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On the Small-time Local Controllability*

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The convexity of a set of "control variations" is one of the crucial properties needed to prove sufficient controllability conditions or necessary optimality conditions. Heuristically, if one can construct control variations in all possible directions, then the considered control system is small-time locally controllable. As it was shown in [7], the cones generated by needle variations may fail to be convex. The purpose of the present paper is to define a convex set of high-order control variations and to prove a sufficient controllability condition. The proof is based on a general Lie series formalism. One illustrative example is also presented.

Keywords: High-order control variations, smooth control systems, small-time local controllability

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1. Introduction

Let us consider the following control system :

$$\dot{x}(t) = f_0(x(t)) + u(t)\ f_1(x(t)), \quad x(0) = x_0, \tag{1}$$

where the state variable x belongs to a smooth n-dimensional manifold M, x_0 is a fixed point of M, f_0 and f_1 are smooth vector fields with $f_0(x_0) = 0$.

Admissible controls are the Lebesgue integrable functions u whose domain is a compact interval of the form $[0, T]$, $T > 0$, and $u(t) \in [-1, 1]$ for almost every t from $[0, T]$. By $\mathcal{U}$ we denote the set of all admissible controls. A trajectory of (1) defined on $[0, T]$, starting from x_0 and corresponding to some admissible control u, is an absolutely continuous function $x(t)$, $t \in [0, T]$, satisfying (1) for almost every t from $[0, T]$. The reachable set $\mathcal{R}(x_0, T)$ of (1) from x_0 at time $T > 0$ is defined as the set of all points that can be reached in time T by means of trajectories of (1) starting from x_0. The system (1) is said to be small-time locally controllable (STLC) at x_0, if x_0 belongs to the interior of the set $\mathcal{R}(x_0, T)$ for each $T > 0$.

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The small-time local controllability of the control system (1) is a local property for the case of bounded controls. Thus, we can take a local point of view, i.e. we assume that $x_0 \in I\!\!R^n$ and the vector fields f_0 and f_1 are defined on some compact neighbourhood of the point x_0.

There are many possible approaches to study the small-time local controllability, leading to different results and requiring different assumptions. Here we follow a geometrical point of view. The underlying philosophy is that for analytic systems of the form (1) the local properties of the reachable set are determined by the iterated Lie brackets in the vector fields f_0 and f_1, evaluated at the initial point (cf. [14], [25], [26], [28], [30] etc.). These values are in principle easily computable. Thus, it is very natural to look for conditions for small-time local controllability which can be expressed in terms of elements of the Lie algebra $\mathcal{L}(\vec{f})$ generated by f_0 and f_1. Some very general sufficient conditions (cf. [2], [4], [5], [6], [11], [12], [18] [27], [32] etc.) are known as well as some necessary conditions (cf. [16], [20], [29], [31], etc.). But to our knowledge, necessary and sufficient conditions for small-time local controllability are proved only in some special cases (cf. [3], [8], [15], [21], [23], [33], [34] etc.).

We consider the case of single-input control systems using simple input symmetries. But, one can easily extend the main result for the case of multi-input systems. The underlying idea is the following: We consider the family $\mathcal{W} = \{f_0 + \alpha f_1 : \alpha \in [-1, 1]\}$ of smooth vector fields. It is easy to verify that the Lie algebra generated by the elements of $\mathcal{W}$ is $\mathcal{L}(\vec{f})$. Forgetting about rigor, we can think about the "Lie group" $\mathcal{G}(\vec{f})$ with Lie algebra $\mathcal{L}(\vec{f})$, and obtain an "action" of $\mathcal{G}(\vec{f})$ on the phase space. Every element g of $\mathcal{G}(\vec{f})$ is a product of exponentials of the type $\exp(t_i\, g_i)$. Hence, the result $g \cdot x_0$ of acting on x_0 by g is a point obtained by starting from the point x_0 and moving along integral trajectories of the elements of the set $\mathcal{W}$, with switching of vector fields allowed, and with motion "backwards in time" also permitted. All finite products of exponentials of the type $\exp(t_i\, g_i)$ with $t_i > 0$ and $g_i \in \mathcal{W}$ are true trajectories of the considered control system and constitute a subsemigroup $\mathcal{S}(\vec{f})$ of $\mathcal{G}(\vec{f})$. Then the reachable set from the point x_0 contains $\mathcal{S}(\vec{f}) \cdot x_0$. Let $\mathcal{H}_{x_0}(\vec{f})$ denote the usotropy subgroup of $\mathcal{G}(\vec{f})$ at the point x_0, i.e. $\mathcal{H}_{x_0}(\vec{f}) := \{g \in \mathcal{G}(\vec{f}) \,:\, g \cdot x_0 = x_0\}$. Clearly, if the interior of $\mathcal{S}(\vec{f})$ contains an element of $\mathcal{H}_{x_0}(\vec{f})$, then x_0 will be an interior point of $\mathcal{S}(\vec{f}) \cdot x_0$.

To make this rigorous, we use an abstract exponential Lie series formalism developed in [2], [31] and [32]. It is needed to surround the obstacles arising from the fact that $\mathcal{L}(\vec{f})$ is, in general, infinite dimensional, and hence $\mathcal{G}(\vec{f})$ is not a well defined Lie group. Rather then work with $\mathcal{L}(\vec{f})$ we work formally with a free Lie algebra $\mathcal{L}(\vec{X})$ generated by the indeterminates X_0 and X_1 and with his completion, the Lie algebra $\widetilde{\mathcal{L}}(\vec{X})$ of formal Lie series in X_0 and X_1. Then the controls can be embedded in $\widetilde{\mathcal{G}}(\vec{X})$ as a semigroup $\mathcal{S}(\vec{X})$ by means of a map that assigns to each control a noncommutative formal power series, obtained by solving the differential equation of the system formally using indeterminates instead the original vector fields (cf., for example, [31] and [32]). Since $\widetilde{\mathcal{G}}(\vec{X})$ is not a true Lie group, we use its nilpotent approximation $\mathcal{G}^N(\vec{X})$ obtained by killing all Lie brackets of length greater than N. Now $\mathcal{G}^N(\vec{X})$ is a Lie group in the usual sense. We denote by $\mathcal{S}^N(\vec{X})$ the corresponding approximating semigroup.

To get local controllability, we use the fact that the interior of $\mathcal{S}^N(\vec{X})$ contains an exponential of a linear combination of "bad Lie brackets". These Lie brackets are the possible obstruction for small-time local controllability. So, if each bad Lie bracket can be "neutralized" (i.e. each bad Lie bracket can be expressed as a linear combination of Lie brackets of lower "r-weight" (the notion of r-weight is defined in the next section)), then the control system is small-time local controllable at x_0. In fact, this is the idea of the general sufficient controllability condition proved by Sussmann in [32]. In order to "realize" the r-weight of the Lie brackets, we use families of admissible controls parameterized by the amplitude and by the length of the time interval where these controls are defined.

However, as it was shown in [17], there exist control systems that are small-time locally controllable but only by using increasingly faster switching controls, i.e. controls whose number of switchings increase to infinity as the length of the time interval goes to zero. To handle this "fast switching phenomenon", we use the idea proposed in [2] to study flows generated by semi-groups of diffeomorphisms of special kind: We choose an arbitrary set Π of Lie brackets of $\mathcal{L}^N(\vec{X})$ such that the free Lie algebra generated by the elements of Π contains all bad Lie brackets. Using the fact that an exponential of a linear combination Θ of bad Lie brackets belongs to the interior of $\mathcal{S}^N(\vec{X})$, we obtain that the exponential of the sum of Θ and suitable linear combinations of the elements of Π also belong to $\mathcal{S}^N(\vec{X})$. We denote by $\mathcal{S}^N_\Pi(\vec{X})$ the semigroup consisting of all finite products of these exponentials. Clearly, $\mathcal{S}^N_\Pi(\vec{X}) \subseteq \mathcal{S}^N(\vec{X})$. We study the semigroup $\mathcal{S}^N_\Pi(\vec{X})$ and are able to prove that an exponential of a linear combination of new "Π-bad Lie brackets" belongs to the the set $\mathcal{S}^N_\Pi(\vec{X})$. If each of these Π-bad Lie bracket can be "Π-neutralized", i.e. each Π-bad Lie bracket can be expressed as a linear combination of Lie brackets of lower "(r,σ)-weight" (the notion of (r,σ)-weight is defined in the next section), then we obtain a set of "control variations" to the reachable set. Heuristically, if we can construct control variations in all possible directions, then the reachable set has to be a full neighbourhood of the starting point (note that the cones generated by needle variations, for example, may fail to be convex as it was shown in [7] by presenting a series of purposefully constructed examples). We would like to point out that to "realize" the (r,σ)-weight of the Lie brackets, we use families of admissible controls parameterized by the amplitude, by the length of the time interval and by the number of switchings.

