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On Approximation by Δ-Convex Polyhedron Support Functions and the Dual of $\mathrm{cc}(X)$ and $\mathrm{wcc}(X)$

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The classical Weierstrass theorem states that every continuous function f defined on a compact set $\Omega \subset \mathbb{R}^n$ can be uniformly approximated by polynomials. In this paper we show first that it is again valid if Ω is a compact Hausdorff metric space, i.e., it holds in the following sense: there exists a surjective isometry T from a compact set K_Ω of a Banach sequence space S to Ω, such that for every $\varepsilon > 0$ there is an n variable polynomial p satisfying

$$|f(T(s)) - p(s_1, s_2, \cdots, s_n)| < \varepsilon, \quad \forall s = (s_j) \in K_\Omega.$$

We prove also that for any *weak* (w^*, resp.) continuous positively homogenous function f defined on a (dual, resp.) Banach space X (X^*, resp.) then for all $\varepsilon > 0$ and for every weakly compact set $K \subset X$ (w^* compact set $K \subset X^*$), there exist $\phi_i \in X^*$ (X, resp.) for $i = 1, 2, \cdots, m$, and $\psi_j \in X^*$ (X, resp.) for $j = 1, 2, \cdots, n$ such that

$$|f(x) - [(\phi_1 \vee \phi_2 \vee \cdots \vee \phi_m)(x) - (\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n)(x)]| < \varepsilon$$

uniformly for $x \in K$. Let $\mathrm{cc}(X)$ ($\mathrm{wcc}(X)$, resp.) be the norm semigroup consisting of all nonempty (weakly, resp.) compact convex sets of the space X. As its application, we give two representation theorems of the duals of $\mathrm{cc}(X)$ and $\mathrm{wcc}(X)$.

Keywords: Weierstrass theorem, function approximation, weakly continuous function, weakly compact set, normed semigroup, Δ convex polyhedron support function

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1. Introduction

It is always fascinating to have an account of the steps in the development of a significant theorem. The result that every continuous function f defined on a compact

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set $K \subset \mathbb{R}^n$ can be uniformly approximated by polynomials, known as the classical Weierstrass theorem, which is due to Weierstrass [25], and it has become standard in approximation theory and which has been usefully extended in the study of density of subalgebras and subspaces in a continuous function space $C(\Omega)$ for some compact Hausdorff space Ω. Most interesting generalizations of the Weierstrass theorem are due to Stone [22] (see, also [23]) (known as the Stone-Weierstrass theorem), S. Kakutani [16] and M. Krein and S. Krein [19] (known as the Kakutani-Krein theorem). First, we recall them as follows.

Theorem 1.1 (Stone-Weierstrass theorem). *Let Ω be a compact Hausdorff space and $C(\Omega)$ the space of all real valued continuous functions on Ω endowed with the* sup-*norm. Let $\mathcal{A} \subset C(\Omega)$ satisfy the three conditions:*

(*i*) *$\mathcal{A}$ is an algebra;*
(*ii*) *the constant function 1 is in $\mathcal{A}$, and*
(*iii*) *$\mathcal{A}$ separates points of Ω.*

Then $\mathcal{A}$ is dense in $C(\Omega)$.

Theorem 1.2 (Kakutani-Krein theorem). *Let Ω be a compact Hausdorff space and $C(\Omega)$ the space of real valued continuous functions on Ω endowed with the* sup-*norm. Let $\mathcal{A} \subset C(\Omega)$ satisfy the conditions:*

(*i*) *$\mathcal{A}$ is a vector lattice of $C(\Omega)$;*
(*ii*) *constant function $1 \in \mathcal{A}$, and*
(*iii*) *$\mathcal{A}$ separates points of Ω.*

Then $\mathcal{A}$ is dense in $C(\Omega)$.

There are three gaols of this paper: (1) giving an extension of the classical approximation Weierstrass theorem to continuous functions defined on a compact metric space; (2) establishing an approximation theorem for weakly (w^*, resp.) continuous positively homogenous functions defined on a (dual, resp.) Banach space; and as its application, (3) showing two representation theorems of the duals $\mathrm{cc}(X)^*$ and $\mathrm{wcc}(X)^*$ of $\mathrm{cc}(X)$ and $\mathrm{wcc}(X)$ for a Banach space X. They are done by making use of the Stone-Weierstrass theorem (Theorem 1.1), and other properties about Banach space theory, isometric embedding, compact and weakly compact sets, Fréchet differentiability of convex functions and their duality theory.

