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Stability in Regularized Quasi-Variational Settings

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First, new upper stability results are obtained for parametric quasi-variational and linearized quasi-variational problems using an extension of the classical Minty lemma. Then, we show that lower stability results cannot be achieved in general, even on very restrictive conditions, so we introduce approximate solutions for the above problems and we investigate their lower and upper stability properties.

Keywords: Quasi-variational, pseudomonotone, approximate solution, closed map, lower semicontinuous map

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1. Introduction

The term "quasi-variational" identifies a class of variational problems having constraint sets depending on their own solutions and including several problems, among which

- Variational Inequality (VI), [5],
- Complementarity Problem (CP), [6],
- Implicit Variational Problem (IVP), [37],
- Quasi-Variational Inequality (QVI), [5],
- Generalized Variational Inequality (GVI), [11],
- Generalized Quasi-variational Inequality ($GQVI$), [18],
- Equilibrium Problem (EP), [7],
- Social (or Generalized) Nash Equilibrium Problem ($SNEP$), [13],
- Mixed Quasivariational-like Inequality (MQI), [9].

All these theoretical problems play an important role in concrete engineering or economic problems such as electric power market modelization [19], optimal shape

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design [17], topology optimization in structural mechanics [14], traffic equilibrium [12], transportation network congestion [8], financial derivative models [16], social and economic networks modelization [38],...
A more general formulation, considered in [37], [5], [23], [25], is the following one. Given a real Banach space U with dual U^*, let K be a nonempty closed and convex subset of U, let f be a real-valued function defined in $U \times U$ and S be a set-valued map from K to K with nonempty values.
Then, the quasi-variational problem (QVP) (called in [39] and in [1] *quasi-equilibrium* problem) looks for the solution set $\mathcal{Q}$ defined by

$$u \in \mathcal{Q} \quad \Longleftrightarrow \quad u \in S(u) \quad \text{and} \quad f(u,w) \leq 0 \ \ \forall\, w \in S(u).$$

Each one of the above problems can be described considering an appropriate function and/or a set-valued map:

- (VI) consider $S(u) = K$ and $f(u,w) = \langle Au, u - w\rangle$ where $A : U \to U^*$ is an operator,
- (CP) consider $S(u) = C$, where C is a convex, closed cone with apex in the origin 0 and $f(u,w) = \langle Au, u - w\rangle$, where $A : U \to U^*$ is an operator,
- (QVI) consider $f(u,w) = \langle Au, u - w\rangle$, where $A : U \to U^*$ is an operator,
- (IVP) consider $S(u) = K$ and $f(u,w) = g(u,w) + \phi(u,u) \mathbin{\dot{-}} \phi(u,w)$ where $g : U \times U \to \mathbb{R}$, $\phi : U \times U \to \mathbb{R} \cup \{+\infty\}$ and $(+\infty) \mathbin{\dot{-}} (+\infty) = -\infty$,
- (GVI) consider $S(u) = K$ and $f(u,w) = \min_{u^* \in T(u)} \langle u^*, u - w\rangle$ where T is a set-valued operator from U to U^*,
- $(GQVI)$ consider $f(u,w) = \min_{u^* \in T(u)} \langle u^*, u - w\rangle$ where T is a set-valued operator from U to U^*,
- (EP) consider $S(u) = K$,
- $(SNEP)$ consider $f(u,w) = J_1(u_1,u_2) + J_2(u_1,u_2) - J_1(u_1,w_2) - J_2(w_1,u_2)$ where J_1 and J_2 are functions from $Y_1 \times Y_2$ to $\mathbb{R}$ and Y_1 and Y_2 are respectively nonempty subsets of E_1 and E_2, real normed vector spaces, $S(u) = S(u_1,u_2) = Q_1(u_2) \times Q_2(u_1)$ where Q_1 and Q_2 are set-valued functions from Y_2 to Y_1 and from Y_1 to Y_2 respectively,
- $(MQVLI)$ consider $f(u,w) = \min_{u^* \in T(u)} \langle u^*, \eta(u,w)\rangle + h(u) - h(w)$ where T is a set-valued operator from U to U^*, $\eta : K \times K \to K$ and $h : K \to \mathbb{R}$ are functions.

Classically, in order to avoid very restrictive assumptions in the investigation of variational inequalities in infinite dimensional spaces, the following *linearized* variational inequality (also called *Minty variational inequality*) is considered [32]:

$$(LVI) \quad \text{find } u \in K \text{ such that } \langle Av, u - v\rangle \leq 0 \ \ \forall\, v \in K$$

The equivalence between the problems (VI) and (LVI) is provided by the Minty Lemma [32] which represents the prototype to obtain analogous results for most of the problems listed before (see Section 2). Therefore, concerning quasi-variational problems (QVP), it looks natural to introduce the linearized quasi-variational problem

$$(LQVP) \quad \text{find } u \in S(u) \text{ such that } f(w,u) \geq 0 \ \ \forall\, w \in S(u)$$

whose solution set is denoted by $\mathcal{LQ}$.
A quasi-variational (respectively linearized quasi-variational) problem depending on a parameter t is denoted by

$$(QVP)(t) \quad \text{find } u \in S(t,u) \text{ such that } f(t,u,w) \leq 0 \ \ \forall\, w \in S(t,u)$$

$$(\text{respectively } (LQVP)(t) \quad \text{find } u \in S(t,u) \text{ such that } f(t,w,u) \geq 0 \ \ \forall\, w \in S(t,u))$$

and we aim to investigate the *stability* properties of the solution sets $\mathcal{Q}(t)$ and $\mathcal{LQ}(t)$ for t belonging to a topological space (T,τ).
In this paper, we are interested in the sequential *upper* and/or *lower* stability of $\mathcal{Q}(t)$ (resp. of $\mathcal{LQ}(t)$), meaning respectively that for every sequence $(t_n)_n$ converging to t in T one has

$$\mathcal{Q}(t) \subseteq \liminf_{n\to+\infty} \mathcal{Q}(t_n) \quad \text{and/or} \quad \limsup_{n\to+\infty} \mathcal{Q}(t_n) \subseteq \mathcal{Q}(t),$$

$$(\text{resp. } \mathcal{LQ}(t) \subseteq \liminf_{n\to+\infty} \mathcal{LQ}(t_n) \quad \text{and/or} \quad \limsup_{n\to+\infty} \mathcal{LQ}(t_n) \subseteq \mathcal{LQ}(t),$$

