Journal of Convex Analysis
Volume 19 (2012), No. 3, 795–813

Existence Results for a Non Coercive Homogeneous Nonlinear Elliptic Equation

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Dedicated to Arrigo Cellina on the occasion of his 70th birthday.

Received: February 28, 2011
Revised manuscript received: October 25, 2011

Let $a : \mathbb{R}^n \to \mathbb{R}^n$ be monotonic, but unnecessarily strictly monotonic. We study the existence of Lipschitz or locally Lipschitz solutions to the equation $\operatorname{div} a(\nabla u) = 0$ when the Lipschitz boundary datum fulfills some recent unilateral Bounded Slope Conditions.

Keywords: Lipschitz, nonlinear, degenerate, elliptic, pde, convex

1991 Mathematics Subject Classification: 35J70, 35J15, 49K20

1. Introduction

The aim of this paper is to establish two existence results concerning the solutions to $\operatorname{div} a(\nabla u) = 0$, where $a : \mathbb{R}^n \to \mathbb{R}^n$ is a monotonic and continuous vector field, i.e.

$$\forall p, q \in \mathbb{R}^n \quad (a(q) - a(p)) \cdot (q - p) \geq 0. \tag{1}$$

By a solution to $\operatorname{div} a(\nabla u) = 0$ we mean a function u in a suitable Sobolev space satisfying

$$\forall \eta \in \mathcal{C}_c^\infty(\Omega) \quad \int_\Omega a(\nabla u(x)) \cdot \nabla \eta(x)\, dx = 0, \tag{2}$$

where Ω is a prescribed open and bounded subset of $\mathbb{R}^n$. The interest in looking for (possibly) degenerate equations is motivated by the Euler equation for functionals that arise from non convex problems of the calculus of variations: indeed the convexified problem in this case has typically a non strictly convex lagrangean.

The reference model here and main source of inspiration is a result of Hartman and Stampacchia [7] stating that if the boundary datum ϕ satisfies the *Bounded Slope Condition* (BSC) then (2) has a solution u that is Lipschitz and equal to ϕ on $\partial\Omega$.

*Partially supported by the “Progetto di ateneo dell’Università degli Studi di Padova” n. CPDA081458/08 and PRIN “Equazioni e sistemi di tipo ellittico: stime a priori, esistenza, regolarità” (2008).

ISSN 0944-6532 / \$ 2.50

We recall that ϕ is said to satisfy the (BSC) if, at every point of the graph of ϕ there are two hyperplanes containing the point and that bound the graph of ϕ from below and from above; it is required moreover that the slopes of these hyperplanes are uniformly bounded. The (BSC) is a quite restrictive condition that forces for instance the boundary datum to be affine on the flat parts of $\partial\Omega$. This existence result is obtained even for a more general class of non homogeneous nonlinear elliptic equations of the form

$$-\operatorname{div} a(\nabla u) = F[u] \tag{3}$$

by assuming, instead of (1) and among other requirements on F, a uniform ellipticity condition on the field a like

$$\forall p,\, q \in \mathbb{R}^n \quad (a(q) - a(p)) \cdot (q - p) \geq \nu |q - p|^2 \tag{4}$$

for some $\nu > 0$. It must be said that, in the homogeneous case, the result is obtained by a perturbation of the field a so it becomes uniformly elliptic, and the conclusion is achieved by a limiting process. Under some similar assumptions, Bousquet proved in [1] that the equation (3) has a solution in $\phi + W_0^{1,2}(\Omega)$ that is *locally Lipschitz* whenever ϕ satisfies just the Lower (BSC), a unilateral version of the original (BSC) introduced in [4] by Clarke, see Definition 6.1. This new condition is less restrictive than the (BSC); it is worth mentioning that ϕ satisfies the Lower (BSC) if and only if it is the restriction of a convex function. The Lipschitz constant of the solution on any compact subset of the domain and many intermediate estimates in the proof of [1, Theorem 2.1] depend strongly on the constant ν that appears in (4), giving a priori little chance to raise a conclusion when the field a is not uniformly elliptic by means of a limiting process.

We encompass the difficulty of the lack of strict monotonicity by means of the notion of quasi–solutions to $\operatorname{div} a(\nabla u) = 0$; we refer to [7] for the main results on the subject. Of course the comparison principle among quasi–solutions, a main tool in the existence theory of [7], fails in general without assuming the strict monotonicity of the operator a. However, similarly to what has been recently established in [8] for minimizers of integrals functions that are not strictly convex, we establish a comparison principle among some particular quasi–solutions, together with the proof of their existence. Under just (1), Theorem 6.3 shows that if ϕ satisfies the Lower or Upper (BSC) then there is a function u that is locally Lipschitz and solves (2); moreover u is locally a uniform limit of quasi–solutions u^K to $\operatorname{div} a(\nabla u) = 0$ among Lipschitz functions of constant less than K and that are equal to ϕ on $\partial\Omega$; the notion of quasi–solution recalled in Definition 3.1 is classical in the field of the variational inequalities, we refer to [7] for the main results on the subject. This solution turns out to belong to $W^{1,p}(\Omega)$ as soon as the field a satisfies growth assumptions from below and from above. The same conclusion is obtained by weakening the unilateral (BSC) with a unilateral Generalized (BSC) obtained by replacing the affine functions in the (BSC) with a new class of functions introduced by Cellina in [3]: in the case of a non strictly monotonic operator this Generalized unilateral (BSC) is satisfied by a wider class of boundary data than the unilateral (BSC): for instance if the level set $\{q : a(q) = a(0)\}$ of a is not reduced to 0 then *every* Lipschitz function of rank suitably small satisfies both the Generalized Lower and Upper (BSC). Of course anything comes free of charge: as a counterpart we require here a further mild ellipticity assumption, namely that

$$\forall p,\, q \in \mathbb{R}^n \quad (a(q) - a(p)) \cdot (q - p) = 0 \Longleftrightarrow a(q) = a(p) : \tag{5}$$

This assumption is fulfilled for instance in the case the equation $\operatorname{div} a(\nabla u) = 0$ is variational, i.e. when a is strictly monotonic or if a is the gradient of a convex and smooth function; more details are presented in [10] where condition (5) was introduced. The proof of Theorem 6.3 follows narrowly that of [1], by trying to encompass the difficulty that the field a is not uniformly elliptic: this is realized in two different ways depending whether the boundary datum satisfies either Clarke's Lower (BSC) or the Generalized Lower (BSC). In the first version of this paper condition (5) appeared among the assumptions of Theorem 6.3 even in the case of a boundary datum satisfying the Lower (BSC): I thank Pierre Bousquet for suggesting me that in this case condition (5) could be thrown away.

As a byproduct of our methods we prove, again under (5), the existence of a Lipschitz solution to $\operatorname{div} a(\nabla u) = 0$ if the boundary datum ϕ satisfies the Generalized (BSC), thus extending the result of Hartman–Stampacchia quoted above.

The methods involved in [7] were certainly inspired by the paper [12] of Stampacchia concerning the minimizers of an integral functional. Similarly, the techniques here arise from [2], [4], [8] for problems of the calculus of variations, and rely mostly on some comparison principles for quasi–solutions to $\operatorname{div} a(\nabla u) = 0$ on spaces of Lipschitz functions. A similar approach in dealing with solutions to $\operatorname{div} a(\nabla u) = 0$ in Sobolev spaces instead, as here, of quasi–solutions among Lipschitz functions, was first used in [10] in a joint work with Treu where we study the problem of the Hölder regularity of solutions to (2). We establish there some comparison principles among solutions to (2) in Sobolev spaces; the existence of solutions is ensured in [10] thanks to the fact that the field a satisfies some growth conditions that are not assumed here.