This paper is organized as follows. In the following section, we show that there exists a Banach sequential space $S \subset \mathbb{R}^{\mathbb{N}}$, which is linearly isometric to $C[0,1]$, such that for any compact metric space Ω there exists a surjective isometric mapping $T : K_\Omega \to \Omega$ for some compact set $K_\Omega \subset S$ satisfies that for every $f \in C(\Omega)$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a polynomial $p = p(s_1, s_2, \cdots, s_n)$ such that

$$|f(T(s)) - p(s_1, s_2, \cdots, s_n)| < \varepsilon, \quad \forall s = (s_j) \in K_\Omega.$$

In Section 3, we show that if f is a weakly (w^*, resp.) continuous positively homogenous function defined on a Banach space X (the dual X^* of a Banach space X, resp.), then for every weakly (w^*, resp.) compact set $K \subset X$ (X^*, resp.) and for all $\varepsilon > 0$, there exist $\phi_i \in X^*$ (X, resp.) for $i = 1, 2, \cdots, m$, and $\psi_j \in X^*$ (X, resp.) for $j = 1, 2, \cdots, n$ such that

$$|f(x) - [(\phi_1 \vee \phi_2 \vee \cdots \vee \phi_m)(x) - (\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n)(x)]| < \varepsilon$$

uniformly for $x \in K$. In the last section, making use of the conclusion above, we show that for every Banach space X, if we denote by $\mathrm{cc}(X)$ and $\mathrm{wcc}(X)$ the normed semigroups consisting of all nonempty compact convex sets and all nonempty weakly compact convex sets, respectively, then we obtain the the following representation theorems of their duals:

$$\mathrm{cc}(X)^* \cong C_{\mathrm{PH}}(B_{w^*})^* \quad \text{and} \quad \mathrm{wcc}(X)^* \cong C_{\Delta SFD}(B^*)^*,$$

where $B^*_{w^*}$ denotes the closed unit ball of X^* endowed with the w^* topology of X^*, $C_{\mathrm{PH}}(B_{w^*})$ the Banach space of all positively homogenous w^* continuous functions restricted to the closed unit ball B^* of X^* equipped with the sup-norm, and $C_{\Delta SFD}(B^*)$ denotes the Banach space of the closure of all continuous functions f on B^* admitting a decomposition $f = \sigma_C - \sigma_D$ for some weakly compact sets $C, D \subset X$ such that σ_C^2 and σ_D^2 are Fréchet differentiable equipped with the sup-norm.

In this paper, the letter X will always be a real Banach space and X^* its dual. B_X (B_{X^*}, resp.) stands for the closed unit ball of X (X^*, resp.); if there is no confusion arising, we simply denote by B (B^*, resp.) for B_X (B_{X^*}, resp.). S_X (S_{X^*}, resp.) represents the unit sphere of X (X^*, resp.). We denote by Ω a compact Hausdorff space, and by $C(\Omega)$ the Banach space of all real-valued continuous functions defined on Ω endowed with the sup-norm. For a subset $A \subset X$, σ_A stands for the support function of A, i.e., $\sigma_A(x^*) = \sup_{x \in A}\langle x^*, x\rangle$, and A^0 for the annihilator of A, i.e., $A^0 = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for all } x \in A\}$.

2. An extension of the classical Weierstrass theorem

In this section, we shall give an extension of the classical Weierstrass approximation theorem to continuous functions defined on a compact metric space. First, we recall some notions which will be used in this sequel.

Let $G \subset C(\Omega)$ be a set, and $\mathcal{A}_G$ be the algebra generated by G, i.e., $\mathcal{A}_G$ is the smallest set containing $G \cup \{1\}$, and closed under both linear and multiplication operations. A function f is said to be a G-polynomial provided $f \in \mathcal{A}_G$. For example, f is called an n variable polynomial if and only if $f \in \mathcal{A}_{1,s_1,s_2,\cdots,s_n}$.

The following notions about Schauder bases of Banach spaces are taken from Lindenstrauss and Tzafriri [15].

A sequence $\{x_n\}$ in a Banach space X is said to be a Schauder basis of X if for every $x \in X$ there exists a unique sequence of scalars $\{a_n\}$ so that $x = \sum_{n=1}^\infty a_n x_n$. Let $\{x_n\}$ be a basis of X. Then for every $n \in \mathbb{N}$ the linear functional x_n^* defined by $\langle x_n^*, \sum_{j=1}^\infty a_j x_j\rangle = a_n$ is in X^*. These functionals $\{x_n^*\}$, which are characterized by the relation $\langle x_n^*, x_m\rangle = \delta_n^m$, are called the bi-orthogonal functionals associated to the basis $\{x_n\}$. Let $\{P_n\}$ be the natural projections associated to the basis, i.e., $P_n(\sum_{j=1}^\infty a_j x_j) = \sum_{j=1}^n a_j x_j$. Since $\lim_n \|P_n x - x\| = 0$, for every $x \in X$, we get that, in the sense of convergence in the w^* topology, $x^* = \sum_{n=1}^\infty \langle x^*, x\rangle x_n^*$ for every $x^* \in X^*$.

Now, we are ready to state and prove the main theorem of this section.

