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Convex Integrals on Sobolev Spaces

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Let $j_0, j_1 : \mathbb{R} \mapsto [0, \infty)$ denote convex functions vanishing at the origin, and let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ. This paper is devoted to the study of the convex functional $J(u) = \int_\Omega j_0(u)d\Omega + \int_\Gamma j_1(\gamma u)d\Gamma$ on the Sobolev space $H^1(\Omega)$. We describe the convex conjugate J^* and the subdifferential ∂J. It is shown that the action of ∂J coincides pointwise a.e. in Ω with $\partial j_0(u(x))$, and a.e on Γ with $\partial j_1(u(x))$. These conclusions are nontrivial because, although they have been known for the subdifferentials of the functionals $J_0(u) = \int_\Omega j_0(u)d\Omega$ and $J_1(u) = \int_\Gamma j_1(\gamma u)d\Gamma$, the lack of any growth restrictions on j_0 and j_1 makes the sufficient *domain condition* for the sum of two maximal monotone operators ∂J_0 and ∂J_1 infeasible to verify directly.

The presented theorems extend the results in [6] and fundamentally complement the emerging research literature addressing supercritical damping and sources in hyperbolic PDE's. These findings rigorously confirm that a *combination* of supercritical interior and boundary damping feedbacks can be modeled by the subdifferential of a suitable convex functional on the state space.

1. Introduction

Wave equations under the influence of nonlinear damping and nonlinear sources have generated considerable interest over recent years. As the linear theory has been substantially developed, many problems for systems with high-order nonlinearities remain open. The well-posedness results for semilinear evolution PDE's rely heavily on nonlinear semigroups and the theory of monotone operators (see for instance [3, 4, 5, 9]). From the viewpoint of stability, the presence of non-dissipative sources

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of critical or supercritical order necessitates the introduction of interior and/or boundary damping. On the other hand, it is well known that nonlinear dissipation in hyperbolic and hyperbolic-like dynamics has been a source of many technical difficulties, especially when both interior g_0 and boundary g_1 feedbacks are present. For instance, such difficulties are manifested when one studies a system of the form:

$$\begin{cases} w_{tt} - \Delta w + g_0(w_t) = F_0(w), & \text{in } \Omega \times (0,T), \\ \partial_\nu w + w + g_1(w_t) = F_1(w), & \text{in } \partial\Omega \times (0,T), \end{cases}$$

where F_0, F_1 are strong sources of supercritical order, and g_0, g_1 represent the interior and boundary damping, respectively. One of the main challenges here is the simple fact that the sum of two maximal monotone operators (one is originating from the interior and the other from the boundary damping) is not necessarily maximal monotone, unless one is able to check the validity of a certain "domain condition", or an equivalent thereof, which in most cases, is too difficult to verify directly when the damping functions are arbitrary continuous monotone graphs.

The typical strategy for well-posedness of such PDEs with interior and boundary damping, would be to introduce the nonlinear semigroup generator by defining the subdifferential of the sum of the convex functionals which correspond to the (antiderivatives) of the two nonlinear damping maps. However, in this context one has to verify *whether the subdifferential does indeed coincide pointwise a.e. with the sum of the damping in the interior and that on the boundary.* The latter conclusion does not trivially follow from monotone operator theory. Techniques pertinent to this issue are scattered in the literature on convex analysis, in particular [1, 2, 6, 7]. To our knowledge none of the existing arguments thoroughly address the subdifferential of the sum of functionals, which are defined on spaces related by a trace map. A famous result in this direction was established by H. Brézis in [6] dealing, however, with a single functional and one underlying spatial domain; this scenario cannot be directly applied to the present case. The approach given here follows and extends the techniques in [6].

It is important to stress that we resolve this question of identification of the subdifferential of a sum of convex functionals without imposing any growth restrictions on them (as, for example, in the treatment of boundary damping in [9]). The results intend to fundamentally complement the existence theory for nonlinear wave equations with interior and boundary damping, e.g., [3, 4, 5]. All of the theorems in this paper are formulated for arbitrary non-negative convex functionals j_0 and j_1 that vanish at 0. In the aforementioned context of hyperbolic PDE's with nonlinear interior $g_0(s)$ and boundary $g_1(s)$ damping feedback maps (each continuous monotone increasing, and vanishing at the origin) one could represent $j_k(s) = \int_0^s g_k(\tau)d\tau$, $k = 0, 1$. One of the main features of the presented analysis is that it **does not** depend on the growth rate of each $j_k(s)$, hence we impose no restriction on the growth rates of each $g_k(s)$.

2. Main results

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with a C^2-boundary Γ. Let j_0, $j_1 : \mathbb{R} \to [0, +\infty)$ be convex functions vanishing at 0. Note that, since j_0 and j_1 are convex functions and finite everywhere, then they are continuous on $\mathbb{R}$. Let $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ denote the trace map, and define the functional $J : H^1(\Omega) \to [0, +\infty]$ by

$$J(u) = \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma. \tag{1}$$

Clearly, J is convex and lower semicontinuous with its domain given by

$$D(J) = \{u \in H^1(\Omega) : j_0(u) \in L^1(\Omega) \text{ and } j_1(\gamma u) \in L^1(\Gamma)\}. \tag{2}$$

As usual, $D(\partial J)$ represents the set of all functions $u \in H^1(\Omega)$ for which $\partial J(u)$ is nonempty. It is well known that $D(\partial J)$ is a dense subset of $D(J)$. The convex conjugate of J is defined by

$$J^*(T) = \sup\{\langle T, u\rangle - J(u) : u \in D(J)\} \text{ for } T \in (H^1(\Omega))', \tag{3}$$

where, here and later, $(H^1(\Omega))'$ denotes the dual space of $H^1(\Omega)$. Similarly, the convex conjugate of j_k, $k = 0, 1$; is given by

$$j_k^*(x) = \sup\{xy - j_k(y) : y \in \mathbb{R}\}, \quad x \in \mathbb{R}. \tag{4}$$

H. Brézis [6] studied the convex functional $J_0(u) = \int_\Omega j_0(u)dx$ on $H_0^1(\Omega)$ and characterized its conjugate J_0^* and its subdifferential ∂J_0. The main Theorems presented here generalize the results in [6] to the functional J. The strategy of the proof is conceptually similar to the one by Brézis, however our conclusions cannot be directly derived from the work in [6], and necessitate a number of nontrivial technical auxiliary results.

Our main findings are stated in the following theorems.

