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Best Approximation Problems in Compactly Uniformly Rotund Spaces*

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We investigate under what geometric conditions the best approximation problem to a nonempty closed subset of a real Banach space is generalized well-posed, or, more generally, the problem either has no solution or is generalized well-posed, for the majority of the points in the space. "Majority" is understood as a set whose complement in the space is σ-porous or σ-cone supported. Analogously to the case when uniqueness of the best approximation is considered, it turns out that certain local uniform, or uniform, properties of the norm of the underlying space have to be required.

Keywords: Best approximation, metric projection, well-posedness, approximative compactness, Baire category, porous sets, cone supported sets, compact uniform rotundity

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1. Introduction

Let X be a real Banach space and A be a nonempty and closed subset of X. Starting with the paper of Stečkin [16] interest has been paid to the study of the structure of the set of points $x \in X$ for which the best approximation problem for x to A has no more than one solution. Stečkin proved that in the case of locally uniformly rotund Banach spaces (for precise definitions see next sections) this set is big from the Baire category point of view in the sense that its complement in X is of the first Baire category in X. Subsequently, this result has been confirmed for other classes of spaces with rotund norms by Zajíček [19], Lau [12], Konyagin [9], Zhivkov [22, 23], Fabian and Preiss [4] and generalizations can be found in [2]. It is still

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an open question whether the original Stečkin result is true in any rotund Banach space. Stečkin also proved in [16] that in the case of uniformly rotund spaces X (see also the paper of Edelstein [3]) the set of those points $x \in X$ for which the best approximation to an arbitrary nonempty closed set A of X is unique has a complement of the first Baire category in the space X.

Meanwhile, investigation from other points of view has been added to the above problem. One of them was to study the set of points for which the best approximation problem is with unique (or compact set of) solution(s) and, moreover, is well-posed. Results in this direction from Baire category point of view as above have been obtained by Lau [12] in reflexive Banach spaces with special norm (see also Konyagin [9]) and by De Blasi, Myjak and Papini [1] in uniformly rotund spaces.

Another direction of research related to the above problem has been to sharpen the notion of smallness of the set of "bad" points, by replacing the notion of the first Baire category by stronger notions of smallness of a set. A candidate for such a sharpening turned out to be the concept of *porosity*, a geometric notion introduced in metric spaces by Zajíček (see, e.g. the surveys [20, 21]): a set is porous in X if, roughly speaking, it has wholes in any ball around a point from the set in the shape of balls with proportional radii. Sigma porous sets are defined as countable unions of porous sets. Every such set is of the first Baire category. The interest to this notion lies, for example, in the following facts: in finite dimensions, it is a stronger property than a set to be both of Lebesgue measure zero and of the first Baire category; moreover in any Banach space there are sets which are of the first Baire category but not sigma porous.

Results with this new notion of smallness concerning the set of points with no more than one best approximation have been obtained by Zajíček [19] (in separable rotund spaces), De Blasi Myjak and Papini [1] (in uniformly rotund spaces) and in our recent paper [14] in the case of locally uniformly rotund spaces, where we also consider a further sharpening of the notion of smallness (see below the definition of cone supported set).

The aim of this article is to investigate what kind of geometric properties of the underlying space are needed if we want to have similar results, as the latter ones, in the case where only the existence of a best approximation is required. It turns out that, in order to have such results, we need to require properties which are generalizations of the (local) uniform rotundity of the space. These are the so-called compactly (locally) uniform rotund Banach spaces, where the classical rotundity property is replaced by (local) compact rotundity. These notions are introduced and studied in Section 2, where some of their basic properties are obtained as well as a comparison with other similar ones are presented. In Section 3 we prove a Stečkin type lens lemma in the spaces we consider. With these notions and auxiliary results in hand, we prove in our Section 4 (Theorem 4.1) that when the space X is compactly locally uniformly rotund, then the set of points x in X for which the best approximation problem to a nonempty closed set A of X is either empty or generalized well-posed has a complement which is sigma cone supported. This is a multivalued version of a previous result of the authors [14, Theorem 3.1], concerning the case of uniqueness of the best approximations. In the smaller class of compactly uniformly rotund spaces our main result from Section 5 (Theorem 5.1) asserts that the set of points x for which the best approximation problem to a nonempty closed set A is generalized well-posed has a complement which is sigma-porous. The latter is a generalization of a result of De Blasi, Myjak and Papini [1], proved in the setting of uniformly rotund spaces.