where the lim inf and lim sup denote the lower and the upper limit in the sense of Painlevé-Kuratowski [4] of a family of sets, whose definitions will be recalled in Section 2.
To our knowledge, the first upper stability results for quasi-variational problems have been established by the authors in [23] when T is the set of positive integers $\mathbb{N}$. For other upper stability results on quasi-variational problems see, among others, [10], [35], [36], [1], [2]. In Section 2 we will show that upper stability results for the solution maps $\mathcal{Q}$ and $\mathcal{LQ}$ can be obtained under mild assumptions on the data while the lower stability of $\mathcal{Q}$ and $\mathcal{LQ}$ may not be achieved in general, even in very restrictive conditions. This lack of stability of the exact solutions motivates to introduce approximate solutions that can be simultaneously lower and upper stable. It is worth mentioning that the lower stability property plays a fundamental role in the investigation of hierarchical problems. Indeed, some examples show that the optimal solutions to perturbed bilevel problems, as well as the optimal values, may not be stable (see Example 4.1 in [24], Example 2.3 in [26]). Therefore, regularized models have been investigated when the lower level is described by an Optimization problem [28], [29], [24], by generalized saddle point equilibria [33], generalized Nash equilibria [34] or Nash equilibria in mixed strategies [31], considering approximate solutions to the lower level problem which satisfy the lower stability property. This approach has been proved to be fruitful, for instance, when applied to a class of bilevel optimization problems arising in structural optimization [14]. Then, in this paper we aim to investigate approximate solutions for quasi-variational problems (QVP) that turn out to be lower stable. In literature, for all of the problems listed at the beginning, several concepts of approximate solutions have been defined with different motivations and purposes. In particular we mention the papers by Lucchetti-Patrone [30], Revalski [40], Lignola-Morgan [22] and [27], related to approximate solutions for variational inequalities, Morgan-Raucci [34], related to approximate social Nash equilibria, and the papers by Lignola [20] and Ceng-Hadjisavvas-Schaible-Yao [9] concerning approximate solutions for quasi-variational inequalities and mixed quasi-variational-like inequalities. Inspired by these papers,

we define in Section 3 two concepts of approximate solutions for (QVP)'s and two concepts for $(LQVP)$'s investigating, for each of them, upper and lower stability properties. Previously, in Section 2, after preliminaries and notations, an overview of the behavior of the exact solutions sets is given.

2. Basic notations and background

The investigation of problems of variational or quasi-variational nature in infinite dimensional spaces needs some continuity and monotonicity properties, [4] and [18], in order to avoid very restrictive assumptions. So, we recall here the notions for bivariate functions and for set-valued maps that will be used throughout the paper. We denote by w and s, respectively, the weak and the strong convergence on a normed space U; by $\text{int}\, H$ the strong interior of a set H; by $G(F)$ the graph of a set-valued map $F : U \to V$, where V is a topological space, i.e. the set $\{(y,v) \in U \times V \ : \ v \in F(y)\}$ and, given a positive number r, by $B(H,r)$ the closed ball around H, i.e. the set $\{u \in E \ : \ d(u,H) \leq r\}$.

A function $f : U \times U \to \mathbb{R}$ is said to be: *monotone* if $f(u,w) + f(w,u) \geq 0$, *pseudomonotone* if $f(u,w) \leq 0$ implies $f(w,u) \geq 0$, *coercive* if every net $(u_\alpha, v_\alpha)_\alpha$, such that $f(u_\alpha, v_\alpha) \leq k$ for every α, has a convergent subnet.

A set-valued map $F : (X,\tau) \to (Y,\sigma)$, where (X,τ) and (Y,σ) are topological spaces, is said to be (τ,σ)-*lower semicontinuous* at $x_o \in X$ if for every $y_o \in F(x_o)$ and every neighborhhood I of y_o there exists a neighborhood Q of x_o such that $F(x) \cap I \neq \emptyset$ for all $x \in Q$; F is said to be (τ,σ)-*closed* at x_o if for every $y \notin F(x_o)$ there exist a neighborhood I of y_o and a neighborhood Q of x_o such that $F(x) \cap I = \emptyset$ for all $x \in Q$; F is said to be (τ,σ)-*subcontinuous* at x_o if given a net $(x_\alpha)_{\alpha \in \mathcal{A}}$ converging to x_o, every net $(y_\alpha)_{\alpha \in \mathcal{A}}$ with $y_\alpha \in F(x_\alpha)$ has a convergent subnet; F is said to be (τ,σ)-lower semicontinuous (respectively closed or subcontinuous) over a set $H \subseteq X$ if it is (τ,σ)-lower semicontinuous (respectively closed or subcontinuous) at x for every $x \in H$. If τ and σ are first countable then the above properties can be caracterized as follows: F is (τ,σ)-lower semicontinuous at x_o iff for every sequence $(x_n)_n$ τ-converging to x_o in X and every $y_o \in F(x_o)$ there exists a sequence $(y_n)_n$ σ-converging to y_o in Y such that $y_n \in F(x_n)$ for sufficiently large $n \in \mathbb{N}$; F is (τ,σ)-closed at x_o iff for every $y_o \notin F(x_o)$ there exist a sequence $(x_n)_n$ τ-converging to x_o in X and a sequence $(y_n)_n$ σ-converging to y_o in Y such that $y_n \notin F(x_n)$ for sufficiently large $n \in \mathbb{N}$; F is (τ,σ)-subcontinuous at x_o iff, given a sequence $(x_n)_n$ τ-converging to x_o in X, every sequence $(y_n)_n$ such that $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ has a σ-convergent subsequence.

Let $(H_n)_n$ be a sequence of subsets of Y. The Painlevé-Kuratowski upper and lower limit of the sequence $(H_n)_n$ are defined as follows.

- $z \in \sigma\text{-}\limsup_n H_n$ if there exists a sequence $(z_k)_k$ σ-converging to z in Y such that $z_k \in H_{n_k}$, for a subsequence (H_{n_k}) of $(H_n)_n$ and for each $k \in \mathbb{N}$;
- $z \in \sigma\text{-}\liminf_n H_n$ if there exists a sequence $(z_n)_n$ σ-converging to z in Y and such that $z_n \in H_n$ for n sufficiently large.

During the whole paper, we will assume that the set K is nonempty, closed and convex and that the following assumptions are satisfied

$$(\Xi) \qquad f(u,u) = 0 \quad \forall\, u \in K,$$

and, for parametric problems,

$$(\Xi_t) \qquad f(t,u,u) = 0 \quad \forall\, t \in T \text{ and } \forall\, u \in K.$$

3. Stability of exact solutions

We start this section extending to quasi-variational problems the classical Minty lemma [32].

