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Semi-Infinite Programming: Strong Stability implies EMFCQ*

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In this paper we consider strongly stable stationary points of semi-infinite programming problems. The concept of strong stability was introduced by Kojima for finite programming problems and it refers to the local existence and uniqueness of a stationary point for each sufficiently small perturbed problem where perturbations up to second order are allowed. Under the extended Mangasarian-Fromovitz constraint qualification (EMFCQ) strong stability can be characterized algebraically by the first and second derivatives of the describing functions. In this paper we show that strong stability implies that EMFCQ holds at the stationary point under consideration.

Keywords: Semi-infinite programming, strongly stable stationary point, extended Mangasarian-Fromovitz constraint qualification

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1. Introduction

As a starting point of this paper we consider a semi-infinite programming problem of the form

$$SIP(f, H, G) \qquad \text{minimize } f(x) \text{ subject to } x \in M[H, G], \tag{1}$$

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where the feasible set is given as

$$M[H,G] = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ i \in I,\ G(x,y) \geq 0,\ y \in Y\},$$

and where $I = \{1,..,m\}$, $H = (h_1,..,h_m)$, $Y \subset \mathbb{R}^r$ is an infinite compact index set and the real-valued functions f, h_i, $i \in I$ and G are twice continuously differentiable. *Semi-infinite* means that there are finitely many decision variables $x \in \mathbb{R}^n$ and infinitely many inequality constraints: each $\bar{y} \in Y$ represents a corresponding constraint $G(\cdot, \bar{y}) \geq 0$. Semi-infinite programming has a wide range of applications and it became a vivid research topic within mathematical programming over the last two decades. For more details, we refer to the pioneering survey paper [6], the monographs [2, 15, 18], and the edited compilations [3, 16].

In this paper we consider the concept of strong stability of a stationary point which was introduced by Kojima [12] for finite programming problems (having finitely many equality and inequality constraints) and which refers to the local existence and uniqueness of a stationary point for each sufficiently small perturbed problem where perturbations up to second order are allowed. In case that the well-known Mangasarian-Fromovitz constraint qualification (MFCQ) holds at the strongly stable stationary point under consideration, Kojima [12] provided an algebraic characterization of strong stability by using the first and second order derivatives of the objective function and the constraints.

Later, in [17], the concept of strong stability was generalized to semi-infinite problems of the type (1) and a corresponding algebraic characterization was provided under the extended Mangasarian-Fromovitz constraint qualification (EMFCQ). The strong stability of stationary points, both for finite and semi-infinite problems, plays a crucial role for several areas in mathematical programming such as sensitivity analysis, parametric optimization, structural stability and others (for more details see e.g. [4, 8, 9, 10, 11, 13]).

The remaining question was whether (E)MFCQ is a necessary condition for the strong stability of a stationary point. An affirmative answer for finite problems was given in [5]. In this paper we show for semi-infinite problems that the strong stability of a stationary point $\bar{x} \in M[H,G]$ of $SIP(f,H,G)$ implies that EMFCQ holds at $\bar{x}$.

The paper is organized as follows. Section 2 contains definitions and basic results. In Section 3 we present all new results and, in particular, we prove the main theorem.

2. Definitions and basic results

Throughout this paper let $C^2(\mathbb{R}^n,\mathbb{R})$ be the space of twice continuously differentiable real-valued functions defined on $\mathbb{R}^n$. The product space is defined as $C^2(\mathbb{R}^n,\mathbb{R})^k = C^2(\mathbb{R}^n,\mathbb{R}) \times \cdots \times C^2(\mathbb{R}^n,\mathbb{R})$ (k-times). If $g \in C^2(\mathbb{R}^n,\mathbb{R})$, then the *row vector* $Dg(\bar{x})$ ($D_{x'}g(\bar{x})$) represents the gradient (partial derivative with respect to the subvector x' of x) of g at $\bar{x}$. Second derivatives are analogously defined. The set *support of* g ($\mathrm{supp}(g)$) denotes the closure of the set $\{x \in \mathbb{R}^n \mid g(x) \neq 0\}$. For $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$ let $B_\delta(\bar{x}) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| < \delta\}$, where $\|\cdot\|$ denotes the Euclidean norm. For a set $V \subset \mathbb{R}^n$ let $\mathrm{cl}\, V$ and $\mathrm{conv}(V)$ denote the closure of

V and the convex hull of V, respectively. If Z is a symmetric (n,n)-matrix, then $\|Z\| = \max\{\|Zx\| \mid x \in \mathbb{R}^n, \|x\| = 1\}$.