Theorem 2.1. *Suppose $T \in (H^1(\Omega))'$ such that $J^*(T) < +\infty$. Then T is a signed Radon measure on $\overline{\Omega}$ and there exist $T_a \in L^1(\Omega)$ and $T_{\Gamma,a} \in L^1(\Gamma)$ such that*

$$\langle T, v\rangle = \int_\Omega T_a v dx + \int_\Gamma T_{\Gamma,a}\gamma v d\Gamma, \quad \text{for all } v \in C(\overline{\Omega}). \tag{5}$$

Moreover,

$$J^*(T) = \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma.$$

Theorem 2.2. *Let $u \in H^1(\Omega)$. If $T \in (H^1(\Omega))'$ such that $T \in \partial J(u)$, then T is a signed Radon measure on $\overline{\Omega}$ and there exist $T_a \in L^1(\Omega)$, $T_{\Gamma,a} \in L^1(\Gamma)$ such that T satisfies* (5). *Moreover, T, T_a, $T_{\Gamma,a}$ verify the following:*

- $T_a \in \partial j_0(u)$ *a.e. in* Ω *and* $T_{\Gamma,a} \in \partial j_1(\gamma u)$ *a.e. on* Γ, (6)
- $T_a u \in L^1(\Omega)$ *and* $T_{\Gamma,a}\gamma u \in L^1(\Gamma)$, (7)
- $\langle T, u\rangle = \int_\Omega T_a u dx + \int_\Gamma T_{\Gamma,a}\gamma u d\Gamma.$ (8)

Conversely, if $T \in (H^1(\Omega))'$ such that there exist $T_a \in L^1(\Omega)$, $T_{\Gamma,a} \in L^1(\Gamma)$ satisfying (5) *and* (6), *then $T \in \partial J(u)$.*

Assume for the moment that Theorem 2.2 has been proven. Define the functionals J_0 and $J_1 : H^1(\Omega) \to [0, +\infty]$ by

$$J_0(u) = \int_\Omega j_0(u)dx \quad \text{and} \quad J_1(u) = \int_\Gamma j_1(\gamma u)d\,\Gamma.$$

Then, following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $u \in H^1(\Omega)$. Then,*

- *if $j_1 = 0$ (i.e., $J = J_0$), then*

$$\partial J_0(u) = \{T \in (H^1(\Omega))' \cap L^1(\Omega) : T \in \partial j_0(u) \text{ a.e. in } \Omega\}. \tag{9}$$

- *if $j_0 = 0$ (i.e., $J = J_1$), then*

$$\begin{aligned}\partial J_1(u) = \{T \in (H^1(\Omega))' : T = \gamma^* T_\Gamma,\ \text{where } T_\Gamma \in H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma) \\ \text{such that } T_\Gamma \in \partial j_1(\gamma u) \text{ a.e. on } \Gamma\}.\end{aligned} \tag{10}$$

Proof. The first statement of the Corollary is clear from Theorem 2.2. As for the second statement, first assume that $T \in (H^1(\Omega))'$ such that $T = \gamma^* T_\Gamma$ where $T_\Gamma \in H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma)$ with $T_\Gamma \in \partial j_1(\gamma u)$ a.e. on Γ. Note for all $w \in C^1(\overline{\Omega})$,

$$\langle T, w\rangle = \langle \gamma^* T_\Gamma, w\rangle = \langle T_\Gamma, \gamma w\rangle = \int_\Gamma T_\Gamma \gamma w d\Gamma.$$

Let $v \in C(\overline{\Omega})$, then there exists a sequence $w_n \in C^1(\overline{\Omega})$ such that $w_n \to v$ in $C(\overline{\Omega})$. Then, it follows easily from the Lebesgue Dominated Convergence Theorem that we may extend T to a bounded linear functional on $C(\overline{\Omega})$ via

$$\langle T, v\rangle = \lim_{n\to\infty} \langle T, w_n\rangle = \lim_{n\to\infty} \int_\Gamma T_\Gamma \gamma w_n d\Gamma = \int_\Gamma T_\Gamma \gamma v d\Gamma.$$

Therefore, by Theorem 2.2, with $j_0 = 0$, we obtain $T \in \partial J_1(u)$.

Conversely, if $T \in (H^1(\Omega))'$ such that $T \in \partial J_1(u)$, then by Theorem 2.2, with $j_0 = 0$, T is a Radon measure on $\overline{\Omega}$ and there exists $T_\Gamma \in L^1(\Gamma)$ such that $T_\Gamma \in \partial j_1(\gamma u)$ and

$$\langle T, v\rangle = \int_\Gamma T_\Gamma \gamma v d\Gamma \quad \text{for all } v \in C(\overline{\Omega}). \tag{11}$$

Since $T_\Gamma \in L^1(\Gamma)$, we have $T_\Gamma \in (C(\Gamma))'$ such that $\langle T_\Gamma, \phi\rangle = \int_\Gamma T_\Gamma \phi d\Gamma$, for all $\phi \in C(\Gamma)$. Note, for any $\psi \in H^{\frac{1}{2}}(\Gamma)$, there exists a sequence $\phi_n \in C^1(\Gamma)$ such that $\phi_n \to \psi$ in $H^{\frac{1}{2}}(\Gamma)$. Since $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ is surjective and has a continuous linear right inverse γ^{-1}, then clearly $|\langle T, \gamma^{-1}\psi\rangle| \le \|T\|\,\|\gamma^{-1}\psi\|_{H^1(\Omega)} < \infty$, for all $\psi \in H^{\frac{1}{2}}(\Gamma)$. Therefore, we can extend T_Γ to a bounded linear functional on $H^{\frac{1}{2}}(\Gamma)$ as follows:

$$\langle T_\Gamma, \psi\rangle = \lim_{n\to\infty} \langle T_\Gamma, \phi_n\rangle = \lim_{n\to\infty} \int_\Gamma T_\Gamma \phi_n d\Gamma = \lim_{n\to\infty} \langle T, \gamma^{-1}\phi_n\rangle = \langle T, \gamma^{-1}\psi\rangle,$$

for all $\psi \in H^{\frac{1}{2}}(\Gamma)$, where we have used (11). Hence, $T_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$ such that $\langle T_\Gamma, \gamma v\rangle = \int_\Gamma T_\Gamma \gamma v d\Gamma = \langle T, v\rangle$ for all $v \in C^1(\overline{\Omega})$. Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, we obtain $\langle T_\Gamma, \gamma v\rangle = \langle T, v\rangle$ for all $v \in H^1(\Omega)$, i.e., $T = \gamma^* T_\Gamma$. □

