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On Some Properties of Pettis Integrable Multifunctions

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We study Aumann-Pettis integrable multifunctions on 2^X, where X is a separable Banach space and their integrals. We prove the existence of a weakly compact convex-valued Pettis integrable multifunction F for a closed, convex, decomposable and weakly sequentially compact subset K of $P_1(\mu, X)$, the space of all Pettis integrable functions on X such that K coincides with S_F^P, the collection of all Pettis integrable selectors of F. We also study weak compactness property of S_F^P.

Keywords: Aumann-Pettis integrable multifunctions, Pettis integrable multifunctions, Aumann-Pettis integral, Pettis integral, weak convergence

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1. Introduction

This paper may be considered as a continuation of our work in [8].

In this paper, we study several aspects of the general theory of Aumann-Pettis integrable multifunctions.

The organization of the paper is as follows. In Section 2, we give necessary notations, definitions and preliminaries. In Section 3, we discuss our main results.

In 1977, Hiai and Umegaki [11] made significant contributions to the study of closed-valued Aumann integrable (and integrably bounded) multifunctions. In fact, apart from the book of Castaing and Valadier [7], this paper may be considered as fundamental in the sense that most of the subsequent works, in some way or other, depend on this paper so far as the integration theory of closed-valued measurable multifunctions is concerned. In [10, Lemma 23, p. 7], Godet-Thobie and Satco extended Lemma 1.1 of [11] to closed-valued Aumann-Pettis integrable multifunctions. The main objective of Section 3.1 of our main results is to generalize the results of

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Section 1 of [11] to the closed-valued Aumann-Pettis integrable multifunctions with the help of Lemma 23 of [10].

In Section 3.2 of our main results, we extend some of the results of Section 4 of [11] to Aumann-Pettis integrable multifunctions.

In [11, Theorem 3.1, p. 158], Hiai and Umegaki discussed the existence of a closed-valued Aumann integrable multifunction F for a nonempty closed and decomposable subset K of $L_1(\mu, X)$, the space of all Bochner integrable functions taking values on a separable Banach space X, such that $K = S^1_F$, the collection of all Bochner integrable selectors of F. Assuming norm separability of the space $P_1(\mu, X)$, Godet-Thobie and Satco [10, Theorem 25, p. 9] showed that the nonempty norm-closed decomposable subset K of $P_1(\mu, X)$ coincides with the adherence of S^P_F in $P_1(\mu, X)$ for an Aumann-Pettis integrable multifunction F. Imposing more conditions on the Banach space X as well as on the subset K of $L_1(\mu, X)$, Papageorgiou [16, Theorem 5.2, p. 252] further extended the result of Hiai and Umegaki [11, Theorem 3.1, p. 158] in which he proved the existence of a weakly compact convex-valued integrably bounded multifunction F such that $K = S^1_F$. Here in Section 3.3 we generalize Theorem 5.2 of [16] to $P_1(\mu, X)$, which is an improvement of Theorem 25 of [10, p. 9].

The study of weakly compact subsets of $P_1(\mu, X)$ was initiated by Brooks and Dinculeanu in [4, 5]. In [1], Amrani and Castaing studied weak compactness property in $P_1(\mu, X)$ by using the method of Grothendieck's interchangeable double limits property and James's theorem [13]. In [8], we studied weak compactness in $P_1(\mu, X)$ by using regular methods of summability. Barcenas and Urbina proved a necessary and sufficient condition for the weak compactness of S^1_F for a closed convex-valued integrably bounded multifunction F [3, Theorem 3.2, p. 1215]. In [15, Theorem 3.1, p. 187], Papageorgiou proved the sufficiency part of [3, Theorem 3.2, p. 1215] using a separate method. In this section we generalize Theorem 3.2 of [3] to S^P_F for a closed, convex-valued Pettis integrable multifunction F. Castaing [6, Theorem 3.2, p. 413] also proved the sufficiency condition for the weak compactness of S^P_F for a closed convex-valued Pettis integrable multifunction F on X. But our proof is different and it is based on Theorem 3.6 of [8, p. 102].