Applying the main result of [24], one can construct suitable sets of the so called S-control variations (also defined in [24]). In the present paper we extend the class of S-control variations and define a more general class of I-control variations that contain the class of S-control variations. We prove that the sets of I-control variations are convex in some extended sense and obtain as a corollary a more general sufficient STLC condition.

The paper is organized as follows: We present an abstract exponential Lie series formalism, formulate the main results and give an illustrative example in Section 2. The corresponding proofs can be found in Section 3.

2. Preliminaries and statement of the main results

Using an abstract exponential Lie series formalism, we study the problem of small-time controllability of semi-groups of diffeomorphisms of special kind. Following [2], [31] and [32], we introduce some notations: Let X_0 and X_1 be two symbols (called "indeterminates"). We set $\vec{X} = (X_0, X_1)$ and fix a sufficiently large positive integer N. By $\mathcal{A}^N(\vec{X})$ we denote the free nilpotent associative algebra of order $N+1$: If $I = (i_1, \dots, i_k)$ is any finite sequence with $i_j \in \{0, 1\}$, then we denote by $\|I\|$ its length k and set $X_I := X_{i_1} \cdots X_{i_k}$. We let $X_\emptyset := 1$. If $I \circ J$ denotes the concatenation of I and J, then the multiplication in $\mathcal{A}^N(\vec{X})$ is given by $X_I\, X_J := X_{I\circ J}$ whenever $\|I\| + \|J\| \le N$. If $\|I\| + \|J\| > N$, then the product $X_I\, X_J$ is set equal to zero. Then the basis of $\mathcal{A}^N(\vec{X})$ consists of all monomial X_I of length less than or equal to N.

We denote by $\mathcal{L}^N(\vec{X})$ the nilpotent Lie subalgebra of $\mathcal{A}^N(\vec{X})$ generated by X_0 and X_1 with the Lie bracket defined by

$$[P, Q] := PQ - QP.$$

The elements of $\mathcal{L}^N(\vec{X})$ will be referred to as *Lie polynomials* in X_0 and X_1. We apply very often the *Campbell-Baker-Hausdorff formula (C-B-H formula)* (cf., for example, [1], [9] and [10]) which says that if A and B are Lie polynomials, then there exists a Lie polynomial C such that

$$\exp(A)\exp(B) = \exp(C).$$

Here $\exp(P) := 1 + \sum_{i=1}^N \frac{P^i}{i!}$ for each Lie polynomial P. The C-B-H formula up to order three is

$$C = A + B + \frac{1}{2}\,[A, B] + \frac{1}{12}\,[A, [A, B]] + \frac{1}{12}\,[B, [B, A]] + \cdots$$

Let us define $\mathcal{G}^N(\vec{X})$ to be the set

$$\mathcal{G}^N(\vec{X}) = \left\{ \exp(A) : \; A \in \mathcal{L}^N(\vec{X}) \right\}.$$

Then, because of the C-B-H formula, $\mathcal{G}^N(\vec{X})$ is a group.

Following [31], we consider the following control system on $\mathcal{A}^N(\vec{X})$:

$$\dot{S}(t) = S(t)(X_0 + u(t)X_1), \quad \text{where } u(t) \in \mathcal{U} \text{ and } S(0) = 1. \tag{2}$$

Let us remind that by $\mathcal{U}$ we have denoted the set of all admissible controls, i.e. the set of all Lebesgue integrable functions u whose domain is a compact interval of the form $[0,T]$, $T>0$, and $u(t)\in[-1,1]$ for almost every t from $[0,T]$. The time T will be referred to as the terminal time of u and will be denoted by $T(u)$. If $u_i:[0,T(u_i)]\to[-1,1]$, $i=1,2$, are admissible controls, then by $u_2\circ u_1$ we denote an element of $\mathcal{U}$ with $T(u_2\circ u_1)=T(u_2)+T(u_1)$ and defined as follows:

$$u_2\circ u_1(t):=\begin{cases} u_1(t) & \text{for } t\in[0,T(u_1)),\\ u_2(t-T(u_1)) & \text{for } t\in[T(u_1),T(u_1)+T(u_2)). \end{cases} \tag{3}$$

It is proved in [31] that for each control $u\in\mathcal{U}$ which is defined on the interval $[0,T(u)]$, the solution $S(u)$ of (2) satisfying $S(u)(0)=1$ is well defined on $[0,T(u)]$ and

$$S(u)(t)=\sum_{\|I\|\le N} s_I(u)(t)X_I,\quad \forall\, t\in[0,T(u)],$$

where $s_\emptyset(u)(t):=1$ and for each $I=(i_1,i_2,\dots,i_k)$ with $i_j\in\{0,1\}$, $j=1,\dots,k$,

$$s_I(u)(t):=\int_0^t\int_0^{t_k}\int_0^{t_{k-1}}\cdots\int_0^{t_2} u^{i_k}(\tau_k)u^{i_{k-1}}(\tau_{k-1})\cdots u^{i_2}(\tau_2)u^{i_1}(\tau_1)\,d\tau_1\cdots d\tau_k$$

(here $u^0(t)=1$ and $u^1(t)=u(t)$ for each $t\in[0,T(u)]$). We define $\mathrm{Ser}(u)$ to be $S(u)(T(u))$.

The reachable set $\mathcal{R}^N_{\vec{X}}(T)$ of (2) at time $T>0$ is defined as the set of all points of $\mathcal{A}^N(\vec{X})$ that can be reached in time T by means of solutions of (2) starting from 1. Some properties of the control system (2) are presented in more details in [31]. Here, we shall remind only one corollary of Lemma 3.1 in [31]:

$$\mathrm{Ser}\ (u_1\circ u_2)=\ \mathrm{Ser}\ (u_1)\ \mathrm{Ser}\ (u_2) \tag{4}$$

for every two admissible controls u_1 and u_2. Also,

$$\begin{aligned}&\text{if } \exp(A_i)\in\mathcal{R}^N_{\vec{X}}(T_i)\ \text{ for } i=1,\dots,k,\\ &\qquad\text{then } \exp(A_1)\cdot\exp(A_2)\cdots\exp(A_k)\in\mathcal{R}^N_{\vec{X}}(T_1+T_2+\dots+T_k).\end{aligned}$$

Let us remind that $\mathcal{L}(\vec{X})$ denotes the free Lie algebra generated by the indeterminates X_0 and X_1, and let Λ be a Lie bracket belonging to $\mathcal{L}(\vec{X})$. We denote by $\Lambda(\vec{f})$ that Lie bracket in f_0 and f_1 which is obtained from Λ by substituting each X_0 and X_1 by f_0 and f_1, respectively. Also, we set $(\sum_{i=1}^k\alpha_i\Lambda_i)(\vec{f}):=\sum_{i=1}^k\alpha_i\Lambda_i(\vec{f})$ for each Lie brackets Λ_i in X_0 and X_1 and each real numbers α_i, $i=1,\dots,k$. If S is a subset of $\mathcal{L}(\vec{X})$, then by $\operatorname{span}S$ we denote the minimal linear subspace of $\mathcal{L}(\vec{X})$ containing the elements of S,

$$S(\vec{f}):=\left\{\Lambda(\vec{f}):\Lambda\in S\right\}\quad\text{and}\quad S(\vec{f})(x_0):=\left\{\Lambda(\vec{f})(x_0):\Lambda\in S\right\}.$$

Let Π^1 and Π^2 be two sets of Lie brackets from $\mathcal{L}(\vec{X})$. We set

$$\left[\Pi^1,\Pi^2\right]:=\left\{[\pi_1,\pi_2]:\ \pi_1\in\Pi^1,\ \pi_2\in\Pi^2\right\}.$$

At last, by $\mathcal{B}(\vec{X})$ we denote the set of all Lie brackets in X_0 and X_1 of odd length in which X_1 appears an even number of times. We call the elements of $\mathcal{B}(\vec{X})$ "bad Lie brackets". The main idea of the obtained sufficient conditions in [2], [5], [31] and [32] is that the elements of $\mathcal{B}(\vec{X})$ have to be "neutralized" in order not to be obstructions for small-time local controllability. This idea is also realized in the present paper but for a different set of bad Lie brackets: Let Π be an arbitrary set of Lie brackets from $\mathcal{L}(\vec{X})$. By $\mathcal{L}(\Pi)$ we denote the Lie algebra freely generated by the elements of Π.