In the following we recall several definitions and basic results which will be used later.

Fritz-John points and stationary points of $SIP(f,H,G)$

Let $SIP(f,H,G)$ be given as in (1) and define for a feasible point $\bar{x} \in M[H,G]$ the set of active inequality constraints as

$$Y_0(\bar{x}) = \{y \in Y \mid G(\bar{x},y) = 0\}.$$

Obviously, $Y_0(\bar{x})$ is a compact set. The following definition is well-known ([6, 7]).

Definition 2.1. (i) A point $\bar{x} \in M[H,G]$ is called a *Fritz-John point* (*FJ point*) of $SIP(f,H,G)$ if there exist finitely many $y^j \in Y_0(\bar{x})$, $j = 1,\dots,q$ and corresponding multipliers $\mu_j \geq 0$, $j = 0,\dots,q$, $\lambda_i \in \mathbb{R}$, $i \in I$ – not all vanishing – such that

$$\mu_0 Df(\bar{x}) + \sum_{i \in I} \lambda_i Dh_i(\bar{x}) - \sum_{j=1}^{q} \mu_j D_x G(\bar{x}, y^j) = 0. \tag{2}$$

(ii) A point $\bar{x} \in M[H,G]$ is called a *stationary point* of $SIP(f,H,G)$ if $\bar{x}$ is a FJ point of $SIP(f,H,G)$ and in (2) one can choose $\mu_0 > 0$.

Note that if $\bar{x}$ is a local minimizer of $SIP(f,H,G)$, then $\bar{x}$ is also a FJ point of $SIP(f,H,G)$ ([6, 7]).

Strong stability of a stationary point

As mentioned in Section 1, Kojima [12] established the strong stability of a stationary point of a finite programming problem which refers to the local existence and uniqueness of a stationary point for each sufficiently small perturbed problem where perturbations up to second order are allowed. We recall the following definition from [17] where this concept was generalized to a semi-infinite problem.

For $U \subset \mathbb{R}^n$ and $(\tilde{f}, \tilde{H}, \tilde{G}) \in C^2(\mathbb{R}^n, \mathbb{R})^{1+m} \times C^2(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R})$ we define the seminorm

$$\operatorname{norm}((\tilde{f}, \tilde{H}, \tilde{G}), U) = \max \left\{ \begin{array}{l} \sup\limits_{x \in U} \max\{|\tilde{f}(x)|,\ \|D\tilde{f}(x)\|,\ \|D^2\tilde{f}(x)\|\}, \\ \sup\limits_{i \in I} \sup\limits_{x \in U} \max\{|\tilde{h}_i(x)|,\ \|D\tilde{h}_i(x)\|,\ \|D^2\tilde{h}_i(x)\|\}, \\ \sup\limits_{y \in Y} \sup\limits_{x \in U} \max\{|\tilde{G}(x,y)|,\ \|D\tilde{G}(x,y)\|,\ \|D^2\tilde{G}(x,y)\|\} \end{array} \right\}.$$

Definition 2.2. A stationary point $\bar{x}$ of $SIP(f,H,G)$ is called *strongly stable* if there exists a real number $\delta^* > 0$ such that for each $\delta \in (0, \delta^*]$ there is a real number $\gamma > 0$ and $\operatorname{cl} B_\delta(\bar{x})$ contains a stationary point of the problem $SIP(f+\tilde{f}, H+\tilde{H}, G+\tilde{G})$ which is unique in $\operatorname{cl} B_{\delta^*}(\bar{x})$ whenever norm $((\tilde{f}, \tilde{H}, \tilde{G}), \operatorname{cl} B_{\delta^*}(\bar{x})) \leq \gamma$.