2. Some preliminary notions and results

In the sequel $(X, \|\cdot\|)$ will designate a real Banach space with topological dual X^*. The norm in X^* will be denoted again by $\|\cdot\|$ and the pairing between X^* and X by $\langle\cdot,\cdot\rangle$. The symbols B_X and S_X will stay for the closed unit ball and unit sphere in X respectively. More generally, $B[x,r]$ and $S(x,r)$ will be used for the closed ball centered at $x \in X$ and radius $r > 0$ and the corresponding sphere of this ball. As usual $B(x,r)$ is reserved for the open ball centered at x and with radius $r > 0$.

A Banach space X, $(\dim X \geq 2)$, is called *compactly locally uniformly rotund* (in brief CLUR) space if whenever $x \in S_X$ and $(x_n)_n \subset S_X$ are such that $\lim_n \|x+x_n\| = 2$ then $(x_n)_n$ has a convergent subsequence. The CLUR property of a Banach space has been employed in [17] and [13] for the study of properties of metric projections.

Let us remind that, in the particular case when each sequence $(x_n)_n$ as above converges to x, for any $x \in S_X$, the space X is called *locally uniformly rotund* (briefly LUR) space. The space X is called *rotund* (or, equivalently, *strictly convex*) if the unit sphere S_X does not contain line segments. It is easily seen that a space which is compactly locally uniformly rotund and strictly convex is, in fact, locally uniformly rotund. For further comparison of these concepts of rotundity, also with other similar ones, see Examples 2.8 and 2.9 below. Let us also mention that, in the above definitions, the term "rotund" is frequently substitute by the synonym term "convex".

Our next result is a characterization of the CLUR property used in the sequel. To this end, let us first recall that the *Kuratowski index of non-compactness* $\alpha(A)$ for a set $A \subset X$ is the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of sets with diameters less than ε. The index of non-compactness is non-decreasing, i.e. $\alpha(A_1) \leq \alpha(A_2)$ whenever $A_1 \subset A_2$. It can be easily verified that $\alpha(A + \varepsilon B_X) \leq \alpha(A) + 2\varepsilon$ for $A \subset X$, $\varepsilon > 0$. Moreover, $\alpha(A) = 0$ exactly when A is relatively compact. Whenever a sequence of nonempty closed nested sets $(A_n)_n$ in a Banach space X is given, so that $\alpha(A_n) \to 0$, then the intersection $\cap_n A_n$ is a nonempty compact set of X (the latter is the well-known generalized Cantor lemma).

Let now $x, y \in X$, $r := \|y - x\| > 0$, and $\delta \in [0, r]$. Define

$$\mathrm{Cap}[x, y, \delta] = \left\{ z \in S(x,r) \colon \left\| \frac{z+y}{2} - x \right\| \geq r - \delta \right\},$$

i.e., this is the (non empty) set of points z on the sphere $S(x,r)$ such that the mid-points of the segments $[z,y]$ are not deeper inside $B(x,r)$ than $r - \delta$. Certainly,

$\mathrm{Cap}[x, y, \delta]$ is a closed subset of $S(x, r)$. The following elementary properties of 'caps' are used in the sequel:

$$z + \mathrm{Cap}[x, y, \delta] = \mathrm{Cap}[z + x, z + y, \delta],$$
$$\lambda\, \mathrm{Cap}[x, y, \delta] = \mathrm{Cap}[\lambda x, \lambda y, \lambda\delta],$$
$$\mathrm{Cap}[x, y, \delta_1] \subset \mathrm{Cap}[x, y, \delta_2] \ \text{ whenever } \delta_1 \leq \delta_2.$$

Lemma 2.1. *Let X be a CLUR Banach space, x, y, r, and δ be as above. Then* $\lim_{\delta\downarrow 0} \alpha(\mathrm{Cap}[x, y, \delta]) = 0$.