2. Notations, Definitions and Preliminaries

Throughout this paper, unless otherwise stated, (Ω, Σ, μ) is a complete finite positive measure space and X is a separable Banach space with dual X^*. The closed unit ball of X (respectively X^*) is denoted by B_X (resp. B_{X^*}). If C is a subset of X, then $\operatorname{co} C$ and $\overline{\operatorname{co}} C$ denote the convex hull of C and the closed convex hull of C respectively. $CL(X)$, $C(X)$, $CB(X)$, $WK(X)$ and $CWK(X)$ denote the non-empty closed, closed convex, closed convex bounded, weakly compact and weakly compact convex subsets of X respectively. The symbol $L_1(\mu, X)$, denotes the Banach space of all equivalence classes of Bochner integrable functions $f : \Omega \to X$ with respect to the measure μ equipped with the norm

$$\|f\|_1 = \int_\Omega \|f\| d\mu.$$

A measurable function $f : \Omega \to X$ is said to be scalarly (or weakly) integrable if for each $x^* \in X^*, \langle x^*, f\rangle$ is a member of $L_1(\mu)$, the set of all of μ-integrable real valued functions. A scalarly integrable function is also called Dunford integrable. It is well known that given a scalarly integrable function f and a member $A \in \Sigma$, there exists $x_A^{**} \in X^{**}$, the bidual of X, such that $\langle x^*, x_A^{**}\rangle = \int_A \langle x^*, f\rangle d\mu$, for all $x^* \in X^*$. x_A^{**} is called the Dunford integral of f for all $A \in \Sigma$ and is denoted by $D - \int_A f d\mu$. The scalarly integrable function f is said to be Pettis integrable if for every $A \in \Sigma$, there exists $x_A \in X$ such that $\langle x^*, x_A\rangle = \int_A \langle x^*, f\rangle d\mu$, for all $x^* \in X^*.x_A$ is called the Pettis integral of f over A and is denoted by $P - \int_A f d\mu$ (or simply by $\int_A f d\mu$, if no confusion arises).

We denote by $P_1(\mu, X)$, the space of all scalarly equivalence classes of X-valued Pettis integrable functions $f : \Omega \to X$, equipped with the semivariation norm

$$\|f\|_P = \sup\left\{\int_\Omega |\langle x^*, f\rangle| d\mu;\ x^* \in B_{X^*}\right\}.$$

It is well known that $P_1(\mu, X)$ is a normed linear space which, in general, is not a Banach space.

We can define another topology on $P_1(\mu, X)$ induced by the duality $(P_1(\mu, X), L_\infty(\mu)$ $\otimes X^*)$, since the operation

$$\langle v \otimes x^*, f\rangle = \int_\Omega v(\omega)\langle x^*, f(\omega)\rangle d\mu,$$

where $v \in L_\infty(\mu)$ and $x^* \in X^*$, is a bilinear form. This topology is known as weak topology of $P_1(\mu, X)$ [10, p. 3].

A subset K of $L_1(\mu, X)$ is said to be

(i) uniformly integrable if

$$\lim_{c\to\infty} \sup_{f\in K} \int_{[\|f\|\geq c]} \|f\| d\mu = 0;$$

(ii) equi-integrable if

$$\lim_{\mu(A)\to 0} \sup_{f\in K} \int_A \|f\| d\mu = 0.$$

It is well known that K is uniformly integrable iff it is equi-integrable and bounded. The two concepts of uniform integrability and equi-integrability coincide if the measure space (Ω, Σ, μ) is non-atomic [10, p. 2–3].

A subset K of $P_1(\mu, X)$ is said to be Pettis uniformly integrable (PUI) if, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $\mu(A) < \delta_\varepsilon$ implies that $\| \int_A h d\mu\| < \varepsilon$, for all $h \in K$. It is not difficult to see that a subset K of $P_1(\mu, X)$ is Pettis uniformly integrable iff $\{\langle x^*, h\rangle; x^* \in B_{X^*}, h \in K\}$ is equi-integrable [10, p. 4].

A family K of scalarly integrable functions is said to be weakly Pettis uniformly integrable ($WPUI$) if, for each $x^* \in B_{X^*}$, the family $\{\langle x^*, h\rangle; h \in K\}$ is equi-integrable [10, p. 4].

It is obvious that $PUI \Rightarrow WPUI$, but the reverse implication may not be true. Sufficient conditions for the reverse implication have been studied in [10].

Let K be a set of measurable functions $f : \Omega \to X$. We call K decomposable with respect to Σ, if f_1 and $f_2 \in K$ and $A \in \Sigma$ imply $\chi_A f_1 + \chi_{A^c} f_2 \in K$. It is clear that if K is decomposable, then $\sum_{i=1}^n \chi_{A_i} f_i \in K$, for each finite partition $\{A_1, A_2, \ldots, A_n\}$ of Ω and $\{f_1, f_2, \ldots, f_n\} \in K$.