Lemma 3.1. *If f is pseudomonotone on K, then every solution u_o to the quasi-variational problem (QVI) is also a solution to the problem*

$$(LQVP) \quad \text{find } u \in S(u) \text{ such that } f(w,u) \geq 0 \quad \forall\, w \in S(u).$$

If $f(\cdot, w)$ is lower semicontinuous on the segments of K for every $w \in K$, $f(u, \cdot)$ is concave on K for every $u \in K$ and S is convex-valued and closed-valued, then every solution u_o to the linearized quasi-variational problem $(LQVP)$ is also a solution to the problem (QVP).

Proof. The proof of the first part is straightforward, so it is omitted.
Let $u_o \in S(u_o)$ such that

$$f(w, u_o) \geq 0 \quad \forall\, w \in S(u_o)$$

and let $w_o \in S(u_o)$ such that $w_o \neq u_o$. For every $\lambda \in [0,1]$ consider $u_\lambda = \lambda u_o + (1 - \lambda) w_o$. Since $f(\cdot, w_o)$ is lower semicontinuous on the segments of K one has

$$f(u_o, w_o) \leq \liminf_{\lambda \to 1} f(u_\lambda, w_o),$$

so, in order to prove that $f(u_o, w_o) \leq 0$, it is sufficient to prove that $f(u_\lambda, w_o) \leq 0$ for every $\lambda \in]0,1[$. This inequality follows from the concavity of f in the second variable and observing that $\lambda f(u_\lambda, u_o) \geq 0$:

$$\begin{aligned} f(u_\lambda, w_o) &\leq f(u_\lambda, w_o) + \lambda f(u_\lambda, u_o) + \lambda f(u_\lambda, w_o) - \lambda f(u_\lambda, w_o) \\ &\leq f(u_\lambda, u_\lambda) + \lambda f(u_\lambda, w_o) = \lambda f(u_\lambda, w_o). \end{aligned}$$

The above lemma can be suitably used to get analogous statements for generalized quasi-variational or variational inequalites, for implicit variational problems and for equilibrium problems.
The next two results, that can be proved using standard arguments, concern the topological properties of the solution set-valued maps $\mathcal{Q}: \ t \in T \to \mathcal{Q}(t) \subseteq U$ and $\mathcal{LQ}: \ t \in T \to \mathcal{LQ}(t) \subseteq U$.

Proposition 3.2. *Let $t \in T$.*
If $f(t, \cdot, \cdot)$ is lower semicontinuous on $K \times K$and $S(t, \cdot)$ is lower semicontinuous and closed-valued, then the set $\mathcal{Q}(t)$ is closed.
If f is lower semicontinuous on $T \times K \times K$ and S is closed and lower semicontinuous on $T \times K$, then the set-valued map $\mathcal{Q}$ is closed.
If $f(\cdot, \cdot, w)$ is coercive on $T \times K$, for every $w \in K$, then the set-valued map $\mathcal{Q}$ is subcontinuous.

Proposition 3.3. *Let $t \in T$.*
If $f(t,\cdot,\cdot)$ is upper semicontinuous on $K \times K$ and $S(t,\cdot)$ is lower semicontinuous and closed-valued, then the set $\mathcal{LQ}(t)$ is closed.
If f is upper semicontinuous on $T \times K \times K$ and S is closed and lower semicontinuous on $T \times K$, then the set-valued map $\mathcal{LQ}$ is closed.
If $-f(\cdot,\cdot,w)$ is coercive on $T \times K$, for every $w \in K$, then the set-valued map $\mathcal{LQ}$ is subcontinuous.

Unfortunately, both propositions contain a semicontinuity assumption on the function f at the couple (u,w), that could be a very restrictive assumption in the case where $f(u,w) = \langle Au, u-w\rangle$. For instance, if U is an infinite dimensional Hilbert space and $\langle\cdot,\cdot\rangle$ denotes the scalar product in U, it is known that the function $f(u,w) = \langle u, u-w\rangle$ is not weakly upper semicontinuous on the unitary ball.
Therefore, results avoiding a so restrictive assumption would be desirable. To this end, we recall the following lemma [21] concerning lower convergent sequences of convex sets having nonempty interior.

Lemma 3.4 ([21], Lemma 3.1). *Let $(H_n)_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of nonempty subsets of a Banach space E such that:*

i) H_n *is convex for every* $n \in N$*;*
ii) $H_o \subseteq \operatorname{Liminf}_n H_n$;
iii) *there exists* $m \in N$ *such that* $\operatorname{int} \bigcap_{n\geq m} H_n \neq \emptyset$.

Then, for every $u \in \operatorname{int} H_o$ there exists a positive real number δ such that

$$B(u,\delta) \subseteq H_n \quad \forall\, n \geq m.$$

If E is a finite dimensional space, then assumption iii) can be substituted by:
iii′) $\operatorname{int} H_o \neq \emptyset$.

Now, we present closedness results for the solution maps $\mathcal{Q}$ and $\mathcal{LQ}$ under "nicer" assumptions.

Proposition 3.5. *Assume that the following assumptions hold:*

i) *S is convex-valued, $(\tau \times s, s)$-lower semicontinuous and $(\tau \times s, s)$-closed on $T \times K$;*
ii) *$f(t,u,\cdot)$ is concave on K for every $t \in T$ and $u \in K$;*
iii) *$f(t,\cdot,w)$ is lower semicontinuous on the segments of K for every $t \in T$ and $w \in K$;*
iv) *for every $(t,u,w) \in T \times K \times K$, for every sequence $(t_n,u_n,w_n)_n$ such that $(t_n)_n$ τ-converges to t, $(u_n,w_n)_n$ $(s \times s)$-converges to (u,w) one has*

$$-f(t,w,u) \leq \liminf_n f(t_n,u_n,w_n).$$

Then, the set-valued map $\mathcal{Q}$ is (τ,s)-closed.