Proof. Assume the contrary: There is $\varepsilon_0 > 0$ such that $\lim_{\delta\downarrow 0} \alpha(\mathrm{Cap}[x, y, \delta]) > \varepsilon_0$. Find inductively a sequence $(z_n)_n \subset S(x, r)$ such that

(i) $z_n \in \mathrm{Cap}[x, y, r/n]$ for every $n \in \mathbb{N}$;
(ii) $\|z_{n+1} - z_j\| > \varepsilon_0/2$, for every $n \in \mathbb{N}$ and every $j = 1, \ldots, n$.

Then, by (i) and the CLUR assumption, $(z_n)_n$ has a convergent subsequence, but this contradicts (ii). □

Remark 2.2. It can be seen (by using the generalized Cantor lemma) that the converse of the above lemma is also true, and thus it is characteristic for CLUR Banach spaces.

Remark 2.3. Obviously, in a CLUR Banach space X, $\mathrm{Cap}[x,y,0] = \cap_{0<\delta\leq r} \mathrm{Cap}[x,y,\delta]$ is non empty and compact for every x, $y \in X$, $x \neq y$, $r = \|x - y\|$.

The above characterization of CLUR Banach spaces suggests the following

Definition 2.4. The Banach space X is called *compactly uniformly rotund*, if for any x, y, r, and δ as above we have $\lim_{\delta\downarrow 0} \alpha(\mathrm{Cap}[x, y, \delta]) = 0$ uniformly on $y \in S(x, r)$.

It is clear that according to the properties of 'caps' mentioned above, it is enough to give the definition only for the unit sphere in the space X (in this case we omit the origin in the notation of the cap and write simply $\mathrm{Cap}[y, \delta]$ instead of $\mathrm{Cap}[\theta, y, \delta]$). Obviously any compactly uniformly rotund Banach space is also a CLUR space. It is clear also that any finite dimensional normed space is compactly uniformly rotund. It can be seen that any uniformly rotund space is compactly uniformly rotund as well: we recall that $(X, \|\cdot\|)$ is *uniformly rotund* if for any $\varepsilon \in (0, 2]$ there is some $\delta \in (0, 1)$ so that $x, y \in S_X$ with $\|x - y\| > \varepsilon$ implies $\|x + y\| > 2(1 - \delta)$.

In order to present several properties of the compactly (locally) uniformly rotund spaces, let us recall a piece of terminology coming from optimization: given a continuous function $h : X \to \mathbb{R}$ and a nonempty closed set $C \subset X$, such that h is bounded below (resp. above) on C, the problem to minimize (resp. maximize) h on C is called *generalized well-posed* if any minimizing (resp. maximizing) sequence for h in C has a convergent subsequence. In such a case, the set of minimizers (resp. maximizers) of h in C is nonempty and compact. A well-known result of Furi and Vignoli [6] asserts that the problem to minimize (maximize) h on C is generalized well-posed, if and only if, the measure of non compactness of the level sets $\{x \in C : h(x) \leq \inf_C h + \varepsilon\}$ (resp. $\{x \in C : h(x) \geq \sup_C h - \varepsilon\}$) converges to zero as $\varepsilon \downarrow 0$.

The following property is proved as its version when the norm is LUR. For the sake of completeness, we sketch its proof.

Proposition 2.5. *Let X be a CLUR Banach space. Then for any functional $x^* \in S_{X^*}$ which attains its norm on B_X, the problem to maximize x^* on B_X is generalized well-posed.*

Proof. Let $x^* \in S_{X^*}$ and let $x \in S_X$ be such that $\langle x^*, x\rangle = 1 = \|x^*\|$. Let $(x_n)_n \subset B_X$ be a maximizing sequence for x^* on B_X, that is $\langle x^*, x_n\rangle \to 1 = \langle x^*, x\rangle$. We may think that $x_n \neq 0$ for all n. Then the sequence $z_n = x_n/\|x_n\|$, $n = 1, 2 \ldots$ is a well defined sequence on the sphere S_X and verifies $\langle x^*, z_n\rangle \leq 1$ which entails $\liminf_n \|x_n\| \geq 1$. Since, on the other hand $\limsup_n \|x_n\| \leq 1$ we conclude that $\lim_n \|x_n\| = 1$ and thus $(z_n)_n$ is also a maximizing sequence for x^* on B_X belonging to the unit sphere. We have

$$2 \geq \|x + z_n\| \geq \langle x^*, x + z_n\rangle, \quad n = 1, 2, \ldots,$$

which, because $(z_n)_n$ is maximizing for x^* on B_X, shows that $\|x + z_n\| \to 2$. Since X is a CLUR space the latter entails that $(z_n)_n$, and therefore, also $(x_n)_n$, has a convergent subsequence. □