For every $C \in CL(X)$, the support function of C is denoted by $\sigma(., C)$ and defined on X^* by

$$\sigma(x^*, C) = \sup\{\langle x^*, x\rangle; x \in C\}, \quad \text{for all } x^* \in X^*.$$

A multifunction $F : \Omega \to CL(X)$ is said to be weakly measurable (or simply measurable) if for every open subset V of X, the set $\{\omega \in \Omega; F(\omega) \cap V \neq \emptyset\}$ belongs to Σ.

The reader is referred to Theorem 1.0 of [11] for different notions of measurability of a multifunction and their equivalences.

A function $f : \Omega \to X$ is said to be a selector of $F : \Omega \to CL(X)$ if $f(\omega) \in F(\omega)$, μ-a.e. The collection of all measurable selectors of F is denoted by S_F. S_F^1 (respectively S_F^P) denotes the family of all Bochner (resp. Pettis) integrable selectors of the measurable multifunction F.

A measurable multifunction $F : \Omega \to CL(X)$ is said to be scalarly integrable if the scalar function $\sigma(x^*, F(\cdot))$ is integrable with respect to μ, for each $x^* \in X^*$.

A measurable multifunction $F : \Omega \to CL(X)$ is said to be Aumann-Pettis integrable (respectively Aumann integrable or simply integrable) if S_F^P (resp. S_F^1) is non-empty. In this case we denote the Aumann-Pettis integral of F over $A \in \Sigma$ by $I_A(F)$ and is defined by $I_A(F) = \left\{\int_A f d\mu; f \in S_F^P\right\}.I_\Omega(F)$ is simply denoted by $I(F)$ [2, p. 341].

A measurable multifunction $F : \Omega \to C(X)$ is said to be Pettis integrable if F is scalarly integrable and for each $A \in \Sigma$, there exists $C_A(F) \in C(X)$ such that

$$\sigma(x^*, C_A(F)) = \int_A \sigma(x^*, F) d\mu, \quad \text{for each } x^* \in X^*.$$

$C_A(F)$ is called the Pettis integral of F over $A \in \Sigma$ and is denoted by $\int_A F d\mu$.

If $F : \Omega \to CWK(X)$ is a scalarly integrable multifunction then it follows from [2, Theorem 5.4, p. 352] or [17, Theorem 3.2, p. 126] or [18, Theorem 1, p. 228] that F is Aumann-Pettis integrable iff it is Pettis integrable and in this case $I_A(F) = C_A(F) \in CWK(X)$, for each $A \in \Sigma$.

The set of all Pettis integrable multifunctions is denoted by $\mathcal{P}_1(\mu, X)$.

A multifunction $F : \Omega \to CL(X)$ is said to be integrably bounded if there is a μ-integrable the real valued function h with $\|F(\omega)\| \leq h(\omega)$, μ a.e.

3. Main Results

3.1. Aumann-Pettis integrable multifunctions

We define the following two operations in the family of Aumann-Pettis integrable multifunctions.

1. Addition: $(F_1 \dot{+} F_2)(\omega) = \{F_1(\omega) + F_2(\omega)\}$, for all $w \in \Omega$, where $F_1 : \Omega \to CWK(X)$ and $F_2 : \Omega \to CWK(X)$ are two Aumann-Pettis integrable multifunctions.

2. Closed Convex hull: $(\overline{\text{co}}F)(\omega) = \overline{\text{co}}F(\omega)$, the closed convex hull in X, for all $w \in \Omega$, where $F : \Omega \to CL(X)$ is an Aumann-Pettis integrable multifunction such that $F(\omega)$ is $WK(X)$-valued for all $\omega \in \Omega$.