At this time, few comments are in order. In applications to semilinear PDE's, for instance the wave equation with nonlinear interior and boundary damping:

$$\begin{cases} w_{tt} - \Delta w + g_0(w_t) = 0, & \text{in } \Omega \times (0,T), \\ \partial_\nu w + w + g_1(w_t) = 0, & \text{in } \partial\Omega \times (0,T), \end{cases}$$

we define $j_k(s) = \int_0^s g_k(\tau)d\tau$, $k = 0, 1$. The feedback maps g_0, g_1 are continuous monotone increasing and equal zero at the origin. Consequently, ∂j_0 and ∂j_1 are single-valued. In this context, the following observation is of interest: the interior measure $T_a \in L^1(\Omega)$ from Theorem 2.2 corresponds pointwise a.e. to the values of the subdifferential associated with the homogeneous Dirichlét problem

$$\begin{cases} w_{tt} - \Delta w + g_0(w_t) = 0, & \text{in } \Omega \times (0,T), \\ w = 0, \quad \text{in } \partial\Omega \times (0,T). \end{cases}$$

More precisely, $S \in \partial J_0(u)$ for $u \in H_0^1(\Omega)$, if and only if $S = T_a \in L^1(\Omega)$ a.e. and $T_a \in \partial J(u)$. This result readily follows from Theorems 2.1 and 2.2 because when the domains are restricted to $H_0^1(\Omega)$ functions, and ∂j_0 is single-valued, then $\partial J_0(u)$ is a singleton $\{T_a\}$ and defines an element of $H^{-1}(\Omega)$.

2.1. Open question

An interesting problem related to Theorem 2.2 is the following: under what additional conditions is it possible to prove that $T_a \in (H^1(\Omega))'$ and $T_{\Gamma,a} \in H^{-\frac{1}{2}}(\Gamma)$? In this case we conclude that the monotone graphs ∂J and $\partial J_0 + \partial J_1$ have identical domains and, therefore, coincide.

A sufficient condition would be to have $|\partial j_0(s)|$ or $|\partial j_1(s)|$ bounded by $C_1|s|^\alpha + C_2$ where α respectively does not exceed the critical Sobolev exponent for the embedding $H^1(\Omega) \to L^\alpha(\Omega)$ (namely, $\alpha \le \frac{2n}{n-2}$) or $H^1(\Omega) \to L^\alpha(\Gamma)$ (that is, $\alpha \le \frac{2(n-1)}{(n-1)-1}$). Then it will follow that the domain of either ∂J_0 or ∂J_1 is all of $H^1(\Omega)$, hence the domain condition for the sum of maximal monotone operators holds and $\partial J_0 + \partial J_1 = \partial(J_0 + J_1)$.

Without growth restrictions, the answer to the question is likely “No", in general. For, if $u \in H^1(\Omega)$ then despite, say, $\partial j_0(u) \in L^1$, one may not be able to guarantee that u has any higher boundary regularity than $L^4(\Gamma)$. However, it would be interesting to establish whether some alternative structural assumptions on j_0 and j_1 may provide that.

3. Proofs of the main results

3.1. Approximation Results

In order to prove the main theorems, we shall need several approximation lemmas. Throughout the paper, $C_0(\Omega)$ denotes the space of continuous functions with compact support in Ω.

Lemma 3.1. *If $u \in D(J)$, then there exists a sequence $v_n \in H^2(\Omega)$ such that $v_n \to u$ in $H^1(\Omega)$, $j_0(v_n) \to j_0(u)$ in $L^1(\Omega)$ and $j_1(\gamma v_n) \to j_1(\gamma u)$ in $L^1(\Gamma)$.*

Proof. We consider the functional $\varphi : L^2(\Omega) \to [0, +\infty]$ defined by

$$\varphi(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + j_0(v) \right) dx + \int_\Gamma j_1(\gamma v) d\Gamma, \tag{12}$$

if $v \in H^1(\Omega)$, $j_0(v) \in L^1(\Omega)$, $j_1(\gamma v) \in L^1(\Gamma)$; otherwise $\varphi(v) = +\infty$. Clearly, the functional φ is convex and lower semicontinuous. By Corollary 13 in [7, p. 115] it follows that, $\partial\varphi : L^2(\Omega) \to L^2(\Omega)$ is given by

$$\partial\varphi(v) = \{w \in L^2(\Omega) : w + \Delta v \in \partial j_0(v) \text{ a.e. in } \Omega\}$$

with its domain

$$D(\partial\varphi) = \left\{ v \in H^2(\Omega) : -\frac{\partial v}{\partial \nu} \in \partial j_1(v) \text{ a.e. on } \Gamma \right\}.$$

Next, fix $u \in D(J) \subset H^1(\Omega)$ and put:

$$v_n = \left(I + \frac{1}{n} \partial\varphi \right)^{-1} u. \tag{13}$$

Since $\partial\varphi$ is maximal monotone then $(I + \frac{1}{n}\partial\varphi) : D(\partial\varphi) \subset L^2(\Omega) \to L^2(\Omega)$ is one-to-one, onto, and $v_n \to u$ in $L^2(\Omega)$. Also notice that, $v_n \in D(\partial\phi) \subset H^2(\Omega)$.

Let us first show that,

$$\lim_{n\to\infty} \varphi(v_n) = \varphi(u). \tag{14}$$

To see this, note that (13) implies $\frac{1}{n}\partial\varphi(v_n) = u - v_n$. So, by the definition of subdifferential, we have

$$\frac{1}{n} \|\partial\varphi(v_n)\|^2_{L^2(\Omega)} = \left(\partial\varphi(v_n), \frac{1}{n}\partial\varphi(v_n) \right) = (\partial\varphi(v_n), u - v_n) \le \varphi(u) - \varphi(v_n).$$

Consequently $\varphi(v_n) \le \varphi(u)$. Since φ is lower semicontinuous and $v_n \to u$ in $L^2(\Omega)$, we have $\liminf_{n\to\infty} \varphi(v_n) \ge \varphi(u)$, and so (14) holds.

Our next step is to show that

$$v_n \to u \text{ strongly in } H^1(\Omega). \tag{15}$$

Indeed,

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla(v_n - u)|^2 dx &= \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \nabla v_n \cdot \nabla u dx \\ &= \varphi(v_n) - \varphi(u) - \int_\Omega j_0(v_n) dx - \int_\Gamma j_1(\gamma v_n) d\Gamma \\ &\quad + \int_\Omega j_0(u) dx + \int_\Gamma j_1(\gamma u) d\Gamma - \int_\Omega \nabla(v_n - u) \cdot \nabla u dx. \end{aligned} \tag{16}$$

The fact that $u \in D(J)$ (whence $\varphi(u) < +\infty$), the definition of φ (12), and the convergence result (14) imply that $\{\|\nabla v_n\|_{L^2(\Omega)}\}$ is bounded. Also, since $\{v_n\}$ is bounded in $L^2(\Omega)$, we infer that $\{v_n\}$ is bounded in $H^1(\Omega)$ and so, on a subsequence labeled by $\{v_n\}$, we have

$$v_n \to u \quad \text{weakly in } H^1(\Omega). \tag{17}$$

Now, since the embedding $H^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$ is compact for $0 < \epsilon < 1$, then on a subsequence, $v_n \to u$ strongly in $H^{1-\epsilon}(\Omega)$ (for sufficiently small $\epsilon > 0$) and therefore $\gamma v_n \to \gamma u$ in $L^2(\Gamma)$.