In the following definition we refer to the seminorm given above.

Definition 2.3. Let $\{f^\nu, H^\nu, G^\nu\}_{\nu\in\mathbb{N}} \subset C^2(\mathbb{R}^n,\mathbb{R})^{1+m} \times C^2(\mathbb{R}^n \times \mathbb{R}^r,\mathbb{R})$ be a sequence of function vectors. We say that $\{f^\nu, H^\nu, G^\nu\}_{\nu\in\mathbb{N}}$ converges to $(\bar{f}, \bar{H}, \bar{G})$ *in the seminorm with respect to* U (notation: $\{f^\nu, H^\nu, G^\nu\} \xrightarrow{U} (\bar{f}, \bar{H}, \bar{G})$) if norm $((f^\nu - \bar{f}, H^\nu - \bar{H}, G^\nu - \bar{G}), U) \to 0$ as $\nu \to \infty$.

2.1. Constraint qualification

In the following we recall the extension of the well-known Mangasarian-Fromovitz constraint qualification to the semi-infinite case ([7, 11]).

The *extended Mangasarian-Fromovitz constraint qualification (EMFCQ)* is said to hold at $\bar{x} \in M[H,G]$ if

(i) the gradients $Dh_i(\bar{x})$, $i \in I$ are linearly independent and
(ii) there exists $\xi \in \mathbb{R}^n$ such that

$$\begin{aligned} Dh_i(\bar{x})\xi = 0, \quad i \in I, \\ D_xG(\bar{x},y)\xi > 0, \quad y \in Y_0(\bar{x}). \end{aligned} \tag{3}$$

Note that EMFCQ is invariant under local C^1-coordinate transformations [11]. Furthermore, if $\bar{x}$ is a local minimizer of $SIP(f,H,G)$ and EMFCQ holds at $\bar{x}$, then $\bar{x}$ is a stationary point of $SIP(f,H,G)$ [6, 7]. As already mentioned in Section 1, in [17] an equivalent algebraic characterization of the strong stability of a stationary point $\bar{x}$ of a semi-infinite problem is given under the assumption that EMFCQ holds at $\bar{x}$. The next lemma presents for EMFCQ a corresponding dual formulation.

Lemma 2.4.

(i) [1] *Let $V \subset \mathbb{R}^n$ be a nonempty compact set. Then, the system*

$$v^\top \xi > 0 \quad \text{for all } v \in V$$

has a solution $\xi \in \mathbb{R}^n$ if and only if $0 \notin \operatorname{conv}(V)$.

(ii) *The condition EMFCQ does not hold at $\bar{x} \in M[H,G]$ if and only if there exist finitely many $y^j \in Y_0(\bar{x})$, $j = 1,\dots,q$ and multipliers $\lambda_i \in \mathbb{R}$, $i \in I$, $\mu_j \geq 0$, $j = 1,\dots,q$ - not all vanishing - such that*

$$\sum_{i\in I} \lambda_i Dh_i(\bar{x}) - \sum_{j=1}^{q} \mu_j D_xG(\bar{x},y^j) = 0. \tag{4}$$

Proof of (ii)**.** Suppose first that EMFCQ does not hold at $\bar{x}$. If the gradients $Dh_i(\bar{x})$, $i \in I$ are linearly dependent, then, obviously, a combination (4) with $\mu_j = 0$, $j = 1,\dots,q$ exists. If the gradients $Dh_i(\bar{x})$, $i \in I$ are linearly independent, then, by applying an appropriate C^1-coordinate transformation, assume without loss of generality that $I = \emptyset$. Since EMFCQ does not hold at $\bar{x}$ there does not exist a vector $\xi \in \mathbb{R}^n$ satisfying (3) and, by (i), we get

$$0 \in \operatorname{conv}(\{D_xG(\bar{x},y),\ y \in Y_0(\bar{x})\}).$$

The latter means that a combination (4) exists.