Lemma 3.1. *If $F : \Omega \to CL(X)$ is Aumann-Pettis integrable, then there exists a sequence $\{g_n\}$ contained in S_F^P such that*

$$F(\omega) = c1\{g_n(\omega); n \geq 1\}, \quad \mu \text{ a.e.}$$

Proof. The proof is given in [10, Lemma 23, p. 7]. □

Corollary 3.2. *Let $F_1 : \Omega \to CL(X)$ and $F_2 : \Omega \to CL(X)$ be two measurable multifunctions such that $S_{F_1}^P$ and $S_{F_2}^P$ are both nonempty. If $S_{F_1}^P = S_{F_2}^P$, then $F_1(\omega) = F_2(\omega)$, μ a.e.*

Proof. The proof is easy and so omitted. □

Lemma 3.3. *Let $F : \Omega \to CL(X)$ be an Aumann-Pettis integrable multifunction. Let $\{f_i\}$ be a sequence in S_F^P such that $F(\omega) = \text{cl}\{f_i(\omega)\}$, μ a.e. Then for each $f \in S_F^P$ and $\varepsilon > 0$, there exists a finite measurable partition $\{A_1, A_2, \ldots, A_n\}$ of Ω such that*

$$\left\| f - \sum_{i=1}^{n} \chi_{A_i} f_i \right\|_P < \varepsilon.$$

Proof. The proof is contained in the first part of Theorem 24 of [10, p. 8]. □

Note. The following theorem is generalisation of [11, Theorem 1.4, p. 153] and [12, Proposition 3.28, p. 184] for Aumann-Integrable multifunctions to the case of Aumann-Pettis Integrable multifunctions. The proof follows by necessary modification. So we omit the proof.

Theorem 3.4. *Let $F_1 : \Omega \to CWK(X)$ and $F_2 : \Omega \to CWK(X)$ be two Aumann-Pettis integrable multifunctions and let $F : \Omega \to 2^X \setminus \{\emptyset\}$ be defined as $F(\omega) = \{F_1(\omega) + F_2(\omega)\}$ for all $\omega \in \Omega$. Then F is a $CWK(X)$-valued Aumann-Pettis integrable multifunction such that*

$$S_F^P = \text{cl}\{S_{F_1}^P + S_{F_2}^P\} \quad \text{in } P_1(\mu, X).$$

Theorem 3.5. *Let $F : \Omega \to WK(X)$ be a measurable multifunction such that S_F^P is non-empty. Let $(\overline{\text{co}}F)(\omega) = \overline{\text{co}}F(\omega)$, the closed convex hull in X, for all $\omega \in \Omega$. Then $\overline{\text{co}}F$ is a $CWK(X)$-valued Aumann-Pettis integrable multifunction such that*

$$S_{\overline{\text{co}}F}^P = \overline{\text{co}}S_F^P \quad \text{in } P_1(\mu, X).$$

Proof. Put $G = \overline{\text{co}}F$.

Since S_F^P is nonempty, so by Lemma 3.1, there exists a sequence $\{f_i\}$ in S_F^P such that

$$F(\omega) = \text{cl}\{f_i(\omega)\}, \quad \mu \text{ a.e.}$$

Define $U = \{g; g = \sum_{i=1}^m \alpha_i f_i, \alpha_i \geq 0 \text{ rational}, \sum_{i=1}^m \alpha_i = 1, m \geq 1\}$

Then U is a countable subset of measurable functions such that

$$G(\omega) = \text{cl}\{g(\omega), g \in U\}, \quad \mu \text{ a.e.}$$

Hence by [11, Theorem 1.0(V), p. 151], G is measurable multifunction from Ω to $C(X)$.

Again since F is Aumann-Pettis integrable, so is G.

As $F(\omega)$ is $WK(X)$-valued, by Krein-Smulian theorem [9, Theorem 11, p. 51], $G(\omega)$ is $CWK(X)$-valued.

So G is a $CWK(X)$-valued Aumann-Pettis integrable multifunction. Therefore by [10, Corollary 17, p. 6], S_G^P is $\|\cdot\|_P$-norm closed and it is also convex as $G(\omega)$ is convex, μ a.e.

Now since

$$S_F^P \subseteq S_G^P, \quad \text{we have } \overline{\text{co}}S_F^P \subseteq \overline{\text{co}}S_G^P = S_G^P \tag{1}$$

To prove the converse, let $f \in S_G^P$ and $\varepsilon > 0$, then by Lemma 3.3, we can choose a finite measurable partition $\{A_1, A_2, \ldots, A_n\}$ of Ω and $g_1, g_2, \ldots, g_n \in S_G^P$ such that

$$\left\| f - \sum_{i=1}^n \chi_{A_i} g_i \right\|_P < \varepsilon.$$

Now proceeding as in the proof of Theorem 1.5 of [11, p. 154], we can show that $\sum_{i=1}^n \chi_{A_i} g_i$ is a convex combination of elements of S_F^P.