Remark 2.1. For simplicity of the notations, we accept the following convention: Usually, we consider the elements of the set Π as indeterminates that generate the Lie algebra $\mathcal{L}(\Pi)$. For example, let $Y_1 := X_0$, $Y_2 := [X_0, X_1]$ and $Y_3 := [X_0, [X_0, X_1]]$ belong to the set Π. Then $[Y_1, Y_2]$ and Y_3 are different when they are considered as elements of $\mathcal{L}(\Pi)$. On the other hand, we also consider the Lie brackets in the elements of Π as elements of $\mathcal{L}(\vec{X})$. For example, $[Y_1, Y_2]$ and Y_3 coincide when they are considered as elements of $\mathcal{L}(\vec{X})$. We use one and the same notations in both cases. It will be clear from the context whether a Lie bracket in the elements of Π is considered as an element of $\mathcal{L}(\Pi)$ or as an element of $\mathcal{L}(\vec{X})$.

For each $k = 1, 2, \dots$ we define recurrently the sets

$$\Pi^1 := \Pi, \qquad \Pi^{k+1} := \left[\Pi^1,\ \Pi^k\right], \qquad \Pi^\infty := \bigcup_{k=1}^{\infty} \Pi^k,$$

Let us fix a vector $r = (r_0, r_1)$ whose components are positive integers such that $1 \le r_0 \le r_1$, and let σ be a positive number not greater than r_0. We set

$$\|\Lambda\|_r := r_0|\Lambda|_0 + r_1|\Lambda|_1, \qquad \|\Lambda\|_r^\sigma := \|\Lambda\|_r \ - \ \sigma k \ \text{ for every } \Lambda \in \Pi^k,$$

where the number of times that X_i, $i = 0, 1$, appears in Λ is denoted by $|\Lambda|_i$. Here, the relation $\Lambda \in \Pi^k$ implies that the Lie bracket Λ is considered as an element of $\mathcal{L}(\Pi)$. On the other hand, to calculate the numbers $|\Lambda|_0$ and $|\Lambda|_1$, we consider Λ as an element of the Lie algebra $\mathcal{L}^N(\vec{X})$ (cf. Remark 2.1). The number $|\Lambda|_i$ is called degree of Λ with respect to X_i, $i = 0, 1$. Clearly, the length $\|\Lambda\|$ of Λ is equal to $|\Lambda|_0 + |\Lambda|_1$. The positive numbers $\|\Lambda\|_r$ and $\|\Lambda\|_r^\sigma$ are called "*r-weight*" and "(r, σ)-weight" of the Lie bracket Λ, respectively.

For each positive number δ we define the sets

$$\overline{\mathcal{L}}_{r,\sigma}^{\delta}(\Pi) = \left\{\Lambda \in \Pi^\infty : \ \|\Lambda\|_r^\sigma = \delta\right\}.$$

and

$$\mathcal{L}_{r,\sigma}^{\delta}(\Pi) = \left\{\Lambda \in \Pi^\infty : \ \|\Lambda\|_r^\sigma \le \delta\right\}.$$

We say that the (r, σ)-weight $\|\cdot\|_r^\sigma$ is regular if the set $\{\delta : \ \overline{\mathcal{L}}_{r,\sigma}^{\delta}(\Pi) \neq \emptyset\}$ has no limit points.

Also, we define a set of "good" elements of the set Π as follows:

$$\mathrm{Good}\,(\Pi) := \Pi \setminus \mathcal{B}(\vec{X}),$$

i.e. good elements of the set Π are those elements of Π that are not bad Lie brackets. Let Λ be a Lie bracket belonging to Π^∞. We say that Λ belongs to $\mathcal{B}_\Pi(\vec{X})$ iff $\Lambda \in \Pi^{2i+1}$ for some nonnegative integer i and each element of Good (Π) appears an even number of times in Λ. We call the elements of $\mathcal{B}_\Pi(\vec{X})$ Π-*bad Lie brackets*. Let Λ_0 belong to $\mathcal{B}_\Pi(\vec{X})$. Because each good element of Π appears an even number of times in Λ_0, the total number of times that elements of the set $\Pi \cap \mathcal{B}(\vec{X})$ appears in Λ_0 is odd. Since each of these elements is of odd degree with respect to X_0 and of even degree with respect X_1, we may conclude that Λ_0 is of odd degree with respect to X_0 and of even degree with respect X_1, i.e. $\Lambda_0 \in \mathcal{B}(\vec{X})$. It is said that Λ_0 can be Π-*neutralized* if

$$\Lambda_0(\vec{f}(x_0)) \in \operatorname{span}\left\{\Lambda(\vec{f})(x_0) : \ \Lambda \in \Pi^\infty \text{ with } \|\Lambda\|_r^\sigma < \|\Lambda_0\|_r^\sigma\right\}.$$

Also, if $\Lambda_0(\vec{f})(x_0) = 0$, then Λ_0 is Π-neutralized.

To formulate our main result, we define a set of a high-order control variations:

Definition 2.2. Let $\mathcal{V} = \left\{V^1, \dots, V^k\right\}$ be a finite set of Lie brackets in X_0 and X_1. It is said that $\mathcal{V}$ is a set of I-control variations of the control system (2) at the point x_0 if there exist positive numbers λ, γ_0 and γ_1 with $\gamma_0 < \gamma_1 < (N+1)\,\lambda$, elements $\Theta^i \in \mathcal{L}^N(\vec{X})$ with $\Theta^i(\vec{f})(x_0) = 0$, $i = 1, \dots, d^\theta$, and $\Delta^j \in \mathcal{L}^N(\vec{X})$, $j = 1, \dots, d^\Delta$, such that for each vector $s = (s^1, \dots, s^k)$ whose components belong to the interval $[0, 1]$ and for each γ from the open interval (γ_0, γ_1) the following inclusion holds true

$$\exp\left(\eta^{-\gamma}\sum_{l=1}^{k} s^l \alpha_l(\gamma,\eta)V^l + \sum_{i=1}^{d^\theta}\theta_i(\gamma,\eta)\Theta^i + \sum_{j=1}^{d^\Delta}\delta_j(\gamma,\eta)\Delta^j\right) \in \mathcal{R}^N_{\vec{X}}(p_{s,\gamma}(\eta)) \quad (5)$$

for each sufficiently large positive integer η, where:

i) $\alpha_l : (\gamma_0, \gamma_1) \times [0, +\infty) \to [0, +\infty)$, $l = 1, \dots, k$ are functions with $\alpha_l(\gamma, \eta) \geq 1$ for each $\gamma \in (\gamma_0, \gamma_1)$ and for each positive integer η and $\lim_{\eta\to+\infty} \alpha_l(\gamma, \eta) = 1$ for each $\gamma \in (\gamma_0, \gamma_1)$;

ii) $\theta_i : (\gamma_0, \gamma_1) \times [0, +\infty) \to [0, +\infty)$, $i = 1, \dots, d^\theta$, are functions such that $\lim_{\eta\to+\infty} \theta_i(\gamma, \eta) = 0$ for each $\gamma \in (\gamma_0, \gamma_1)$;

iii) $\delta_j : (\gamma_0, \gamma_1) \times [0, +\infty) \to [0, +\infty)$, $j = 1, \dots, d^\Delta$, are functions such that $\lim_{\eta\to+\infty} \eta^\gamma\, \delta_j(\gamma, \eta) = 0$ for each $\gamma \in (\gamma_0, \gamma_1)$;

iv) $p_{s,\gamma} : [0, +\infty) \to [0, +\infty)$ is a function with $p_{s,\gamma}(\eta) < \nu\, \eta^{-\lambda}$ for each positive integer η, where ν is a positive constant.

We denote by $E_I^+(x_0)$ the set of all sets of I-control variations at x_0.

Proposition 2.3. *Let* $\mathcal{V}_1 = \left\{V_1^1, \dots, V_1^{k_1}\right\}$ *and* $\mathcal{V}_2 = \left\{V_2^1, \dots, V_2^{k_2}\right\}$ *be two sets of I-control variations at* x_0*. Then* $\mathcal{V} = \left\{V_1^1, \dots, V_1^{k_1}, V_2^1, \dots, V_2^{k_2}\right\}$ *is also a set of I-control variations at* x_0*.*

A class of S-control variations is defined in [24]. One can easily verify that each set of S-control variations is also a set of I-control variations. More about high-order control variations can be found in [24]. To formulate the main result, we need also the following

Definition 2.4. Let V belong to $\mathcal{L}(\vec{X})$. It is said that V is an H-control variation of the control system (2) at the point x_0 if there exist positive numbers ν, γ, $\gamma_1, \ldots, \gamma_{d^\Delta}$, $\theta_1, \ldots, \theta_{d^\Theta}$, and λ with

$$\gamma < \widetilde{\gamma} := \min\left(\gamma_1, \ldots, \gamma_{d^\Delta}, (N+1)\,\lambda\right),$$

elements $\Theta^i \in \mathcal{L}^N(\vec{X})$ with $\Theta^i(\vec{f})(x_0) = 0$, $i = 1, \ldots, d^\theta$, and $\Delta^j \in \mathcal{L}^N(\vec{X})$, $j = 1, \ldots, d^\Delta$, and a continuous function $p : R_+ \to R_+$ with $p(t) < \nu\, t^\lambda$ such that for each sufficiently small positive $t > 0$

$$\exp\left(t^\gamma V + \sum_{i=1}^{d^\theta} t^{\theta_i}\Theta^i + \sum_{j=1}^{d^\Delta} t^{\gamma_j}\Delta^j\right) \in \mathcal{R}^N_{\vec{X}}(p(t)),$$

We denote by $E^+_H(x_0)$ the set of all H-control variations at x_0.