A Banach space which has the property from the above proposition, that is, every $x^* \in S_{X^*}$ which attains its norm is generalized well-posed on B_X, is called sometimes *nearly strongly convex space* – see e.g. [7, 18]. It is a routine matter to check that every nearly strongly convex Banach space satisfies also the *Kadec-Klee (also called (H)) property*: every sequence $(x_n)_n \subset S_X$ which weakly converges to $x \in S_X$, converges to x also for the norm. Thus, according to the above proposition, in particular, every CLUR Banach space is nearly strongly convex and satisfies the Kadec-Klee property as well.

Our list of preliminary results will be completed by showing that, as in the classical case of uniform rotundity, the compactly uniformly rotund Banach spaces are reflexive Banach spaces. We also compare the latter notion with other similar ones. In order to do this, we recall first some notions given by Huff in [8]:

A Banach space X is *nearly uniformly convex* (in brief, NUC) whenever, for every $\varepsilon > 0$ there is $\delta \in (0, 1)$ such that for every sequence $(x_n)_n \subset B_X$ with $\text{sep}(x_n) \geq \varepsilon$ it follows $\text{co}(x_n) \cap (1 - \delta)B_X \neq \emptyset$. Here, $\text{sep}(x_n)$ is defined as

$$\text{sep}(x_n) := \inf\{\|x_m - x_l\| : m \neq l\},$$

and $\text{co}(x_n)$ stands for the convex hull of the elements of the sequence $(x_n)_n$.

A Banach space X is called *uniformly Kadec-Klee* (in brief, UKK) whenever, for every $\varepsilon > 0$ there is $\delta \in (0, 1)$ such that for every sequence $(x_n)_n \subset B_X$ with $x_n \to x$ weakly and $\text{sep}(x_n) \geq \varepsilon$ it follows $x \in (1 - \delta)B_X$.

A theorem of Huff from [8] states that a Banach space X is NUC if and only if it is UKK and reflexive. We will show that compactly uniformly rotund Banach spaces are a subclass of the NUC spaces:

Proposition 2.6. *Any compactly uniformly rotund Banach space X is nearly uniformly convex.*

Proof. First, let us mention that, without loss of generality, the test whether a space X is NUC can be done only with sequences $(x_n)_n$ which lie on the unit sphere S_X of the space. Having this remark, let $\varepsilon > 0$ be arbitrary. According to Definition 2.4, there is $\delta > 0$ so that $\alpha(\mathrm{Cap}[x, \delta]) < \varepsilon/2$ for any $x \in S_X$. Let $(x_n)_n \subset S_X$ be a sequence such that $\mathrm{sep}(x_n) \geq \varepsilon$. We will prove that $\mathrm{co}(x_n) \cap (1-\delta)B_X \neq \emptyset$ which will show that X is nearly uniformly convex. Assuming, this is not true, then for any $j \in \mathbb{N}$ we have $(x_n)_n \subset \mathrm{Cap}[x_j, \delta]$. Thus from one side $\alpha(\mathrm{Cap}[x_j, \delta]) < \varepsilon/2$, but on the other side the measure of noncompactness of an ε-separated sequence is not less than $\varepsilon/2$ and then $\varepsilon/2 \leq \alpha((x_n)_n) \leq \alpha(\mathrm{Cap}[x_j, \delta]) < \varepsilon/2$ which is a contradiction. $\square$

An immediate corollary from the previous proposition and the theorem of Huff [8] is:

Corollary 2.7. *Any compactly uniformly rotund Banach space X is reflexive and uniformly Kadec-Klee.*

At the end of this section we give two examples which additionally clarify the place of the class of compactly uniform rotund spaces. First of all, let us mention that a compactly (locally) rotund space needs not be rotund: for instance, it is enough to consider the sup-norm in any finite dimensional space $\mathbb{R}^n$. More generally, one can have the same phenomena in infinite dimensional Hilbert spaces as the following example shows. Below, $\|\cdot\|_2$ denotes the usual Hilbert norm in the sequence space l_2 and B_{l_2} the unit ball with respect to this norm.