Theorem 2.1. *There exists a Banach sequential space $S \subset \mathbb{R}^{\mathbb{N}}$, which is linearly isometric to $C[0,1]$, such that for any compact metric space Ω, there exists a surjective isometric mapping $T : K_\Omega \to \Omega$ for some compact set $K_\Omega \subset S$ satisfies that for every $f \in C(\Omega)$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a polynomial $p = p(s_1, s_2, \cdots, s_n)$ such that*

$$|f(T(s)) - p(s_1, s_2, \cdots, s_n)| < \varepsilon, \quad \forall s = (s_j) \in K_\Omega.$$

Proof. Let $\{x_n\}$ be a normalized Schouder basis of $C[0,1]$, and $\{x_n^*\}$ be the associated bi-orthogonal functionals to $\{x_n\}$. We define a sequential space S by

$$S = \left\{ s = (s_n) : s_n \in \mathbb{R} \text{ with } \|s\| \equiv \left\| \sum_{n=1}^{\infty} s_n x_n \right\| < \infty \right\}.$$

It is clear that the sequential space S is isometric onto $C[0,1]$, and the standard unit vectors $\{e_n\}$ of S form a normalized basis of S. Let $\{e_n^*\}$ be the associated bi-orthogonal functionals to $\{e_n\}$.

Assume that Ω is a nonempty compact metric space. Then Ω is separable, and which is, in turn, isometric to a (compact) subset Ω_1 of ℓ_∞ (see, for instance, [2], pp. 2–3). Let Y be the closure of $\operatorname{span}\Omega_1$ in ℓ_∞. Then Y is separable. According to the Banach-Mazur theorem ([1], Chap. XI, Sec. 8), Y is linearly isometric to a linear subspace of $C[0,1]$, hence, a subspace of the space S. Therefore, we have shown that Ω is isometric to a compact subset K_Ω of S. Let $T : K_\Omega \to \Omega$ be a surjective isometry.

Note that the algebra $\mathcal{A}_{1,e_1^*,\cdots,e_n^*,\cdots}$ generated by $\{1, e_1^*, \cdots, e_n^*, \cdots\}$ separates points of K_Ω. By the Stone-Weierstrass theorem (Theorem 1.1), $\mathcal{A}_{1,e_1^*,\cdots,e_n^*,\cdots}$ is dense in $C(K_\Omega)$. Thus, for every $f \in C(\Omega)$, and for every $\varepsilon > 0$, there exists an n polynomial $p = p(e_1^*, e_2^*, \cdots, e_n^*)(\cdot) \in \mathcal{A}_{1,e_1^*,\cdots,e_m^*,\cdots}$ such that

$$|f(T(s)) - p(s)| < \varepsilon, \quad \text{uniformly for } s \in K_\Omega.$$

It is trivial to see that $p(s) = p(s_1, s_2, \cdots, s_n)$. $\square$

3. Approximation of weakly continuous positive homogenous functions

In this section, we shall study approximation property of weakly (w^*, resp.) continuous positively homogenous functions defined on a (dual, resp.) Banach space X (X^*, resp.) restricted to some weakly (w^*, resp.) compact set approximated by convex polyhedron support functions. To begin with, we recall again some notions.

A subset $A \subset X$ is said to be an n-simplex provided there exist $n+1$ affinely independent vectors $x_0, x_1, \cdots, x_n \in A$ (that is, the n vectors $x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0$ are linearly independent) such that $A = \operatorname{co}\{x_0, x_1, \cdots, x_n\}$. The set A is called a convex polyhedron if it is convex and is the union of finitely many n-simplexes for some $n \in \mathbb{N}$. A function f on a Banach space X is said to be a convex polyhedron support function if there exists a convex polyhedron $A \subset X^*$ such that $f = \sigma_A$. For $f, g \in C(\Omega)$ and $\omega \in \Omega$, let

$$(f \vee g)(\omega) = \max(f(\omega), g(\omega)) \quad \text{and} \quad (f \wedge g)(\omega) = \min(f(\omega), g(\omega)).$$

A subspace $M \subset C(\Omega)$ is a vector lattice, if it is closed under the operations of " $\vee$ " and " $\wedge$ ".

Proposition 3.1. *A function f on a Banach space X is a convex polyhedron support function if and only if there exist n linear functionals $\phi_1, \phi_2, \cdots, \phi_n \in X^*$ such that $f = \phi_1 \vee \phi_2 \vee \cdots \vee \phi_n$.*

Assume that X is a Banach space. Put

$$\mathrm{cp}_0(X) = \{K \in \mathrm{cc}(X) : K \text{ is a convex polydedron with } 0 \in K\};$$
$$P_{\mathrm{cp}_0(X)} = \{\sigma_K : K \in \mathrm{cp}_0(X)\}$$
$$M_{\mathrm{cp}_0}(X) = \{\sigma_{K_1} - \sigma_{K_2} : K_1, K_2 \in pc_0(X)\}.$$

Lemma 3.2. *Suppose that X is a Banach space. Then $M_{\mathrm{cp}_0(X)}$ is a vector lattice of $C(B^*_{w^*})$.*