Remark 2.5. We would like to point out that the inclusion (5) in Definition 2.2 holds true only for countably many values of η. Also, high-order variations similar to those defined in Definition 2.2 (Definition 2.4) are used in [2], [31], [32] etc. ([13], [22], [23], [33], [34] etc.) But there exist different definitions of high-order variations (cf., for example, [6], [7], [11] etc.).

For example, the sufficient controllability conditions proved in [2], [31] and [32] ensure elements of the set $E^+_S(x_0)$. Also, the following proposition provides constructions of elements of the set $E^+_H(x_0)$.

Proposition 2.6. *The following assertions holds true:*

(*i*) *the set $E^+_H(x_0)$ is a convex cone;*
(*ii*) *X_1 and $-X_1$ belong to $E^+_H(x_0)$;*
(*iii*) *if V and $-V$ belong to $E^+_H(x_0)$, then $\pm[X_0, V] \in E^+_H(x_0)$.*

The idea of the proof of this proposition can be found, for example, in [13], [22], [23], [33], [34] etc. Let us denote by int S and co S the interior and the convex hull of the set S, respectively.

Next we formulate the main result:

Theorem 2.7. *Let $\mathcal{V}_i = \left\{V_i^1, \ldots, V_i^{k_i}\right\}$, $i = 1, \ldots, \mu$, be sets of I-control variations at x_0, let $W^1, \ldots, W^k \in H^+(x_0)$ and let the origin belong to the interior of the convex hull of the set*

$$\begin{gathered}\Big\{V_1^1(\vec{f})(x_0), \ldots, V_1^{k_1}(\vec{f})(x_0)(x_0), \ldots, V_\mu^1(\vec{f})(x_0)(x_0), \ldots, \\ V_\mu^{k_\mu}(\vec{f})(x_0)(x_0), W_1(\vec{f})(x_0)(x_0), \ldots, W_k(\vec{f})(x_0)(x_0)\Big\}.\end{gathered} \tag{6}$$

Then the control system (1) *is STLC at the point x_0.*

In order to apply Theorem 2.7, we can use, for example, the main result of [24]. This result shows how to construct suitable high-order control variations and it can be formulated as follows:

Theorem 2.8. *Let $r = (r_0, r_1)$ be a vector whose components are positive integers satisfying the inequalities $1 \le r_0 \le r_1$, and σ be a positive number not greater than r_0 such that the (r, σ)-weight $\|\cdot\|_r^\sigma$ is regular. Let Π be an arbitrary set of Lie brackets belonging to $\mathcal{L}(\vec{X})$. Suppose that*

i) *the set of all bad Lie brackets $\mathcal{B}(\vec{X})$ is a subset of $\mathcal{L}(\Pi)$;*

ii) *there exists $\delta > 0$ such that every Lie bracket from $\mathcal{B}_\Pi(\vec{X}) \bigcap \mathcal{L}^\delta_{r,\sigma}(\Pi)$ can be Π-neutralized.*

Then there exists a set $\{\pm\Lambda_1, \dots, \pm\Lambda_{i_0}\}$ of S-control variations at x_0 such that each Λ_i is a Lie bracket in the elements of the set Π and

$$\operatorname{span}\left\{\Lambda_1(\vec{f})(x_0), \dots, \Lambda_{i_0}(\vec{f})(x_0)\right\} \equiv \operatorname{span} \mathcal{L}^\delta_{r,\sigma}(\Pi)(\vec{f})(x_0).$$

Remark 2.9. To apply Theorem 2.8, the so called Elimination theorem is quite useful (to our knowledge, its importance for studying the STLC property is shown at first in [2]): Let S be an arbitrary subset of a Lie algebra and let s_0 belong to S. Then the linear hull of all elements of S except the element s_0, and of all Lie brackets in the elements of S is a Lie algebra freely generated by the set

$$\left\{\left(\mathrm{ad}^i\ s_0, s\right) : s \in S \setminus \{s_0\},\ i = 0, 1, \dots\right\}.$$

To illustrate the application of Theorem 2.7, we consider a nonlinear example. To the author knowledge, no one of the published sufficient conditions (cf., for example, [2], [24]) and [32]) implies that the considered control system is STLC at the origin.

Example 2.10. Let us consider the following control system

$$\left|\begin{array}{ll} \dot{x}_1 = u, & x_1(0) = 0 \\ \dot{x}_2 = x_1, & x_2(0) = 0 \\ \dot{x}_3 = x_1^3, & x_3(0) = 0 \\ \dot{x}_4 = x_1^5, & x_4(0) = 0 \\ \dot{x}_5 = x_1^7, & x_5(0) = 0 \\ \dot{x}_6 = x_2, & x_6(0) = 0 \\ \dot{x}_7 = x_6, & x_7(0) = 0 \end{array}\right. \quad \left|\begin{array}{ll} \dot{x}_8 = x_7, & x_8(0) = 0, \quad u \in [-1, 1], \\ \dot{x}_9 = x_8, & x_9(0) = 0 \\ \dot{x}_{10} = x_9, & x_{10}(0) = 0, \\ \dot{x}_{11} = x_{10}, & x_{11}(0) = 0, \\ \dot{x}_{12} = x_{11}, & x_{12}(0) = 0 \\ \dot{x}_{13} = x_{12}, & x_{13}(0) = 0 \\ \dot{x}_{14} = x_2 x_5 + x_2^6, & x_{14}(0) = 0 \\ \dot{x}_{15} = x_3 x_{13} + x_2^6, & x_{15}(0) = 0 \end{array}\right.$$

We set $f_0 := (0, x_1, x_1^3, x_1^5, x_1^7, x_2, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_2 x_5 + x_2^6, x_3 x_{13} + x_2^6)$ and $f_1 := (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. One can check directly that the following Lie brackets in f_0 and f_1 are nonvanishing at the origin:

$$f_1, \qquad [f_1, f_0], \qquad \left(\mathrm{ad}^3\ f_1, f_0\right), \qquad \left(\mathrm{ad}^5\ f_1, f_0\right), \qquad \left(\mathrm{ad}^7\ f_1, f_0\right),$$
$$\left(\mathrm{ad}^i\ f_0, [f_1, f_0]\right),\quad i = 1, \dots, 8, \qquad \left[\left(\mathrm{ad}^7\ f_1, f_0\right), [[f_1, f_0], f_0]\right],$$
$$\left[\left(\mathrm{ad}^3\ f_1, f_0\right), \left[\left(\mathrm{ad}^8\ f_0, [f_1, f_0]\right), f_0\right]\right] \quad \text{and} \quad \left(\mathrm{ad}^6\ [f_1, f_0], f_0\right).$$

We set $S := S_1 \cup S_2$, where

$$S_1 := \left\{ f_1,\ [f_1, f_0],\ \left(\mathrm{ad}^3\ f_1, f_0\right),\ \left(\mathrm{ad}^5\ f_1, f_0\right),\ \left(\mathrm{ad}^7\ f_1, f_0\right) \right\}$$

and

$$S_2 := \left\{ \left(\mathrm{ad}^i\ f_0, [f_1, f_0]\right),\ i = 1, \ldots, 8 \right\}.$$

The Lie brackets

$$Z_G^1 := \left[\left(\mathrm{ad}^7\ X_1, X_0\right), [[X_1, X_0], X_0] \right], \qquad Z_B := \left(\mathrm{ad}^6\ [X_1, X_0], X_0\right)$$

and $Z_G^2 := \left[\left(\mathrm{ad}^3\ X_1, X_0\right), \left[\left(\mathrm{ad}^8\ X_0, [X_1, X_0]\right), X_0 \right] \right]$ belong to $\mathcal{B}(\vec{X})$. Their corresponding values $Z_G^1(\vec{f})(0)$, $Z_G^2(\vec{f})(0)$ and $Z_B(\vec{f})(0)$ at the origin can be not represented as a linear combination of the remainder Lie brackets evaluated also at the origin. Hence we can not apply the sufficient conditions proved in [2] and [32]. To prove that this control system is STLC at the origin, we set

$$\Pi = \left\{ \left(\mathrm{ad}^i\ X_1, X_0\right) :\ i = 0, 1, 2, \ldots \right\}.$$

