(T)) \neq \emptyset. \tag{13}$$

Using Lemma 4.3 and Lemma 3.1 we conclude that $\mathrm{int}(D(T))$ is nonempty. Therefore, using [13, Theorem 1] we conclude that $\mathrm{int}(D(T))$ is convex and

$$\mathrm{cl}\, D(T) = \mathrm{cl}\, \mathrm{int}(D(T)). \tag{14}$$

In particular, $\mathrm{cl}\, D(T)$ is a closed convex set. Moreover, assumption (13) also implies that

$$\mathrm{aff}(\mathrm{cl}(D(T))) = \mathrm{aff}(D(T)) = X.$$

Hence, applying Corollary 3.4 to $A = \mathrm{cl}(D(T))$ we have

$$\mathrm{ai}(\mathrm{cl}\, D(T)) = \mathrm{int}(\mathrm{cl}\, D(T)).$$

Therefore, using the above identity and (14) we obtain,

$$\mathrm{ai}(D(T)) \subset \mathrm{ai}(\mathrm{cl}\, D(T)) = \mathrm{int}(\mathrm{cl}\, D(T)) = \mathrm{int}(\mathrm{cl}\, \mathrm{int}(D(T))) = \mathrm{int}(D(T))$$

where the last identity follows from the fact that $\mathrm{int}(D(T))$ is open, nonempty and convex. It proves the first statement of the theorem. The second one follows from (14) and (12). □

Corollary 4.5. *Let $T : X \rightrightarrows X^*$ be maximal monotone with* $\mathrm{aff}(D(T))$ *closed. Then*

$$\mathrm{rai}(D(T)) = \mathrm{ri}(D(T)), \tag{15}$$

and both sets in (15) *are convex. Moreover, if the sets in* (15) *are nonempty, then*

$$\mathrm{cl}\, D(T) = \mathrm{cl}\, \mathrm{ri}(D(T)) = \mathrm{cl}\, \mathrm{rai}(D(T)).$$

Proof. First assume that

$$0 \in D(T)$$

and let

$$V := \operatorname{aff}(D(T)).$$

Then V is a closed subspace. We will consider V as a linear space endowed with the natural norm, that is, the norm obtained by the restriction of the norm of X to V. Then V is a Banach space. Define

$$\tilde{T} : V \rightrightarrows V^*, \qquad \tilde{T}(v) = \{x^*|_V \mid x^* \in T(v)\}.$$

Monotonicity of $\tilde{T}$ follows trivially from the monotonicity of T. To prove that $\tilde{T}$ is maximal monotone, suppose that $\{(v_0, v_0^*)\} \cup \tilde{T}$ is monotone. Using Hahn-Banach Lemma we conclude that there exists $x_0^* \in X^*$ such that $x_0^*|_V = v_0^*$. Take an arbitrary $(x, x^*) \in T$. Then $x \in V$, $(x, x^*|_V) \in \tilde{T}$ and

$$\begin{aligned}\langle x - v_0, x^* - x_0^*\rangle &= \langle x - v_0, x^*|_V - x_0^*|_V\rangle_V \\ &= \langle x - v_0, x^*|_V - v_0^*\rangle_V \geq 0,\end{aligned}$$

where $\langle \cdot, \cdot \rangle_V$ stands for the duality product in $V \times V^*$. Hence, using the maximal monotonicity of T we conclude that $(v_0, x_0^*) \in T$ and so that

$$(v_0, x_0^*|_V) = (v_0, v_0^*) \in \tilde{T}.$$

Trivially, $D(T) = D(\tilde{T})$, $\operatorname{ri}(D(T)) = \operatorname{int}(D(\tilde{T}))$ and $\operatorname{rai}(D(T)) = \operatorname{ai}(D(\tilde{T}))$. Therefore, applying Theorem 4.4, we conclude that the conclusion of the corollary holds if $0 \in D(T)$.