Now, suppose that a combination (4) exists and assume that EMFCQ holds at $\bar{x}$. By multiplying the equation (4) with the vector $\xi \in \mathbb{R}^n$ from (3) we obtain a contradiction. $\square$

The latter lemma implies that if EMFCQ does not hold at $\bar{x} \in M[H, G]$, then $\bar{x}$ is a FJ point of $SIP(f, H, G)$.

Some auxiliary results

Lemma 2.5 (Caratheodory's theorem, see e.g. [6, 7]). *Let $V \subset \mathbb{R}^n$ be a non-empty set and $x \in \operatorname{conv}(V) \setminus \{0\}$. Then, there exist linearly independent vectors $x^i \in V$, $i = 1, \dots, p$ and real numbers $\mu_i > 0$, $i = 1, \dots, p$ such that*

$$x = \sum_{i=1}^{p} \mu_i x^i.$$

Lemma 2.6 ([8]). *Let $w^i \in \mathbb{R}^n$, $i = 0, \dots, p$. Then, exactly one of the alternatives (i), (ii) holds:*

(*i*) *The system*

$$(w^0)^\top \xi > 0, \ (w^i)^\top \xi \geq 0, \quad i = 1, \dots, p$$

has a solution $\xi \in \mathbb{R}^n$.

(*ii*) *There exist real numbers $\mu_0 > 0$, $\mu_i \geq 0$, $i = 1, \dots, p$ such that $\sum_{i=0}^{p} \mu_i w^i = 0$.*

Lemma 2.7 ([8]). *Let $V \subset \mathbb{R}^n$ be a closed set and U an open neighbourhood of V. Then, there exists a function $\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with the following properties:*

(*i*) $\varepsilon(x) \in [0, 1]$ *for all* $x \in \mathbb{R}^n$.

(*ii*) $\varepsilon(x) = 1$ *on a neighbourhood of* V.

(*iii*) $\operatorname{supp}(\varepsilon) \subset U$.

Remark 2.8. In the following we will sometimes use the expression "we perturb locally around a point $\bar{x}$ the function g as follows: $g(x) + \tilde{g}(x)$". By this expression we mean that we actually add the function $\varepsilon(x)\tilde{g}(x)$ to g where the function ε has the properties as described in Lemma 2.7 and V is a neighbourhood of $\bar{x}$. If the words "on the set U" are added to this expression, then $\operatorname{supp}(\varepsilon)$ has to be contained in U.

Partition of unity

We recall the well-known definition of a partition of unity (cf. e.g. [8]).

Definition 2.9. Let $\{U^j, j \in J\}$ be an open covering of $\mathbb{R}^n$ where J is an arbitrary index set. A family of C^∞-functions $\Psi^j : \mathbb{R}^n \to \mathbb{R}$, $j \in J$ is called a (C^∞)-*partition of unity subordinate to* $\{U^j, j \in J\}$ if:

- $\operatorname{supp}(\Psi^j) \subset U^j$,
- $\{\operatorname{supp}(\Psi^j), j \in J\}$ is locally finite, that means that for any $\bar{x} \in \mathbb{R}^n$ there is a neighbourhood U of $\bar{x}$ such that the set $\{j \in J \mid U \cap \operatorname{supp}(\Psi^j) \neq \emptyset\}$ is finite,
- $\Psi^j(x) \geq 0$, $j \in J$ for all $x \in \mathbb{R}^n$,

- $\sum_{j \in J} \Psi^j(x) = 1$ for all $x \in \mathbb{R}^n$.

By [14], for each open covering $\{U^j, j \in J\}$ of $\mathbb{R}^n$ there exists a (C^∞-) partition of unity subordinate to $\{U^j \mid j \in J\}$.

3. Results

Throughout this section let $\bar{x} \in M[H, G]$ be our feasible point under consideration. The following proposition is a generalization of [5, Perturbation Lemma].