Then

$$\mathrm{Good}\,(\Pi) = \left\{ \left(\mathrm{ad}^{2i+1}\ X_1, X_0\right) :\ i = 0, 1, 2, \ldots \right\}.$$

For $r_1 := (r_0^1, r_1^1)$ with $\sigma_1 \geq 1$, $r_0^1 \geq 2\sigma_1 + 1$ and $r_1^1 := 2(r_0^1 - \sigma_1) - 1$, we obtain that

$$\left\| Z_G^1 \right\|_{r_1}^{\sigma_1} = 19(r_0^1 - \sigma_1) - 8, \qquad \left\| Z_G^2 \right\|_{r_1}^{\sigma_1} = 19(r_0^1 - \sigma_1) - 4,$$
$$\left\| Z_B \right\|_{r_1}^{\sigma_1} = 19(r_0^1 - \sigma_1) - 6,$$

while for $r_2 := (r_0^2, r_1^2)$ with $\sigma_2 \geq 1$, $r_0^2 \geq 2\sigma_1 - 1$ and $r_1^2 := 2(r_0^2 - \sigma_2) + 1$, we obtain that

$$\left\| Z_G^1 \right\|_{r_2}^{\sigma} = 19(r_0^1 - \sigma_1) + 8, \qquad \left\| Z_G^2 \right\|_{r_2}^{\sigma} = 19(r_0^1 - \sigma_1) + 4,$$
$$\left\| Z_B \right\|_{r_2}^{\sigma} = 19(r_0^1 - \sigma_1) + 6.$$

The Elimination theorem implies that the set $\mathcal{B}(\vec{X})$ is a subset of $\mathcal{L}(\Pi)$. The Lie bracket Z_G^1 is of first degree with respect to $\left(\mathrm{ad}^5\ X_1, X_0\right)$ and to $[X^1, X^0]$ (the last two Lie brackets belong to the set $\mathrm{Good}\,(\Pi)$). Hence, it does not belong to $\mathcal{B}_\Pi(\vec{X})$. One can directly check that all Lie brackets belonging to the set $\mathcal{B}_\Pi(\vec{X}) \bigcap \mathcal{L}_{r_1,\sigma_1}^{19(r_0^1 - \sigma_1) - 8}(\Pi)$ vanish at the origin. Applying Theorem 2.8, we obtain that the set

$$-S \cup S \cup \left\{ \pm Z_G^1 \right\} \subset \mathcal{L}_{r_1,\sigma_1}^{19(r_0^1 - \sigma_1) - 8}(\Pi)$$

is a set of S-control variations.

Analogously, the Lie bracket Z_G^2 is of first degree with respect to $\left(\mathrm{ad}^3\ X_1, X_0\right)$ and to $[X^1, X^0]$ (the last two Lie brackets belong to the set $\mathrm{Good}\,(\Pi)$). Hence, it does not belong to $\mathcal{B}_\Pi(\vec{X})$. One can directly check that all Lie brackets belonging to the set $\mathcal{B}_\Pi(\vec{X}) \bigcap \mathcal{L}_{r_2,\sigma_2}^{19(r_0^2 - \sigma_2) + 4}(\Pi)$ vanish at the origin. Applying Theorem 2.8, we obtain that the set

$$-S \cup S \cup \left\{ \pm Z_G^2 \right\} \subset \mathcal{L}_{r_2,\sigma_2}^{19(r_0^2 - \sigma_2) + 4}(\Pi)$$

is a set of S-control variations. is a set of S-control variations. According to (*ii*) of Proposition 2.6, X_1 and $-X_1$ are H-control variation. Since the origin belongs to the interior of the convex hull of the vectors

$$\begin{gathered}\pm f_1(0), \qquad \pm [f_1, f_0](0), \qquad \pm \left(\mathrm{ad}^3\, f_1, f_0\right)(0), \qquad \pm \left(\mathrm{ad}^5\, f_1, f_0\right)(0),\\ \pm \left(\mathrm{ad}^7\, f_1, f_0\right)(0), \qquad \pm \left(\mathrm{ad}^i\, f_0, [f_1, f_0]\right)(0), \quad i = 1, \dots, 8,\\ \pm \left[\left(\mathrm{ad}^5\, f_1, f_0\right), \left[\left(\mathrm{ad}^4\, f_0, [f_1, f_0]\right), f_0\right]\right](0),\\ \pm \left[\left(\mathrm{ad}^3\, f_1, f_0\right), \left[\left(\mathrm{ad}^8\, f_0, [f_1, f_0]\right)(0), f_0\right]\right](0) \quad \text{and} \quad \pm \left(\mathrm{ad}^6\, [f_1, f_0], f_0\right)(0)\end{gathered}$$

we obtain according to Theorem 2.7 that this control system is STLC at the origin.

We would like to note that only for $r = (r_0, r_1)$ with $\sigma \geq 1$, $r_1 = 2(r_0 - \sigma)$ and $r_0 \geq 2\sigma$ we have that

$$\left\|Z_G^1\right\|_r^\sigma = \left\|Z_G^2\right\|_r^\sigma = \|Z_B\|_r^\sigma = 19(r_0 - \sigma).$$

Unfortunately, we can not apply Theorem 2.8 for this choice of r because the Lie Bracket Z_B is not Π-neutralized.

3. Proofs

Here we present the proofs of Proposition 2.3 and Theorem 2.7.

Proof of Proposition 2.3. According to Definition 2.2 there exist positive numbers λ_ι, γ_0^ι and γ_1^ι with $\gamma_0^\iota < \gamma_1^\iota < (N+1)\,\lambda_\iota$, elements $\Theta_\iota^i \in \mathcal{L}^N(\vec{X})$ with $\Theta_\iota^i(\vec{f})(x_0) = 0$, $i = 1, \dots, d^\theta$, and $\Delta^j \in \mathcal{L}^N(\vec{X})$, $j = 1, \dots, d^\Delta$, such that for each vector $s_\iota = (s_\iota^1, \dots, s_\iota^k)$ whose components belong to the interval $[0,1]$ and for each γ_ι from the open interval $(\gamma_0^\iota, \gamma_1^\iota)$ the following inclusion holds true

$$\exp\left(\eta^{-\gamma}\sum_{l=1}^{k} s_\iota^l \alpha_l^\iota(\gamma_\iota, \eta) V_\iota^l + \sum_{i=1}^{d^\theta} \theta_i^\iota(\gamma, \eta)\Theta_\iota^i + \sum_{j=1}^{d^\Delta} \delta_j^\iota(\gamma, \eta)\Delta_\iota^j\right) \in \mathcal{R}_{\vec{X}}^N(p_{s_\iota,\gamma_\iota}^\iota(\eta)) \quad (7)$$

for each sufficiently large positive integer η, where:

i) $\alpha_l^\iota : (\gamma_0^\iota, \gamma_1^\iota) \times [0,+\infty) \to [0,+\infty)$, $l = 1, \dots, k$ are functions with $\alpha_l^\iota(\gamma, \eta) \geq 1$ for each $\gamma_\iota \in (\gamma_0, \gamma_1)$ and for each positive integer η and $\lim_{\eta\to+\infty} \alpha_l^\iota(\gamma, \eta) = 1$ for each $\gamma_\iota \in (\gamma_0^\iota, \gamma_1^\iota)$;

ii) $\theta_i^\iota : (\gamma_0^\iota, \gamma_1^\iota) \times [0,+\infty) \to [0,+\infty)$, $i = 1, \dots, d_\iota^\theta$, are functions such that $\lim_{\eta\to+\infty} \theta_i^\iota(\gamma_\iota, \eta) = 0$ for each $\gamma_\iota \in (\gamma_0^\iota, \gamma_1^\iota)$;

iii) $\delta_j^\iota : (\gamma_0^\iota, \gamma_1^\iota) \times [0,+\infty) \to [0,+\infty)$, $j = 1, \dots, d_\iota^\Delta$, are functions such that $\lim_{\eta\to+\infty} \eta^{\gamma_\iota}\delta_j^\iota(\gamma_\iota, \eta) = 0$ for each $\gamma_\iota \in (\gamma_0^\iota, \gamma_1^\iota)$;

iv) $p_{s,\gamma}^\iota : [0,+\infty) \to [0,+\infty)$ is a function with $p_{s,\gamma}^\iota(\eta) < \nu_\iota\, \eta^{-\lambda_\iota}$ for each positive integer η, where ν_ι is a positive constant.