Proof. Let $(t_n)_n$ and $(u_n)_n$ be sequences converging to t_o and u_o, respectively in T

and in K, such that for every $n \in \mathbb{N}$ $u_n \in \mathcal{Q}(t_n)$, that is

$$u_n \in S(t_n, u_n) \quad \text{and} \quad f(t_n, u_n, w) \leq 0 \ \ \forall\, w \in S(t_n, u_n).$$

Since the closedness of S implies that $u_o \in S(t_o, u_o)$, in order to prove that $u_o \in \mathcal{Q}(t_o)$ it suffices only to prove that for every $w \in S(t_o, u_o)$ one has $f(t_o, u_o, w) \leq 0$. Being S lower semicontinuous, given $w \in S(t_o, u_o)$, there exists a sequence $(w_n)_n$ converging to w such that $w_n \in S(t_n, u_n)$ for n sufficiently large and, by condition *iv*), one gets $-f(t_o, w, u_o) \leq 0$.
Therefore $u_o \in \mathcal{LQ}(t_o)$ and the proof can be completed adapting the proof of Lemma 3.1 to parametric quasi-variational problems.

Proposition 3.6. *Assume that the following assumptions hold:*

- *i*) *S is convex-valued, $(\tau \times s, s)$-lower semicontinuous and $(\tau \times s, s)$-closed on $T \times K$;*
- *ii*) *$f(\cdot, u, \cdot)$ is upper semicontinuous on $T \times K$ for every $u \in K$;*
- *iii*) *$f(t, \cdot, w)$ is upper semicontinuous on the segments of K for every $t \in T$ and $w \in K$;*
- *iv*) *for every sequence $(t_n, u_n)_n$, $t_n \in T$ and $u_n \in K$ for all $n \in \mathbb{N}$, such that $(t_n)_n$ τ-converges in T and $(u_n)_n$ s-converges in K, there exists $m \in \mathbb{N}$ such that*

$$\operatorname{int} \bigcap_{n \geq m} S(t_n, u_n) \neq \emptyset.$$

Then, the set-valued map $\mathcal{LQ}$ is (τ, s)-closed.
If U is a finite dimensional space, then assumption iv) can be substituted by:

iv′) *for every $t \in T$ and $u \in K$, $\operatorname{int} S(t, u) \neq \emptyset$.*

Proof. Let $(t_n)_n$ and $(u_n)_n$ be sequences converging to t_o and u_o, respectively in T and in K, such that for every $n \in \mathbb{N}$

$$u_n \in S(t_n, u_n) \quad \text{and} \quad -f(t_n, w, u_n) \leq 0 \ \ \forall\, w \in S(t_n, u_n).$$

Since the closedness of S implies that $u_o \in S(t_o, u_o)$, in order to prove that $u_o \in \mathcal{LQ}(t_o)$ it suffices only to prove that for every $w \in S(t_o, u_o)$ one has $-f(t_o, w, u_o) \leq 0$.
Given $w \in \operatorname{int} S(t_o, u_o)$, the lower semicontinuity of S, assumption *iv*) and Lemma 3.4 imply that $w \in \operatorname{int} S(t_n, u_n)$ for n sufficiently large and, by condition *ii*), one gets $-f(t_o, w, u_o) \leq 0$.
When $w \in S(t_o, u_o) \setminus \operatorname{int} S(t_o, u_o)$, being $S(t_o, u_o)$ a convex set, there exists a sequence $(w_n)_n$ converging to w along a segment such that $w_n \in \operatorname{int} S(t_o, u_o)$.
Therefore $-f(t_o, w_n, u_o) \leq 0$ and assumption *iii*) implies that $u_o \in \mathcal{LQ}(t_o)$. $\square$

Remark 3.7. Propositions 3.5 and 3.6 also provide a (τ, w)-closedness result for the solution maps $\mathcal{Q}$ and $\mathcal{LQ}$ if in *iv*) the weak convergence of the sequence $(u_n)_n$ is required instead of the strong convergence and in *i*) the set-valued map S is assumed to be convex-valued, $(\tau \times w, s)$-lower semicontinuous and $(\tau \times w, w)$-closed on $T \times K$.

Remark 3.8. The assumptions of Proposition 3.5 imply that $\mathcal{Q}(t)=\mathcal{LQ}(t)$ for every $t \in T$, since Lemma 3.1 holds for the function $f(t,\cdot,\cdot)$. Thus, Proposition 3.5 gives also a $\tau \times s$-closedness result for the map $\mathcal{LQ}$ that is not comparable with Proposition 3.6 in which assumption $iv)$ on the constraint map S (that is not present in Proposition 3.5) plays an essential role.

As announced in the Introduction, the maps $\mathcal{LQ}$ and $\mathcal{Q}$ may fail to be lower semicontinuous even in presence of very regular data.

Example 3.9. Let $T=[-1,1]$, $U=R$, $S(t,u)=K=[-1,1]$, $f(t,u,w)=t(u-w)$, for each $t \in T$. With such data, the problem $(QVP)(t)$ consists of finding $u \in [-1,1]$ such that $t(u-w) \leq 0$ for any $w \in [-1,1]$. So, the solutions map $\mathcal{Q}$ is

$$\mathcal{Q}(t)=\begin{cases}\{1\} & \text{if } t<0\\ [-1,1] & \text{if } t=0\\ \{-1\} & \text{if } t>0\end{cases}$$

which is not lower semicontinuous at $t=0$. More precisely, there exist a sequence $(t_n)_n$ converging to 0 and an element $u \in \mathcal{Q}(0)$, for example $u=0$, such that every sequence $(u_n)_n$, $u_n \in \mathcal{Q}(t_n)$ for n large, does not converge to u. Observing that $-f(t,w,u)=f(t,u,w)$, one gets $\mathcal{Q}(t)=\mathcal{LQ}(t)$ for every $t \in T$, so also the set-valued map $\mathcal{LQ}$ is not lower semicontinuous at $t=0$.

This lack of lower semicontinuity leads us to introduce suitable concepts of approximate solution maps for quasi-variational problems.

4. Approximate solutions and upper stability

Given a positive real number r, consider the set-valued maps defined on the parameters set T by

$$\begin{aligned}\mathcal{Q}_r(t) &= \{u \in K \ : \ u \in B(S(t,u),r) \text{ and } f(t,u,w) \leq r \ \forall\, w \in S(t,u)\},\\ \mathcal{T}_r(t) &= \{u \in K \ : \ u \in B(S(t,u),r)\\ &\qquad \text{and } f(t,u,w) \leq r\|u-w\| \ \forall\, w \in B(S(t,u),r)\},\\ \mathcal{LQ}_r(t) &= \{u \in K \ : \ u \in B(S(t,u),r) \text{ and } -f(t,w,u) \leq r \ \forall\, w \in S(t,u)\},\\ \mathcal{LT}_r(t) &= \{u \in K \ : \ u \in B(S(t,u),r)\\ &\qquad \text{and } -f(t,w,u) \leq r\|u-w\| \ \forall\, w \in B(S(t,u),r)\},\end{aligned}$$

for every $t \in T$.
Arguing as in Proposition 3.2 (respectively Proposition 3.3), one proves that the maps $\mathcal{Q}_r$ and $\mathcal{T}_r$ (respectively $\mathcal{LQ}_r$ and $\mathcal{LT}_r$) are "upper stable".