Proposition 3.1. *Assume that EMFCQ does not hold at $\bar{x}$. Then, there exists a function vector $(\tilde{f}, \tilde{H}, \tilde{G})$ arbitrarily close to (f, H, G) in the seminorm with respect to any neighbourhood U of $\bar{x}$ such that $\bar{x}$ is a stationary point of $SIP(\tilde{f}, \tilde{H}, \tilde{G})$.*

Proof. By Lemma 2.4 (ii), there exists a combination (4) with

$$\sum_{i \in I} \lambda_i^2 + \sum_{j=1}^{q} \mu_j^2 = 1.$$

Let $\varepsilon > 0$ and perturb locally around $\bar{x}$ the functions $h_i, i \in I$ as follows:

$$h_{i,\varepsilon}(x) = h_i(x) - \varepsilon \lambda_i Df(\bar{x})(x - \bar{x}), \quad i \in I.$$

Now, we choose open neighbourhoods $W^j \subset \mathbb{R}^r$ of $y^j, j = 1, \ldots, q$ with $W^i \cap W^j = \emptyset$ whenever $i \neq j$ and select on W^j the multiplier $\mu_j, j = 1, \ldots, q$ and on $\mathbb{R}^r \setminus \{y^1, \ldots, y^q\}$ the scalar $0 \in \mathbb{R}$. Then, by using a partition of unity subordinate to the open covering $\{W^1, \ldots, W^q, \mathbb{R}^r \setminus \{y^1, \ldots, y^q\}\}$ of $\mathbb{R}^r$ we obtain a C^∞-function $\bar{\mu} : y \in \mathbb{R}^r \mapsto \bar{\mu}(y) \in \mathbb{R}$ with $\bar{\mu}(y^j) = \mu_j, j = 1, \ldots, q$. Perturb locally around $\bar{x}$ the function G as follows:

$$G_\varepsilon(x, y) = G(x, y) + \varepsilon \bar{\mu}(y) Df(\bar{x})(x - \bar{x}).$$

Note that $(H_\varepsilon, G_\varepsilon) \xrightarrow{U} (H, G)$ as $\varepsilon \to 0$ for any neighbourhood U of $\bar{x}$ (where $H_\varepsilon = (h_{1,\varepsilon}, \ldots, h_{m,\varepsilon})$). Then,

$$Df(\bar{x}) + \frac{1}{\varepsilon}\left[\sum_{i \in I} \lambda_i Dh_{i,\varepsilon}(\bar{x}) - \sum_{j=1}^{q} \mu_j D_x G_\varepsilon(\bar{x}, y^j)\right] = 0,$$

and, hence, $\bar{x}$ is a stationary point of $SIP(f, H_\varepsilon, G_\varepsilon)$. □

The next proposition states that locally around a strongly stable stationary point each sufficiently small perturbed problem has a uniquely determined FJ point (which is also a stationary point).

Proposition 3.2. *Let $\bar{x}$ be a strongly stable stationary point of $SIP(f, H, G)$ and $\delta^* > 0$ be defined as in Definition 2.2. Then, for each $\delta \in (0, \delta^*]$ there is a real number $\gamma(\delta) > 0$ and $\operatorname{cl} B_\delta(\bar{x})$ contains a FJ point of $SIP(f + \tilde{f}, H + \tilde{H}, G + \tilde{G})$ which is unique in $\operatorname{cl} B_{\delta^*}(\bar{x})$ whenever norm $(\tilde{f}, \tilde{H}, \tilde{G}, \operatorname{cl} B_{\delta^*}(\bar{x})) \leq \gamma(\delta)$. Furthermore, this unique FJ point is also a stationary point of $SIP(f + \tilde{f}, H + \tilde{H}, G + \tilde{G})$.*