Hence

$$f \in \overline{\text{co}}S_F^P, \quad \text{and so } S_G^P \subseteq \overline{\text{co}}S_F^P \tag{2}$$

Combining (1) and (2), we have

$$S_{\overline{\text{co}}F}^P = \overline{\text{co}}S_F^P. \qquad \square$$

Corollary 3.6. *Let $F : \Omega \to WK(X)$ be an Aumann-Pettis integrable multifunction, then S_F^P is convex if and only if $F(\omega)$ is convex, μ a.e.*

Proof. It immediately follows from Theorem 3.5 and Corollary 3.2. $\square$

3.2. Integrals of Aumann-Pettis integrable multifunctions

Theorem 3.7. *Let $F_1 : \Omega \to CWK(X)$ and $F_2 : \Omega \to CWK(X)$ be two Aumann-Pettis integrable multifunctions. Then*

$$\int_\Omega (F_1 \dot{+} F_2)(\omega) d\mu = \left\{ \int_\Omega F_1(\omega) d\mu + \int_\Omega F_2(\omega) d\mu \right\}$$

Proof. Since F_1 and F_2 are $CWK(X)$-valued Aumann-Pettis integrable, by [2, Theorem 3.7, p. 341], they are Pettis integrable. so for any A in Σ, there exists two sets $C_A(F_1)$ and $C_A(F_2)$ in $CWK(X)$ such that for any x^* in X^*

$$\sigma(x^*, C_A(F_1)) = \int_A \sigma(x^*, F_1(\omega) d\mu \tag{3}$$

$$\sigma(x^*, C_A(F_2)) = \int_A \sigma(x^*, F_2(\omega) d\mu \tag{4}$$

Now by Theorem 3.4, $F_1 \dot{+} F_2$ is a $CWK(X)$-valued Aumann-Pettis integrable multifunction and hence Pettis integrable (see [2, Theorem 3.7, p. 341]). So for any A in Σ, there exists $C_A(F_1 \dot{+} F_2)$ in $CWK(X)$ such that for any x^* in X^*

$$\begin{aligned} \sigma(x^*, C_A(F_1) + C_A(F_2)) &= \sigma(x^*, C_A(F_1)) + \sigma(x^*, C_A(F_2)) \\ &= \int_A \sigma(x^*, F_1(\omega) d\mu + \int_A \sigma(x^*, F_1(\omega) d\mu \\ &= \int_A \{\sigma(x^*, F_1(\omega) + \sigma(x^*, F_2(\omega) d\mu\} \\ &= \int_A \sigma(x^*, F_1 \dot{+} F_2) d\mu \\ &= \sigma(x^*, C_A(F_1 \dot{+} F_2)) \end{aligned}$$

So, for any A in Σ we have, $C_A(F_1) + C_A(F_2) = C_A(F_1 \dot{+} F_2)$

Or,

$$\int_A F_1(\omega) d\mu + \int_A F_2(\omega) d\mu = \int_A (F_1 \dot{+} F_2) d\mu,$$

as F_1, F_2, $(F_1 \dot{+} F_2)$ are $CWK(X)$ Pettis integrable multifunctions, by [2, Theorem 5.4, p. 352], their Pettis integrals and their Aumann-Pettis integral are same. Hence we have,

$$\int_\Omega F_1(\omega) d\mu + \int_\Omega F_2(\omega) d\mu = \int_\Omega (F_1 \dot{+} F_2) d\mu. \qquad \square$$

Note. The following theorem follows from Corollary 3.10 of [2, p. 345]. However, we give an independent proof of this result.

Theorem 3.8. *If $F : \Omega \to WK(X)$ is an Aumann-Pettis integrable multifunction, then*

$$\int_\Omega (\overline{\text{co}} F)(\omega) d\mu = \overline{\text{co}} I(F).$$

Proof. By Theorem 3.5, $\overline{\text{co}} F$ is a $CWK(X)$-valued Aumann-Pettis integrable multifunction and $S^P_{\overline{\text{co}} F} = \overline{\text{co}} S^P_F$. Hence by [2, Theorem 3.7, p. 341], $\overline{\text{co}} F$ is Pettis integrable and

$$I(F) \subseteq \int_\Omega (\overline{\text{co}} F)(\omega) d\mu.$$

Also $\int_\Omega (\overline{\text{co}} F)(\omega) d\mu$ is $CWK(X)$-valued by [2, Theorem 5.4, p. 352].