Proposition 4.1. *If f is lower semicontinuous on $T \times K \times K$ and S is closed and lower semicontinuous on $T \times K$, then the set-valued maps $\mathcal{Q}_r$ and $\mathcal{T}_r$ are closed.*
If f is upper semicontinuous on $T \times K \times K$ and S is closed and lower semicontinuous on $T \times K$, then the set-valued maps $\mathcal{LQ}_r$ and $\mathcal{LT}_r$ are closed.

However, a result in line with Proposition 3.5 cannot be expected for the map $\mathcal{Q}_r$ since, in general, a Minty Lemma type does not hold for $\mathcal{Q}_r$ and $\mathcal{L}_r$.

Example 4.2. Consider $U = \mathbb{R}$, $S(t,u) = K = [0,1]$ and $f(u,w) = u(u-w)$. Then one easily cheks that $\mathcal{Q}_r = [0,\sqrt{r}] \subset \mathcal{LQ}_r = [0,2\sqrt{r}]$.

Instead, the maps $\mathcal{T}_r$ and $\mathcal{LT}_r$ satisfy the next Minty type lemma.

Lemma 4.3. *Consider $t \in T$. If $f(t,\cdot,\cdot)$ is monotone on $K \times K$, then*

$$\mathcal{Q}_r(t) \subseteq \mathcal{LQ}_r(t) \quad \textit{and} \quad \mathcal{T}_r(t) \subseteq \mathcal{LT}_r(t).$$

If $f(t,\cdot,w)$ is lower semicontinuous on the segments of K for every $w \in K$, $f(t,u,\cdot)$ is concave on K for every $u \in K$ and $S(t,\cdot)$ is convex-valued and closed-valued, then

$$\mathcal{LT}_r(t) \subseteq \mathcal{T}_r(t).$$

Proof. The proof of the first part is straightforward. In order to prove the second part, let $u \in K$ be such that $d(S(t,u),u) \leq r$ and

$$-f(t,w,u) \leq r||u-w|| \quad \forall\, w \in B(S(t,u),r).$$

Let $w' \neq u$ be a point belonging to $B(S(t,u),r)$ and, for every $\lambda \in [0,1]$, consider the point $u_\lambda = \lambda u + (1-\lambda)w'$. Since $f(t,\cdot,w')$ is lower semicontinuous on the segments of K one has

$$f(t,u,w') \leq \liminf_{\lambda \to 1} f(t,u_\lambda,w'),$$

so it is sufficient to prove that $f(t,u_\lambda,w') \leq r||u-w'||$ for every $\lambda \in]0,1[$; this inequality follows from the concavity of f in the last variable and observing that $f(t,u_\lambda,u_\lambda) = 0$ and $\lambda f(t,u_\lambda,u) + \lambda r||u-u_\lambda|| \geq 0$, since $u_\lambda \in B(S(t,u),r)$. Indeed one has

$$\begin{aligned} f(t,u_\lambda,w') &\leq f(t,u_\lambda,w') + \lambda f(t,u_\lambda,w') - \lambda f(t,u_\lambda,w') + \lambda f(t,u_\lambda,u) + \lambda r||u_\lambda - u|| \\ &\leq f(t,u_\lambda,u_\lambda) + \lambda f(t,u_\lambda,w') + \lambda(1-\lambda) r||u-w'|| \\ &= \lambda f(t,u_\lambda,w') + \lambda(1-\lambda) r||u-w'||. \end{aligned}$$

Now, we give upper stability results for the maps $\mathcal{T}_r$, $\mathcal{LT}_r$ and $\mathcal{LQ}_r$ under "nicer" assumptions.

Proposition 4.4. *Assume that the following assumptions hold:*

i) *S is convex-valued, $(\tau \times s, s)$-lower semicontinuous, $(\tau \times s, s)$-closed and $(\tau \times s, s)$-subcontinuous on $T \times K$;*

ii) *$f(t,u,\cdot)$ is concave on K for every $t \in T$ and $u \in K$;*

iii) *$f(t,\cdot,w)$ is lower semicontinuous on the segments of K for every $t \in T$ and $w \in K$;*

iv) *for every $(t,u,w) \in T \times K \times K$, for every sequence $(t_n,u_n,w_n)_n$ such that $(t_n)_n$ τ-converges to t, $(u_n,w_n)_n$ $(s \times s)$-converges to (u,w) one has*

$$-f(t,w,u) \leq \liminf_n f(t_n,u_n,w_n).$$

Then, the set-valued map $\mathcal{T}_r$ is (τ, s)-closed.

Proof. Let $(t_n)_n$ and $(u_n)_n$ be sequences converging, respectively in T and in K, to t_o and u_o, such that for every $n \in \mathbb{N}$

$$d(u_n, S(t_n, u_n)) \leq r \quad \text{and} \quad f(t_n, u_n, w) \leq r||u_n - w|| \;\; \forall\, w \in B(S(t_n, u_n), r). \quad (1)$$

The closedness and the subcontinuity of S imply that $u_o \in B(S(t_o, u_o), r)$.
Indeed, if we assume that $d(u_o, S(t_o, u_o)) > a > r \geq d(u_n, S(t_n, u_n))$ for every $n \in \mathbb{N}$, there exists a sequence $(v_n)_n$ such that $v_n \in S(t_n, u_n)$ and $||u_n - v_n|| < a$ for every $n \in \mathbb{N}$. A subsequence $(v_{n_k})_k$ must strongly converge to a point $v_o \in S(t_o, u_o)$ and this leads to a contradiction.
Therefore, in order to prove that $u_o \in \mathcal{T}_r(t_o)$ it suffices only to prove that for every $w \in B(S(t_o, u_o), r)$ one has $f(t_o, u_o, w) \leq r||u_o - w||$.
We claim that for every $w \in B(S(t_o, u_o), r)$ there exists a sequence $(w_n)_n$ converging to w such that $w_n \in B(S(t_n, u_n), r)$ for n sufficiently large.
Indeed, if $d(w, S(t_o, u_o)) < r$, there exists $z \in S(t_o, u_o)$ such that $||w - z|| < r$. Since S is lower semicontinuous, there exists a sequence $(z_n)_n$ converging to z such that $z_n \in S(t_n, u_n)$ for n large and the sequence defined by $w_n = w + z_n - z$ satisfies the required conditions.
If $d(w, S(t_o, u_o)) = r$, then w may be approximated by a sequence (v_n) wholly contained in $\operatorname{int} B(S(t_o, u_o), r)$. Then, for every $n \in \mathbb{N}$ there exists a sequence $(\tilde{v}_k^n)_k$ such that

$$\lim_k \tilde{v}_k^n = v_n \quad \text{and} \quad d(\tilde{v}_k^n, S(t_k, u_k)) < r \;\; \forall\, k \in \mathbb{N}.$$