2. Basic assumptions and preliminary results

Here Ω is an open and bounded subset of $\mathbb{R}^n$; its boundary is denoted by Γ and its closure by $\overline{\Omega}$. The scalar product in $\mathbb{R}^n$ is denoted by "$\cdot$". If u and v are functions then $u \wedge v$ (resp. $u \vee v$) stands for the pointwise minimum (resp. maximum) of u and v; $u^+ = u \vee 0$ is the positive part of u.

Definition 2.1. A subset X of $W^{1,1}(\Omega)$ is a *sublattice* of $W^{1,1}(\Omega)$ if $u \wedge v \in X$ and $u \vee v \in X$ whenever both $u, v \in X$. If $\phi \in W^{1,1}(\Omega)$ we denote by X_ϕ the set

$$X_\phi = \{u \in X : \, u - \phi \in W_0^{1,1}(\Omega)\}$$

Example 2.2. We will be mostly concerned here with $X = \mathcal{K}^K(\Omega)$ (or simply $\mathcal{K}^K$), the set of Lipschitz functions on Ω of rank less or equal to $K > 0$ or with $X = W^{1,1}(\Omega)$.

Definition 2.3 (Monotonicity assumptions). The field $a : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and *monotonic*:

$$\forall p,\, q \in \mathbb{R}^n \quad (a(q) - a(p)) \cdot (q - p) \geq 0. \tag{6}$$

We will sometimes consider the following mild ellipticity condition:

$$\forall p,\, q \in \mathbb{R}^n \quad (a(q) - a(p)) \cdot (q - p) = 0 \Longleftrightarrow a(q) = a(p). \tag{7}$$

In what follows, for $p \in \mathbb{R}^n$, the sets of the form

$$F_p = \{q \in \mathbb{R}^n : \, a(q) = a(p)\}$$

will be called the *level sets* of a.

Remark 2.4. Condition (7) is of course satisfied if the field a is strictly monotonic. As it is pointed out in [10], conditions (6) and (7) are both satisfied if for instance $a = \nabla f$ where $f : \mathbb{R}^n \to \mathbb{R}$ is convex of class $\mathcal{C}^1$.

As it is pointed out in [10] we have the following simple result, whose short proof is given here for the convenience of the reader:

Lemma 2.5. *Assume* (6) *and* (7)*. The level sets of the field a are convex.*

Proof. Assume $a(q) = a(p)$ for some $p, q \in \mathbb{R}^n$ and let $\zeta = \lambda p + (1 - \lambda)q$ with $\lambda \in]0, 1[$. Then

$$0 \le (a(\zeta) - a(p)) \cdot (\zeta - p) = (1 - \lambda)(a(\zeta) - a(p)) \cdot (q - p)$$

and

$$0 \le (a(\zeta) - a(q)) \cdot (\zeta - q) = -\lambda(a(\zeta) - a(p)) \cdot (q - p)$$

so that $(a(\zeta) - a(p)) \cdot (\zeta - p) = 0$ whence the conclusion. $\square$

3. Comparison Principles for quasi–solutions to $\operatorname{div} a(\nabla u) = 0$

Definition 3.1. Let X be a sublattice of $W^{1,1}(\Omega)$ and $\phi \in W^{1,1}(\Omega)$. A *quasi–solution* to $\operatorname{div} a(\nabla u) = 0$ in X_ϕ is a function $u \in X_\phi$ satisfying

$$\forall v \in X_\phi \quad a(\nabla u) \cdot \nabla(v - u) \in L^1(\Omega), \qquad \int_\Omega a(\nabla u) \cdot \nabla(v - u)\, dx \ge 0.$$

We will also say that u is a *quasi–solution* to $\operatorname{div} a(\nabla u) = 0$ in X if u is a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in X_u.

Example 3.2. It is an obvious but useful fact that constants and affine functions in X are quasi–solutions, due to the fact that the integral of a partial derivative of a function in $W^{1,1}_0(\Omega)$ is equal to zero. Moreover if u is a quasi–solution in X_ϕ and $c \in \mathbb{R}$ then $u + c$ is still a quasi–solution in $X_{\phi+c}$: indeed $\nabla(u + c) = \nabla u$.

The forthcoming results are based in the following well known existence theorem of Hartman-Stampacchia [7].

Theorem 3.3. *Let a be continuous and satisfy the monotonicity assumption* (6)*. Let $\phi \in \mathcal{K}^K$. There exists at least one quasi–solution u to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K_\phi$.*

If the monotonicity assumption (6) is not strict, the quasi–solutions to the equation $\operatorname{div} a(\nabla u) = 0$ are not unique; the following lattice property turns out to be useful.

Theorem 3.4 (Lattice property for quasi–solutions). *Assume that the field a satisfies* (6) *and* (7). *Let X be a sublattice of $W^{1,1}(\Omega)$ and ϕ_1, ϕ_2 be two Lipschitz functions on $\overline{\Omega}$ with $\phi_1 \le \phi_2$ on Γ. Let u_i be a quasi–solution to* $\operatorname{div} a(\nabla u) = 0$ *in X_{ϕ_i}, $i = 1, 2$. Then $u_1 \wedge u_2$ (resp. $u_1 \vee u_2$) is a quasi–solution to* $\operatorname{div} a(\nabla u) = 0$ *in X_{ϕ_1} (resp. X_{ϕ_2}).*

The proof of Theorem 3.4 is postponed after the following lemma.

Lemma 3.5. *Under the assumptions of Theorem* 3.4 *we have $a(\nabla u_1(x)) = a(\nabla u_2(x))$ a.e. on the "bad set" $\{u_1 > u_2\} \doteq \{x \in \Omega :\ u_1(x) > u_2(x)\}$ and moreover*

$$\int_{\{u_1>u_2\}} a(\nabla u_i) \cdot \nabla(u_1 - u_2)\, dx = 0 \quad (i = 1, 2) \tag{8}$$

In particular if $\phi_1 = \phi_2$ then $a(\nabla u_1(x)) = a(\nabla u_2(x))$ a.e. on Ω.

Proof of Lemma 3.5. We know that

$$\forall v \in X_{\phi_i} \quad \int_\Omega a(\nabla u_i) \cdot \nabla(v - u_i)\, dx \ge 0 \quad (i = 1, 2)$$

so that by taking $v = \min\{u_1, u_2\} \in X_{\phi_1}$ for $i = 1$ we get

$$\int_{\{u_1>u_2\}} a(\nabla u_1) \cdot \nabla(u_1 - u_2)\, dx \le 0 \tag{9}$$

whereas by taking $v = \max\{u_1, u_2\} \in X_{\phi_2}$ for $i = 2$ we get

$$\int_{\{u_1>u_2\}} a(\nabla u_2) \cdot \nabla(u_1 - u_2)\, dx \ge 0. \tag{10}$$

Now (9) and (10) give

$$\int_{\{u_1>u_2\}} (a(\nabla u_1) - a(\nabla u_2)) \cdot \nabla(u_1 - u_2)\, dx \le 0;$$

the monotonicity condition (6) then implies that

$$(a(\nabla u_1) - a(\nabla u_2)) \cdot \nabla(u_1 - u_2) = 0 \ \text{ a.e. on } \{u_1 > u_2\} :$$

the weak non degeneracy condition (7) yields that $a(\nabla u_1) = a(\nabla u_2)$ on $\{u_2 > u_1\}$; (8) follows then directly from (9)–(10). □