Let us fix two arbitrary vectors $s_\iota = (s_\iota^1, \dots, s_\iota^k)$, $\iota = 1, 2$, whose components belong to the interval $[0,1]$. Two cases are possible: a) $(\gamma_0^1, \gamma_1^1) \cap (\gamma_0^2, \gamma_1^2) \neq \emptyset$; b) $(\gamma_0^1, \gamma_1^1) \cap (\gamma_0^2, \gamma_1^2) = \emptyset$;

Let the case a) hold true. We set $\gamma_0 := \max(\gamma_0^1, \gamma_0^2)$ and $\gamma_1 := \min(\gamma_1^1, \gamma_1^2)$. Then (7) holds true for each $\iota = 1, 2$, for each $\gamma \in (\gamma_0, \gamma_1)$ and for each sufficiently large positive integer η.

Let us assume that the case b) holds true. Without loss of generality we may assume that $\gamma_0^1 \geq \gamma_1^2$. Let us fix an arbitrary positive number γ_2 from (γ_0^2, γ_1^2), and set $\gamma_0 := \max(\gamma_0^1, \gamma_2\gamma_1^1/\gamma_1^2)$ and $\gamma_1 := \gamma_1^1$. Let us fix an arbitrary $\gamma \in (\gamma_0, \gamma_1)$. Then the inequalities $\gamma_2 < \gamma_1^2 \leq \gamma_0 < \gamma$ imply that for each positive integer η there exists a positive integer $\mu = \mu(\eta) \geq \eta$ such that the following inequalities hold true:

$$\left(\frac{1}{\mu(\eta)}\right)^{\gamma_2} \geq \left(\frac{1}{\eta}\right)^{\gamma} > \left(\frac{1}{\mu(\eta)+1}\right)^{\gamma_2}.$$

These inequalities imply that

$$1 \leq \frac{\eta^\gamma}{(\mu(\eta))^{\gamma_2}},$$

$$\lim_{\eta\to+\infty} \frac{\eta^\gamma}{(\mu(\eta))^{\gamma_2}} \leq \lim_{\eta\to+\infty} \frac{(\mu(\eta)+1)^{\gamma_2}}{(\mu(\eta))^{\gamma_2}} = \lim_{\eta\to+\infty} \left(1 + \frac{1}{(\mu(\eta))}\right)^{\gamma_2} = 1,$$

and hence

$$\lim_{\eta\to+\infty} \frac{\eta^\gamma}{(\mu(\eta))^{\gamma_2}} = 1. \tag{8}$$

We set $\tilde\alpha_2(\gamma,\eta) := \frac{\eta^\gamma}{(\mu(\eta))^{\gamma_2}}\alpha_2(\gamma_2, \mu(\eta))$. Clearly $\tilde\alpha_2(\gamma,\eta) \geq 1$ for each positive integer η and $1 \leq \lim_{\eta\to+\infty} \tilde\alpha_2(\gamma,\eta) = \lim_{\eta\to+\infty} \alpha_2(\gamma_2, \mu(\eta)) \lim_{\eta\to+\infty} \frac{\eta^\gamma}{(\mu(\eta))^{\gamma_2}} = 1$. Then the inclusion (7) (for $\iota = 2$ and $\eta := \mu(\eta)$) can be written as follows

$$\exp\left(\eta^{-\gamma}\tilde\alpha_2(\gamma,\eta)\sum_{j=1}^{k_2} s^j V_2^j + \sum_{i=1}^{d_2^\theta} \tilde\theta_i^2(\gamma,\eta)\Theta_2^i + \sum_{j=1}^{d_2^\Delta} \tilde\delta_j^2(\gamma,\eta)\Delta_2^j\right) \in \mathcal{R}_{\vec X}^N(\tilde p_{s,\gamma}^2(\eta)), \tag{9}$$

where $\tilde\theta_i^2(\gamma,\eta) := \theta_i^2(\gamma_2, \mu(\eta))$, $i = 1, \ldots, d_2^\theta$, $\tilde\delta_j^2(\gamma,\eta) := \delta_j^2(\gamma_2, \mu(\eta))$, $j = 1, \ldots, d_2^\Delta$, $\tilde p_{s,\gamma}^2(\eta) := p_{s,\gamma_2}^2(\mu(\eta))$. Our choice of $\mu(\eta)$, the properties of θ_i^2, $i = 1, \ldots, d_2^\theta$, and δ_j^2, $j = 1, \ldots, d_2^\Delta$ and (8) imply that

$$\lim_{\eta\to+\infty} \tilde\theta_i^2(\gamma,\eta) = \lim_{\eta\to+\infty} \theta_i^2(\gamma_2, \mu(\eta)) = 0,$$

$$\begin{aligned}\lim_{\eta\to+\infty} \eta^\gamma \tilde\delta_j^2(\gamma,\eta) &= \lim_{\eta\to+\infty} \eta^\gamma \delta_j^2(\gamma_2, \mu(\eta)) \\ &= \lim_{\eta\to+\infty} (\mu(\eta))^{\gamma_2}\delta_j^2(\gamma_2, \mu(\eta)) \lim_{\eta\to+\infty} \frac{\eta^\gamma}{(\mu(\eta))^{\gamma_2}} = 0.\end{aligned}$$

Also, according to (8), we have that

$$\tilde p_{s,\gamma}^2(\eta) := p_{s,\gamma_2}^2(\mu(\eta)) \leq \nu_2\ (\mu(\eta))^{-\lambda_2} = \nu_2 \left(\frac{\eta^{\gamma/\gamma_2}}{\mu(\eta)}\right)^{\lambda_2} \eta^{-(\lambda_2\gamma)/\gamma_2} \leq 2\nu_2\eta^{-(\gamma\lambda_2)/\gamma_2}$$

for all sufficiently large positive integers η.

Then

$$\mathcal{E}(s_1, s_2, \eta, \gamma) := \exp\left(\eta^{-\gamma}\alpha_1(\gamma,\eta)\sum_{j=1}^{k} s_1^j V_1^j + \sum_{i=1}^{d_1^\theta}\theta_i^1(\gamma,\eta)\Theta_1^i + \sum_{j=1}^{d_1^\Delta}\gamma_j^1(\gamma,\eta)\Delta_1^j\right)$$

$$\exp\left(\eta^{-\gamma}\sum_{j=1}^{k_2}\tilde{\alpha}_2^j(\gamma,\eta)s_2^j V_2^j + \sum_{i=1}^{d_2^\theta}\tilde{\theta}_i^2(\gamma,\eta)\Theta_2^i + \sum_{j=1}^{d_2^\Delta}\tilde{\delta}_j^2(\gamma,\eta)\Delta_2^j\right) \in \mathcal{R}_{\vec{X}}^N(p_{s_1,\gamma}^2(\eta) + \tilde{p}_{s_2,\gamma}^2(\eta)),$$

for all sufficiently large positive integers η. Applying the Campbell-Baker-Hausdorff formula we obtain that $\mathcal{E}(s_1, s_2, \eta, \gamma) =$

$$\exp\left(\eta^{-\gamma}\left(\sum_{\iota=1}^{k_2}\alpha_1^\iota(\gamma,\eta)s_1^\iota V_1^\iota + \sum_{\iota=1}^{k_2}\bar{\alpha}_2^\iota(\gamma,\eta)s_2^\iota V_2^\iota\right) + \sum_{i=1}^{d^\theta}\theta_i(\gamma,\eta)\bar{\Theta}^i + \sum_{j=1}^{d^\Delta}\delta_j(\gamma,\eta)\bar{\Delta}^j\right)$$

where each $\bar{\Theta}^i$, $i = 1,\ldots,d^\theta$ is a Lie bracket of Θ_1^i, $i = 1,\ldots,d_1^\theta$, and Θ_2^i, $i = 1,\ldots,d_2^\theta$. Hence $\bar{\Theta}^i(\vec{f})(x_0) = 0$, $i = 1,\ldots,d^\theta$. Also, each function $\theta_i(\gamma,\eta)$, $i = 1,\ldots,d^\theta$, is proportional to a product of some of the functions θ_i^1, $i = 1,\ldots,d_1^\theta$, and $\tilde{\theta}_i^2$, $i = 1,\ldots,d_2^\theta$. Thus, $\lim_{\eta\to+\infty}\theta_i(\gamma,\eta) = 0$, $i = 1,\ldots,d^\theta$. Analogously, each $\bar{\Delta}^j$, $i = 1,\ldots,d^\Delta$ is a Lie bracket of V_1^ι, $\iota = 1,\ldots,k_1$, V_2^ι, $\iota = 1,\ldots,k_2$, Θ_1^i, $i = 1,\ldots,d_1^\theta$, Θ_2^i, $i = 1,\ldots,d_2^\theta$, Δ_1^j, $j = 1,\ldots,d_1^\Delta$, and Δ_2^j, $j = 1,\ldots,d_2^\Delta$ and at last one factor is V_1^ι, V_2^ι, Δ_1^j or Δ_2^j. The corresponding function δ_j is proportional to the product of the functions multiplying the corresponding factors. Thus each function δ_j is a product of at last two of the before written functions and contains as a factor at last one function $\eta^{-\gamma}\alpha_1^\iota(\gamma,\eta)$, $\eta^{-\gamma}\tilde{\alpha}_2^\iota(\gamma,\eta)$, $\delta_j^1(\gamma,\eta)$ or $\tilde{\delta}_j^2\gamma,\eta)$. Thus $\lim_{\eta\to+\infty}\eta^{-\gamma}\delta_i(\gamma,\eta) = 0$. We have prove in this way that $\{V_1^\iota : \iota = 1,\ldots,k_1\}\bigcup\{V_2^\iota : \iota = 2,\ldots,k_2\}$ is a set of I-control variations. The case a) is simpler and can be treat in the same way. This completes the proof. □