Applying a diagonalization argument (see Corollary 1.18 in [3]), there exists an increasing sequence $(k(n))_n$ such that the sequence $(w_n)_n$, defined by $w_n = \tilde{v}_{k(n)}^n$, converges to w and $w_n \in B(S(t_{k(n)}, u_{k(n)}), r)$ for n large, so conditions (1) and *iv)* imply that $-f(t_o, w, u_o) \leq r||u_o - w_o||$.
Therefore $u_o \in \mathcal{LT}_r(t_o)$ and the result follows from Lemma 4.3.

Proposition 4.5. *Assume that the following assumptions hold:*

i) S *is convex-valued,* $(\tau \times s, s)$*-lower semicontinuous,* $(\tau \times s, s)$*-closed and* $(\tau \times s, s)$*-subcontinuous on* $T \times K$*;*

ii) $f(\cdot, u, \cdot)$ *is upper semicontinuous on* $T \times K$ *for every* $u \in K$*;*

iii) $f(t, \cdot, w)$ *is upper semicontinuous on the segments of* K *for every* $t \in T$ *and* $w \in K$*;*

iv) *for every sequence* $(t_n, u_n)_n$, $t_n \in T$ *and* $u_n \in K$ *for all* $n \in \mathbb{N}$*, such that* $(t_n)_n$ τ*-converges in* T *and* $(u_n)_n$ s*-converges in* K*, there exists* $m \in \mathbb{N}$ *such that*

$$\operatorname{int} \bigcap_{n \geq m} S(t_n, u_n) \neq \emptyset.$$

Then, the set-valued map $\mathcal{LQ}_r$ *and* $\mathcal{LT}_r$ *are* (τ, s)*-closed.*
If U *is a finite dimensional space, then assumption iv) can be substituted by:*

iv′) *for every* $t \in T$ *and* $u \in K$*,* $\operatorname{int} S(t, u) \neq \emptyset$*.*

Proof. Let $(t_n)_n$ and $(u_n)_n$ be sequences converging, respectively in T and in K to t_o and u_o, such that for every $n \in \mathbb{N}$

$$u_n \in B(S(t_n, u_n), r) \quad \text{and} \quad -f(t_n, w, u_n) \leq r||u_n - w|| \;\; \forall\, w \in B(S(t_n, u_n), r).$$

As in Proposition 4.4 one can prove that $u_o \in B(S(t_o, u_o), r)$. Then, in order to show that $u_o \in \mathcal{LT}_r(t_o)$ it suffices to prove that for every $w \in B(S(t_o, u_o), r)$ one has $-f(t_o, w, u_o) \leq r||u_o - w||$.
If $w \in \operatorname{int} B(S(t_o, u_o), r)$ one has that $w \in \operatorname{int} B(S(t_n, u_n), r)$ for n sufficiently large since the lower semicontinuity of S allows to apply Lemma 3.4 taking $H_n = B(S(t_n, u_n), r)$ for $n \in \mathbb{N}$ and $H_o = B(S(t_o, u_o), r)$. Therefore, for such indexes n, $-f(t_n, w, u_n) \leq r||u_n - w||$ and condition *ii*) implies that $-f(t_o, w, u_o) \leq r||u_o - w||$.
If $w \in B(S(t_o, u_o), r) \setminus \operatorname{int} B(S(t_o, u_o), r)$, there exists a sequence $(w_n)_n$ converging to w such that $w_n \in \operatorname{int} B(S(t_o, u_o), r)$ for every $n \in \mathbb{N}$ and one gets $-f(t_o, w_n, u_o) \leq r||u_o - w_n||$. Hence, assumption *iii*) implies that $u_o \in \mathcal{LT}_r(t_o)$. The proof for the set-valued map $\mathcal{LQ}_r$ is similar and is omitted.

As observed in Remark 3.7, results concerning the $(\tau \times w)$-closedness of the set-valued maps $\mathcal{LQ}_r$, $\mathcal{LT}_r$ and $\mathcal{T}_r$ can be also achieved.

5. Lower stability of approximate solutions

In this section we investigate the lower stability of the approximate solution maps and we start proving a lower semicontinuity result for a kind of approximate fixed points.

Proposition 5.1. *Assume that U is a reflexive Banach space and the following assumptions hold:*

i) *the set-valued map S is closed-valued, convex-valued and $(\tau \times s, s)$-lower semicontinuous on $T \times K$;*

ii) *for every sequence $(t_n, u_n)_n$, $t_n \in T$ and $u_n \in K$ for all $n \in \mathbb{N}$, such that $(t_n)_n$ τ-converges in T and $(u_n)_n$ s-converges in K, there exists $m \in \mathbb{N}$ such that*

$$\operatorname{int} \bigcap_{n \geq m} S(t_n, u_n) \neq \emptyset.$$

Then, for every $t \in T$, for every sequence $(t_n)_n$ τ-converging to t, every $u \in K$ such that $u \in B(S(t, u), r)$, there exists a sequence $(u_n)_n$ strongly converging to u such that $u_n \in \operatorname{int} B(S(t_n, u_n), r)$ for n sufficiently large and the set-valued map

$$\mathcal{F}_r \;:\; t \in T \to \{u \in K : d(u, S(t, u)) \leq r\}$$

is lower semicontinuous.
If U is a finite dimensional space, then assumption ii) can be substituted by:

ii') *for every $t \in T$ and $u \in K$, $\operatorname{int} S(t, u) \neq \emptyset$.*

