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A Maximal Monotone Operator of Type (D) for which Maximal Monotone Extension to the Bidual is Not of Type (D)

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We define a family of linear type (D) operators for which the inverse of their maximal monotone extensions to the bidual are not of type (D) and provide an example of an operator in this family.

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1. Introduction

Let U, V arbitrary sets. A *point-to-set* (or multivalued) operator $T : U \rightrightarrows V$ is a map $T : U \to \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of V. Given $T : U \rightrightarrows V$, the *graph* of T is the set

$$\mathrm{Gr}(T) := \{(u, v) \in U \times V \mid v \in T(u)\},$$

the *domain* and the *range* of T are, respectively,

$$\mathrm{dom}(T) := \{u \in U \mid T(u) \neq \emptyset\}, \qquad \mathrm{R}(T) := \{v \in V \mid \exists u \in U,\ v \in T(u)\},$$

and the *inverse* of T is the point-to-set operator $T^{-1} : V \rightrightarrows U$,

$$T^{-1}(v) = \{u \in U \mid v \in T(u)\}.$$

A point-to-set operator $T : U \rightrightarrows V$ is called *point-to-point* if for every $u \in \mathrm{dom}(T)$, $T(u)$ has only one element and in this case we use the notation $T : U \to V$ (which *does not* mean that $\mathrm{dom}(T) = U$). Trivially, a point-to-point operator is injective if, and only if, its inverse is also point-to-point.

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Let X be a real Banach space. We use the notation X^* for the topological dual of X. We will identify X with its canonical injection into $X^{**} = (X^*)^*$ and we will use the notation $\langle \cdot, \cdot \rangle$ for the duality product,

$$\langle x, x^* \rangle = \langle x^*, x \rangle = x^*(x), \quad x \in X, \ x^* \in X^*.$$

A point-to-set operator $T : X \rightrightarrows X^*$ (respectively $T : X^{**} \rightrightarrows X^*$) is *monotone*, if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \mathrm{Gr}(T),$$

(resp. $\langle x^* - y^*, x^{**} - y^{**} \rangle \geq 0$, $\forall (x^{**}, x^*), (y^{**}, y^*) \in \mathrm{Gr}(T)$), and it is *maximal monotone* if it is monotone and maximal in the family of monotone operators in $X \times X^*$ (resp. $X^{**} \times X^*$) with respect to the order of inclusion of the graphs.

Let X be a *non-reflexive* real Banach space and $T : X \rightrightarrows X^*$ be maximal monotone. Since $X \subset X^{**}$, the point-to-set operator T can also be regarded as an operator from X^{**} to X^*. We denote $\widehat{T} : X^{**} \rightrightarrows X^*$ as the operator such that

$$\mathrm{Gr}(\widehat{T}) = \mathrm{Gr}(T).$$

If $T : X \rightrightarrows X^*$ is maximal monotone then $\widehat{T}$ is (still) trivially monotone but, in general, not maximal monotone. Direct use of the Zorn's Lemma shows that $\widehat{T}$ has a maximal monotone extension. So it is natural to ask if such maximal monotone extension to the bidual is unique. Gossez [1, 2, 3, 4] gave a sufficient condition for uniqueness of such an extension.

Definition 1.1 ([1]). Gossez's *monotone closure* (with respect to $X^{**} \times X^*$) of a maximal monotone operator $T : X \rightrightarrows X^*$, is the point-to-set operator $\widetilde{T} : X^{**} \rightrightarrows X^*$ whose graph $\mathrm{Gr}\left(\widetilde{T}\right)$ is given by

$$\mathrm{Gr}\left(\widetilde{T}\right) = \{(x^{**}, x^*) \in X^{**} \times X^* \mid \langle x^* - y^*, x^{**} - y \rangle \geq 0, \ \forall (y, y^*) \in T\}.$$

A maximal monotone operator $T : X \rightrightarrows X^*$, is of *Gossez type (D)* if for any $(x^{**}, x^*) \in \mathrm{Gr}\left(\widetilde{T}\right)$, there exists a bounded net $\left((x_i, x_i^*)\right)_{i \in}$ in $\mathrm{Gr}(T)$ which converges to (x^{**}, x^*) in the $\sigma(X^{**}, X^*)\times$strong topology of $X^{**} \times X^*$.