Proof. Assume the contrary: there exist $\delta \in (0, \delta^*]$ and sequences $\{\gamma^\nu\}_{\nu\in N}, \gamma^\nu > 0, \gamma^\nu \to 0$ and $\{f^\nu, H^\nu, G^\nu\}_{\nu\in N}$ with norm $((f^\nu, H^\nu, G^\nu), \operatorname{cl} B_{\delta^*}(\bar{x})) \to 0$ such that $\operatorname{cl} B_{\delta^*}(\bar{x})$ contains two different FJ points of $SIP^\nu = SIP(f + f^\nu, H + H^\nu, G + G^\nu)$ (note that, without loss of generality, $\gamma^\nu < \gamma$, where γ is chosen as in Definition 2.2). By the strong stability of $\bar{x}$, only one of them can be a stationary point, say x^ν, which is unique in $\operatorname{cl} B_{\delta^*}(\bar{x})$. Let $\bar{x}^\nu \in \operatorname{cl} B_{\delta^*}(\bar{x}) \setminus \{x^\nu\}$ be the other FJ point of SIP^ν and choose a neighbourhood U^ν of $\bar{x}^\nu$ with $x^\nu \notin U^\nu$. As in the proof of Proposition 3.1, locally around $\bar{x}^\nu$ on U^ν we can perturb (H^ν, G^ν) in such a way that we get for the perturbed function $(\bar{H}^\nu, \bar{G}^\nu)$ that:

- $(f, \bar{H}^\nu, \bar{G}^\nu) \to (f, H, G)$,
- x^ν and $\bar{x}^\nu$ are stationary points of $SIP(f, \bar{H}^\nu, \bar{G}^\nu)$.

However, the latter two properties contradict the strong stability of $\bar{x}$. This completes the proof. □

Proposition 3.3. *Consider a sequence $\{f^\nu, H^\nu, G^\nu\}_{\nu\in\mathbb{N}}$ of functions and assume that*

- $(f^\nu, H^\nu, G^\nu) \xrightarrow{U} (f, H, G)$ *for some neighbourhood U of $\bar{x}$ and*
- $x^\nu \in U$ *is a FJ point of $SIP(f^\nu, H^\nu, G^\nu)$ with $x^\nu \to \tilde{x}$, $\tilde{x} \in U$.*

Then, $\tilde{x}$ is a FJ point of $SIP(f, H, G)$.

Proof. Since x^ν is a FJ point of $SIP(f^\nu, H^\nu, G^\nu)$ there exist combinations

$$\mu_0^\nu Df^\nu(x^\nu) + \sum_{i\in I} \lambda_i^\nu Dh_i^\nu(x^\nu) - \sum_{j=1}^{q^\nu} \mu_j^\nu D_x G^\nu(x^\nu, y^{j,\nu}) = 0$$

with $\sum_{i\in I} |\lambda_i^\nu| + \sum_{j=0}^{q^\nu} \mu_j^\nu = 1$ and $y^{j,\nu} \in Y_0(x^\nu), j = 1, \ldots, q^\nu$.

By Lemma 2.5 and after restricting to a subsequence we can assume without loss of generality that

- q^ν is fixed for all $\nu \in \mathbb{N}$, say $q^\nu = q$,
- $\mu_j^\nu \to \mu_j,\ j = 0, \ldots, q,\ \lambda_i^\nu \to \lambda_i,\ i \in I\ \left(\sum_{i\in I} |\lambda_i| + \sum_{j=0}^{q} \mu_j = 1\right)$,
- $y^{j,\nu} \to y^j,\ j = 1, \ldots, q$ (since Y is compact).

Since $(f^\nu, H^\nu, G^\nu) \xrightarrow{U} (f, H, G)$ we get $y^j \in Y_0(\tilde{x})$ and

$$\mu_0 Df(\tilde{x}) + \sum_{i\in I} \lambda_i Dh_i(\tilde{x}) - \sum_{j=1}^{q} \mu_j D_x G(\tilde{x}, y^j) = 0.$$

Therefore, $\tilde{x}$ is a FJ point of $SIP(f, H, G)$. □

Now, we state our main result.

Theorem 3.4. *Let $\bar{x}$ be a strongly stable stationary point of $SIP(f, H, G)$. Then, EMFCQ holds at $\bar{x}$.*

Proof. Suppose that EMFCQ does not hold at $\bar{x}$ and that $\bar{x}$ is a strongly stable stationary point of $SIP(f, H, G)$ where δ^* is chosen as in Definition 2.2. In the following we will generalize some of the arguments used in [5, Theorem 2.7]. We distinguish two cases.

Case 1. The vectors $Dh_i(\bar{x})$, $i \in I$ are linearly dependent.