Proof. We start considering $u \in \operatorname{int} B(S(t, u), r)$, i.e. $d(u, S(t, u)) < r$.
Let $z \in S(t, u)$ be such that $||z - u|| < r$. There exists a sequence $(z_n)_n$ converging

to z such that $z_n \in S(t_n, u)$; since $||z_n - u|| < r$ for n sufficiently large, one can put $u_n = u$ for every $n \in \mathbb{N}$.
Now, we assume that $d(u, S(t,u)) = r$. Since $S(t,u)$ is convex and U is a reflexive Banach space, there exists $z \in S(t,u)$ such that $||u - z|| = r$.
Assume that $z \in \operatorname{int} S(t,u)$ and let $(\lambda_n)_n$ be a sequence of nonnegative real numbers in $]0,1]$ converging to 0. The sequence obtained setting $u_n = \lambda_n z + (1-\lambda_n)u$ converges to u, $||u_n - z|| = (1-\lambda_n)||u-z|| < r$ for every $n \in \mathbb{N}$ and $z \in \operatorname{int} S(t_n, u_n)$ for n large in light of condition *ii)* and Lemma 3.4.
Finally, assume that $z \in S(t,u) \setminus \operatorname{int} S(t,u)$. Let $(z_n)_n$ be a sequence strongly converging to z wholly contained in $\operatorname{int} S(t,u)$. Then, for every $n \in \mathbb{N}$ there exists a sequence $(\widetilde{u}^{\,n}_k)_k$ strongly converging to z_n and such that $d(\widetilde{u}^{\,n}_k, S(t_k, \widetilde{u}^{\,n}_k)) < r$ for every $k \in \mathbb{N}$. Applying a diagonalization argument (see [3], Corollary 1.18), there exists an increasing sequence $(k(n))n$ such that $(\widetilde{u}^n_{k(n)})_n$ converges to u and $d(\widetilde{u}^{\,n}_{k(n)}, S(t_{k(n)}, \widetilde{u}^{\,n}_{k(n)})) < r$.

The next two propositions are concerned with the lower semicontinuity of the set-valued map $\mathcal{LQ}_r$ and $\mathcal{Q}_r$.

Proposition 5.2. *Assume that the following assumptions hold:*

i) *the set-valued map S has convex graph;*
ii) *the set-valued map S is $(\tau \times s, s)$-lower semicontinuous, $(\tau \times s, s)$-closed and $(\tau \times s, s)$-subcontinuous on $T \times K$;*
iii) *the function $f(t,\cdot,\cdot)$ is strictly quasi-concave on $K \times K$, for every $t \in T$;*
iv) *for every $(t,u) \in T \times K$, for every sequence $(t_n)_n$ converging to t in τ, there exists a sequence $(u'_n)_n$ which strongly converges to u in K such that for every $w \in K$ and every sequence $(w_n)_n$ strongly converging to w in K one has*

$$f(t,w,u) \le \liminf_n f(t_n, w_n, u'_n);$$

v) *for every $t \in T$ there exists $z \in K$ such that*

$$d(z, S(t,z)) < r \quad \text{and} \quad -f(t,w,z) < r \quad \forall\, w \in S(t,z).$$

Then, the set-valued map $\mathcal{LQ}_r$ is (τ, s)-lower semicontinuous on T.

Proof. The proof consists in two steps.

Step 1. For every $t \in T$,

$$\mathcal{LQ}_r(t) \subseteq \operatorname{cl} \widetilde{\mathcal{LQ}}_r(t),$$

where

$$\widetilde{\mathcal{LQ}}_r(t) = \{u \in K : d(u, S(t,u)) < r \text{ and } -f(t,w,u) < r\ \forall\, w \in S(t,u)\}.$$

Assume that there exist $t_o \in T$ and $u_o \in \mathcal{LQ}_r(t_o)$ such that $u_o \notin \operatorname{cl} \widetilde{\mathcal{LQ}}_r(t_o)$. Assumption *v)* says that there exists $z_o \in K$ such that

$$d(z_o, S(t_o, z_o)) < r \quad \text{and} \quad -f(t_o, w, z_o) < r \quad \forall\, w \in S(t_o, z_o).$$

Given a sequence $(\lambda_n)_n$ converging to 0 in [0,1], consider, for every $n \in \mathbb{N}$, the point

$$u_n = \lambda_n z_o + (1 - \lambda_n) u_o$$

and observe that, in light of assumptions $i)$ and $ii)$, one has $d(u_n, S(t_o, u_n)) < r$, since, for every $t \in T$, the function $u \in K \to d(u, S(t, u))$ turns to be convex (see, for example, [15]).
Moreover, for every $n \in \mathbb{N}$, $S(t_o, u_n) \subseteq \lambda_n S(t_o, z_o) + (1 - \lambda_n) S(t_o, u_o)$ so, if $w_n \in S(t_o, u_n)$, there exist $p_n \in S(t_o, z_o)$ and $q_n \in S(t_o, u_o)$ such that $w_n = \lambda_n p_n + (1 - \lambda_n) q_n$. Therefore, being $u_o \neq z_o$ since $u_o \notin \widetilde{\mathcal{LQ}}_r(t_o)$, assumption $iii)$ implies that

$$-f(t_o, w_n, u_n) < \max\{-f(t_o, p_n, z_o), -f(t_o, q_n, u_o)\} \leq r$$

and one gets a contradiction considering that $u_n \in \widetilde{\mathcal{LQ}}_r(t_o)$ for every $n \in \mathbb{N}$ and that $u_o = \lim_n u_n \in \operatorname{cl} \widetilde{\mathcal{LQ}}_r(t_o)$.