Proof of Theorem 3.4. Let $v \in X_{\phi_1}$. Then

$$I \doteq \int_\Omega a(\nabla(u_1 \wedge u_2)) \cdot \nabla(v - (u_1 \wedge u_2))\, dx = A + B$$

where

$$A = \int_{\{u_1 \le u_2\}} a(\nabla u_1) \cdot \nabla(v - u_1)\, dx, \qquad B = \int_{\{u_1 > u_2\}} a(\nabla u_2) \cdot \nabla(v - u_2)\, dx.$$

Writing that

$$A = \int_\Omega a(\nabla u_1) \cdot \nabla(v - u_1)\, dx - \int_{\{u_1 > u_2\}} a(\nabla u_1) \cdot \nabla(v - u_1)\, dx$$

the fact that u_1 is a quasi–solution yields

$$A \ge -\int_{\{u_1 > u_2\}} a(\nabla u_1) \cdot \nabla(v - u_1)\, dx$$

so that

$$I \ge \int_{\{u_1 > u_2\}} a(\nabla u_2) \cdot \nabla(v - u_2) - a(\nabla u_1) \cdot \nabla(v - u_1)\, dx.$$

Now by Lemma 3.5 we have $a(\nabla u_1) = a(\nabla u_2)$ a.e. on $\{u_1 > u_2\}$, thus

$$I \ge \int_{\{u_1 > u_2\}} a(\nabla u_2) \cdot \nabla(u_1 - u_2)\, dx = 0$$

thanks to (8), proving that $u_1 \wedge u_2$ is a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in X_{ϕ_1}; the other part of the claim follows similarly. □

We need to compare quasi-solutions upon their behavior at the boundary of the domain. The difficulty here is that the field a is not strictly monotonic so that the comparison principle does not hold in general.

Definition 3.6 (Minimal and maximal of the quasi–solutions). Let X be a sublattice of $W^{1,1}(\Omega)$ and $\phi \in W^{1,1}(\Omega)$. A quasi–solution u to $\operatorname{div} a(\nabla u) = 0$ in X_ϕ is said to be the minimal (resp. maximal) one if $u \le v$ ($u \ge v$) on Ω for every other quasi–solution v to $\operatorname{div} a(\nabla u) = 0$ in X_ϕ.

The next example will be used in the sequel.

Example 3.7. Let ϕ be Lipschitz in Ω and u be a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K_\phi(\Omega)$. Let $D \subset \Omega$ be open. The restriction $u_{|D}$ of u to D is a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K(D)$. Indeed any $v \in \mathcal{K}^K_{u_{|D}}(D)$ can be extended to a function $\tilde v$ in $\mathcal{K}^K_\phi(\Omega)$ with $\tilde v = u$ out of D and thus

$$\int_D a(\nabla u_{|D}) \cdot \nabla(v - u_{|D})\, dx = \int_\Omega a(\nabla u) \cdot \nabla(\tilde v - u)\, dx \ge 0.$$

Notice that if u is maximal in $\mathcal{K}^K_\phi(\Omega)$ then $u_{|D}$ is maximal in $\mathcal{K}^K_{u_{|D}}(D)$.

The proof of the next result is inspired by that of [10, Theorem 3.2] for weak solutions instead of quasi–solutions; however there is no need here to assume that the field a satisfies some further growth conditions, due to the fact that we deal with Lipschitz functions whose rank is bounded.

Proposition 3.8 (Existence of the minimal and the maximal quasi–solution in $\mathcal{K}_\phi^K$). *Assume that the field a satisfies* (6) *and* (7). *Let $K > 0$ and ϕ be Lipschitz of rank less or equal than K. The minimal and the maximal quasi–solutions to* $\operatorname{div} a(\nabla u) = 0$ *in $\mathcal{K}_\phi^K$ do exist.*

Proof. Let $\mathcal{S}$ be the set of the quasi–solutions to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}_\phi^K$; we prove the existence of a maximal element in $\mathcal{S}$. Notice first that $\mathcal{S}$ *is closed in* $W^{1,1}(\Omega)$. Indeed if $u_k \in \mathcal{S}$ converges to u in $W^{1,1}(\Omega)$ then, modulo a subsequence, ∇u_k converges a.e. to ∇u; therefore for $v \in \mathcal{K}_\phi^K$ we have

$$\lim_{k\to+\infty} a(\nabla u_k)\cdot(\nabla v - \nabla u_k) = a(\nabla u)\cdot(\nabla v - \nabla u) \text{ a.e.}$$

and $a(\nabla u_k)\cdot(\nabla v - \nabla u_k)$ remains bounded in $L^\infty(\Omega)$ since $|\nabla v| \le K$ and $|\nabla u_k| \le K$ a.e. in Ω for all k: the dominated convergence theorem then implies that

$$\int_\Omega a(\nabla u)\cdot(\nabla v - \nabla u)\,dx = \lim_k \int_\Omega a(\nabla u_k)\cdot(\nabla v - \nabla u_k)\,dx \ge 0.$$

Moreover $\mathcal{S}$ *is convex.* Indeed assume $u_1, u_2 \in \mathcal{S}$ and let $u = \lambda u_1 + (1-\lambda)u_2$ for some $\lambda \in [0,1]$. By Lemma 3.5 we have $a(\nabla u_1) = a(\nabla u_2)$ a.e.; Lemma 2.5 implies that $a(\nabla u) = a(\nabla u_1) = a(\nabla u_2)$ a.e. so that for $v \in \mathcal{K}_\phi^K$ we get

$$\begin{aligned}&\int_\Omega a(\nabla u)\cdot(\nabla v - \nabla u)\,dx\\ &= \lambda\int_\Omega a(\nabla u_1)\cdot(\nabla v - \nabla u_1)\,dx + (1-\lambda)\int_\Omega a(\nabla u_2)\cdot(\nabla v - \nabla u_2)\,dx \ge 0\end{aligned}$$

proving that u is a quasi–solution to our problem.

Since $W^{1,1}(\Omega)$ is separable and $\mathcal{S}$ is closed there is a family v_k ($k \in \mathbb{N}$) of functions that is dense in $\mathcal{S}$ for the topology in $W^{1,1}$. The lattice property Theorem 3.4 implies that $u_k = v_1 \vee ... \vee v_k \in \mathcal{S}$ for any k: let u be the pointwise limit of u_k. We claim that u is the maximal quasi–solution we are looking for. Of course $u \in \mathcal{K}_\phi^K$ and $u \ge v$ a.e. for any $v \in \mathcal{S}$, due to the fact that a convergent sequence in L^1 has a subsequence that converges a.e.; it remains to show that u is a quasi–solution. Since the functions u_k are bounded in $W^{1,\infty}(\Omega)$ we may assume that they weakly converge in $W^{1,2}(\Omega)$, necessarily to u. Mazur's Lemma then yields a sequence in the convex hull of $\{u_k : k \in \mathbb{N}\}$ that strongly converges to u in $W^{1,1}(\Omega)$: the closure of $\mathcal{S}$ yields the conclusion. □