Example 2.8. There is an equivalent renorming of the Hilbert space $(l_2, \|\cdot\|_2)$ which is compactly uniformly rotund but not strictly convex. Indeed, consider the equivalent norm $\|\cdot\|$ in l_2 whose unit ball is $B = C \cap B_{l_2}$, where $C = \{x = (x_n)_{n\geq 1} \in l_2 \colon \sum_{n=2}^{\infty} x_n^2 \leq 1/4\}$. That is,

$$\|x\| = \max\left\{ 2\left(\sum_{n=2}^{\infty} x_n^2\right)^{\frac{1}{2}}, \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} \right\}, \quad x = (x_n)_{n\geq 1} \in l_2.$$

This norm is not strictly convex: for example, consider $x = (1/2, 1/2, \ldots)$ and $y = (-1/2, 1/2, \ldots)$ for which we have $\|x\| = \|y\| = \|(x+y)/2\| = 1$. Further, using the definition, it can be seen that the space $X = (l_2, \|\cdot\|)$ is a CLUR space. Let us mention that (this is true in any CLUR space) the convergence of $\alpha(\mathrm{Cap}[x, \delta])$ as $\delta \to 0$ is uniform on the compact subsets of the unit sphere S_X of the space X. Finally, to pass from the latter to the uniform convergence of the caps on the sphere, one has to use two things: to consider an arbitrary two-dimensional section S of S_X through the points $e^1 = (1, 0, \ldots)$ and $-e^1$; and second, to check that

$\alpha(\mathrm{Cap}[x,\delta]) = \alpha(\mathrm{Cap}[y,\delta])$, whenever $x = (x_n)_n \subset S$, $y = (y_n)_n \subset S_X$ are so that $|x_1| = |y_1|$.

The next example shows that a nearly uniformly convex norm need not be CLUR. In particular, it follows that the class of compactly uniform rotund Banach spaces is a proper subclass of the nearly uniform convex spaces.

Example 2.9. There is a nearly uniformly convex renorming of the Hilbert spaces $(l_2, \|\cdot\|_2)$ which is not CLUR: For $x = (x_n)_{n\geq 1} \in l_2$ define $\|x\| := |x_1| + (\sum_{n=2}^{\infty} x_n^2)^{\frac{1}{2}}$. This is an equivalent norm which is known to be a non CLUR norm. To see the latter, take as a reference point $x = e^1$ and consider the sequence $(e^k)_{k\geq 2}$, where e^k are the usual unit vectors in l_2. It is seen that $\|e^1 + e^k\| = 2$ for all $k \geq 2$, while the sequence $(e^k)_{k\geq 2}$ has no convergent subsequence.

On the other hand, the space $X = (l_2, \|\cdot\|)$ is nearly uniformly convex. This can be derived, for example, from some more general results due to Huff [8]. A sketch of a possible direct verification is the following: Let $\varepsilon > 0$ and consider a sequence $(x^k)_k \subset B_X$ with $x^k = (x_n^k)_{n\geq 1}$, $k \geq 1$, and such that $\mathrm{sep}(x^k) \geq \varepsilon$. It is seen that the set $\{x \in X : |x_1| > 1 - \varepsilon/3\}$ contains no more than two elements of the sequence $(x^k)_k$. Thus, without loss of generality after passing to a subsequence, we may suppose that $|x_1^k| \leq 1 - \varepsilon/3$ for any $k \geq 1$, and $\lim_k x_1^k = \bar{x}_1$. There is no loss of generality to assume also that $(\sum_{n\geq 2}(x_n^k - x_n^l)^2)^{1/2} \geq \varepsilon/3$ for $k \neq l$, since $(x_1^k)_k$ is convergent. In this case, the sequence $(\bar{x}^k)_k$ of the projections $\bar{x}^k = (\bar{x}_1, x_2^k, \ldots)$ of x^k, $k = 1, 2\ldots$, is $(\varepsilon/3)$-separated in the hyperplane κ orthogonal to e^1 and passing through the point $\bar{x}_1 e^1$. Also, $(\bar{x}^k)_k$ is contained in an l_2-ball of radius not less than $\varepsilon/3$ in κ. Finally, one has to make use of this and the nearly uniform convexity of the Hilbert norm $\|\cdot\|_2$ to conclude that $\|\cdot\|$ is NUC.