Gossez proved [4] that a maximal monotone operator $T : X \rightrightarrows X^*$ of type (D) has a unique maximal monotone extension to the bidual, namely, its Gossez's monotone closure $\widetilde{T} : X^{**} \rightrightarrows X^*$. Beside this fact, maximal monotone operators of type (D) share many properties with maximal monotone operators defined in *reflexive* Banach spaces, as for example, convexity of the closure of the domain and convexity of the closure of the range [1].

If $f : X \to [-\infty, \infty]$ is a proper convex and lower semi-continuous function, then $\partial f : X \rightrightarrows X^*$ is of type (D) and the unique maximal monotone extension of ∂f to $X^{**} \times X^*$ is $(\partial f^*)^{-1}$, which is the inverse of a new type (D) operator, namely $\partial f^* : X^* \rightrightarrows X^{**}$. It is natural to ask if this "hereditary" property of the subdifferential is shared by other maximal monotone operators of type (D). Explicitly, if $T : X \rightrightarrows X^*$ is of type (D), is $(\widetilde{T}\,)^{-1} : X^* \rightrightarrows X^{**}$ also of type (D)? The answer to this question is negative in general. We will provide an example of a type (D) operator $T : X \rightrightarrows X^*$ for which the inverse of its (unique) maximal monotone extension to $X^{**} \times X^*$ is *not* of type (D).

We will need the following characterization of maximal monotone operators of type (D), provided in [6, eq. (5) and Theorem 4.4, item 2].

Theorem 1.2. *A maximal monotone operator $T : X \rightrightarrows X^*$ is of type (D) if and only if*

$$\sup_{(y,y^*)\in T} \langle y, x^*\rangle + \langle y^*, x^{**}\rangle - \langle y^*, y\rangle \geq \langle x^*, x^{**}\rangle$$

for any $(x^*, x^{**}) \in X^* \times X^{**}$.

2. A special family of linear type (D) operators

In this section we will characterize a family of maximal monotone operators of type (D) for which the inverse of its maximal monotone extensions to the bidual are not of type (D).

Theorem 2.1. *Let $A : X^* \to X$ be linear, monotone, injective and everywhere defined, and let*

$$T = A^{-1} : R(A) \subset X \to X^*.$$

If there exists $x_0^{**} \in X^{**} \setminus X$ such that

$$\sup_{x^*\in X^*} \langle x^*, x_0^{**}\rangle - \langle A(x^*), x^*\rangle < \infty \tag{1}$$

then

1. *T is maximal monotone of type (D).*
2. *$\widetilde{T} = \widehat{T}$.*
3. *$(\widetilde{T}\,)^{-1} : X^* \rightrightarrows X^{**}$ is not of type (D) on $X^* \times X^{**}$.*

Proof. Since A is injective, linear and monotone, the operator $T = A^{-1} : R(A) \to X^*$ is point-to-point, linear and monotone. Moreover

$$\mathrm{R}(T) = \mathrm{dom}(A) = X^*.$$

Therefore, using Theorem 6.7 of [5] we conclude that

a) T is maximal monotone of type (D) in $X \times X^*$,

b) $\widehat{T}$ is maximal monotone in $X^{**} \times X^*$,

which proves item *1* and *2*.

To prove item *3*, let

$$\beta = \sup_{x^*\in X^*} \langle x^*, x_0^{**}\rangle - \langle A(x^*), x^*\rangle < \infty.$$

As X is a closed subspace of X^{**}, and $x_0^{**} \in X^{**} \setminus X$, there exists $x_0^{***} : X^{**} \to \mathbb{R}$, such that

$$x_0^{***} \in X^{***}, \qquad x_0^{***}(x) = 0 \;\; \forall x \in X, \;\; \langle x_0^{**}, x_0^{***}\rangle > \beta.$$

Since $\widetilde{T} = \widehat{T}$, using the definition of $\widehat{T}$ we have

$$(\widetilde{T}\,)^{-1}(x^*) = (\widehat{T}\,)^{-1}(x^*) = T^{-1}(x^*) = A(x^*) \in X,$$

for any $x^* \in X^*$. Therefore $(\,\widehat{T}\,)^{-1}$ is point-to-point and

$$\begin{aligned}
&\sup_{(x^*,x^{**})\in(\widetilde{T})^{-1}} \langle x^*, x_0^{**}\rangle + \langle x^{**}, x_0^{***}\rangle - \langle x^*, x^{**}\rangle \\
&= \sup_{x^*\in X^*} \langle x^*, x_0^{**}\rangle + \langle A(x^*), x_0^{***}\rangle - \langle A(x^*), x^*\rangle \\
&= \sup_{x^*\in X^*} \langle x^*, x_0^{**}\rangle - \langle A(x^*), x^*\rangle \\
&= \beta < \langle x_0^{**}, x_0^{***}\rangle.
\end{aligned}$$