Theorem 3.9 (Comparison Principle for quasi–solutions). *Assume that the field a satisfies* (6) *and* (7). *Let X be a sublattice of $W^{1,1}(\Omega)$ and ϕ_1, ϕ_2 be two Lipschitz functions on the closure of Ω with $\phi_1 \le \phi_2$ on Γ. Let u_i be a quasi–solution to* $\operatorname{div} a(\nabla u) = 0$ *in X_{ϕ_i}, $i = 1, 2$ and u_1 be minimal or u_2 be maximal. Then $u_1 \le u_2$ a.e. on Ω.*

Proof. It follows from Theorem 3.4 that $u_1 \wedge u_2$ is a quasi–solution to the equation $\operatorname{div} a(\nabla u) = 0$ in X_{ϕ_1} and $u_1 \vee u_2$ is a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in X_{ϕ_2}. If u_1 is the minimal quasi–solution then $u_1 \le u_1 \wedge u_2$; analogously if u_2 is the maximal quasi–solution then $u_1 \vee u_2 \le u_2$. In both cases the last inequalities yield $u_1 \le u_2$ a.e.. □

The next result is a version for quasi–solutions to $\operatorname{div} a(\nabla u) = 0$ of the Haar–Radò type results proven in [9] in the framework of minimizers of the calculus of variations and in [10] concerning weak solutions in Sobolev spaces to the same equation. It will be used in the proof of Theorem 5.1.

Theorem 3.10 (A Haar–Radò type result for quasi–solutions). *Assume that the field a satisfies* (6) *and* (7). *Let ϕ be Lipschitz of rank less or equal than K and u be the minimal or the maximal quasi–solution to* $\operatorname{div} a(\nabla u) = 0$ *in* $\mathcal{K}^K_\phi$. *Assume that there is $K_1 \ge K$ such that*

$$\forall \gamma \in \Gamma,\, \forall x \in \Omega \quad |u(x) - \phi(\gamma)| \le K_1 |x - \gamma|. \tag{11}$$

Then the Lipschitz constant of u is less or equal than K_1.

We need the following simple lemma concerning quasi–solutions to $\operatorname{div} a(\nabla u) = 0$.

Lemma 3.11. *Let $K > 0$. If u is a quasi–solution in $\mathcal{K}^K(\Omega)$ then, for every $h \in \mathbb{R}^n$, the function $u(x-h)$ is a quasi–solution in $\mathcal{K}^K(h+\Omega)$.*

Proof. Let $v \in \mathcal{K}^K(h+\Omega)$ with $v(y) = u(y-h)$ for each $y \in \partial(h+\Omega) = h + \Gamma$. Then the change of variables $y = x + h$ yields

$$\begin{aligned}&\int_{h+\Omega} a(\nabla u(y-h)) \cdot (\nabla v(y) - \nabla u(y-h))\, dy \\ &= \int_\Omega a(\nabla u(x)) \cdot (\nabla v(x+h) - \nabla u(x))\, dx \ge 0\end{aligned}$$

since $v(x+h) \in \mathcal{K}^K(\Omega)$ and $v(x+h) = u(x)$ for every $x \in \Gamma$, proving that $u(y-h)$ is a quasi–solution. □

Proof of Theorem 3.10. By extending u out of Ω to a Lipschitz function of rank less or equal than K we may assume that (11) holds for every x in $\mathbb{R}^n$. Fix $h \in \mathbb{R}^n$ such that $\Omega \cap (h+\Omega) \ne \emptyset$. Then $u(x-h) \le u(x) + K_1|h|$ on $\partial(\Omega \cap (h+\Omega))$. Indeed let $r \in \partial(\Omega \cap (h+\Omega))$: if $r = \gamma \in \Gamma$ then

$$u(r-h) - u(r) = u(\gamma - h) - u(\gamma) = u(\gamma-h) - \phi(\gamma) \le K_1|h|;$$

if $r = h + \gamma \in h + \Gamma$ then

$$u(r-h) - u(r) = u(\gamma) - u(h+\gamma) = \phi(\gamma) - u(h+\gamma) \le K_1|h|.$$

It follows from Lemma 3.11 and Example 3.2 that both $u(x-h) - K_1|h|$ and u are quasi–solutions in $\mathcal{K}^K(\Omega \cap (h+\Omega))$. If u is maximal in $\mathcal{K}^K(\Omega)$ then u is still maximal in $\mathcal{K}^K(\Omega\cap(h+\Omega))$; if instead u is minimal in $\mathcal{K}^K(\Omega)$ then $u(x-h)-K_1|h|$ is minimal in $\mathcal{K}^K(\Omega\cap(h+\Omega))$: Theorem 3.9 implies that $u(x-h)-K_1|h|\le u(x)$ on $\Omega\cap(h+\Omega)$. Now if $a,b\in\Omega$ set $h=a-b$: then

$$u(b)-u(a)=u(a-h)-u(a)\le K_1|h|=K_1|b-a|. \qquad \square$$

Remark 3.12. A slight modification in the proofs above shows that the claims of Theorem 3.4 and Theorem 3.9 hold also for quasi–solutions to the equation $\operatorname{div} a(\nabla u)+b(x,u)=0$, where b is a Carathéodory map and $u\mapsto b(x,u)$ is monotonic increasing for a.e. x; a more subtle modification as in [10, §3.2] is needed to adapt Theorem 3.10. We do not formulate these results explicitly since we do not use them here.

4. A generalized Bounded Slope Condition and an existence result

The functions introduced below were first introduced by Cellina in the framework of the calculus of variations; they play the role of the affine functions in minimization problems in the case where the integrand is not strictly convex.

Definition 4.1. Whenever F is a compact and convex subset of $\mathbb{R}^n$ and $x_0\in\mathbb{R}^n$ let

$$h^+_{F,x_0}(x)=\max\{\xi\cdot(x-x_0):\ \xi\in F\}$$

$$h^-_{F,x_0}(x)=\min\{\xi\cdot(x-x_0):\ \xi\in F\}=-h^+_{-F,x_0}(x).$$

Example 4.2. Let F be the unit ball. Then $h^+_{F,x_0}(x)=|x-x_0|$ and $h^-_{F,x_0}(x)=-|x-x_0|$ for all x_0.

It is worth mentioning that the functions just defined are Lipschitz of rank less or equal than L_F defined by

$$L_F\doteq\max\{|\xi|:\xi\in F\},$$

that $\nabla h^\pm_{F,x_0}\in F$ a.e. and that $h^\pm_{F,x_0}(x)=\nabla h^\pm_{F,x_0}(x)\cdot(x-x_0)$ a.e.: this follows easily from the properties of the support function to a set [11] or see [3] for a direct proof; these functions are nothing more than affine when F is reduced to a single point. We prove here that these functions are quasi–solutions to $\operatorname{div} a(\nabla u)=0$ and that they satisfy the Comparison Principle.