So

$$\overline{\text{co}}I(F) \subseteq \int_\Omega (\overline{\text{co}}F)(\omega)d\mu \tag{5}$$

To prove the reverse inclusion, let $x \in \int_\Omega (\overline{\text{co}}F)(\omega)d\mu$. Then there exists an $f \in S^P_{\overline{\text{co}}F}$ such that

$$x = \int_\Omega f(\omega)d\mu.$$

Since $f \in S^P_{\overline{\text{co}}F}$ and since by Theorem 3.5,

$$S^P_{\overline{\text{co}}F} = \overline{\text{co}}S^P_F,$$

for an arbitrary positive number ε, there exist $f_1, f_2, \ldots, f_n \in S^P_F$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n \lambda_i = 1$, such that

$$\left\| f - \sum_{i=1}^n \lambda_i f_i \right\|_P < \varepsilon$$

Since $\| \ \|_P$-norm is equivalent to norm $\sup_{A\in\Sigma} \left\| \int_A f d\mu \right\|$ [10, p. 3], it follows that

$$\sup_{A\in\Sigma} \left\| \int_A (f(\omega) - \sum_{i=1}^n \lambda_i f_i(\omega))d\mu \right\| \leq \left\| f - \sum_{i=1}^n \lambda_i f_i \right\|_P < \varepsilon$$

or,

$$\left\| \int_A (f(\omega) - \sum_{i=1}^n \lambda_i \int_A f_i(\omega)d\mu \right\| < \varepsilon, \quad \text{for all } A \in \Sigma$$

and hence,

$$\left\| \int_\Omega f(\omega) - \sum_{i=1}^n \lambda_i \int_\Omega f_i(\omega)d\mu \right\| < \varepsilon$$

or,

$$\left\| x - \sum_{i=1}^n \lambda_i \int_\Omega f_i(\omega)d\mu \right\| < \varepsilon \tag{6}$$

Also,

$$\sum_{i=1}^n \lambda_i \int_\Omega f_i d\mu \in \text{co}\, I(F). \tag{7}$$

Hence from (6) and (7), we have $x = \int_\Omega f(\omega)d\mu \in \overline{\text{co}}I(F)$.

That is,

$$\int_\Omega (\overline{\text{co}}F)(\omega)d\mu. \subseteq \overline{\text{co}}I(F). \tag{8}$$

Combining (5) and (8) it follows that,

$$\int_\Omega (\overline{\text{co}}F)(\omega)d\mu = \overline{\text{co}}I(F). \qquad \square$$

3.3. Existence of Pettis Integrable Multifunction

Theorem 3.9. *Assume that $P_1(\mu, X)$ is norm separable. Let K be any non-empty closed, decomposable, convex subset of $P_1(\mu, X)$. Also assume that $K(\omega)$, where $K(\omega) = \{f(\omega); f \in K\}$, is weakly compact for all $\omega \in \Omega$. Then there exists a Pettis integrable multifunction $G : \Omega \to CWK(X)$ such that*

$$K = S_G^P$$

Proof. Since $P_1(\mu, X)$ is norm separable and K is a non-empty closed, decomposable subset of $P_1(\mu, X)$, it follows by [10, Theorem 25, p. 9] that there exists a measurable multifunction $F : \Omega \to CL(X)$ such that,

$$K = \overline{S_F^P}^{\|\cdot\|_P}$$

Since $\overline{S_F^P}^{\|\cdot\|_P} = K$ is nonempty, so is S_F^P. Hence F is Aumann-Pettis integrable.

Therefore, by Lemma 3.1, there exists a sequence $\{g_n\} \in S_F^P \subset K$ such that

$$F(\omega) = \mathrm{cl}\{g_n(\omega);\ n \geq 1\}, \quad \mu \text{ a.e.}$$

Let us now define a multifunction $G : \Omega \to 2^X \setminus \{\emptyset\}$ as

$$G(\omega) = \overline{\mathrm{co}} F(\omega).$$

Then,

$$G(\omega) = \overline{\mathrm{co}}\, \mathrm{cl}\{g_n(\omega); n \geq 1\} = \overline{\mathrm{co}}\{g_n(\omega); n \geq 1\}, \quad \mu \text{ a.e.}$$