Proof. It is clear that $M_{\mathrm{cp}_0(X)}$ is a linear space since $P_{\mathrm{cp}_0(X)}$ is a cone and since $M_{\mathrm{cp}_0(X)} = P_{\mathrm{cp}_0(X)} - P_{\mathrm{cp}_0(X)}$. Note that $(f \wedge g)(\omega) = \min(f(\omega), g(\omega)) = -\max(-f(\omega), -g(\omega)) = -(-f \vee -g)(\omega)$. To show that $M_{\mathrm{cp}_0(X)}$ is a vector lattice, it suffices to prove that $M_{\mathrm{cp}_0(X)}$ is closed under the operation of " $\vee$ ". Indeed, $\forall\, f, g \in M_{\mathrm{cp}_0(X)}$, let $C_1, C_2, D_1, D_2 \in \mathrm{cp}_0(X)$ such that $f = \sigma_{C_1} - \sigma_{D_1},\ g = \sigma_{C_2} - \sigma_{D_2}\}$. Then

$$\begin{aligned} f \vee g = &\ \max\{f, g\} = \max\{\sigma_{C_1} - \sigma_{D_1}, \sigma_{C_2} - \sigma_{D_2}\} \\ &= [(\sigma_{C_1} - \sigma_{D_1}) - (\sigma_{C_2} - \sigma_{D_2})] \vee 0 + (\sigma_{C_2} - \sigma_{D_2}) \\ &= (\sigma_{C_1+D_2} - \sigma_{C_2+D_1}) \vee 0 + (\sigma_{C_2} - \sigma_{D_2}) \\ &= \sigma_{\mathrm{co}[(C_1+D_2)\bigcup(C_2+D_1)]} - \sigma_{C_2+D_1} + (\sigma_{C_2} - \sigma_{D_2}) \\ &= \sigma_{\mathrm{co}[(C_1+D_2)\bigcup(C_2+D_1)+C_2]} - \sigma_{C_2+D_1+D_2} \\ &= \sigma_{\mathrm{co}[(C_1+D_2)\bigcup(C_2+D_1)]} - \sigma_{D_1+D_2} \end{aligned}$$

Therefore, $f \vee g = \max\{f, g\} \in M_{\mathrm{cp}_0(X)}$. □

For a real Banach space X, let $B^*_{w^*}$ denote the closed unit ball of X^* endowed with the w^* topology, and let $C_{\mathrm{PH}}(B^*_{w^*})$ be the Banach space of all the w^* continuous positively homogenous functions on X^* restricted to B^*.

Lemma 3.3. *Let X be a real Banach space. Then $M_{\mathrm{cp}_0(X)}$ satisfies the condition that for every $h \in C_{\mathrm{PH}}(B^*)$, and for all $x^*, y^* \in B^*$, $x^* \neq y^*$ there exist $C, D \in$ $\mathrm{cp}_0(X)$ such that*

$$(\sigma_C - \sigma_D)(x^*) = h(x^*) \quad \text{and} \quad (\sigma_C - \sigma_D)(y^*) = h(y^*).$$

Proof. Since h and all functions in $M_{\mathrm{cp}_0(X)}$ are w^* continuous and positively homogenous, we can assume that $x^*, y^* \in S_{X^*}$. Let $h(x^*) = \alpha$, $h(y^*) = \beta$, and $\alpha \geq \beta$.

Case (i)*:* $x^* = -y^*$. we choose $x \in X$ such that $x^*(x) = 1$. If $\alpha \geq \beta \geq 0$, then we take $C = \mathrm{co}\{-\beta x, \alpha x\}$ and $D = \{0\}$; if $\alpha \geq 0 \geq \beta$, then we set $C = \mathrm{co}\{0, \alpha x\}$ and $D = \mathrm{co}\{0, \beta x\}$; if $0 \geq \alpha \geq \beta$, we put $C = \{0\}$ and $D = \mathrm{co}\{\beta x, -\alpha x\}$.

Case (ii)*:* $x^* \neq -y^*$. Obviously, x^* and y^* are linearly independent. So we can choose $y \in \ker(x^*)$ and $x \in \ker(y^*)$ such that $y^*(y) = 1$ and $x^*(x) = 1$. If $\alpha \geq \beta \geq 0$, then we take $C = \mathrm{co}\{\alpha x, \beta y, 0\}$ and $D = \{0\}$; if $\alpha \geq 0 \geq \beta$, we set $C = \mathrm{co}\{0, \alpha x\}$ and $D = \mathrm{co}\{0, -\beta y\}$; if $0 \geq \alpha \geq \beta$, we put $C = \{0\}$ and $D = \mathrm{co}\{-\alpha x, -\beta y, 0\}$. □

Now, we apply Lemmas 3.2 and 3.3 to prove the following result.