Proposition 4.3. *Assume that a satisfies the monotonicity assumptions* (6) *and* (7)*. Let F be a compact level set for a and let $K\ge L_F=\max\{|\xi|:\xi\in F\}$. The following assumptions hold:*

i) *For every $x_0\in\mathbb{R}^n$ and $c\in\mathbb{R}$ the functions $c+h^\pm_{F,x_0}$ are quasi–solutions to $\operatorname{div} a(\nabla u)=0$ in $\mathcal{K}^K$;*

ii) *If $x_0\notin\Omega$ then $c+h^\pm_{F,x_0}$ satisfy the Comparison Principle: if u is a quasi–solution to $\operatorname{div} a(\nabla u)=0$ in $\mathcal{K}^K$ and $u\le c+h^+_{F,x_0}$ on Γ then $u\le c+h^+_{F,x_0}$ on Ω (C.P. from above); analogously if $u\ge c+h^-_{F,x_0}$ on Γ then $u\ge c+h^-_{F,x_0}$ on Ω (C.P. from below).*

Proof. *i*) We assume without restriction that $c = 0$. Assume that $F = \{q \in \mathbb{R}^n : a(q) = a(p)\}$ for some p in $\mathbb{R}^n$. The fact that $c + h^{\pm}_{F,x_0}$ are quasi–solutions follows immediately since $\nabla h^{\pm}_{F,x_0} \in F$ a.e. and thus

$$a(\nabla(h^{\pm}_{F,x_0})) = a(p)$$

is a constant.

ii) We consider now $h \doteq h^{+}_{F,x_0}$ and we prove that h satisfies the Comparison Principle from above; the proof of the claim concerning h^{-}_{F,x_0} is similar. It follows from Theorem 3.9 that it is enough to prove that h is a maximal quasi–solution. Let u be a quasi–solution of $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K_h$. It follows from Lemma 3.5 that $a(\nabla u) = a(\nabla h)$ a.e. on Ω so that $\nabla u \in F$ a.e. whence

$$\nabla u(x) \cdot (x - x_0) \leq \max\{\xi \cdot (x - x_0) : \xi \in F\} = h(x) = \nabla h(x) \cdot (x - x_0) \text{ a.e..}$$

Extend u with $u = h$ out of Ω. For $x \in \Gamma$ set

$$g_x(t) = (h - u)(x_0 + t(x - x_0)), \quad t \in \mathbb{R}.$$

Since $h - u$ is Lipschitz then it is differentiable on a.e. line through x_0 and for a.e. $x \in \Gamma$ (for the $(n-1)$–dimensional Hausdorff measure) we have

$$g'_x(t) = \nabla(h - u)(x_0 + t(x - x_0)) \cdot (x - x_0) \ \text{ a.e. } t.$$

Since $x_0 \notin \Omega$ then $g_x(0) = h(x_0) - u(x_0) = 0$; moreover

$$g'_x(t) = \nabla(h - u)(x_0 + t(x - x_0)) \cdot (x - x_0) \geq 0$$

so that g_x is decreasing and $g_x(1) = h(x) - u(x) \geq 0$, proving that $u(x) \leq h(x)$ for a.e. (and thus every) $x \in \Gamma$. □

An analogous of the next variant of the (BSC) was formulated in [3] in the framework of the calculus of variations: there the level sets of a that appear below in our definition are replaced by some sets that depend on the faces of a convex lagrangean.

Definition 4.4 (Generalized (BSC)). The pair (ϕ, a) satisfies the Generalized (BSC) of rank $Q > 0$ if for every $\gamma \in \partial\Omega$:

i) there exists a level set F^-_γ for a, contained in a ball of radius Q, such that

$$\forall \gamma' \in \partial\Omega \ \ \phi(\gamma) + h^{-}_{F^-_\gamma,\gamma}(\gamma') \leq \phi(\gamma'); \tag{12}$$

ii) there exists a level set F^+_γ for a, contained in a ball of radius Q, such that

$$\forall \gamma' \in \partial\Omega \ \ \phi(\gamma) + h^{+}_{F^+_\gamma,\gamma}(\gamma') \geq \phi(\gamma'). \tag{13}$$

Example 4.5. The fact that the field a has some non trivial level sets increases the chances that a given function ϕ satisfies the Generalized (BSC): assume that a level set F of a contains a closed ball $B(p, R]$ of center p and radius R and let ψ be *any* Lipschitz function of rank less than R; then $\psi(x) + p \cdot x$ fulfills the generalized (BSC). Indeed for any $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$ we have

$$h^+_{F,\gamma}(x) \geq \max\{\xi \cdot (x-\gamma) : \xi \in B(p,R]\} = p \cdot (x-\gamma) + R|x-\gamma|,$$

$$h^-_{F,\gamma}(x) \leq \min\{\xi \cdot (x-\gamma) : \xi \in B(p,R]\} = p \cdot (x-\gamma) - R|x-\gamma|.$$

Thus if $\phi(x) = p \cdot x + \psi(x)$ we have

$$\phi(x) - \phi(\gamma) = p \cdot (x-\gamma) + \psi(x) - \psi(\gamma)$$

so that, if ψ is Lipschitz of rank less than R,

$$\phi(x) - \phi(\gamma) \leq p \cdot (x-\gamma) + R|x-\gamma| \leq h^+_{F,\gamma}(x)$$

and, analogously,

$$\phi(\gamma) - \phi(x) \geq p \cdot (x-\gamma) - R|x-\gamma| \geq h^-_{F,\gamma}(x).$$

Remark 4.6. It is an easy matter to show that the (BSC) of rank Q implies the Generalized (BSC) of rank Q' if $|q| \leq Q'$ whenever $a(q) = a(p)$ for some $|p| \leq Q$.

Example 4.7. The Generalized (BSC) is strictly more general than the (BSC): when the field a is not uniformly elliptic the class of functions that satisfy the Generalized (BSC) is wider than the class of functions that satisfy the (BSC). For instance if

$$a(\xi) = \nabla f(\xi), \qquad f(\xi) = \begin{cases} (|\xi|^2-1)^2 & \text{if } |\xi| \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

then the level set F of a containing the origin is the closed unit ball. It follows from Example 4.2 that $h^+_{F,x_0}(x) = |x-x_0|$ and $h^-_{F,x_0}(x) = -|x-x_0|$. Therefore any Lipschitz function ϕ of rank less or equal than 1 is such that (ϕ, a) satisfies the Generalized (BSC); note that the domain may be not convex whereas the validity of the (BSC) implies the convexity of the domain.

5. Existence of a Lipschitz solution to $\operatorname{div} a(\nabla u) = 0$ under the Generalized (BSC)

The next result generalizes [7, Theorem 13.1], where the authors establish the existence of a Lipschitz weak solution to $\operatorname{div} a(\nabla u) = 0$ under the (BSC); its proof is based on the original one, modulo the obvious changes.

Theorem 5.1 (Existence of a Lipschitz solution to $\operatorname{div} a(\nabla u) = 0$ under the Generalized (BSC)). *Assume that a satisfies the monotonicity assumptions* (6) *and* (7). *Let ϕ be Lipschitz and that (ϕ, a) satisfies the Generalized (BSC) of rank Q. There is at least one Lipschitz function u of rank less or equal than Q and equal to ϕ on $\partial\Omega$ satisfying*

$$\forall \eta \in W^{1,2}_0(\Omega) \quad \int_\Omega a(\nabla u) \cdot \nabla \eta \, dx = 0$$

Proof. For every $K > Q$ there is, from Proposition 3.8 a quasi–solution in $\mathcal{K}_\phi^K$ to $\operatorname{div} a(\nabla u) = 0$ that is either minimal or maximal: in any case we call it u^K. For every $\gamma \in \Gamma$ conditions (12) and (13) together with Proposition 4.3 imply that

$$\forall x \in \Omega \quad \phi(\gamma) + h^-_{F^-_\gamma,\gamma}(x) \le u^K(x) \le \phi(\gamma) + h^+_{F^+_\gamma,\gamma}(x)$$

so that in particular

$$\forall \gamma \in \Gamma \ \forall x \in \Omega \quad |u^K(x) - \phi(\gamma)| \le Q|x - \gamma|. \tag{14}$$