3. Best approximation problems and Stečkin type lens lemma

In the beginning of this section, we recall some notions related to best approximation problems. Let be given a nonempty closed set $A \subset X$ of a real Banach space X. The standard notation for the metric projection and the distance function generated by A is as follows

$$P_A(x) = \{y \in A \colon \|x - y\| = d(x, A)\}, \qquad d(x, A) := \inf\{\|x - z\| \colon z \in A\}, \quad x \in X.$$

The metric projection $P_A(x)$ contains all solutions to the best approximation problem generated by x and A (the latter problem will be denoted sometimes by the couple (x, A)).

Given a point $x \in X$, the set A is called *approximatively compact for* $x \in X$ whenever every minimizing sequence $(y_n)_n \subset A$ for the best approximation problem (x, A), i.e. $\lim_n \|x - y_n\| = d(x, A)$, has a convergent subsequence. Equivalently, A is approximatively compact for $x \in X$ if the problem to minimize the function $\|x - \cdot\|$ on A is generalized well-posed. The set A is called *approximatively compact* whenever it is so for every $x \in X$. We say also (with some abuse on terminology) that P_A is *approximatively compact for* x whenever A is approximatively compact for x. An approximatively compact set is obviously closed. Denote

$$K_A = \{x \in X \colon A \text{ is approximatively compact for } x\}.$$

Certainly, $P_A(x)$ is nonempty and compact, whenever $x \in K_A$. Finally, a set A is called *proximinal* if for any $x \in X$ there is at least one best approximation for x in A. As it is shown in [7] in Banach spaces which are nearly strongly convex proximinal sets turn out also to be approximatively compact (in fact the latter is a characterization of the property of being nearly strongly convex).

For $A \subset X$, $x \in X$ and $\varepsilon > 0$, the ε-level set in A of the distance function is

$$\operatorname{Lev}(A, x, \varepsilon) = \{z \in A \colon \|x - z\| \le d(x, A) + \varepsilon\}.$$

The following fact is a consequence of the theorem of Furi and Vignoli [6] mentioned in the previous section:

Fact 3.1. *Let X be a Banach space and $A \subset X$ be a nonempty closed set. Then A is approximatively compact for $x \in X$ if and only if,* $\lim_{\varepsilon \downarrow 0} \alpha(\operatorname{Lev}(A, x, \varepsilon)) = 0$.

Given $x \in X$, $r > 0$, $y \in B(x, r/2)$ with $y \ne x$ and $\sigma \in (0, 2\|y - x\|)$ consider the *lens* determined by x, r, y and σ

$$\operatorname{Lens}(x, y, r, \sigma) = B[y, r - \|y - x\| + \sigma] \backslash B(x, r).$$

The following is easy to verify (below $]x, z[$ means the open segment generated by $x, z \in X$).

Fact 3.2. *If $x \notin A$, $z \in P_A(x)$, $y \in]x, z[\cap B(x, d(x, A)/2)$, then for $\sigma \in (0, 2\|y - x\|)$*

$$\operatorname{Lev}(A, v, \sigma/3) \subset \operatorname{Lens}(x, y, d(x, A), \sigma), \quad \textit{whenever } v \in B(y, \sigma/3).$$

Stečkin [16, Lemma 2] proved that in a LUR space the diameters of the lenses above go to zero as σ goes to zero. De Blasi, Myjak and Papini [1, Lemma 2.1] proved an explicit estimation of the diameters of the lenses in the case of uniformly convex spaces (see also, the LUR case in [14, Lemma 2.1]). Here we give the analogous estimate in the terms of the non-compactness indexes.