Theorem 3.4. *$M_{\mathrm{cp}_0(X)}$ is a dense subspace of $C_{\mathrm{PH}}(B^*_{w^*})$.*

Proof. Let $h \in C_{\mathrm{PH}}(B^*_{w^*})$. We want to show that for every $\varepsilon > 0$ there exists $f \in M_{\mathrm{cp}_0(X)}$ such that

$$\mid f(u) - h(u) \mid < \epsilon, \quad \text{for all } u \in B^*.$$

Let $t \neq s \in B^*$. By Lemma 3.3, we assert that there exists $f_{st} \in M_{\mathrm{cp}_0(X)}$ such that $f_{st}(s) = h(s)$, and $f_{st}(t) = h(t)$. Given $\epsilon > 0$ and a point $t \in B^*$, then, for each $s \in B^*$, there is a w^* open neighborhood $U(s)$ of s such that $f_{st}(u) > h(u) - \epsilon$ whenever $u \in U(s)$. w^*-compactness of B^* entails that there exist n such w^* open neighborhoods $\{U(s_i)\}_{i=1}^n$ which form a covering of B^*. We define f_t by $f_t = f_{s_1t} \vee \cdots \vee f_{s_nt}$. Then, by Lemma 3.2, $f_t \in M_{\mathrm{cp}_0(X)}$ and $f_t(u) > h(u) - \epsilon$ for all $u \in B^*$. We also have $f_t(t) = h(t)$ since $f_{s_it}(t) = h(t)$. Hence there is a w^* open neighborhood $V(t)$ of t such that $f_t(u) < h(u) + \varepsilon$ whenever $u \in V(t)$. Let $\{V(t_j)\}_{j=1}^m$ be a w^* open covering of B^*. Let $f = f_{t_1} \wedge \cdots \wedge f_{t_m}$. Then (again by Lemma 3.2) $f \in M_{\mathrm{cp}_0(X)}$ and $f(u) < h(u) + \epsilon$ for all $u \in B^*$. On the other hand, we have $f(u) > h(u) - \epsilon$ for all $u \in B^*$ because that $f_{t_j}(u) > h(u) - \epsilon$ and for all $1 \leq j \leq m$. Therefore, $\mid f(u) - h(u) \mid < \epsilon$, for all $u \in B^*$. □

Corollary 3.5. *Suppose that f is w^*-continuous positively homogenous function defined on X^*. Then for all $\varepsilon > 0$ and for every nonempty bounded subset $A \subset X^*$ there are convex polyhedrons $C, D \subset X$ such that*

$$\sup_{x^* \in A} |f(x^*) - (\sigma_C(x^*) - \sigma_D(x^*))| < \varepsilon.$$

Let K_w be a weakly compact set of a Banach space X endowed with the weak topology, and let $C_{\mathrm{PH}}(K_w)$ be the Banach space of all the weakly continuous positively homogenous functions on X restricted to K_w. Then we have analogously the following theorem.

Theorem 3.6. *Suppose that K_w is a weakly compact set of a Banach space X. Then $M_{\mathrm{cp}_0(X^*)}$ is a dense subspace of $C_{\mathrm{PH}}(K_w)$, that is, for all $\varepsilon > 0$ and for every weakly continuous positively homogenous function f on X there are convex polyhedrons $C, D \subset X^*$ such that*

$$\sup_{x \in K_w} |f(x) - (\sigma_C(x) - \sigma_D(x))| < \varepsilon.$$

4. On representation of $\mathrm{cc}(X)^*$ and $\mathrm{wcc}(X)^*$

There is a large number of articles concerning fixed point property and applications of the hyperspaces $\mathrm{cc}(X)$ and $\mathrm{wcc}(X)$ endowed with the Hausdorff metric. See, for example, [8], [11], [12] and [13]. In this section, we shall give representation of their dual spaces in view of normed semigroup. We recall first that the definition of normed semigroup. Let G be an Abelian semigroup and let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. G is said to be a module if there are two operations $(x, y) \in G \times G \to x + y \in G$, and $(\alpha, x) \in (\mathbb{F} \times G) \to \alpha x \in G$ satisfying

$$(\lambda\mu)g = \lambda(\mu g), \quad \forall \lambda, \mu \in \mathbb{F} \text{ and } g \in G;$$
$$\lambda(g_1 + g_2) = \lambda g_1 + \lambda g_2, \quad \forall \lambda \in \mathbb{F} \text{ and } g_1, g_2 \in G;$$

and

$$1g = g \text{ and } 0g = 0 \quad \forall g \in G.$$

A module G endowed with a norm is called a normed semigroup.

A function ϕ on a normed semigroup G is called a linear functional if it satisfies that

$$\phi(\alpha g_1 + \beta g_2) = \alpha\phi(g_1) + \beta\phi(g_2), \quad \forall \alpha, \beta \in \mathbb{R}^+ \text{ and } g_1, g_2 \in G.$$

It is said to be bounded provided $\|\phi\| = \sup\{\phi(g) : g \in G, \|g\| \leq 1\} < \infty$. We denote by G^* the Banach space of all bounded functionals on G, and call it the dual of G.