Subcase 1.1. The vectors $Dh_i(\bar{x})$, $i \in I$ span a subspace $W \subset \mathbb{R}^n$ whose dimension is smaller than n.

Let $\varepsilon > 0$ and perturb locally around $\bar{x}$ the functions f and G as follows:

- $f_\varepsilon(x) = f(x) + \varepsilon v^\top (x - \bar{x})$, where v is chosen in such a way that $Df(\bar{x}) + v \notin W$;
- $G_\varepsilon(x, y) = G(x, y) + \varepsilon$ (hence, the corresponding active index set $Y_\varepsilon(\bar{x})$ is empty).

We have that $(f_\varepsilon, H, G_\varepsilon) \xrightarrow{B_{\delta^*}(\bar{x})} (f, H, G)$ as $\varepsilon \to 0$ and that $\bar{x}$ is a FJ point – but not a stationary point – of $SIP(f_\varepsilon, H, G_\varepsilon)$. By Proposition 3.2, this contradicts the strong stability of $\bar{x}$.

Subcase 1.2. The vectors $Dh_i(\bar{x})$, $i \in I$ span the whole $\mathbb{R}^n$.

Without loss of generality assume that the vectors $Dh_i(\bar{x})$, $i = 1, \dots, n$ are linearly independent. Then, $\bar{x}$ is an isolated point of the set

$$\tilde{M} = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ i = 1, \dots, n\}$$

which means that there exists $\beta \in (0, \delta^*)$ such that

$$B_\beta(\bar{x}) \cap \tilde{M} = \{\bar{x}\}. \tag{5}$$

Now, let $\varepsilon > 0$ and perturb locally around $\bar{x}$ the function h_{n+1} as follows:

$$h_{n+1,\varepsilon}(x) = h_{n+1}(x) + \varepsilon.$$

By (5), we obtain for $H_\varepsilon = (h_1, \dots, h_n, h_{n+1,\varepsilon}, h_{n+2}, \dots, h_m)$ that

$$B_\beta(\bar{x}) \cap M[H_\varepsilon, G] = \emptyset.$$

Since $\bar{x}$ is a strongly stable stationary point of $SIP(f, H, G)$, the problem $SIP(f, H_\varepsilon, G)$ has a uniquely determined stationary point $x(\varepsilon) \in \operatorname{cl} B_{\delta^*}(\bar{x})$ (for all ε sufficiently small) and – by considering an appropriate subsequence – $x(\varepsilon) \to \tilde{x} \in \operatorname{cl} B_{\delta*}(\bar{x}) \setminus B_\beta(\bar{x})$ as $\varepsilon \to 0$. By Proposition 3.3, $\tilde{x}$ is a FJ point of $SIP(f, H, G)$ and, by Proposition 3.2 and $\tilde{x} \neq \bar{x}$, we obtain a contradiction.

Case 2. The vectors $Dh_i(\bar{x})$, $i \in I$ are linearly independent.

By applying an appropriate C^1-coordinate transformation, we assume without loss of generality that $I = \emptyset$. By Lemmas 2.4 (*ii*) and 2.5, there exists a combination

$$\sum_{j=1}^{q} \mu_j D_x G(\bar{x}, y^j) = 0 \tag{6}$$

with $y^j \in Y_0(\bar{x})$, $\mu_j > 0, j = 1, \ldots, q$ such that the vectors $D_x G(\bar{x}, y^j), j = 1, \ldots, q$ span a $(q-1)$-dimensional subspace $W \subset \mathbb{R}^n$.

Subcase 2.1. $q \leq n$.

Choose open neighbourhoods $W^j \subset \mathbb{R}^r$ of $y^j, j = 1, \ldots, q$ with $W^i \cap W^j = \emptyset$ whenever $i \neq j$ and select on W^j the function $(y - y^j)^T (y - y^j), j = 1, \ldots, q$ and on $\mathbb{R}^r \setminus \{y^1, \ldots, y^q\}$ the constant function with value one. Then, by using a partition of unity subordinate to the open covering $\{W^1, \ldots, W^q, \mathbb{R}^r \setminus \{y^1, \ldots, y^q\}\}$ of $\mathbb{R}^r$ we obtain a C^∞-function $g : y \in \mathbb{R}^r \mapsto g(y)$ with $g(y^j) = 0, j = 1, \ldots, q$ and $g(y) > 0$ elsewhere.