Lemma 3.3. *Let X be an arbitrary Banach space,* $\dim X \ge 2$, *and let $x \in X$, $r > 0$, $y \in B(x, r/2)$ with $y \ne x$ and $\sigma \in (0, 2\|y - x\|)$. Then*

(*i*) $\operatorname{Lens}(x, y, r, \sigma) \subset y + (r - \|y - x\|) \operatorname{Cap}\left[\frac{y-x}{\|y-x\|}, \frac{\sigma}{2\|y-x\|}\right] + \sigma B$

(*ii*) $\alpha(\operatorname{Lens}(x, y, r, \sigma)) \le (r - \|y - x\|)\alpha\left(\operatorname{Cap}\left[\frac{y-x}{\|y-x\|}, \frac{\sigma}{2\|y-x\|}\right]\right) + 2\sigma.$

Proof. Take $z \in \operatorname{Lens}(x, y, r, \sigma)$ and put for brevity $s := \|y - x\|$. Further, put

$$\bar{y} = y + (r - s)\frac{y - x}{s},$$

$$\bar{z} = y + (r - s)\frac{z - y}{\|z - y\|}.$$

Obviously, $\|z-\bar{z}\| \le \sigma$. We have

$$\begin{aligned} z-x &= (z-\bar{z})+(\bar{z}-y)+(y-x) \\ &= (z-\bar{z})+\left(1-\frac{s}{r-s}\right)(\bar{z}-y)+\frac{s}{r-s}(\bar{z}-y+\bar{y}-y). \end{aligned}$$

Then

$$r \le \|z-x\| \le \sigma+r-2s+\frac{s}{r-s}\left\|\bar{z}-y+\bar{y}-y\right\|,$$

whence

$$1-\frac{\sigma}{2s} \le \frac{1}{2}\left\|\frac{\bar{z}-y}{r-s}+\frac{\bar{y}-y}{r-s}\right\|$$

which implies

$$\frac{\bar{z}-y}{r-s} \in \operatorname{Cap}\left[\frac{\bar{y}-y}{r-s}, \frac{\sigma}{2s}\right]$$

Therefore,

$$\bar{z} \in y+(r-s)\operatorname{Cap}\left[\frac{\bar{y}-y}{r-s}, \frac{\sigma}{2s}\right]$$

and (i) is proved. The assertion (ii) follows immediately from (i). $\square$

Remark 3.4. One can put another assertion in the previous lemma, viz.

(iii) $\operatorname{diam}(\operatorname{Lens}(x,y,r,\sigma)) \le (r-\|y-x\|)\operatorname{diam}\left(\operatorname{Cap}\left[\frac{y-x}{\|y-x\|}, \frac{\sigma}{2\|y-x\|}\right]\right)+2\sigma,$

corresponding to the statements of the analogous lemmas in [1] and [14].

4. Best approximations in CLUR spaces

In this (and the next) section we will study the structure of some sets related to best approximation problems. More precisely, given a nonempty closed subset A of a Banach space X, we will be interested in the structure of the complements of the following sets: the set Q_A of points $x \in X$ so that the metric projection $P_A(x)$ is no more than a singleton; the set K_A of points x at which the set A is approximatively compact (or, equivalently, the best approximation problem to A for x is generalized well-posed). Putting, $\Phi_A = \{x \in X \colon P_A(x) = \emptyset\}$ a further question will be to study the complement of the set $\Phi_A \cup K_A$. In all cases, the main idea is to prove that the complements of the above sets are small (in appropriate sense) in the space X.

As we mentioned in the introduction, Stečkin [16] proved that in a LUR Banach space X, the complement of Q_A is of the first Baire category in X (a refinement of this result is in [22], where the set $\Phi_A \cup K_A$ is considered). The conclusion in the Stečkin result has been confirmed in other settings e.g., by [19] (separable rotund spaces), Zhivkov [22, 23] (for example, for Asplund rotund spaces) and for another large class of spaces by Fabian and Preiss [4]. We will be interested in the sequel how we can substantially sharpen the notion of smallness of the above complements. For this we need to recall some notions of smallness of sets which have been introduced by Zajíček (for the definitions and the properties mentioned below the reader can consult the papers of Zajíček [19, 20, 21]):

Given an element $h \in S_X$ and $\lambda \in (0,1)$, denote by $C(h,\lambda)$ the cone $\cup_{s>0} sB(h,\lambda)$. A set A is called *cone supported at* $x \in A$ if there is $r_0 > 0$, $h \in S_X$ and $\lambda \in (0,1)$ so that $A \cap B(x,r_0) \cap \{x + C(h,\lambda)\} = \emptyset$. A is *cone supported* if it is cone supported at any $x \in A$. The set A is *σ-cone supported* if A is a countable union of cone supported sets.