The Haar–Radò type Theorem 3.10 yields that u^K is Lipschitz of rank less or equal than Q. We have thus obtained a sequence of Lipschitz functions u^K whose Lipschitz rank is bounded by the same constant Q such that, for each $K > Q$,

$$\forall v \in \mathcal{K}_\phi^K \quad \int_\Omega a(\nabla u^K) \cdot (\nabla v - \nabla u^K)\, dx \ge 0.$$

We thus proceed as in [7, Lemma 12.3]: we may assume modulo a subsequence that u^K converges uniformly and weakly in $W^{1,2}(\Omega)$ to a Lipschitz function u, of rank less or equal than Q. Let η be a smooth function with compact support in Ω. Since for any $K > Q$ the function $u + \eta$ is Lipschitz of rank less than $2Q$, for K sufficiently large we have

$$\int_\Omega a(\nabla u^K) \cdot (\nabla(u + \eta) - \nabla u^K)\, dx \ge 0$$

so that, by the monotonicity assumption (6),

$$\int_\Omega a(\nabla(u + \eta)) \cdot (\nabla(u + \eta - u^K)\, dx \ge 0.$$

Now ∇u^K converges weakly to ∇u in L^2 and $a(\nabla(u + \eta))$ is bounded; passing to the limit as $K \to +\infty$ in the latter inequality we obtain

$$\int_\Omega a(\nabla(u + \eta)) \cdot \nabla \eta\, dx \ge 0.$$

By taking $t\eta$ instead of η, for $t \to 0^+$ and to 0^- we obtain

$$\forall \eta \in \mathcal{C}_c^\infty(\Omega) \quad \int_\Omega a(\nabla u) \cdot \nabla \eta\, dx = 0;$$

the conclusion follows by the density of smooth functions with compact support in $W_0^{1,2}(\Omega)$ and the fact that $a(\nabla u)$ is bounded, u being Lipschitz. $\square$

6. The existence of a solution to $\operatorname{div} a(\nabla u) = 0$ under the (LBSC) and its generalized version

We recall here a notion introduced by Clarke in [4].

Definition 6.1 (LBSC). A Lipschitz function $\phi : \overline{\Omega} \to \mathbb{R}$ is said to satisfy the *Lower Bounded Slope Condition*, (shortly the (LBSC)) of rank Q if given any point $\gamma \in \Gamma$ there exists an affine function $\zeta_\gamma \cdot (y - \gamma) + \phi(\gamma)$ with $|\zeta_\gamma| \leq Q$ such that

$$\forall \gamma' \in \Gamma \quad \phi(\gamma') \geq \zeta_\gamma \cdot (\gamma' - \gamma) + \phi(\gamma). \tag{15}$$

It can be shown that ϕ satisfies the (LBSC) of some rank if and only if ϕ is the restriction of a convex function.

By replacing the affine functions with the class of functions introduced in Definition 4.1 we obtain the obvious generalization of the (LBSC), depending of course on the monotone operator a.

Definition 6.2 (Generalized (LBSC)). The pair (ϕ, a) is said to satisfy the *Generalized* (LBSC) of rank $Q > 0$ if (12) holds for every $\gamma \in \partial\Omega$, i.e. there exists a level set F_γ^- for a, contained in a ball of radius Q, such that

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + h^-_{F_\gamma^-,\gamma}(\gamma') \leq \phi(\gamma')$$

We prove that if ϕ satisfies the (LBSC), or the Generalized (LBSC) jointly with the weak ellipticity condition (7), then there exists a solution u to (16) that on every compact set is Lipschitz and a uniform limit of functions in $\mathcal{K}_\phi^K$. Moreover if a fulfills the natural growth conditions then u agrees with the boundary datum in the trace sense.

Theorem 6.3 (Existence of a locally Lipschitz solution to $\operatorname{div} a(\nabla u) = 0$**).**[1] *Assume that Ω is convex. Let a be a monotonic field, i.e. such that* (6) *holds, and assume that one of the following conditions holds:*

A) *the function ϕ satisfies the* (*LBSC*) *of rank Q, or*

B) *the operator a satisfies the weak ellipticity condition* (7)*, the level set $F_0 = \{\eta \in \mathbb{R}^n : a(\eta) = a(0)\}$ of the field a is bounded and ϕ satisfies the Generalized* (*LBSC*) *of rank Q.*

Then:

i) *There exists a sequence of functions $(u^K)_{K \in \mathbb{N}}$ with $u^K \in \mathcal{K}_\phi^K$ and a function u such that:*

- *a)* *For every K, u^K is a quasi–solution to* $\operatorname{div} a(\nabla u) = 0$ *in $\mathcal{K}_\phi^K$;*
- *b)* *the sequence u^K converges uniformly to u and ∇u^K converges to ∇u for the weak* topology of L^∞ on every compact subset of Ω;*
- *c)* *u is a locally Lipschitz solution to* $\operatorname{div} a(\nabla u) = 0$ *in the sense of distributions, i.e.*

$$\forall \eta \in \mathcal{C}_c^\infty(\Omega) \quad \int_\Omega a(\nabla u(x)) \cdot \nabla \eta(x)\, dx = 0. \tag{16}$$

ii) *If, moreover, there are $\alpha > 0$, $\beta > 0$ and $r \in \mathbb{R}$ satisfying*

$$\forall \xi \in \mathbb{R}^n \quad a(\xi) \cdot \xi \geq \alpha |\xi|^p, \quad |a(\xi)| \leq \beta |\xi|^{p-1} + r \quad (p > 1) \tag{17}$$

[1] I thank again here Pierre Bousquet for pointing out that there is no need to assume (7) in order to obtain the conclusion of Theorem 6.3 under Assumption A).

then the locally Lipschitz function u defined in i) is a weak solution to $\operatorname{div} a(\nabla u) = 0$ *in* $W^{1,p}_\phi(\Omega)$, *i.e.* $u \in W^{1,p}_\phi(\Omega)$, $a(\nabla u) \in L^{p'}(\Omega)$ $(1/p' + 1/p = 1)$ *and*

$$\forall \eta \in W^{1,p}_0(\Omega) \quad \int_\Omega a(\nabla u(x)) \cdot \nabla \eta(x)\, dx = 0.$$

Remark 6.4. The conclusion of *ii*), more precisely the existence of a weak solution to $\operatorname{div} a(\nabla u) = 0$ assuming that the field a satisfies the growth assumptions (17) and ϕ satisfies the (LBSC) is obtained also in [10] with a different approach, by using Minty's theorem for the existence of a solution in Sobolev spaces.

The proof of Theorem 6.3 is based on the following Lemma. We recall that by [7, Lemma 12.1] a solution to $\operatorname{div} a(\nabla u) = 0$ exists in $\mathcal{K}^K_\phi$ whenever K is greater than the Lipschitz constant of ϕ.

Lemma 6.5. *Let* $K > Q > 0$. *Under the assumptions A) or B) of Theorem* 6.3 *there exist a quasi–solution* u^K *to* $\operatorname{div} a(\nabla u) = 0$ *in* $\mathcal{K}^K_\phi$ *and constants* T, C *depending only on* a, ϕ, Ω *(in any case not on* K*), satisfying*

$$\forall x \in \Omega \quad \|u_K\|_\infty \le T, \quad |\nabla u^K(x)| \le \frac{C}{\operatorname{dist}(x, \Gamma)} \tag{18}$$

Moreover, when a *is strictly monotonic, then* $T = \|\phi\|_\infty$ *and* C *does not depend on* a.