For proving the general case, take $x' \in D(T)$, define $T'(x) = T(x + x')$ and use the above result for T'. $\square$

5. On the qualification conditions

In this section we discuss the relation between Attouch-Riahi-Thera's QC (2) and the QC (7) of Lemma 2.2.

Proposition 5.1. *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators, $h_1 \in \mathcal{F}_{T_1}$, $h_2 \in \mathcal{F}_{T_2}$ and*

$$V := \bigcup_{\lambda > 0} \lambda \left(D(T_1) - D(T_2) \right).$$

If V is a closed subspace, then

$$V = \bigcup_{\lambda > 0} \lambda \left(\operatorname{Pr}_X(D(h_1)) - \operatorname{Pr}_X(D(h_2)) \right).$$

In particular, if V is a closed subspace, then

$$\bigcup_{\lambda > 0} \lambda \left(\operatorname{Pr}_X(D(h_1)) - \operatorname{Pr}_X(D(h_2)) \right)$$

is a closed subspace.

Proof. Since V is a subspace which contains $D(T_1) - D(T_2)$, we have

$$\mathrm{conv}(D(T_1)) - \mathrm{conv}(D(T_2)) \subset V.$$

Using the assumption of V being closed and Proposition 4.1 we conclude that

$$\mathrm{Pr}_X(D(h_1)) - \mathrm{Pr}_X(D(h_2)) \subset V.$$

Therefore (since V is a linear subspace)

$$V = \bigcup_{\lambda>0} \lambda\,(D(T_1) - D(T_2)) \subset \bigcup_{\lambda>0} \lambda\,(\mathrm{Pr}_X(D(h_1)) - \mathrm{Pr}_X(D(h_2))) \subset V,$$

where the first inclusion follows from the inclusions $D(T_i) \subset \mathrm{Pr}_X(D(h_i))$, $i = 1, 2$. $\square$

Corollary 5.2. *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone. If*

$$\mathrm{aff}(D(T_1)), \qquad \mathrm{aff}(D(T_2)) \quad \textit{and} \quad \mathrm{aff}(D(T_1) - D(T_2))$$

are closed sets and (additionally)

$$\mathrm{rai}(D(T_1)) \neq \emptyset, \qquad \mathrm{rai}(D(T_2)) \neq \emptyset \quad \textit{and} \quad 0 \in \mathrm{rai}(D(T_1) - D(T_2))$$

then $\mathrm{ri}(D(T_1) \cap D(T_2)) \neq \emptyset$ *and* $\mathrm{aff}(D(T_1) \cap D(T_2))$ *is closed. Moreover, for any* $h_1 \in \mathcal{F}_{T_1}$*,* $h_2 \in \mathcal{F}_{T_2}$*,*

$$\mathrm{ri}(\mathrm{Pr}_X(D(h_1)) \cap \mathrm{ri}(\mathrm{Pr}_X(D(h_2)) \neq \emptyset$$

and hence

$$\mathrm{aff}(\mathrm{Pr}_X(D(h_1)) \cap \mathrm{Pr}_X(D(h_2)))$$

is closed.

Proof. Since $0 \in \mathrm{rai}(D(T_1) - D(T_2))$ we have that

$$\mathrm{aff}(D(T_1) - D(T_2)) = \mathrm{span}(D(T_1) - D(T_2)) = \bigcup_{\lambda>0} \lambda\,(D(T_1) - D(T_2)) \tag{16}$$

and there exists $z \in X$ such that

$$z \in D(T_1) \cap D(T_2). \tag{17}$$

Using Corollary 4.5 we conclude that $\mathrm{ri}(D(T_1))$ and $\mathrm{ri}(D(T_2))$ are nonempty convex sets and

$$\mathrm{ri}\, D(T_1) = \mathrm{rai}\, D(T_1), \qquad \mathrm{cl}\, D(T_1) = \mathrm{cl}\,\mathrm{ri}\, D(T_1), \tag{18}$$

$$\mathrm{ri}\, D(T_2) = \mathrm{rai}\, D(T_2), \qquad \mathrm{cl}\, D(T_2) = \mathrm{cl}\,\mathrm{ri}\, D(T_2). \tag{19}$$