Given a (real) Banach space X, let

$$\mathrm{cc}(X) = \{K \subset X : K \text{ is nonempty convex and compact}\},$$
$$\mathrm{cc}_0(X) = \{K \subset X : K \text{ is convex compact and with } 0 \in K\},$$
$$\mathrm{wcc}(X) = \{K \subset X : K \text{ is nonempty convex and weakly compact}\}.$$

and

$$\mathrm{wcc}_0(X) = \{K \in \mathrm{wcc}(X) \text{ with } 0 \in K\}.$$

Clearly, all the four sets with the usual operations $K_1 + K_2 = \{k_1 + k_2 : k_1 \in K_1$ and $k_2 \in K_2\}$ and $\lambda K = \{\lambda k : k \in K\}$ are modules. We endow the Hausdorff metric d_H on $Z \in \{\mathrm{cc}(X), \mathrm{cc}_0(X), \mathrm{wcc}(X), \mathrm{wcc}_0(X)\}$, i.e.,

$$d_H(K_1, K_2) = \max\left\{\sup_{x \in K_1} d(x, K_2), \sup_{y \in K_2} d(K_1, y)\right\}, \quad \text{for } K_1,\ K_2 \in Z.$$

This metric induces further a norm $\|\cdot\|_H$ for $K \in Z$

$$\|K\|_H = d_H(0, K) = \sup\{\|k\| : k \in K\}.$$

Therefore, $\mathrm{cc}(X)$, $\mathrm{cc}_0(X)$, $\mathrm{wcc}(X)$ and $\mathrm{wcc}_0(X)$, endowed with the norm, are normed semigroups.

We would like to mention two remarkable results concerning embedding of $\mathrm{cc}(X)$ and representation of $\mathrm{cc}(\mathbb{R}^n)^*$. Radstrom [20] showed that $\mathrm{cc}(X)$ is (additivity and nonnegative scalar multiplication preserved) isometric to cone of a real Banach space. While Keimel and Roth [17] proved that $\mathrm{cc}(\mathbb{R}^{n*})^* \simeq C(S_{X^*})^*$, where S_{X^*} denotes the unit sphere of $(\mathbb{R}^n)^*$, and $C(S_{X^*})$ stands for the space of all continuous functions on S_{X^*} equipped with the sup-norm. For a Banach space X, let

$$\begin{gathered} P_{\mathrm{cc}(X)} = \{\sigma_K : K \in \mathrm{cc}(X)\} \\ M_{\mathrm{cc}(X)} = \{\sigma_{K_1} - \sigma_{K_2} : K_1, K_2 \in \mathrm{cc}(X)\} \\ P_{\mathrm{wcc}(X)} = \{\sigma_K : K \in \mathrm{wcc}(X)\} \\ M_{\mathrm{wcc}(X)} = \{\sigma_{K_1} - \sigma_{K_2} : K_1, K_2 \in \mathrm{wcc}(X)\}. \\ P_{\mathrm{wcc}_0(X)} = \{\sigma_K : K \in \mathrm{wcc}_0(X)\} \\ M_{\mathrm{wcc}_0(X)} = \{\sigma_{K_1} - \sigma_{K_2} : K_1, K_2 \in \mathrm{wcc}_0(X)\}. \end{gathered}$$

A function f defined on a convex set $C \subset X$ is said to be Δ-convex provided there are two convex functions f_1, f_2 such that $f = f_1 - f_2$ [3]. f is called Δ-lower semi-continuous if there are lower semi-continuous functions f_1, f_2 such that f admits the decomposition $f = f_1 - f_2$. If f is defined on a convex set $C^* \subset X^*$, then it is said to be a Δ-support function if there are convex sets $C, D \subset X$ such that $f = \sigma_C - \sigma_D$. Clearly, a Δ-support function is necessarily Δ-convex and Δ-w^*-lower semi-continuous.

We also use ΔSFSD to represent "Δ-support functions whose decomposition functions are square Fréchet differentiable", i.e., those functions $f = \sigma_C - \sigma_D$ (for some closed convex sets $C,\ D \subset X$) satisfying that $\sigma_C{}^2$ and $\sigma_D{}^2$ are Fréchet differentiable on B^*, and we denote by $C_{\Delta\mathrm{SFSD}}(B^*)$ the closure of the space of all such Δ-support functions.

In this section, inspired by Kermel and Roth [17], we shall show that for any real Banach space X, we have $\mathrm{cc}(X)^* \simeq C_{\mathrm{PH}}(B^*_{w^*})^*$ and $\mathrm{wcc}(X)^* \simeq C_{\Delta\mathrm{SFSD}}(B^*)^*$, where $B^*_{w^*}$ denotes, as before, the closed unit ball B^* of X^* endowed with the w^* topology of X^*, and $C_{\mathrm{PH}}(B^*_{w^*})$ stands for the Banach space of all positively homogenous w^* continuous functions restricted on B^* equipped with the sup-norm.

Lemma 4.1. *Suppose that X is a Banach space. Then*

(*1*) $\mathrm{cc}(X)$ *is order isometric to* $P_{\mathrm{cc}(X)}$*;*

(*2*) $M_{\mathrm{cc}(X)}$ *is a vector lattice of* $C(B^*_{w^*})$*.*

Proof. (*1*) Since for all $\lambda \geq 0$, and for all $K, K_1, K_2 \in \mathrm{cc}(X)$, $\sigma_{K_1+K_2} = \sigma_{K_1} + \sigma_{K_2}$, $\sigma_{\lambda K} = \lambda\sigma_K$, and since $d_H(K_1, K_2) = \|\sigma_{K_1} - \sigma_{K_2}\|$, $\mathrm{cc}(X)$ is order isometric to $P_{\mathrm{cc}(X)}$.