Step 2. The set-valued map

$$\widetilde{\mathcal{LQ}}_r : \ t \in T \to \widetilde{\mathcal{LQ}}_r(t)$$

is (τ, s)-lower semicontinuous on T.
In fact, assume that the set-valued map $\widetilde{\mathcal{LQ}}_r$ is not lower semicontinuous on T and find $t' \in T$, $u' \in \widetilde{\mathcal{LQ}}_r(t')$ and a sequence $(t'_n)_n$ τ-converging to t' in T such that $u' \notin \liminf_n \widetilde{\mathcal{LQ}}_r(t'_n)$.
Consequently, for the sequence $(u'_n)_n$, corresponding to (t', u') and $(t'_n)_n$ in assumption $iv)$, there exists a subsequence $(u'_{n_k})_k$ such that

$$u'_{n_k} \notin \widetilde{\mathcal{LQ}}_r(t'_{n_k}) \ \ \forall\, k \in \mathbb{N}.$$

Since $d(u', S(t, u')) < r$, from assumption $ii)$ one can infer that $d(u'_{n_k}, S(t'_{n_k}, u'_{n_k})) < r$ for $k \in \mathbb{N}$ sufficiently large, so, for such indexes k there exist $w'_k \in S(t'_{n_k}, u'_{n_k})$ such that $-f(t'_{n_k}, w'_k, u'_{n_k}) \geq r$.
Since the map S is closed and subcontinuous, the sequence $(w'_k)_k$ has a subsequence, still denoted by $(w'_k)_k$, converging to $w' \in S(t', u')$, and, using assumption $iv)$, one gets $-f(t', w', u') \geq r$ that is in contradiction with $u' \in \widetilde{\mathcal{LQ}}_r(t')$.
Finally, whatever is the sequence $(t_n)_n$ τ-converging to $t \in T$, one gets

$$\mathcal{LQ}_r(t) \subseteq \operatorname{cl} \widetilde{\mathcal{LQ}}_r(t) \subseteq \operatorname{cl} \liminf_n \widetilde{\mathcal{LQ}}_r(t_n) = \liminf_n \widetilde{\mathcal{LQ}}_r(t_n) \subseteq \liminf_n \mathcal{LQ}_r(t_n)$$

and the proof is complete.

Similarly one can prove:

Proposition 5.3. *Assume that the following assumptions hold:*

i) *the set-valued map S has convex graph;*

ii) *the set-valued map S is $(\tau \times s, s)$-lower semicontinuous, $(\tau \times s, s)$-closed and $(\tau \times s, s)$-subcontinuous on $T \times K$;*

iii) *the function $f(t,\cdot,\cdot)$ is strictly quasi-convex on $K \times K$, for every $t \in T$;*

iv) *for every $(t,u) \in T \times K$, for every sequence $(t_n)_n$ converging to t in τ, there exists a sequence $(u'_n)_n$ which strongly converges to u in K such that for every $w \in K$ and every sequence $(w_n)_n$ strongly converging to w in K one has*

$$f(t,u,w) \geq \limsup_n f(t_n, u'_n, w_n);$$

v) *for every $t \in T$ there exists $z \in K$ such that*

$$d(z, S(t,z)) < r \quad \text{and} \quad f(t,z,w) < r \quad \forall\, w \in S(t,z).$$

Then, the set-valued map $\mathcal{Q}_r$ is (τ,s)-lower semicontinuous on T.

Remark 5.4. We point out that analogous lower semicontinuity results do not hold, in general, for the approximate solutions maps $\mathcal{T}_r$ and $\mathcal{LT}_r$. Indeed, consider $T = U = \mathbb{R}$, $K = [-1,1]$, $S(t,u) = K$ for every $(t,u) \in \mathbb{R} \times [-1,1]$ and $f(t,u,w) = w - u + t$. Then, one can see that $\mathcal{T}_r(0) = \{1\}$ while $\mathcal{T}_r(1/n) = \emptyset$ for all $n \in \mathbb{N}$ and $r < 1$, so $\mathcal{T}_r(0) \not\subseteq \liminf_n \mathcal{T}_r(1/n)$.

Concluding, among the approximate solution maps considered in this paper, only the maps $\mathcal{Q}_r$ and $\mathcal{LQ}_r$ can be simultaneously upper and lower stable under suitable assumptions, as summarized in the following corollaries.

Corollary 5.5. *Assume that the following assumptions hold:*

i) *the set-valued map S has convex graph;*

ii) *the set-valued map S is $(\tau \times s, s)$-lower semicontinuous, $(\tau \times s, s)$-closed and $(\tau \times s, s)$-subcontinuous on $T \times K$;*

iii) *the function $f(t,\cdot,\cdot)$ is strictly quasi-concave on $K \times K$, for every $t \in T$;*

iv) *for every $(t,u) \in T \times K$, for every sequence $(t_n)_n$ converging to t in τ, there exists a sequence $(u'_n)_n$ which strongly converges to u in K such that for every $w \in K$ and every sequence $(w_n)_n$ strongly converging to w in K one has*

$$f(t,w,u) \leq \liminf_n f(t_n, w_n, u'_n);$$

v) *the function $f(\cdot,u,\cdot)$ is upper semicontinuous on $T \times K$, for every $u \in K$;*

vi) *the function $f(t,\cdot,w)$ is upper semicontinuous on the segments of K, for every $t \in T$ and $w \in K$;*

vii) *for every sequence $(t_n,u_n)_n$, $t_n \in T$ and $u_n \in K$ for all $n \in \mathbb{N}$, such that $(t_n)_n$ τ-converges in and $(u_n)_n$ s-converges in K, there exists $m \in \mathbb{N}$ such that*

$$\operatorname{int} \bigcap_{n \geq m} S(t_n, u_n) \neq \emptyset;$$

viii) *for every $t \in T$ there exists $z \in K$ such that*

$$d(z, S(t,z)) < r \quad \text{and} \quad -f(t,w,z) < r \quad \forall\, w \in S(t,z).$$

Then, the set-valued map $\mathcal{LQ}_r$ is (τ,s)-lower semicontinuous and (τ,s)-closed on T.

Corollary 5.6. *Assume that the following assumptions hold:*

i) *the set-valued map S has convex-graph;*

ii) *the set-valued map S is $(\tau \times s, s)$-lower semicontinuous, $(\tau \times s, s)$-closed and $(\tau \times s, s)$-subcontinuous on $T \times K$;*

iii) *the function $f(t, \cdot, \cdot)$ is strictly quasi-convex on $K \times K$, for every $t \in T$;*

iv) *the function f is lower semicontinuous on $T \times K \times K$;*

v) *for every $(t, u) \in T \times K$, for every sequence $(t_n)_n$ converging to t in τ, there exists a sequence $(u'_n)_n$ which strongly converges to u in K such that for every $w \in K$ and every sequence $(w_n)_n$ strongly converging to w in K one has*

$$f(t, u, w) \geq \limsup_n f(t_n, u'_n, w_n);$$

vi) *for every $t \in T$ there exists $z \in K$ such that*

$$d(z, S(t, z)) < r \quad \text{and} \quad f(t, z, w) < r \quad \forall\, w \in S(t, z).$$

Then, the set-valued map $\mathcal{Q}_r$ is (τ, s)-lower semicontinuous and (τ, s)-closed on T.

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