In particular, $\mathrm{cl}\, D(T_1)$, $\mathrm{cl}\, D(T_2)$ are convex. Take

$$x_1 \in \mathrm{ri}(D(T_1)), \qquad x_2 \in \mathrm{ri}(D(T_2)). \tag{20}$$

By (17) $z - x_1$ and $x_2 - z$ belong to $\operatorname{span}(D(T_1) - D(T_2))$. Therefore, using (16) it follows that there exist $z_1, y_1 \in D(T_1)$, $z_2, y_2 \in D(T_2)$, $\lambda, \mu > 0$ such that

$$\lambda(z - x_1) = z_1 - z_2, \qquad \mu(x_2 - z) = y_1 - y_2.$$

Dividing the above equalities by $1+\lambda$ and $1+\mu$, respectively, we obtain, after direct algebraic manipulations.

$$\frac{\lambda}{1+\lambda}x_1 + \frac{1}{1+\lambda}z_1 = \frac{\lambda}{1+\lambda}z + \frac{1}{1+\lambda}z_2, \tag{21}$$

$$\frac{\mu}{1+\mu}x_2 + \frac{1}{1+\mu}y_2 = \frac{\mu}{1+\mu}z + \frac{1}{1+\mu}y_1. \tag{22}$$

Using the first inclusion in (20), (17), the second equality in (18) and Proposition 3.3 for the convex set $\operatorname{ri} D(T_1)$ we obtain

$$p := \frac{\lambda}{1+\lambda}x_1 + \frac{1}{1+\lambda}z_1 \in \operatorname{ri} D(T_1).$$

On the other hand, since $\operatorname{cl} D(T_2)$ is convex, and $z, z_2 \in D(T_2) \subset \operatorname{cl} D(T_2)$, using (21) we have that

$$p \in \operatorname{cl} D(T_2).$$

Thus, using the two last inclusions, we have

$$p \in \operatorname{ri}(D(T_1)) \cap \operatorname{cl} D(T_2). \tag{23}$$

By the same reasoning

$$q := \frac{\mu}{1+\mu}x_2 + \frac{1}{1+\mu}y_2 \in \operatorname{ri}(D(T_2)) \cap \operatorname{cl} D(T_1). \tag{24}$$

Using Propositon 3.3 again, the fact that $\operatorname{cl} D(T_i) = \operatorname{cl}\operatorname{ri}(D(T_i))$, $i = 1, 2$, and the inclusions in (23) and (24) we have

$$\frac{1}{2}(p+q) \in \operatorname{ri}(D(T_1)) \cap \operatorname{ri}(D(T_2)) \neq \emptyset. \tag{25}$$

Hence $\operatorname{aff}(D(T_1) \cap D(T_2)) = (\operatorname{aff} D(T_1)) \cap (\operatorname{aff} D(T_2))$ is closed and

$$(\operatorname{ri} D(T_1)) \cap (\operatorname{ri} D(T_2)) \subset \operatorname{ri}(D(T_1) \cap D(T_2)) \neq \emptyset.$$

Using Corollary 4.2 we have $\operatorname{ri}(D(T_i)) \subset \operatorname{ri}(\operatorname{Pr}_X(D(h_i)))$, $i = 1, 2$. Combining this fact with (25) we obtain $\operatorname{ri}(\operatorname{Pr}_X(D(h_1))) \cap \operatorname{ri}(\operatorname{Pr}_X(D(h_2))) \neq \emptyset$. It follows then by the same reasoning that $\operatorname{aff}(\operatorname{Pr}_X(D(h_1)) \cap \operatorname{Pr}_X(D(h_2)))$ is closed, which ends the proof. □

6. Sum results in general Banach spaces

Next is our main result. It combines qualification conditions of Theorem 2.1 and Lemma 2.2 to obtain a sum theorem in general Banach spaces.