(*2*) It follows from the same procedure of the proof of Lemma 3.2. □

Theorem 4.2. $\mathrm{cp}_0(X)^* = \mathrm{cc}_0(X)^* = \mathrm{cp}(X)^* = \mathrm{cc}(X)^* \simeq C_{\mathrm{PH}}(B^*_{w^*})^*$.

Proof. By Theorem 3.4, it suffices to note $M_{pc_0(X)} \subset M_{\mathrm{cc}_0(X)} \subset M_{\mathrm{cc}(X)} \subset C_{\mathrm{PH}}(B^*_{w^*})$. □

Similar to Lemma 4.1, we have

Lemma 4.3. *Suppose that X is a Banach space. Then*

(*1*) $\mathrm{wcc}(X)$ *is order isometric to* $P_{\mathrm{wcc}(X)}$*;*

(*2*) $M_{\mathrm{wcc}(X)}$ *is a vector lattice of* $C(B^*)$.
(*3*) $\mathrm{wcc}_0(X)$ *is order isometric to* $P_{\mathrm{wcc}_0(X)}$;
(*4*) $M_{\mathrm{wcc}_0(X)}$ *is a vector lattice of* $C(B^*)$.

To show $\mathrm{wcc}(X)^* \simeq C_{\Delta\mathrm{SFSD}}(B^*)^*$, we need more preparation. The following definition is taken from [7].

Definition 4.4. Suppose that $g : X \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued convex function with its effective domain $\mathrm{dom}\, g \equiv \{x \in X : g(x) < \infty\} \neq \emptyset$.

(1) g is said to be locally uniformly convex provided for every $x \in \mathrm{dom}\, g$ and every sequence $\{x_n\} \subset \mathrm{dom}\, g$,

$$g(x) + g(x_n) - 2g\left(\frac{x + x_n}{2}\right) \to 0 \text{ implies } x_n \to x;$$

We call the norm $\|\cdot\|$ of X is locally uniformly convex if $g \equiv \frac{1}{2}\|\cdot\|^2$ is a locally uniformly convex function.

(2) g is called Fréchet differentiable at $x \in X$ provided that there exists $x^* \in X^*$ such that

$$\limsup_{t\to 0^+,\, y\in B_X} \left[\frac{g(x+ty) - g(x)}{t} - \langle x^*, y\rangle\right] = 0.$$

In this case, we call $d_F g(x) \equiv x^*$ the Fréchet derivative of g at x. A norm is said to be Fréchet smooth if it is everywhere Fréchet differentiable off the origin.

(3) The conjugate function $g^*$of $g : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined for $x^* \in X^*$ by

$$g^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}.$$

In particular, letting $g = \frac{1}{2}\|\cdot\|^2$, then the dual norm of $\|\cdot\|$ on X^* is just $\sqrt{2g^*}$.

The following properties are easy to be obtained.

Proposition 4.5. *Suppose that* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *are two extended real-valued lower semicontinuous convex functions with* $\mathrm{dom}(f + g) \neq \emptyset$.

(*1*) *If* X *is a dual space (in particular, a reflexive space) and* f *is* w^* *lower semicontinuous then there exists an extended real-valued lower semicontinuous convex functions* h *on the predual of* X *such that* $f = h^*$;
(*2*) *If one of* f, g *is locally uniformly convex, then* $f + g$ *is also locally uniformly convex;*
(*3*) *If* f^* *is locally uniformly convex, then* f *is Fréchet differentiable in the interior of* $\mathrm{dom}\, f$.

Recall that an extended real-valued Minkowski functional p on a Banach space X is a non-negative-valued sublinear function, i.e., $p(x) \in \mathbb{R} \cup \{+\infty\}$ with $p(\lambda x) = \lambda p(x)$ for all $x \in X$, $\lambda \geq 0$ and with $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Lemma 4.6. *Suppose that* X *is a Banach space and* $p : X \to \mathbb{R} \cup \{+\infty\}$ *is an extended real-valued lower semicontinuous Minkowski functional with* $p \geq \|\cdot\|$ *on*

X. Let $q_n^2 = p^2 + c_n\|\cdot\|^2$ for all $n \in \mathbb{N}$, where $0 < c_n \to 0$. Then $(q_n^2)^* \to (p^2)^*$ uniformly on B^*.

Proof. By definition of conjugate function, it sufficies to note that for all $x^* \in X^*$,

$$\begin{aligned}\frac{1}{\sqrt{1+c_n}}(p^2)^*(x^*) = \{(1+c_n)p^2\}^*(x^*) \leq (q_n^2)^*(x^*)& \\ = \sup\{\langle x^*, x\rangle - (p^2(x) + c_n\|x\|^2) : x \in \operatorname{dom} p\}& \\ \leq (p^2)^*(x^*) \leq (\|\cdot\|^2)^*(x^*).&\end{aligned}$$

□

The following theorem is due to Troyanski [24], see, also [9] and [14].