Now, let $\varepsilon > 0$ and perturb locally around $\bar{x}$ the functions f and G as follows:

- $f_\varepsilon(x) = f(x) + \varepsilon v^\top (x - \bar{x})$, where v is chosen in such a way that $Df(\bar{x}) + v \notin W$;
- $G_\varepsilon(x, y) = G(x, y) + \varepsilon g(y)$.

Obviously, $(f_\varepsilon, G_\varepsilon) \to (f, G)$ as $\varepsilon \to 0$ and $\bar{x}$ is a FJ point - but not a stationary point - of $SIP(f_\varepsilon, G_\varepsilon)$. By Proposition 3.2, this contradicts the strong stability of $\bar{x}$.

Subcase 2.2. $q > n$.

Obviously, we have $q = n + 1$. First, we show the following

Proposition. *$\bar{x}$ is an isolated point of the set*

$$P = \{x \in \mathbb{R}^n \mid G(x, y^j) \geq 0, \; j = 1, \ldots, n + 1\}.$$

Proof of the proposition. Assume the contrary which means that there exists a sequence $\{x^\nu\}_{\nu \in \mathbb{N}} \subset P$ with $x^\nu \neq \bar{x}$ and $x^\nu \to \bar{x}$. A Taylor expansion provides

$$0 \leq \frac{G(x^\nu, y^j) - G(\bar{x}, y^j)}{\|\bar{x} - x^\nu\|} = D_x G(\bar{x}, y^j) \frac{(\bar{x} - x^\nu)}{\|\bar{x} - x^\nu\|} + \frac{o(\|\bar{x} - x^\nu\|)}{\|\bar{x} - x^\nu\|}$$

and – as $\nu \to \infty$ – on the right-hand-side we get $D_x G(\bar{x}, y^j) u \geq 0, j = 1, \ldots, n + 1$ for some $u \in \mathbb{R}^n$. Since $W = \mathbb{R}^n$, there exists an index $i_0 \in \{1, \ldots, n + 1\}$, say $i_0 = 1$, such that

$$D_x G(\bar{x}, y^1) u > 0, \quad D_x G(\bar{x}, y^j) u \geq 0, \quad j = 2, \ldots, n + 1.$$

By Lemma 2.6, the solvability of the latter system and combination (6) provides a contradiction. This completes the proof of the proposition. □

By the latter proposition, there exists $\beta \in (0, \delta^*)$ such that

$$B_\beta(\bar{x}) \cap M[G] = \{\bar{x}\}. \tag{7}$$

Now, let $\varepsilon > 0$ and perturb locally around $\bar{x}$ the function G as follows

$$G_\varepsilon(x, y) = G(x, y) - \varepsilon.$$

By (7), we obtain $B_\beta(\bar{x}) \cap M[G_\varepsilon] = \emptyset$. Since $(f, G_\varepsilon) \xrightarrow{B_{\delta^*}(\bar{x})} (f, G)$ as $\varepsilon \to 0$ and $\bar{x}$ is a strongly stable stationary point of $SIP(f, G)$, the problem $SIP(f, G_\varepsilon)$ has a uniquely determined stationary point $x(\varepsilon) \in \mathrm{cl}\, B_{\delta^*}(\bar{x}) \setminus B_\beta(\bar{x})$ with – after restricting to an appropriate subsequence – $x(\varepsilon) \to \tilde{x} \in \mathrm{cl}\, B_{\delta^*}(\bar{x}) \setminus B_\beta(\bar{x})$ as $\varepsilon \to 0$. By Proposition 3.3, $\tilde{x}$ is a FJ point of $SIP(f, G)$ and, by Proposition 3.2 and $\tilde{x} \neq \bar{x}$, we obtain a contradiction.

This completes the proof of the theorem.

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