Theorem 6.1. *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators, $h_1 \in \mathcal{F}_{T_1}$, $h_2 \in \mathcal{F}_{T_2}$ and $h := h_1(x,\cdot)\Box h_2(x,\cdot)$ defined as in* (6)*. If*

$$\bigcup_{\lambda>0} \lambda \left(\mathrm{Pr}_X(D(h_1)) - \mathrm{Pr}_X(D(h_2))\right), \tag{26}$$

$$\bigcup_{\lambda>0} \lambda \left((\mathrm{Pr}_X(D(h_1)) \cap \mathrm{Pr}_X(D(h_2))) - \{x_0\}\right) \tag{27}$$

are closed subspaces (for some x_0), then $T := T_1 + T_2$ is maximal monotone,

$$T = \{(x,x^*) \in X \times X^* \,|\, h(x,x^*) = \langle x, x^*\rangle\}$$

and $g(x,x^*) := h^*(x^*,x)$ and $\mathrm{cl}\, h$ are convex representations of T, i.e., $g, \mathrm{cl}\, h \in \mathcal{F}_T$.

Proof. First note that since $h_1 \in \mathcal{F}_{T_1}$ and $h_2 \in \mathcal{F}_{T_2}$ we have

$$h_1(x, x^* - y^*) \geq \langle x, x^* - y^*\rangle, \qquad h_2(x,y^*) \geq \langle x, y^*\rangle \tag{28}$$

and

$$h_1^*(u^*, x) \geq \langle x, u^*\rangle, \qquad h_2^*(x^* - u^*, x) \geq \langle x, x^* - u^*\rangle. \tag{29}$$

Using the above inequalities, the definition of h and Lemma 2.2 we obtain

$$h(x,x^*) \geq \langle x, x^*\rangle, \quad h^*(x^*,x) \geq \langle x, x^*\rangle, \quad \forall (x,x^*) \in X \times X^*.$$

Since

$$\mathrm{Pr}_X(D(h)) = \mathrm{Pr}_X(D(h_1)) \cap \mathrm{Pr}_X(D(h_2)),$$

using Theorem 2.1 we conclude that

$$S := \{(x,x^*) \in X \times X^* \,|\, h^*(x^*,x) = \langle x, x^*\rangle\}$$

is maximal monotone. Since g is convex and l.s.c. we have $g \in \mathcal{F}_T$. Now we will prove that $S = T$. To this end, take first $(x,x^*) \in S$. In this case, $h^*(x^*,x) = \langle x, x^*\rangle$ and using (8) we conclude that there exists $u^* \in X^*$ such that

$$h_1^*(u^*,x) + h_2^*(x^* - u^*, x) = \langle x, x^*\rangle.$$

Using (29) we have $h_1^*(u^*,x) = \langle x, u^*\rangle$ and $h_2^*(x^* - u^*, x) = \langle x, x^* - u^*\rangle$, i.e., $u^* \in T_1(x)$ and $x^* - u^* \in T_2(x)$ which proves that $(x,x^*) \in T$.

Thus $S \subset T$ and since T is monotone and S is maximal monotone we have $S = T$.

Note also that $h(x,x^*) \leq \langle x, x^*\rangle$ whenever $(x,x^*) \in T$. Since $h(x,x^*) \geq \langle x, x^*\rangle$ for any $(x,x^*) \in X \times X^*$ we have

$$T \subset \{(x,x^*) \,|\, h(x,x^*) = \langle x, x^*\rangle\} \subset \{(x,x^*) \,|\, \mathrm{cl}\, h(x,x^*) \leq \langle x, x^*\rangle\}.$$

Since the duality product is continuous in $X \times X^*$ we have $\mathrm{cl}\, h(x,x^*) \geq \langle x, x^*\rangle$ for any $(x,x^*) \in X \times X^*$. Hence, using the above inclusion we conclude that $\mathrm{cl}\, h \in \mathcal{F}_T$ which implies that the above inclusion holds as an equality. $\square$

Direct use of Theorem 6.1 and Proposition 5.1 yields the next result.