To end the proof, use the above inequality and Theorem 1.2 to conclude that $(\,\widetilde{T}\,)^{-1}$ is not of type (D). □

3. The family defined in Theorem 2.1 is non-empty

In this section we will provide an explicit example of an operator in the family described in Theorem 2.1. Form now on, $\ell^1 = \ell^1(\mathbb{N})$ and $\ell^\infty = \ell^\infty(\mathbb{N})$. The family of elements of ℓ^∞ which converges to 0 will be denoted by c_0,

$$c_0 := \left\{ x \in \ell^\infty \;\middle|\; \lim_{i\to\infty} x_i = 0 \right\}. \tag{2}$$

We will use the canonical identifications

$$(c_0)^* = \ell^1, \qquad (\ell^1)^* = \ell^\infty.$$

Gossez defined in [2] the following operator

$$G : \ell^1 \to \ell^\infty, \qquad (Gx)_n := \sum_{k>n} x_k - \sum_{k<n} x_k, \tag{3}$$

which is linear, continuous, anti-symmetric and maximal monotone. Let

$$e := (1, 1, 1, \dots) \in \ell^\infty. \tag{4}$$

Lemma 3.1. *For any $x \in \ell^1$, $G(x) + \langle x, e\rangle e \in c_0$.*

Proof. Direct use of definitions (3) and (4) shows that for any $x \in \ell^1$,

$$\lim_{n\to\infty} (Gx)_n = -\langle x, e\rangle,$$

which trivially implies that $G(x) + \langle x, e\rangle e \in c_0$. □

Proposition 3.2. *Let $X = c_0$. The operator*

$$A : \ell^1 \to c_0, \quad x \mapsto G(x) + \langle x, e \rangle e \tag{5}$$

satisfies the assumptions of Theorem 2.1*. Hence A^{-1} is a maximal monotone operator of type (D) and the inverse of its unique maximal monotone extension to the bidual is not of type (D).*

Proof. By Lemma 3.1, A is everywhere defined. Linearity of A follows from (3) and (4). Since Gossez's operator G is anti-symmetric, for any $x \in \ell^1$

$$\langle A(x), x \rangle = \langle G(x) + \langle x, e \rangle e, x \rangle = \langle \langle x, e \rangle e, x \rangle = \langle e, x \rangle^2 \geq 0, \tag{6}$$

which shows that A is monotone.

To prove that A is injective, it suffices to show that its kernel is trivial. Suppose that

$$x \in \ell^1, \quad A(x) = 0.$$

Using (6) we conclude that $\langle x, e \rangle^2 = \langle A(x), x \rangle = 0$. Therefore, $\langle e, x \rangle = 0$, which combined with (5) yields $G(x) = A(x) = 0$. Using (3) we have

$$0 = (Gx)_n - (Gx)_{n+1} = x_n + x_{n+1}, \quad n = 1, 2, \dots$$

Hence $x_n = (-1)^{n+1} x_1$. Since $x \in \ell^1$, we conclude that $x_n = 0$ for all n.

We claim that

$$\sup_{x \in \ell^1} \langle x, e \rangle - \langle A(x), x \rangle = \frac{1}{4} < \infty. \tag{7}$$

Indeed, since $t - t^2 \leq \frac{1}{4}$, for any $t \in \mathbb{R}$, using (6) we have

$$\langle x, e \rangle - \langle A(x), x \rangle = \langle x, e \rangle - \langle x, e \rangle^2 \leq \frac{1}{4},$$

for any $x \in \ell^1$, with equality at $x = (1/2, 0, 0, \dots)$. To end the proof that A satisfies the assumptions of Theorem 2.1, note that

$$e \in \ell^\infty \setminus c_0 = (c_0)^{**} \setminus c_0.$$

which, combined with (7) shows that (1) is satisfied with $x_0^{**} = e$.

The second part of the propositions follows from the first part and Theorem 2.1. □

For A as in (5), the operator $T : c_0 \times \mathbb{R} \rightrightarrows \ell^1 \times \mathbb{R}$ whose graph is

$$\mathrm{Gr}(T) = \{((A(x^*), t), (x^*, t^3)) \mid x^* \in \ell^1, t \in \mathbb{R}\}$$

is an example of a non-linear type (D) operator for which the inverse of its maximal monotone to the bidual is not of type (D).

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