By extracting a subsequence, still labeled by $\{v_n\}$, one has $v_n \to u$ a.e. in Ω and $\gamma v_n \to \gamma u$ a.e. on Γ. Then, Fatou's lemma gives us

$$\liminf_{n\to\infty} \left(\int_\Omega j_0(v_n)dx + \int_\Gamma j_1(\gamma v_n)d\Gamma \right) \geq \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma. \tag{18}$$

Combining (18), (14) and (17), then from (16) we obtain

$$\liminf_{n\to\infty} \int_\Omega |\nabla(v_n - u)|^2 dx \leq 0.$$

Therefore, on a subsequence one has

$$\lim_{n\to\infty} \int_\Omega |\nabla(v_n - u)|^2 dx = 0. \tag{19}$$

Since $v_n \to u$ in $L^2(\Omega)$, then (15) follows. Moreover, by using (14), (17) and (19), then (16) yields

$$\lim_{n\to\infty} \left(\int_\Omega j_0(v_n)dx + \int_\Gamma j_1(\gamma v_n)d\Gamma \right) = \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma. \tag{20}$$

However, by Fatou's lemma,

$$\liminf_{n\to\infty} \int_\Omega j_0(v_n)dx \geq \int_\Omega j_0(u)dx \quad \text{and} \quad \liminf_{n\to\infty} \int_\Gamma j_1(\gamma v_n)d\Gamma \geq \int_\Gamma j_1(\gamma u)d\Gamma. \tag{21}$$

Hence, it follows from (20)–(21) (by extracting a further subsequence) that

$$\lim_{n\to\infty} \int_\Omega j_0(v_n)dx = \int_\Omega j_0(u)dx \quad \text{and} \quad \lim_{n\to\infty} \int_\Gamma j_1(\gamma v_n)d\Gamma = \int_\Gamma j_1(\gamma u)d\Gamma,$$

which completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $K \subset \mathbb{R}^2$ be a convex closed set containing the origin. Then*

$$\left\{(u,v) \in \left[C_0(\Omega) \cap W^{1,\infty}(\Omega)\right]^2 : (u(x), v(x)) \in K, \text{ for all } x \in \Omega\right\}$$

is dense in

$$\{(u,v) \in L^1(\Omega) \times L^1(\Omega) : (u(x), v(x)) \in K, \text{ a.e. } x \in \Omega\}.$$

Proof. Let u, $v \in L^1(\Omega)$ such that $(u(x), v(x)) \in K$ for a.e. $x \in \Omega$. Since $C_0^1(\Omega)$ is dense in $L^1(\Omega)$, there exist $\tilde{u}$, $\tilde{v} \in C_0^1(\Omega)$ such that

$$\|u - \tilde{u}\|_{L^1(\Omega)} < \epsilon \quad \text{and} \quad \|v - \tilde{v}\|_{L^1(\Omega)} < \epsilon. \tag{22}$$

Let $P : \mathbb{R}^2 \to K \subset \mathbb{R}^2$ be the projection onto the convex closed set K. Put $(\hat{u}(x), \hat{v}(x)) = P(\tilde{u}(x), \tilde{v}(x))$ for all $x \in \Omega$. Since P is a (non-strict) contraction on $\mathbb{R}^2$, then for any x_1, $x_2 \in \Omega$, we have

$$\begin{aligned} &|(\hat{u}(x_1), \hat{v}(x_1)) - (\hat{u}(x_2), \hat{v}(x_2))| \leq |(\tilde{u}(x_1), \tilde{v}(x_1)) - (\tilde{u}(x_2), \tilde{v}(x_2))| \\ &\leq |\tilde{u}(x_1) - \tilde{u}(x_2)| + |\tilde{v}(x_1) - \tilde{v}(x_2)| \leq C|x_1 - x_2|, \end{aligned}$$

where in the last inequality we used the fact $\tilde{u}$, $\tilde{v} \in C_0^1(\Omega)$. Therefore,

$$\begin{aligned} |\hat{u}(x_1) - \hat{u}(x_2)| &\leq C|x_1 - x_2| \quad \text{and} \\ |\hat{v}(x_1) - \hat{v}(x_2)| &\leq C|x_1 - x_2| \quad \text{for any } x_1, x_2 \in \Omega. \end{aligned}$$

That is, $\hat{u}$ and $\hat{v}$ are both Lipschitz continuous on Ω, which is equivalent to $\hat{u}$, $\hat{v} \in W^{1,\infty}(\Omega)$. Moreover, since K contains the origin, one has $P(0,0) = (0,0)$, and therefore $\hat{u}$ and $\hat{v}$ both have compact supports in Ω. Also note,

$$|(u(x), v(x)) - (\hat{u}(x), \hat{v}(x))| \leq |(u(x), v(x)) - (\tilde{u}(x), \tilde{v}(x))| \quad \text{a.e. } x \in \Omega,$$

and so, (22) yields

$$\|u - \hat{u}\|_{L^1(\Omega)} < 2\epsilon \quad \text{and} \quad \|v - \hat{v}\|_{L^1(\Omega)} < 2\epsilon,$$

which completes the proof. □

Proposition 3.3. *Let $j : \mathbb{R} \to [0, \infty)$ be a convex function with $j(0) = 0$. If $u \in L^1(\Omega)$, then*

$$\int_\Omega j^*(u)dx = \sup \left\{ \int_\Omega (uv - j(v))dx : v \in C_0(\Omega) \cap W^{1,\infty}(\Omega) \right\}.$$

Proof. Since $u \in L^1(\Omega)$ and $j^{**} = j$ on $\mathbb{R}$, then by identity (1) in [6] we obtain

$$\int_\Omega j^*(u)dx = \sup \left\{ \int_\Omega (uv - j(v))dx : v \in L^\infty(\Omega) \right\}. \tag{23}$$

So, if we put

$$\theta = \sup \left\{ \int_\Omega (uv - j(v))dx : v \in C_0(\Omega) \cap W^{1,\infty}(\Omega) \right\},$$

then $\theta \leq \int_\Omega j^*(u)dx$.