Lemma 6.2. *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone operators, $h_1 \in \mathcal{F}_{T_1}$, $h_2 \in \mathcal{F}_{T_2}$ and $h := h_1(x,\cdot)\Box h_2(x,\cdot)$ defined as in* (6)*. If*

$$\bigcup_{\lambda>0} \lambda\left(D(T_1) - D(T_2)\right), \tag{30}$$

$$\bigcup_{\lambda>0} \lambda\left((\mathrm{Pr}_X(D(h_1)) \cap \mathrm{Pr}_X(D(h_2))) - \{x_0\}\right) \tag{31}$$

are closed subspaces (for some x_0), then $T := T_1 + T_2$ is maximal monotone,

$$T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^*\rangle\}$$

and $g(x, x^*) := h^*(x^*, x)$ and $\mathrm{cl}\, h$ are convex representations of T, i.e., $g, \mathrm{cl}\, h \in \mathcal{F}_T$.

Note that Lemma 6.2 combined with Corollary 5.2 provides an alternative proof of Corollary 4 in [16].

Acknowledgements. We thanks the anonymous referee for the positive criticism and corrections of the original version of this work.

References

[1] H. Attouch, H. Brézis: Duality for the sum of convex functions in general Banach spaces, in: Aspects of Mathematics and its Applications, J. A. Barroso (ed.), North-Holland Math. Library 24, North-Holland, Amsterdam (1986) 125–133.

[2] H. Attouch, H. Riahi, M. Théra: Somme ponctuelle d'opérateurs maximaux monotones, Serdica Math. J. 22(3) (1996) 165–190.

[3] H. H. Bauschke, X. Wang, L. Yao: On the maximal monotonicity of the sum of a maximal monotone linear relation and the subdifferential operator of a sublinear function, arXiv:1001.0257v1 (2010).

[4] R. S. Burachik, B. F. Svaiter: Maximal monotonicity, conjugation and the duality product, Proc. Amer. Math. Soc. 131(8) (2003) 2379–2383.

[5] S. Fitzpatrick: Representing monotone operators by convex functions, in: Functional Analysis and Optimization, Workshop / Miniconference (Canberra, 1988), Proc. Cent. Math. Anal. Aust. Natl. Univ. 20, Australian National University, Canberra (1988) 59–65.

[6] J.-P. Gossez: Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs, J. Math. Anal. Appl. 34 (1971) 371–395.

[7] L. Yao: The sum of a maximal monotone operator of type (FPV) and a maximal monotone operator with full domain is maximal monotone, Nonlinear Anal. 17 (2011) 6144–6152.

[8] M. Marques Alves, B. F. Svaiter: Brønsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces, J. Convex Analysis 15(4) (2008) 693–706.

[9] M. Marques Alves, B. F. Svaiter: A new old class of maximal monotone operators, J. Convex Analysis 16(3&4) (2009) 881–890.

[10] M. Marques Alves, B. F. Svaiter: Maximal monotonicity, conjugation and the duality product in non-reflexive Banach spaces, J. Convex Analysis 17(2) (2010) 553–563.

[11] M. Marques Alves, B. F. Svaiter: On Gossez type (D) maximal monotone operators, J. Convex Analysis 17(3&4) (2010) 1077–1088.

[12] M. Marques Alves, B. F. Svaiter: On the surjectivity properties of perturbations of maximal monotone operators in non-reflexive Banach spaces, J. Convex Analysis 18(1) (2011) 209–226.

[13] R. T. Rockafellar: Local boundedness of nonlinear, monotone operators, Mich. Math. J. 16 (1969) 397–407.

[14] R. T. Rockafellar: On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75–88.

[15] S. Simons: Pairs of monotone operators, Trans. Amer. Math. Soc. 350(7) (1998) 2973–2980.

[16] M. D. Voisei, C. Zălinescu: Maximal monotonicity criteria for the composition and the sum under weak interiority conditions, Math. Program., Ser. B 123 (2010) 265–283.

[17] M. D. Voisei, C. Zălinescu: Strongly-representable monotone operators, J. Convex Analysis 16(3&4) (2009) 1011–1033.