Proof of Theorem 2.7. According to Proposition 2.3 the set

$$\bigcup_{i=1}^{\mu}\{V_i^\iota : \iota = 1,\ldots,k_i\}$$

is a set of I-control variations. Then there exist positive numbers λ, γ_0 and γ_1 with $\gamma_0 < \gamma_1 < (N+1)\lambda$, elements $\Theta^i \in \mathcal{L}^N(\vec{X})$ with $\Theta^i(\vec{f})(x_0) = 0$, $i = 1,\ldots,d^\theta$, and $\Delta^j \in \mathcal{L}^N(\vec{X})$, $j = 1,\ldots,d^\Delta$, such that for each vector $s = (s_1^1,\ldots,s_1^{k_1},\ldots,s_\mu^1,\ldots,s_\mu^{k_\mu})$ whose components belong to the interval $[0,1]$ and for each γ from the open interval (γ_0,γ_1) the following inclusion holds true

$$\exp\left(\eta^{-\gamma}\sum_{i=1}^{\mu}\sum_{\iota=1}^{k_\iota}s_i^\iota\alpha_\iota^i(\gamma,\eta)V_i^\iota + \sum_{i=1}^{d^\theta}\theta_i(\gamma,\eta)\Theta^i + \sum_{j=1}^{d^\Delta}\gamma_j(\gamma,\eta)\Delta^j\right) \in \mathcal{R}_{\vec{X}}^N(p_{s,\gamma}(\eta)) \quad (10)$$

for each sufficiently large positive integer η, where:

i) $\alpha^i_\iota : (\gamma_0,\gamma_1)\times[0,+\infty)\to[0,+\infty)$, $\iota=1,\dots,k_i$, $i=1,\dots,k$, are functions with $\alpha^i_\iota(\gamma,\eta)\ge 1$ for each $\gamma\in(\gamma_0,\gamma_1)$ and for each positive integer η and $\lim_{\eta\to+\infty}\alpha^i_\iota(\gamma,\eta)=1$ for each $\gamma\in(\gamma_0,\gamma_1)$;

ii) $\theta_i : (\gamma_0,\gamma_1)\times[0,+\infty)\to[0,+\infty)$, $i=1,\dots,d^\theta$, are functions such that $\lim_{\eta\to+\infty}\theta_i(\gamma,\eta)=0$ for each $\gamma\in(\gamma_0,\gamma_1)$;

iii) $\delta_j : (\gamma_0,\gamma_1)\times[0,+\infty)\to[0,+\infty)$, $j=1,\dots,d^\Delta$, are functions such that $\lim_{\eta\to+\infty}\eta^\gamma\,\delta_j(\gamma,\eta)=0$ for each $\gamma\in(\gamma_0,\gamma_1)$;

iv) $p_{s,\gamma}:[0,+\infty)\to[0,+\infty)$ is a function with $p_{s,\gamma}(\eta)<\nu\,\eta^{-\lambda}$ for each positive integer η, where ν is a positive constant.

According to Definition 2.4, for each index $j=1,\dots,k$, there exist positive reals $\hat\nu_j$, γ^j, $\hat\gamma^j_1,\dots,\hat\gamma^j_{d^{\hat\Delta_j}}$, $\hat\theta^j_1,\dots,\hat\theta^j_{d^{\hat\Theta_j}}$ and $\hat\lambda_j$ with

$$\gamma^j<\widehat{\gamma}^j:=\min\left(\hat\gamma^j_1,\dots,\hat\gamma^j_{d^{\hat\Delta_j}},(N+1)\hat\lambda_j\right)$$

elements $\hat\Theta^{j,\alpha}\in\mathcal{L}^N(\vec X)$ with $\hat\Theta^{j,\alpha}(\vec f)(x_0)=0$ for each $\alpha=1,\dots,d^{\widehat\Theta_j}$, and elements $\hat\Delta^{j,\beta}\in\mathcal{L}^N(\vec X)$, $\beta=1,\dots,d^{\widehat\Delta_j}$, and a family of continuous functions $p^j:R_+\to R_+$ with $p^j(\eta)<\hat\nu_j\;\eta^{-\lambda_j}$, $j=1,\dots,k$, such that

$$\exp\left(t^{\gamma_j}W_j+\sum_{\alpha=1}^{d^{\widehat\Theta_j}}t^{\hat\theta^j_\alpha}\hat\Theta^{j,\alpha}+\sum_{\beta=1}^{d^{\widehat\Delta_i}}t^{\hat\gamma^j_\alpha}\hat\Delta^{j,\beta}\right)\in\mathcal{R}^N_{\vec X}(p^j(t)).$$

Let us fix an arbitrary γ from (γ_0,γ_1). We set

$$\bar s:=(\hat s_1,\dots,\hat s_k,s_1,\dots,s_\mu)\quad\text{with } s_i:=(s_i^1,\dots,s_i^{k_i}),$$

where each $s_i^\alpha\in[0,1]$, $\alpha=1,\dots,k_i$, $i=1,\dots,\mu$ and each $\hat s_j\in[0,1]$, $j=1,\dots,k$, Then

$$\mathcal{E}(s,\eta,\gamma):=\exp\left(\eta^{-\gamma}\sum_{i=1}^{\mu}\sum_{\iota=1}^{k_\iota}s_i^\iota\alpha^i_\iota(\gamma,\eta)V_i^\iota+\sum_{i=1}^{d^\theta}\theta_i(\gamma,\eta)\Theta^i+\sum_{j=1}^{d^\Delta}\gamma_j(\gamma,\eta)\Delta^j\right)\cdots$$

$$\cdots\exp\left(\hat s_1\eta^{-\gamma}W_1+\sum_{\alpha=1}^{d^{\widehat\Theta_j}}\left(\hat s_1\eta^{-\gamma}\right)^{\hat\theta^1_\alpha/\hat\gamma_1}\hat\Theta^{1,\alpha}+\sum_{\beta=1}^{d^{\widehat\Delta_i}}\left(\hat s_1\eta^{-\gamma}\right)^{\hat\gamma^1_\alpha/\hat\gamma_1}\hat\Delta^{1,\beta}\right)\cdots$$

$$\cdots\exp\left(\hat s_k\eta^{-\gamma}W_k+\sum_{\alpha=1}^{d^{\widehat\Theta_k}}\left(\hat s_k\eta^{-\gamma}\right)^{\hat\theta^k_\alpha/\hat\gamma_k}\hat\Theta^{k,\alpha}+\sum_{\beta=1}^{d^{\widehat\Delta_k}}\left(\hat s_k\eta^{-\gamma}\right)^{\hat\gamma^k_\beta/\hat\gamma_k}\hat\Delta^{k,\beta}\right)\in\mathcal{R}^N_{\vec X}\left(p_{\bar s,\gamma}(\eta)\right),$$

where

$$p_{\bar s,\gamma}(\eta)=p_{s,\gamma}(\eta)+p^1\left(\left(\hat s_1\eta^{-\gamma}\right)^{1/\hat\gamma_1}\right)+\cdots+p^k\left(\left(\hat s_k\eta^{-\gamma}\right)^{1/\hat\gamma_k}\right).$$