Let $\epsilon > 0$ be given. Then, from (23) there exists $v_0 \in L^\infty(\Omega)$, such that

$$\int_\Omega (uv_0 - j(v_0))dx \geq \int_\Omega j^*(u)dx - \epsilon. \tag{24}$$

Now, put

$$h(r) = \begin{cases} j(r) & \text{if } |r| \le \|v_0\|_{L^\infty(\Omega)}, \\ +\infty & \text{if } |r| > \|v_0\|_{L^\infty(\Omega)}, \end{cases} \tag{25}$$

and consider the set $K = \{(r, \rho) \in \mathbb{R}^2 : \rho \ge h(r)\}$. Note, K is the epigraph of h, and since h is convex, lower semicontinuous and $h(0) = 0$, then K is convex, closed and contains the origin. Since $(v_0(x), h(v_0(x))) \in K$ for all $x \in \Omega$, we may apply Lemma 3.2 to $(v_0, h(v_0)) \in L^1(\Omega) \times L^1(\Omega)$ to obtain sequences $\{v_n\}$, $\{\alpha_n\} \subset C_0(\Omega) \cap W^{1,\infty}(\Omega)$ such that,

$$v_n \to v_0, \quad \alpha_n \to h(v_0) \text{ in } L^1(\Omega), \tag{26}$$

and $\alpha_n \ge h(v_n)$ in Ω. It follows (25) that, $\|v_n\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)}$. In addition, $\alpha_n \to j(v_0)$ in $L^1(\Omega)$ and $\alpha_n \ge j(v_n)$ in Ω.

After extracting a subsequence, we have $v_n \to v_0$. a.e. Ω and, since j is continuous, one obtains $j(v_n) \to j(v_0)$, a.e. Ω. By the Generalized Lebesgue Dominated Convergence Theorem, we infer $j(v_n) \to j(v_0)$ in $L^1(\Omega)$. Since $\int_\Omega (uv_n - j(v_n))dx \le \theta$, we can pass to the limit by the Lebesgue Dominated Convergence Theorem to obtain $\int_\Omega (uv_0 - j(v_0))dx \le \theta$. It follows from (24) that $\int_\Omega j^*(u)dx - \epsilon \le \theta \le \int_\Omega j^*(u)dx$, and therefore, $\int_\Omega j^*(u)dx = \theta$. □

Similar to Proposition 3.3 we can deduce the following result.

Proposition 3.4. *Let $j : \mathbb{R} \to [0, \infty)$ be a convex function with $j(0) = 0$. If $u \in L^1(\Gamma)$, then*

$$\int_\Gamma j^*(u)d\Gamma = \sup\left\{ \int_\Gamma (uv - j(v))d\Gamma : v \in W^{1,\infty}(\Gamma) \right\}.$$

3.2. Proof of Theorem 2.1.

We carry out the proof in three steps.

Step 1: Since j_0 and j_1 are continuous on $\mathbb{R}$, then if $\rho > 0$ is given, then there exists $\eta > 0$ such that $j_0(s), j_1(s) \le \eta$, whenever $|s| \le \rho$. Thus, if $v \in C^1(\overline{\Omega})$ with $\|v\|_{C(\overline{\Omega})} \le \rho$, then $j_0(v(x)) \le \eta$ for all $x \in \Omega$ and $j_1(v(x)) \le \eta$ for all $x \in \Gamma$. Therefore, by Fenchel's inequality

$$\begin{aligned} \langle T, v\rangle \le J^*(T) + J(v) &= J^*(T) + \int_\Omega j_0(v)dx + \int_\Gamma j_1(\gamma v)d\Gamma \\ &\le J^*(T) + \eta(|\Omega| + |\Gamma|) < \infty, \end{aligned} \tag{27}$$

for all $v \in C^1(\overline{\Omega})$ with $\|v\|_{C(\overline{\Omega})} \le \rho$. By Hahn-Banach theorem, we can extend T to be a bounded linear functional on $C(\overline{\Omega})$, and since $C^1(\overline{\Omega})$ is dense in $C(\overline{\Omega})$, the extension is unique, which we still denote it by T. That is, $T \in (C(\overline{\Omega}))'$, and so, T is a signed Radon measure on $\overline{\Omega}$. Then we have the following Radon-Nikodym decomposition of T:

$$T = T_a d\Omega + T_{\Omega,s}, \tag{28}$$

where $T_a \in L^1(\Omega)$ and $T_{\Omega,s}$ is singular with respect to $d\Omega$, the Lebesgue measure on $\overline{\Omega}$.

Now, let $d\Gamma$ denote the Lebesgue measure on $(\Gamma, \mathcal{L}_\Gamma)$ where $\mathcal{L}_\Gamma$ is the class of Lebesgue measurable subset of Γ. We extend $d\Gamma$ to the interior of Ω by defining the measure $d\tilde{\Gamma}$ on $(\overline{\Omega}, \mathcal{L}_{\overline{\Omega}})$ via

$$d\tilde{\Gamma}(A) = d\Gamma(A \cap \Gamma),$$

for $A \in \mathcal{L}_{\overline{\Omega}}$. Notice, $d\tilde{\Gamma}$ is a well-defined measure since one can show that $A \cap \Gamma \in \mathcal{L}_\Gamma$ for all $A \in \mathcal{L}_{\overline{\Omega}}$. Subsequently, we decompose $T_{\Omega,s}$ with respect to $d\tilde{\Gamma}$:

$$T_{\Omega,s} = T_{\Gamma,a} d\tilde{\Gamma} + T_s, \tag{29}$$

where $T_{\Gamma,a} \in L^1(d\tilde{\Gamma})$ and T_s is singular with respect to both $d\tilde{\Gamma}$ and $d\Omega$. It follows from (28)–(29) that,

$$T = T_a d\Omega + T_{\Gamma,a} d\tilde{\Gamma} + T_s. \tag{30}$$

Clearly, $T_{\Gamma,a} \in L^1(\Gamma)$. Thus, for all $v \in C(\overline{\Omega})$, we have

$$\begin{aligned} \langle T, v\rangle &= \int_\Omega T_a v dx + \int_{\overline{\Omega}} T_{\Gamma,a} v d\tilde{\Gamma} + \langle T_s, v\rangle \\ &= \int_\Omega T_a v dx + \int_\Gamma T_{\Gamma,a} \gamma v d\Gamma + \langle T_s, v\rangle. \end{aligned} \tag{31}$$

Step 2: Let $v \in H^2(\Omega)$, then Fenchel's inequality yields:

$$\begin{cases} T_a(x)v(x) - j_0(v(x)) \le j_0^*(T_a(x)) & \text{a.e. } x \in \Omega, \\ T_{\Gamma,a}(x)\gamma v(x) - j_1(\gamma v(x)) \le j_1^*(T_{\Gamma,a}(x)) & \text{a.e. } x \in \Gamma. \end{cases} \tag{32}$$

Integrate the two inequalities in (32) over Ω and Γ, respectively, and add the results to obtain:

$$\langle T, v\rangle - \int_\Omega j_0(v)dx - \int_\Gamma j_1(\gamma v)d\Gamma \le \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma + \langle T_s, v\rangle, \tag{33}$$

where we have used (31).