Theorem 4.7. *Every reflexive Banach space X can be renormed in such a way that X and X^* have Fréchet smooth and locally uniformly convex norms.*

The following lemma was motivated by [4] and [6].

Lemma 4.8. *$P_{\mathrm{wcc}_0(X)}$ is dense in the closure of*

$$F \equiv \{\sigma_K : K \in \mathrm{wcc}_0(X), {\sigma_K}^2 \text{ is Frechet differentiable}\}.$$

Proof. We show first that if $K \subset X$ is a closed convex set with $\sigma_K \subset \overline{F}$, then K is weakly compact. Let $\sigma_n \equiv \sigma_{K_n} \in F$ such that $\sigma_n \to \sigma_K$ in $C(B^*)$. Then, by Theorem 4.6 of [4], K_n are weakly compact for all $n \in \mathbb{N}$. This entails that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $K \subset K_n + \varepsilon B$. According to Grothendiek's lemma, K is weakly compact. Conversely, let $K \in \mathrm{wcc}_0(X)$, and let X_0 be the closure of span K in X. Since K is also weakly compact in X_0, by a revised version of the Davis-Figiel-Johnson-Pelczyski factoring lemma [5], there is a reflexive space $(Y, |\cdot|)$ such that $K \subset B_Y \subset B_{X_0}$. Reflexivity of Y and Theorem 4.7 enable us to assume that the dual norm $|\cdot|^*$ of $|\cdot|$ is both locally uniformly convex and Fréchet smooth on Y^*. By Proposition 4.5, there is an extended real-valued lower semicontinuous Minkowski functional $p \geq |\cdot|$ on Y such that $(p^2)^* = \sigma_K^2$, and $h_n \equiv p^2 + 2^{-n}|\cdot|^2$ are locally uniformly convex on Y. Note that h_n^* are real-valued positively homogeneous of the second degree and w^* lower semi continuous on Y^*. Since Y is reflexive, h_n^* must be everywhere Fréchet differentiable in Y^*. Therefore, within the natural norm-preserved restriction to Y, we obtain $X_0^* \subset Y^*$ and $B_{X_0^*} \subset B_{Y^*}$. These further imply that h_n^* are Fréchet differentiable and w^*-l. s. c. on $X_0^* = X^*/X_0^0$. Now, we define Minkowski functionals $\{q_n\}_{n\in\mathbb{N}}$ on X^* for $x^* \in X^*$ by $q_n(x^*) = \sqrt{h_n^*(Q(x^*))}$, where $Q : X^* \to X^*/X_0^0$ denotes the quotient mapping. Then it is easy to see that q_n^2 are Fréchet differentiable on X^*. By Lemma 4.6, $q_n \to \sigma_K$ uniformly on B_{X^*}. □

Theorem 4.9. *$M_{\mathrm{wcc}(X)}$ is a dense subspace of $C_{\Delta\mathrm{SFSD}}(B^*)$.*

Proof. We show first that $M_{\mathrm{wcc}(X)} \subset C_{\Delta\mathrm{SFSD}}(B^*)$. Since $M_{\mathrm{wcc}(X)} = P_{\mathrm{wcc}(X)} - P_{\mathrm{wcc}(X)}$, it suffices to prove $P_{\mathrm{wcc}(X)} \subset C_{\Delta\mathrm{SFSD}}(B^*)$. By Lemma 4.7, $P_{\mathrm{wcc}_0(X)} \subset \overline{F} \subset C_{\Delta\mathrm{SFSD}}(B^*)$. For every $K \in \mathrm{wcc}(X)$, we choose $x_0 \in K$ and let $K_0 = K - x_0$. Then

$\sigma_K = \sigma_{K_0} - (-x_0) \in \overline{F} - C_{\Delta\mathrm{SFSD}}(B^*) = C_{\Delta\mathrm{SFSD}}(B^*)$. Next, we show density of $M_{\mathrm{wcc}(X)}$ in $C_{\Delta\mathrm{SFSD}}(B^*)$. Given $\varepsilon > 0$ and $f \in C_{\Delta\mathrm{SFSD}}(B^*)$, let $f_\varepsilon = \sigma_{K_1} - \sigma_{K_2}$ such that both $\sigma^2_{K_1}$ and $\sigma^2_{K_2}$ are Fréchet differentiable on X^* satisfying $|f(x^*) - f_\varepsilon(x^*)| < \varepsilon$ uniformly for $x^* \in B_{X^*}$. Again by Theorem 4.6 of [4], $C_1 = \overline{\mathrm{co}}K_1$ and $C_2 = \overline{\mathrm{co}}K_2$ are weakly compact. We complete our proof by noticing that $\sigma_{K_1} = \sigma_{C_1}$ and $\sigma_{K_2} = \sigma_{C_2}$. □

Corollary 4.10. *Suppose that X is a Banach space. Then*

$$\mathrm{wcc}(X)^* = C_{\Delta\mathrm{SFSD}}(B^*)^*.$$

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