Taking into account the estimations for $p_{s,\gamma}(\cdot)$ and for $p^j(\cdot)$, $j=1,\dots,k$, we obtain that

$$p_{\bar s,\gamma}(\eta)<\nu\,\eta^{-\lambda}+\sum_{j=1}^{k}\hat\nu_j\;\left(\hat s_j\eta^{-\gamma}\right)^{\hat\lambda_j/\hat\gamma_j}<\nu\,\eta^{-\bar\lambda},\tag{11}$$

where $\nu := (\nu + \sum_{j=1}^{k} \hat{\nu}_j)$ and $\overline{\lambda} := \min(\lambda, \frac{\gamma\hat{\lambda}_1}{\hat{\gamma}_1}, \dots, \frac{\gamma\hat{\lambda}_k}{\hat{\gamma}_k})$. The inequalities $0 < \gamma < (N+1)\lambda$, and $0 < \hat{\gamma}_j < (N+1)\hat{\lambda}_j$, $j = 1, \dots, k$, imply that $0 < \gamma < (N+1)\overline{\lambda}$. Applying the C-B-H formula, we obtain that

$$\begin{aligned} &\mathcal{E}(s,\eta,\gamma) \qquad\qquad (12)\\ &= \exp\left(\eta^{-\gamma}\left(\sum_{i=1}^{\mu}\sum_{\iota=1}^{k_\iota} s_i^\iota \alpha_\iota^i(\gamma,\eta) V_i^\iota + \sum_{j=1}^{k} \hat{s}_j W_j\right) + \sum_{i=1}^{\hat{d}^\theta} \hat{\theta}_i^{\bar{s}}(\gamma,\eta)\hat{\Theta}^i + \sum_{j=1}^{\hat{d}^\Delta} \hat{\delta}_j^{\bar{s}}(\gamma,\eta)\hat{\Delta}^j\right)\\ &\qquad \in \mathcal{R}_{\vec{X}}^N\left(p_{\bar{s},\gamma}(\eta)\right), \end{aligned}$$

where

i) $\hat{\Theta}^i \in \mathcal{L}^N(\vec{X})$ with $\hat{\Theta}^i(\vec{f})(x_0) = 0$ for each $\alpha = 1, \dots, \hat{d}^\theta$;

ii) $\hat{\theta}_i^{\bar{s}} : (\gamma_0, \gamma_1) \times [0, +\infty) \to [0, +\infty)$, $i = 1, \dots, d^\theta$, are functions such that $\lim_{\eta\to+\infty} \hat{\theta}_i^{\bar{s}}(\gamma, \eta) = 0$ uniformly with respect to $\bar{s}$;

iii) $\hat{\delta}_j^{\bar{s}} : (\gamma_0, \gamma_1) \times [0, +\infty) \to [0, +\infty)$, $j = 1, \dots, d^\Delta$, are functions such that $\lim_{\eta\to+\infty} \eta^\gamma\, \hat{\delta}_j^{\bar{s}}(\gamma, \eta) = 0$ uniformly with respect to $\bar{s}$.

If we expand the right-hand side of (12), it turns out that

$$\mathcal{E}(s,\eta,\gamma) = 1 + \eta^{-\gamma}\left(\sum_{i=1}^{\mu}\sum_{\alpha=1}^{k_i} s_i^j V_i^j + \sum_{j=1}^{k} \hat{s}_j W_j\right) + Y_2(\eta,\bar{s},\gamma) + Y_3(\eta,\bar{s},\gamma),$$

where $Y_2(\eta,\bar{s},\gamma)$ is a sum of powers of $\hat{\theta}_i^{\bar{s}}(\gamma,\eta)\hat{\Theta}^i$ and $Y_3(\eta,\bar{s},\gamma)$ is a sum of products of the factors $\eta^{-\gamma}\big(\sum_{i=1}^{\mu}\sum_{\alpha=1}^{k_i} s_i^j V_i^j + \sum_{j=1}^{k} \hat{s}_j W_j\big)$, $\hat{\theta}_i^{\bar{s}}(\gamma,\eta)\hat{\Theta}^i$ and $\hat{\delta}_j^{bars}(\gamma,\eta)\hat{\Delta}^j$ and at last one factor is $\hat{\delta}_j^{\bar{s}}(\gamma,\eta)\hat{\Delta}^j$ or $\eta^{-\gamma}\sum_{i=1}^{d}\sum_{j=1}^{k_i} s_i^j V_i^j$. Since each Lie bracket $\Theta^i(\vec{f})(x_0) = 0$, then every power of $\Theta^i(\vec{f})(x_0) = 0$ vanishes as well. Hence $Y_2(\eta,\bar{s},\gamma)(\vec{f})(x_0) = 0$. Also, the definition of $Y_3(\eta,\bar{s},\gamma)$ implies that $\lim_{\eta\to\infty} \eta^{-\gamma} Y_3(\eta,\bar{s},\gamma) = 0$ uniformly with respect to $\bar{s}$.

According to (11) and (12), there exists an element $u_{\eta,s,\gamma} \in \mathcal{U}_{pc}$ defined on some interval $[0, T_{\eta,s,\gamma}]$ with $T_{\eta,s,\gamma} < \nu\eta^{-\overline{\lambda}}$ such that Ser $(u_{\eta,s,\gamma}) = \mathcal{E}(s,\eta,\gamma)$. Reminding that $\gamma < (N+1)\overline{\lambda}$ and denoting $x\left(p_{s,\gamma}(\eta), u_{\eta,s,\gamma}, x_0\right)$ by $\hat{\pi}(\eta,s,\gamma)$, we obtain that for each coordinate function ϕ_i, $i = 1, \dots, n$,

$$\phi_i\left(\hat{\pi}(\eta,s,\gamma)\right) = \phi(x_0) + \eta^{-\gamma}\left(\sum_{i=1}^{\mu}\sum_{\alpha=1}^{k_i} s_i^j V_i^j + \sum_{j=1}^{k} \hat{s}_j W_j\right)(\vec{f})(\phi_i)(x_0) + o_s\left(\eta^{-\gamma}\right),$$

where $o_s(t)$ means that $\lim_{t\to 0} \frac{o_s(t)}{t} \to 0$, the convergence being uniform with respect to s. Clearly, $\Lambda(\vec{f})(\phi_j)(x_0)$ is the j-th component of the vector $\Lambda(\vec{f})(x_0)$. Thus we find that

$$\hat{\pi}(\eta,s,\gamma) = x_0 + \eta^{-\gamma}\left(\sum_{i=1}^{\mu}\sum_{\alpha=1}^{k_i} s_i^j V_i^j + \sum_{j=1}^{k} \hat{s}_j W_j\right)(\vec{f})(x_0) + o_s\left(\eta^{-\gamma}\right). \qquad (13)$$

Now, let $H^{-1} : R^n \to R^\chi$ with $\chi := k + k_1 + \cdots + k_\mu$ be a left pseudo inverse of the linear map

$$s \to \left(\sum_{i=1}^{\mu} \sum_{\alpha=1}^{k_i} s_i^j V_i^j + \sum_{j=1}^{k} \hat{s}_j W_j \right) (\vec{f})(x_0),$$

where $s = \left(\hat{s}_1, \ldots, \hat{s}_k, s_1^1, \ldots, s_\mu^{k_\mu}\right)$ belongs to the subset Σ_χ of R^χ consisting of all vectors with nonnegative components and such that the sum of their components is equal to 1, and define the map $\varphi(\eta, y) := \hat{\pi}\left(\eta, H^{-1}(y), \gamma\right) - x_0$ for $y \in H(\Sigma_\chi)$ and each positive integer η. Let B be a compact ball in R^n centered at the origin, and such that $H^{-1}(B) \subset \Sigma_\chi$. Then the maps $\psi_{\eta,\gamma}(y) \to \eta^\gamma \varphi(\eta, y)$ are well defined for $y \in B$ and for each positive integer η, and converge to the identity map of B as $\eta \to \infty$. Hence, $\psi_{\eta,\gamma}(B)$ contains a neighbourhood of the origin if $\eta > 0$ is big enough. Thus, the set $C_\eta := \eta^{-\gamma}\{\psi_{\eta,\gamma}(y) : y \in B\}$ also contains a neighbourhood of the origin, if the positive integer $\eta > 0$ is big enough.

Now, if $z \in C_\eta$, then $x_0 + z = \hat{\pi}(\eta, H^{-1}(y), \gamma)$ for some $y \in B$. We set $s = H^{-1}(y)$ and obtain that $x_0 + z = \hat{\pi}(\eta, s, \gamma)$ for some $s \in \Sigma_\chi$. Therefore $x_0 + z$ is reachable from the point x_0 in time $T_{\eta,s,\gamma}$ with $T_{\eta,s,\gamma} < \nu\eta^{-\overline{\lambda}}$, i.e. $x_0 + z$ is reachable from x_0 in time less than $\nu\eta^{-\overline{\lambda}}$. Since $\mathcal{R}(x_0, t_1) \subseteq \mathcal{R}(x_0, t_2)$ whenever $0 \le t_1 \le t_2$, $x_0 + C_\eta \subseteq \mathcal{R}\left(x_0, \nu\eta^{-\overline{\lambda}}\right)$ and C_η contains a neighbourhood of the origin for all sufficiently small $\eta > 0$, we obtain that $\mathcal{R}(x_0, t)$ contains a neighbourhood of the point x_0 for each $t > 0$. This completes the proof of Theorem 2.7. $\square$

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