Now, notice Lemma 3.1 implies

$$\begin{aligned} J^*(T) &= \sup\{\langle T, v\rangle - J(v) : v \in D(J)\} \\ &= \sup\left\{\langle T, v\rangle - \int_\Omega j_0(v)dx - \int_\Gamma j_1(\gamma v)d\Gamma : v \in H^2(\Omega)\right\}. \end{aligned} \tag{34}$$

Therefore, if we set

$$\begin{aligned} A &= \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma, \\ B &= \sup\{\langle T_s, v\rangle : v \in H^2(\Omega)\}, \end{aligned}$$

then (33) and (34) yield $J^*(T) \le A + B$.

Step 3: Since $T_a \in L^1(\Omega)$, then by Proposition 3.3 there exists a sequence $v_1^n \in C_0(\Omega) \cap W^{1,\infty}(\Omega)$ such that

$$\int_\Omega (T_a v_1^n - j_0(v_1^n))dx \uparrow \int_\Omega j_0^*(T_a)dx, \quad \text{as } n \to \infty. \tag{35}$$

Also, since $T_{\Gamma,a} \in L^1(\Gamma)$, then by Proposition 3.4 there exists a sequence $v_2^n \in W^{1,\infty}(\Gamma)$ such that

$$\int_\Gamma (T_{\Gamma,a} v_2^n - j_1(v_2^n))d\Gamma \uparrow \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma, \quad \text{as } n \to \infty. \tag{36}$$

Since each v_1^n has compact support, let $K_n := \operatorname{supp} v_1^n \subset \Omega$. Put $\alpha_n = \|v_2^n\|_{C(\Gamma)}$ and $\beta_n = \sup\{j_0(s) : |s| \le \alpha_n\}$. Since $T_a \in L^1(\Omega)$, then for each n, there exists an open set E_n with smooth boundary such that, $K_n \subset E_n \subset \overline{E_n} \subset \Omega$ and

$$\int_{\Omega \setminus E_n} (\alpha_n |T_a| + \beta_n)\, dx < \frac{1}{n}. \tag{37}$$

Now, for each n, we can construct a function $v_3^n \in C(\overline{\Omega}) \cap H^1(\Omega)$ as follows:

$$v_3^n = \begin{cases} v_1^n & \text{on } K_n, \\ 0 & \text{on } \overline{E_n} \setminus K_n, \\ \xi^n & \text{in } \Omega \setminus \overline{E_n}, \\ v_2^n & \text{on } \Gamma, \end{cases}$$

where $\xi^n \in C^2(\Omega \setminus \overline{E_n}) \cap C(\overline{\Omega} \setminus E_n) \cap H^1(\Omega \setminus \overline{E_n})$ is the unique solution of the Dirichlét problem:

$$\begin{cases} \Delta \xi^n = 0 & \text{in } \Omega \setminus \overline{E_n}, \\ \xi^n = 0 & \text{on } \partial E_n, \\ \xi^n = v_2^n \in W^{1,\infty}(\Gamma) & \text{on } \Gamma. \end{cases}$$

Notice the regularity of ξ^n follows from Theorem 6.1 (p. 55) and Corollary 7.1 (p. 361) in [8]. By the maximal principle, we know $|\xi^n(x)| \le \alpha_n = \|v_2^n\|_{C(\Gamma)}$ for all $x \in \Omega \setminus \overline{E_n}$. Therefore,

$$\begin{aligned} &\left| \int_\Omega (T_a v_3^n - j_0(v_3^n))dx - \int_\Omega (T_a v_1^n - j_0(v_1^n))dx \right| \\ &\le \int_{\Omega \setminus E_n} (\alpha_n |T_a| + \beta_n)\, dx < \frac{1}{n}. \end{aligned} \tag{38}$$

By combining (35)–(38) together with the fact $\gamma v_3^n = v_2^n$, we have

$$\begin{aligned} &\int_\Omega (T_a v_3^n - j_0(v_3^n))dx + \int_\Gamma (T_{\Gamma,a} \gamma v_3^n - j_1(\gamma v_3^n))d\Gamma \\ &\uparrow \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma = A \quad \text{as } n \to \infty. \end{aligned} \tag{39}$$

Let us also remark here that while each v_3^n belongs to $H^1(\Omega)$, the result in (39) does not require the H^1 norm to be bounded in n, so the blow up of ξ_n in $H^1(\Omega)$ as $n \to \infty$ is irrelevant.

Recall $B = \sup\{\langle T_s, v\rangle : v \in H^2(\Omega)\}$, so there exists a sequence $v_4^n \in H^2(\Omega)$ such that

$$\langle T_s, v_4^n\rangle \uparrow B \quad \text{as } n \to \infty. \tag{40}$$

Since the measure T_s is singular with respect to both $d\Omega$ and $d\tilde{\Gamma}$, there exists a measurable set $S \subset \overline{\Omega}$ such that $\overline{\Omega}\backslash S$ is null for T_s and S is null for both $d\Omega$ and $d\tilde{\Gamma}$. So for any $\delta > 0$, there exists U relatively open in $\overline{\Omega}$ such that $S \subset U$ with

$$\int_U dx < \delta \quad \text{and} \quad \int_U d\tilde{\Gamma} = \int_{\Gamma\cap U} d\Gamma < \delta. \tag{41}$$

We may extend U to U_{ext} such that U_{ext} is open and bounded in $\mathbb{R}^3$, $U \subset U_{ext}$ and $U \cap \overline{\Omega} = U_{ext} \cap \overline{\Omega}$.

Comment 3.5. Note that given any set O we can find a *compact* set $K \subset O$ approximating O in measure; furthermore, by covering the boundary points of K with finitely many open balls we can extend K to an open set $\tilde{K} \subset O$ which is open, approximates O in measure from below, and has a smooth boundary (as the boundary at the *finitely many* cusps on the intersections of the balls can be smoothed out).

Given the preceding general observation we can claim that there exist open sets $\{V_k\}$ and $\{U_k\}$ such that V_k and U_k both having smooth boundaries and satisfy $V_k \subset \overline{V_k} \subset U_k \subset U_{ext}$ with

$$\int_{U_{ext}\backslash V_k} dx < \frac{1}{k}, \qquad \int_{\Gamma\cap(U_{ext}\backslash V_k)} d\Gamma < \frac{1}{k} \quad \text{and} \quad \int_{U\cap(U_{ext}\backslash V_k)} dT_s < \frac{1}{k}. \tag{42}$$

Now fix n; one may extend $v_3^n \in C(\overline{\Omega}) \cap H^1(\Omega)$ and $v_4^n \in H^2(\Omega)$ to functions on $\mathbb{R}^3$, i.e., there exist $\tilde{v}_3^n \in C_0(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ and $\tilde{v}_4^n \in C_0(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ such that $\tilde{v}_3^n|_{\overline{\Omega}} = v_3^n$ and $\tilde{v}_4^n|_{\overline{\Omega}} = v_4^n$.