Proof. *Assume first that Assumption B) holds.* Let u^K be a *maximal* quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K_\phi$. Let u_- be the minimal quasi–solution in $\mathcal{K}^K_{-\|\phi\|_\infty}$ and u_+ be the maximal quasi–solution in $\mathcal{K}^K_{\|\phi\|_\infty}$, where $\|\phi\|_\infty$ is the sup–norm of ϕ; notice that $u_- = -\|\phi\|_\infty$ and $u_+ = \|\phi\|_\infty$ if a is strictly monotonic. Since $-\|\phi\|_\infty \le u^K \le \|\phi\|_\infty$ on Γ then Theorem 3.9 implies that $u_- \le u^K \le u_+$ on Ω. Since constants are quasi–solutions it follows from Lemma 3.5 that $a(\nabla u_-) = a(\nabla(-\|\phi\|_\infty)) = a(0)$ and $a(\nabla u_+) = a(\nabla(\|\phi\|_\infty)) = a(0)$ so that $|\nabla u^\pm| \le M$ where $M = \sup\{|q| : a(q) = a(0)\}$. Thus $u^\pm$ are bounded by a constant T depending only on M, Ω and ϕ (in any case not on K), proving the first part of the claim. Fix now $z \in \Gamma$; for $\lambda \in]0,1]$ we set

$$\Omega_\lambda \doteq \lambda(\Omega - z) + z, \qquad \Gamma_\lambda \doteq \partial\Omega_\lambda$$
$$\forall y \in \Omega_\lambda, \quad u^K_\lambda(y) = \lambda u^K((y - z)/\lambda + z)$$
$$\forall \gamma' \in \Gamma_\lambda, \quad \phi_\lambda(\gamma') = \lambda\phi((\gamma' - z)/\lambda + z)$$

By convexity $\Omega_\lambda \subseteq \Omega$ for every $\lambda \in]0,1]$. Then $u^K_\lambda \in \mathcal{K}^K_{\phi_\lambda}(\Omega_\lambda)$. Notice that since $\nabla u^K_\lambda(x) = \nabla u^K((x - z)/\lambda + z)$ then u^K_λ is a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K_{\phi_\lambda}(\Omega_\lambda)$. Indeed let $v \in \mathcal{K}^K_{\phi_\lambda}(\omega_\lambda)$; we may assume without loosing generality that $z = 0$. By the change of variable $x = y/\lambda$ we have

$$\begin{aligned}\int_{\Omega_\lambda} a(\nabla u^K_\lambda(y)) \cdot \nabla(v - u^K_\lambda)(y)\, dy &= \lambda^n \int_\Omega a(\nabla u^K(x)) \cdot (\nabla v(\lambda x) - \nabla u^K(x))\, dx \\ &= \lambda^n \int_\Omega a(\nabla u^K(x)) \cdot (\nabla v^\lambda(x) - \nabla u^K(x))\, dx\end{aligned} \tag{19}$$

where we set $v^\lambda(x) = \frac{1}{\lambda}v(\lambda x)$. Now if $\gamma \in \Gamma$ then $\lambda\gamma \in \Gamma_\lambda$ so that

$$v^\lambda(\gamma) = \frac{1}{\lambda}v(\lambda\gamma) = \frac{1}{\lambda}\lambda\phi(\gamma) = \phi(\gamma)$$

whence $v^\lambda \in \mathcal{K}^K_\phi$: it follows that

$$\int_\Omega a(\nabla u^K(x)) \cdot (\nabla v^\lambda(x) - \nabla u^K(x))\, dx \geq 0$$

and thus, from (19), we get

$$\int_{\Omega_\lambda} a(\nabla u^K_\lambda(y)) \cdot \nabla(v - u^K_\lambda)(y)\, dy \geq 0,$$

proving that u^K_λ is a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}^K_{\phi_\lambda}(\Gamma_\lambda)$.
We wish now to compare u^K_λ and u^K on Ω_λ. Let $\gamma' = \lambda\gamma \in \Gamma_\lambda$; then

$$u^K_\lambda(\gamma') = \lambda\phi(\gamma), \qquad u^K(\gamma') = u^K(\lambda\gamma). \tag{20}$$

Since u^K is the maximal quasi–solution and, by Proposition 4.3, the function $h_{F^-_\gamma,\gamma}+ \phi(\gamma)$ is a quasi–solution, then by Theorem 3.9 we have

$$h_{F^-_\gamma,\gamma}(x) + \phi(\gamma) \leq u^K(x) \quad \text{on } \Omega$$

so that in particular, for $x = \gamma' = \lambda\gamma$, we get

$$\phi(\gamma) \leq u^K(\gamma') - h_{F^-_\gamma,\gamma}(\lambda\gamma). \tag{21}$$

A direct computation shows that

$$h_{F^-_\gamma,\gamma}(\lambda\gamma) = (\lambda - 1)\inf\{\xi \cdot \gamma : \xi \in F^-_\gamma\}$$

whence

$$|h_{F^-_\gamma,\gamma}(\lambda\gamma)| \leq (1 - \lambda)Q \operatorname{diam}\Omega$$

and thus (21) yields

$$u^K_\lambda(\gamma') \leq u^K(\gamma') + q(1 - \lambda), \tag{22}$$

where $q = Q \operatorname{diam}(\Omega) + \|\phi\|_\infty$ is a constant that does not depend neither on a nor on K. Now, as it was pointed out in Example 3.7, $u^K + q(1 - \lambda)$ restricted to Ω_λ is a maximal quasi–solution in $\mathcal{K}^K(\Omega_\lambda)$, among the functions that share the same boundary datum in Γ_λ: it follows from (22) and Theorem 3.9 that

$$\lambda u^K(x) = u^K_\lambda(y) \leq u^K(y) + q(1 - \lambda) \quad \text{on } \Omega_\lambda, \quad x = \frac{y}{\lambda}.$$

In particular we get

$$\begin{aligned} u^K(x) - u^K(y) &= u^K(x) - \lambda u^K(x) + \lambda u^K(x) - u^K(y) \\ &\leq (1 - \lambda)\|u^K\|_\infty + q(1 - \lambda) \\ &\leq (1 - \lambda)\,(T + q)\,, \end{aligned}$$

Now

$$1-\lambda = \frac{|y-x|}{|x-z|} \le \frac{|y-x|}{\operatorname{dist}(x,\partial\Omega)|}$$

so that combining the two latter inequalities we obtain

$$u^K(x) - u^K(y) \le (T+q)\frac{|y-x|}{\operatorname{dist}(x,\partial\Omega)|},$$

proving that

$$|\nabla u^K(x)| \le \frac{C}{\operatorname{dist}(x,\Gamma)}, \quad C = T+q.$$

Notice that if a is strictly monotonic then $C = \|\phi\|_\infty + q$ does not even depend on a.