So,

$$G(\omega) \subset \overline{\mathrm{co}} K(\omega), \quad \mu \text{ a.e.}$$

Then $G(\omega)$ is $CWK(X)$-valued, μ a.e., as $K(\omega)$ is weakly compact for all $\omega \in \Omega$. Without any loss of generality we may assume that $G(\omega)$ is $CWK(X)$ for all $\omega \in \Omega$. To show that G is measurable, let

$$U = \left\{ h; h = \sum_{j=1}^{m} \lambda_i g_i, \lambda_i \geq 0, \text{ are rational and } \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Then U is countable and $G(\omega) = \mathrm{cl}\{h(\omega), h \in U\}$, μ a.e. So by [11, Theorem 1(V), p. 151], G is measurable. As F is Aumann-Pettis integrable, so is G. We now show that $S_G^P = \overline{\mathrm{co}} S_F^P$, the closed convex hull in $P_1(\mu, X)$. Since G is $CWK(X)$-valued Aumann-Pettis integrable, by [10, Corollary 17, p. 6], S_G^P is $\|\cdot\|_P$-norm closed and convex as $G(\omega)$ is convex, μ a.e.

Now since $S_F^P \subseteq S_G^P$, it follows that

$$\overline{\mathrm{co}} S_F^P \subseteq \overline{\mathrm{co}} S_G^P = S_G^P \tag{9}$$

To prove the converse let $f \in S_G^P$.

Now $G(\omega) = \mathrm{cl}\{h(\omega); h \in U\}$, μ a.e.

So by Lemma 3.3, for an arbitrary $\varepsilon > 0$, there exists a finite measurable partition $\{A_1, A_2, \dots, A_n\}$ and $h_1, h_2, \dots, h_n \in U$ such that

$$\left\| f - \sum_{i=1}^{n} \chi_{A_i} h_i \right\|_P < \varepsilon$$

As $U = \{h; h = \sum_{i=1}^{m} \lambda_i g_i, \lambda_i \geq 0;$ are rational and $\sum_{i=1}^{m} \lambda_i = 1\}$ with $g_i \in S_F^P$, proceeding as in the proof of 1.5 [11, p. 154], we can show that $\sum_{i=1}^{n} \chi_{A_i} h_i$ is a convex combination of elements of S_F^P.

So, $f \in \overline{\text{co}} S_F^P$ and hence

$$S_G^P \subseteq \overline{\text{co}} S_F^P \tag{10}$$

Combining (9) and (10), we have

$$S_G^P = \overline{\text{co}} S_F^P$$

We now show that

$$S_G^P = \overline{S_F^P}^{\|\cdot\|_P} = K.$$

$$S_G^P = \overline{\text{co}} S_F^P \subseteq \overline{\text{co}} \overline{S_F^P}^{\|\cdot\|_P} = \overline{S_F^P}^{\|\cdot\|_P} = K, \tag{11}$$

since by hypothesis $\overline{S_F^P}^{\|\cdot\|_P} = K$ is $\|\cdot\|_P$-norm closed and convex.

To prove the converse, let $f \in \overline{S_F^P}^{\|\cdot\|_P}$.

If $f \in S_F^P$, then $f \in S_G^P$.

If $f \in \overline{S_F^P}^{\|\cdot\|_P} \setminus S_F^P$, then there exists a sequence $\{f_n\}$ in S_F^P which converges to f in $\|\cdot\|_P$-norm.

So, $f \in \overline{\text{co}} S_F^P = S_G^P$.

Therefore, in any case,

$$f \in S_G^P.$$

So,

$$K = \overline{S_F^P}^{\|\cdot\|_P} \subseteq S_G^P. \tag{12}$$

From (11) and (12), we have

$$S_G^P = \overline{S_F^P}^{\|\cdot\|_P} = K.$$

Also G is $CWK(X)$-valued Aumann-Pettis integrable, so it is Pettis integrable by [2, Theorem 3.7, p. 341]. □

3.4. Weak Compactness in $P_1(\mu, X)$

Here in this section we use the following notation. Let $T = C[0,1]$ be the Banach space of real valued continous functions defined on the unit interval $[0,1]$ equipped with the suppremum norm. Suppose T is endowed with fixed recursive presentation. Here each of the followings are subset of $T^{\mathbb{N}}$ defined as

$$B_1 = \{x \in T^{\mathbb{N}} : x \text{ is bounded and weakly precompact}\};$$
$$B_2 = \{x \in T^{\mathbb{N}} : x \text{ is weakly Cauchy sequence}\};$$
$$B_3 = \{x \in T^{\mathbb{N}} : x \text{ converges weakly in T}\};$$
$$B_4 = B_2 \setminus B_3;$$
$$B_5 = \{x \in T^{\mathbb{N}} : x \text{ is relatively weakly compact}\}.$$