For each k, we construct a function $\tilde{w}_k^n \in C_0(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$:

$$\tilde{w}_k^n = \begin{cases} \tilde{v}_3^n & \text{in } \mathbb{R}^3\backslash U_k, \\ \zeta_k^n & \text{in } U_k\backslash \overline{V_k}, \\ \tilde{v}_4^n & \text{in } \overline{V_k}, \end{cases} \tag{43}$$

where $\zeta_k^n \in C^2(U_k\backslash\overline{V_k}) \cap C(\overline{U_k}\backslash V_k) \cap H^1(U_k\backslash\overline{V_k})$ is the unique solution of the Dirichlét problem:

$$\begin{cases} \Delta\zeta_k^n = 0 & \text{in } U_k\backslash\overline{V_k}, \\ \zeta_k^n = \tilde{v}_3^n & \text{on } \partial U_k, \\ \zeta_k^n = \tilde{v}_4^n & \text{on } \partial V_k. \end{cases}$$

Again, notice the regularity of ζ_k^n follows from Theorem 6.1 (p. 55) and Corollary 7.1 (p. 361) in [8].

Define $w_k^n = \tilde{w}_k^n|_{\overline{\Omega}}$, then $w_k^n \in C(\overline{\Omega}) \cap H^1(\Omega)$. By Fenchel's inequality and (31) we obtain

$$\begin{aligned} J^*(T) &\geq \langle T, w_k^n \rangle - J(w_k^n) \\ &= \int_\Omega T_a w_k^n dx + \int_\Gamma T_{\Gamma,a} \gamma w_k^n d\Gamma + \langle T_s, w_k^n \rangle - \int_\Omega j_0(w_k^n) dx - \int_\Gamma j_1(\gamma w_k^n) d\Gamma. \end{aligned} \tag{44}$$

By (43) and the maximum principle, one has $\|w_k^n\|_{C(\overline{\Omega})} \leq \max\{\|\tilde{v}_3^n\|_{C(\mathbb{R}^3)}, \|\tilde{v}_4^n\|_{C(\mathbb{R}^3)}\}$, for all k; and by (42) $w_k^n \to v_4^n$ $|T_s|$−a.e. on $\overline{\Omega}$ as $k \to \infty$, we infer $\lim_{k\to\infty} \langle T_s, w_k^n \rangle = \langle T_s, v_4^n \rangle$. Also, by (42) we know $w_k^n \to v_4^n$ a.e. in U and $\gamma w_k^n \to \gamma v_4^n$ a.e. on $\Gamma \cap U$ as $k \to \infty$, thus the Lebesgue dominated convergence theorem implies

$$\lim_{k\to\infty} \int_\Omega T_a w_k^n dx = \lim_{k\to\infty} \int_U T_a w_k^n dx + \int_{\Omega\setminus U} T_a v_3^n dx = \int_U T_a v_4^n dx + \int_{\Omega\setminus U} T_a v_3^n dx,$$

$$\lim_{k\to\infty} \int_\Gamma T_{\Gamma,a} \gamma w_k^n d\Gamma = \int_{\Gamma\cap U} T_{\Gamma,a} \gamma v_4^n d\Gamma + \int_{\Gamma\setminus U} T_{\Gamma,a} \gamma v_3^n d\Gamma,$$

and

$$\lim_{k\to\infty} \int_\Omega j_0(w_k^n) dx = \int_U j_0(v_4^n) dx + \int_{\Omega\setminus U} j_0(v_3^n) dx,$$

$$\lim_{k\to\infty} \int_\Gamma j_1(\gamma w_k^n) d\Gamma = \int_{\Gamma\cap U} j_1(\gamma v_4^n) d\Gamma + \int_{\Gamma\setminus U} j_1(\gamma v_3^n) d\Gamma.$$

Therefore, taking the limit as $k \to \infty$ in (44) yields

$$\begin{aligned} J^*(T) \geq & \int_\Omega (T_a v_3^n - j_0(v_3^n)) dx + \int_\Gamma (T_{\Gamma,a} \gamma v_3^n - j_1(\gamma v_3^n)) d\Gamma + \langle T_s, v_4^n \rangle \\ &+ \int_U (T_a v_4^n - T_a v_3^n - j_0(v_4^n) + j_0(v_3^n)) dx \\ &+ \int_{\Gamma\cap U} (T_{\Gamma,a} \gamma v_4^n - T_{\Gamma,a} \gamma v_3^n - j_1(\gamma v_4^n) + j_1(\gamma v_3^n)) d\Gamma. \end{aligned}$$

By (41), if we let $\delta \to 0$, then the last two integrals on the right-hand side of the above inequality both converge to zero, hence one has

$$J^*(T) \geq \int_\Omega (T_a v_3^n - j_0(v_3^n)) dx + \int_\Gamma (T_{\Gamma,a} \gamma v_3^n - j_1(\gamma v_3^n)) d\Gamma + \langle T_s, v_4^n \rangle.$$

Finally, we let $n \to \infty$ and use (39)–(40) to obtain $J^*(T) \geq A + B$.

Recall that in *Step 2* we have shown that $J^*(T) \leq A + B$, so $J^*(T) = A + B$. Since $J^*(T) < \infty$ and $A > -\infty$, we know that $B < \infty$, and, being a supremum of a linear functional, must be zero. That is, $B = 0$ and $T_s = 0$. It follows that $J^*(T) = A$ and by (31) we obtain (5). This completes the proof of Theorem 2.1.

3.3. Proof of Theorem 2.2.

First, we assume $T \in \partial J(u)$. Then, Fenchel's equality and the fact that $u \in D(\partial J) \subset D(J)$ yield that $J^*(T) = \langle T, u\rangle - J(u) < +\infty$. Then, by Theorem 2.1, T is a signed Radon measure on $\overline{\Omega}$ and there exist $T_a \in L^1(\Omega)$ and $T_{\Gamma,a} \in L^1(\Gamma)$ such that (5) holds.

Since $u \in D(J)$, by Lemma 3.1 there exists a sequence $v_n \in H^2(\Omega)$ such that $v_n \to u$ in $H^1(\Omega)$ and a.e. in Ω, $\gamma v_n \to \gamma u$ a.e. on Γ, $j_0(v_n) \to j_0(u)$ in $L^1(\Omega)$ and a.e. in Ω, $j_1(\gamma v_n) \to j_1(\gamma u)$ in $L^1(\Gamma)$ and a.e. on Γ.