Assume that A) holds. For every $i = 1, 2, \ldots$ let

$$a_i(\xi) = a(\xi) + \frac{1}{i}\xi.$$

Then a_i is strictly monotonic so that there exists a quasi–solution u_i^K to $\operatorname{div} a_i(\nabla u) = 0$ on $\mathcal{K}_\phi^K$. Then

$$\|u_i^K\|_\infty \le \|\phi\|_\infty, \qquad |\nabla u_i^K| \le \frac{C}{\operatorname{dist}(x,\Gamma)},$$

where C does not depend neither on i nor on K: this follows from [1] or even from the fact that in this strictly monotonic case the $(LBSC)$ is equivalent to the Generalized $(LBSC)$, thus (a_i, ϕ) fulfills Assumption $B)$. The Lipschitz ranks of the u_i^K being bounded by K there exists a subsequence of $(u_i^K)_i$ which converges to some $u^K \in \mathcal{K}_\phi^K$ uniformly on $\overline{\Omega}$. Moreover one can further assume that $(\nabla u_i^K)_i$ converges weakly* to ∇u^K in $L^\infty(\Omega)$. For any i and for any $v \in \mathcal{K}_\phi^K$ we have

$$\int_\Omega a_i(\nabla u_i^K)\cdot(\nabla v - \nabla u_i^K)\,dx \ge 0$$

so that

$$\int_\Omega a(\nabla u_i^K)\cdot(\nabla v - \nabla u_i^K)\,dx \ge -\frac{1}{i}\int_\Omega \nabla u_i^K\cdot(\nabla v - \nabla u_i^K)\,dx$$

implying

$$\int_\Omega a(\nabla v)\cdot(\nabla v - \nabla u_i^K)\,dx \ge -\frac{1}{i}\int_\Omega \nabla u_i^K\cdot(\nabla v - \nabla u_i^K)\,dx.$$

By letting i going to $+\infty$ we get

$$\int_\Omega a(\nabla v)\cdot(\nabla v - \nabla u^K)\,dx \ge 0. \tag{23}$$

Let now $w \in \mathcal{K}_\phi^K$. Fix $t \in [0,1[$ and choose $v = w + t(u^K - w)$ in (23). We get

$$\int_\Omega a((1-t)\nabla w + t\nabla u^K) \cdot (\nabla w - \nabla u^K)\, dx \geq 0,$$

so that by passing to the limit for $t \to 1$ we obtain

$$\int_\Omega a(\nabla u^K) \cdot (\nabla w - \nabla u^K)\, dx \geq 0,$$

proving the claim. $\square$

Proof of Theorem 6.3. *i*) For every $K > Q$ let u^K be a quasi–solution to $\operatorname{div} a(\nabla u) = 0$ in $\mathcal{K}_\phi^K$ satisfying (18) for some constants T, C that do not depend on K: such a function exists thanks to Lemma 6.5. Take an increasing sequence of open subsets of Ω satisfying

$$\Omega_j \subset \overline{\Omega}_j \subset \Omega_{j+1}, \qquad \bigcup_j \Omega_j = \Omega.$$

It follows from (18) that, on Ω_j, the functions u^K are Lipschitz of rank less than a constant K_j depending only on j. Then, up to a subsequence, the functions u^K converge uniformly on every compact subset of Ω to a function u which is Lipschitz of rank K_j on Ω_j. Moreover one may assume that for every j, ∇u^K converges to ∇u for the weak $*$ topology of $L^\infty(\Omega_j)$. The fact that u satisfies (16) follows from [1] by noticing that it is enough that the sequence ∇u^K converges weakly $*$ to ∇u on every compact set, instead of knowing that u^K converges weakly to u in $W^{1,2}(\Omega)$: we check it here for the convenience of the reader. Let $\eta \in \mathcal{C}_c^\infty(\Omega)$; its support is contained in some Ω_j for some j. Let $0 \leq \theta_j \leq 1$ be a smooth function with

$$\theta_j = \begin{cases} 1 & \text{on } \Omega_j \\ 0 & \text{out of } \Omega'_j \end{cases}$$

where Ω'_j is an open set satisfying

$$\overline{\Omega_j} \subset \Omega'_j \subset \Omega_{j+1}, \qquad |\Omega'_j \setminus \Omega_j| \leq \varepsilon$$

for a prescribed $\varepsilon > 0$. Set $\psi^K = u^K + \eta + \theta_j(u - u^K)$: then

$$\psi^K = \begin{cases} u + \eta & \text{on } \Omega_j \\ u^K & \text{out of } \Omega'_j \end{cases}$$

A Lipschitz rank for ψ^K on Ω'_j is $S_j \doteq 3K_{j+1} + \|\nabla\eta\|_\infty + 2\|u - u^K\|_{L^\infty(\Omega_{j+1})}\|\nabla\theta_j\|_\infty$; thus ψ^K is globally Lipschitz of rank K for $K > S_j$ For K big enough we have

$$\int_\Omega a(\nabla u^K) \cdot \nabla(\psi^K - u^K)\, dx \geq 0$$

implying

$$\int_\Omega a(\nabla\psi^K)\cdot\nabla(\psi^K-u^K)\,dx\ge 0.$$

It follows that

$$\begin{aligned}\int_{\Omega_j} a(\nabla(u+\eta))\cdot\nabla(u+\eta-u^K)\,dx &\ge -\int_{\Omega'_j\setminus\Omega_j} a(\nabla\psi^K)\cdot\nabla(\psi^K-u^K)\,dx\\ &\ge -c\varepsilon\end{aligned}\tag{24}$$

where c is a positive constant depending only on a and j. Indeed $a(\nabla\psi^K)$ is bounded by a constant depending on a and S_j, whereas the gradient of ψ^K-u^K is bounded by a constant depending on j. By passing to the limit for $K\to+\infty$ in (24) we obtain

$$\int_{\Omega_j} a(\nabla(u+\eta))\cdot\nabla\eta\,dx\ge -c\varepsilon$$

for every $\varepsilon>0$ so that, by passing to the limit for $\varepsilon\to 0^+$,

$$\int_{\Omega_j} a(\nabla(u+\eta))\cdot\nabla\eta\,dx\ge 0$$

thanks to the fact that ∇u^K converges $*$-weakly to ∇u in Ω_j. In the previous inequality we can replace Ω_j by Ω, the support of η being contained in Ω_j. We conclude by taking $t\eta$ instead of η, and letting $t\to 0$.

ii) Assume now that (17) holds. Since ϕ is Lipschitz of rank less than Q then $\phi\in\mathcal{K}^K_\phi$ for every $K>Q$ and thus

$$\int_\Omega a(\nabla u^K)\cdot(\nabla\phi-\nabla u^K)\,dx\ge 0$$

whence

$$\alpha\int_\Omega|\nabla u^K|^p\,dx\le Q\beta\int_\Omega|\nabla u^K|^{p-1}\,dx+\text{const.}$$

so that Hölder's inequality yields

$$\int_\Omega|\nabla u^K|^p\,dx\le Q\beta\left(\int_\Omega|\nabla u^K|^p\,dx\right)^{1/p'}|\Omega|^{1/p}+\text{const.}$$

implying that ∇u^K is a bounded sequence in $L^p(\Omega)$. Thus, modulo a subsequence, u^K converges weakly to u in $W^{1,p}(\Omega)$; in particular $u\in W^{1,p}(\Omega)$ and therefore $|a(\nabla u)|\le\beta|\nabla u|^{p-1}+r\in L^{p'}(\Omega)$. □

Remark 6.6. Of course one could consider, instead of the *Lower* (BSC), the unilateral (BSC) from above, i.e. *Upper* (BSC) defined in [4]. With the obvious changes one obtains the same conclusion of Theorem 6.3 by replacing the (Generalized) Lower (BSC) with the (Generalized) Upper (BSC).

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