Theorem 3.10. *If $F : \Omega \to CB(X)$ is Aumann-Pettis integrable, then S_F^P is convex and sequentially compact for the weak topology of $P_1(\mu, X)$ if and only if $F(\omega)$ is weakly compact, μ a.e.*

Proof. In order to show the sufficiency, let us consider that F is a μ a.e. weakly compact convex-valued Aumann-Pettis integrable multifunction.
Then by [2, Theorem 3.7, p. 341], it is Pettis integrable. That is, F is a μ a.e. $CWK(X)$-valued Pettis integrable multifunction. Without any loss of generality, we may assume that F is $CWK(X)$-valued for all $\omega \in \Omega$. Hence S_F^P is nonempty.
Also by Corollary 3.1.6, S_F^P is convex as $F(\omega)$ is convex, μ a.e.
Now by [2, Theorem 5.4, p. 352], the set $\{\langle x^*, f\rangle; x^* \in B_{X^*}, f \in S_F^P\}$ is uniformly integrable.
So by [8, Theorem 3.6, p. 102], S_F^P is sequentially compact for the weak topology of $P_1(\mu, X)$.
Hence the condition is sufficient.
In order to prove that the condition is necessary, let us assume that S_F^P is convex and sequentially compact for the weak topology of $P_1(\mu, X)$.
We know that $C[0, 1]$ contains isomorphically any separable Banach space.
Weak compactness, being a topological invariant property, we may assume without loss of generality that $X = C[0, 1]$.
Let $\{x_n\}$ be a sequence in $P_1(\mu, X)$ such that $\{x_n(\omega), n \geq 1\}$ dense in $F(\omega)$, μ a.e.
In order to prove the necessary part, it is sufficient to show that $\{x_n(\omega), n \geq 1\}$ is relatively compact, μ a.e. Let

$$K(\omega) = (x_n(\omega))_n \quad \text{and} \quad A = K^{-1}(X^{\mathbb{N}} \setminus B_5) \tag{13}$$

Assume that the condition is not necessary. Then $F(\omega)$ is not weakly compact, μ a.e. and so $\{x_n(\omega), n \geq 1\}$ is not relatively compact, μ a.e.

$$\text{Hence, } A \text{ is a } \mu - \text{measurable set of positive measure.} \tag{14}$$

Also by [14, Lemma 3.1, p. 310], $K(\omega) \in B_1$, μ a.e.
Hence for almost every $\omega \in A$, $K(\omega) \in B_1 \cap (X^{\mathbb{N}} \setminus B_5)$.
Theorem 3.5 in [14] guarantees the existence of a universally measurable function $s = (s_n)$ from $B_1 \cap (X^{\mathbb{N}} \setminus B_5)$ to B_4 such that $s(x)$ is a subsequence of x.
In other words, or almost every $\omega \in A$, $s_n(K(\omega))$ is a weakly Cauchy sequence which does not converge weakly in X.
Also each function $s_n \odot K$ is easily seen to belong to S_F^P.

Now by the hypothesis, S_F^P is sequentially compact for the weak topology of $P_1(\mu, X)$. So the sequence $s_n(K)$ has a subsequence $s_{n_k}(K)$ and there exists a function $f \in P_1(\mu, X)$ such that $s_{n_k}(K)$ converges weakly to f in $P_1(\mu, X)$.
Also for almost every $\omega \in A$, $s_n(K(\omega))$ is a weakly Cauchy sequence in X. So $s_{n_k}(K(\omega))$ is weakly cauchy in X.
Now by Proposition 3.10 of [8], $s_{n_k}(K(\omega))$ converges weakly to $f(\omega)$, for almost all $\omega \in A$
Hence $\mu(A) = \mu(s \odot K)^{-1}(B_4) = 0$.
This contradicts (14).
Hence $\{x_n(\omega), n \geq 1\}$ is relatively compact, μ a.e and so $F(\omega)$ is weakly compact, μ a.e.
Hence the condition is necessary. □

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