Fenchel's inequality gives

$$\begin{gathered} j_0^*(T_a) + j_0(v_n) - T_a v_n \geq 0 \quad \text{a.e. in } \Omega, \\ j_1^*(T_{\Gamma,a}) + j_1(\gamma v_n) - T_{\Gamma,a}\gamma v_n \geq 0 \quad \text{a.e. on } \Gamma. \end{gathered}$$

Since $T \in (H^1(\Omega))'$, by (5) we have

$$\langle T, u\rangle = \lim_{n\to\infty} \langle T, v_n\rangle = \lim_{n\to\infty} \left(\int_\Omega T_a v_n dx + \int_\Gamma T_{\Gamma,a}\gamma v_n d\Gamma \right).$$

Therefore, Fatou's lemma yields

$$\begin{aligned} &\int_\Omega (j_0^*(T_a) + j_0(u) - T_a u)dx + \int_\Gamma (j_1^*(T_{\Gamma,a}) + j_1(\gamma u) - T_{\Gamma,a}\gamma u)d\Gamma \\ &\leq \liminf_{n\to\infty} \left(\int_\Omega (j_0^*(T_a) + j_0(v_n) - T_a v_n)dx + \int_\Gamma (j_1^*(T_{\Gamma,a}) + j_1(\gamma v_n) - T_{\Gamma,a}\gamma v_n)d\Gamma \right) \\ &= \int_\Omega (j_0^*(T_a) + j_0(u))dx + \int_\Gamma (j_1^*(T_{\Gamma,a}) + j_1(\gamma u))d\Gamma - \langle T, u\rangle \\ &= J^*(T) + J(u) - \langle T, u\rangle = 0 \end{aligned} \tag{45}$$

where we have used Theorem 2.1 and Fenchel's equality, since $T \in \partial J(u)$.

On the other hand, Fenchel's inequality implies

$$\begin{gathered} j_0^*(T_a) + j_0(u) - T_a u \geq 0 \quad \text{a.e. in } \Omega, \\ j_1^*(T_{\Gamma,a}) + j_1(\gamma u) - T_{\Gamma,a}\gamma u \geq 0 \quad \text{a.e. on } \Gamma. \end{gathered}$$

In order for (45) to hold, we must have

$$j_0^*(T_a) + j_0(u) = T_a u \quad \text{a.e. in } \Omega \quad \text{and} \quad j_1^*(T_{\Gamma,a}) + j_1(\gamma u) = T_{\Gamma,a}\gamma u \quad \text{a.e. on } \Gamma.$$

So, $T_a u \in L^1(\Omega)$ and $T_{\Gamma,a}\gamma u \in L^1(\Gamma)$. Also (45) becomes equality, and thus (8) holds. Moreover, since $D(j_0)$ and $D(j_1) = \mathbb{R}$, the converse of Fenchel's equality theorem holds and we infer (6).

Conversely, assume $T \in (H^1(\Omega))'$ such that there exist $T_a \in L^1(\Omega)$, $T_{\Gamma,a} \in L^1(\Gamma)$ satisfying (5) and (6). First, we claim that

$$\langle T, v\rangle = \int_\Omega T_a v dx + \int_\Gamma T_{\Gamma,a}\gamma v d\Gamma \quad \text{for all } v \in H^1(\Omega) \cap L^\infty(\Omega). \tag{46}$$

In fact, if $v \in H^1(\Omega) \cap L^\infty(\Omega)$, then there exists $v_n \in C(\overline{\Omega})$ such that $v_n \to v$ in $H^1(\Omega)$ and a.e. in Ω with $|v_n| \le M$ in Ω for some $M > 0$. By (5) and Lebesgue dominated convergence theorem, we obtain (46).

Since $u \in H^1(\Omega)$, if we set

$$u_n = \begin{cases} n & \text{if } u \ge n \\ u & \text{if } |u| < n \\ -n & \text{if } u \le -n \end{cases}$$

then $u_n \in H^1(\Omega) \cap L^\infty(\Omega)$. So by (46), one has

$$\langle T, u_n \rangle = \int_\Omega T_a u_n dx + \int_\Gamma T_{\Gamma,a} \gamma u_n d\Gamma. \tag{47}$$

Since j_0 and j_1 are convex functions, then it follows from (6) that, for all $v \in H^1(\Omega)$,

$$\begin{aligned} T_a(x)(u(x) - v(x)) &\ge j_0(u(x)) - j_0(v(x)) \quad \text{a.e. in } \Omega, \\ T_{\Gamma,a}(x)(\gamma u(x) - \gamma v(x)) &\ge j_1(\gamma u(x)) - j_1(\gamma v(x)) \quad \text{a.e. on } \Gamma. \end{aligned} \tag{48}$$

If $v = 0$, then $T_a(x)u(x) \ge j_0(u(x)) \ge 0$ a.e. in Ω and $T_{\Gamma,a}(x)\gamma u(x) \ge j_1(\gamma u(x)) \ge 0$ a.e. on Γ. Since $u_n(x)$ and $u(x)$ have the same sign a.e. in Ω, we obtain $T_a(x)u_n(x) \ge 0$ a.e. in Ω. Similarly, one has $T_{\Gamma,a}(x)\gamma u_n(x) \ge 0$ a.e. on Γ.

Since $u_n \to u$ in $H^1(\Omega)$ and a.e. in Ω with $\gamma u_n \to \gamma u$ a.e. on Γ, then by (47) and Fatou's lemma one has

$$\begin{aligned} \langle T, u \rangle = &\lim_{n\to\infty} \langle T, u_n \rangle = \lim_{n\to\infty} \left(\int_\Omega T_a u_n dx + \int_\Gamma T_{\Gamma,a} \gamma u_n d\Gamma \right) \\ &\ge \liminf_{n\to\infty} \int_\Omega T_a u_n dx + \liminf_{n\to\infty} \int_\Gamma T_{\Gamma,a} \gamma u_n d\Gamma \ge \int_\Omega T_a u dx + \int_\Gamma T_{\Gamma,a} \gamma u d\Gamma, \end{aligned}$$

and along with (5) and (48) we obtain for all $v \in H^2(\Omega)$,

$$\begin{aligned} \langle T, u - v \rangle &\ge \int_\Omega T_a(u - v) dx + \int_\Gamma T_{\Gamma,a}(\gamma u - \gamma v) d\Gamma \\ &\ge \int_\Omega (j_0(u) - j_0(v)) dx + \int_\Gamma (j_1(\gamma u) - j_1(\gamma v)) d\Gamma. \end{aligned}$$

By Lemma 3.1 we conclude that, for all $v \in D(J)$

$$\langle T, u - v \rangle \ge \int_\Omega (j_0(u) - j_0(v)) dx + \int_\Gamma (j_1(\gamma u) - j_1(\gamma v)) d\Gamma = J(u) - J(v).$$

Thus, $T \in \partial J(u)$, completing the proof of Theorem 2.2.

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