A less restrictive (but still rather strong) notion of smallness is the notion of porosity: a set $A \subset X$ is *porous at* $x \in A$ if there is some $\lambda(x) > 0$ and $r_0(x) > 0$ so that for any $r \in (0, r_0(x)]$ there is $y \in X$ with the property $B(y, \lambda(x)r) \subset B(x,r) \setminus A$. The set A is called *porous* in X if it is porous at any point of $x \in A$ and it is said to be *σ-porous* in X if A is a countable union of porous sets in X.

It is straightforward to see that every (σ-) cone supported set is (σ-) porous (the converse is not true). On the other hand, every porous set is nowhere dense in X. Thus every σ-cone supported set is both σ-porous and a set of the first Baire category. In finite dimensions the class of σ-porous sets is strictly smaller than the class of sets which are simultaneously of the first Baire category and of Lebesgue measure zero. Another fact that distinguish porosity from first Baire category is that in any Banach space there is a set which is of the first Baire category but not σ-porous. In separable Banach spaces the class of σ-cone supported sets coincides with the class of sets which can be covered by countable many Lipschitz surfaces of codimension 1 ([19, 21]): a set $A \subset X$ is called *Lipschitz surface of codimension 1* (or *Lipschitz hypersurface*) if there are a hyperspace H of X, an element $v \in X \setminus H$ and a Lipschitz function $\varphi : H \to \mathbb{R}$ so that $A = \{x + \varphi(x)v : x \in H\}$. In conclusion, both σ-porosity, and even more, the notion of σ-cone supported set, are substantial strengthenings of the notion of smallness of a set in Banach spaces.

The main result to be proved in this section is that for a nonempty closed subset A of a compactly locally uniformly rotund Banach space X, the set of points $x \in X$ at which the metric projection P_A is empty or the best approximation problem for x to A is generalized well-posed, complements a σ-cone supported set in X. This enhances a similar theorem of Konyagin [9] in which the Baire category is involved.

Namely, putting $V_A = \Phi_A \cup K_A$ we have,

Theorem 4.1. *Let X be a CLUR Banach space with $\dim X \geq 2$ and A be a nonempty closed subset of X. Then $X \backslash V_A$ is σ-cone supported.*

Proof. Consider, for $n = 1, 2, \ldots$, the sets

$$K_n = \left\{ x \in X : \lim_{\varepsilon \downarrow 0} \alpha(\mathrm{Lev}(A, x, \varepsilon)) < \frac{1}{n} \right\}.$$

Due to Fact 3.1

$$K_A = \cap_{n=1}^{\infty} K_n.$$

Thus

$$X \backslash V_A = X \backslash (\Phi_A \cup (\cap_{n=1}^{\infty} K_n)) = X \backslash (\cap_{n=1}^{\infty} (\Phi_A \cup K_n)) = \cup_{n=1}^{\infty} X \backslash (\Phi_A \cup K_n).$$

The theorem will be proved if we show that for every $n = 1, 2, \ldots$, the set $X\backslash(\Phi \cup K_n)$ is cone supported.

Suppose $x \in X\backslash(\Phi \cup K_n)$, n is fixed. Since $x \notin \Phi_A$ there is $\bar{y} \in P_A(x)$. Certainly, $d(x, A) = \|\bar{y} - x\| > 0$ as $\lim_{\varepsilon \downarrow 0} \alpha(\mathrm{Lev}(A, x, \varepsilon)) > 0$. Put $h = (\bar{y} - x)/\|\bar{y} - x\|$ and choose $\lambda \in (0, 1)$ such that

$$\alpha\left(\mathrm{Cap}\left[h, \frac{3}{2}\lambda\right]\right) < \frac{1}{2nd(x, A)}.$$

Such a choice of λ is possible due to Lemma 2.1. Denote

$$\bar{s} = \min\left\{\frac{d(x, A)}{2}, \frac{1}{12\lambda n}\right\},$$

($\bar{s}$ depends on λ but both are fixed.)

Take arbitrary $y \in]x, \bar{y}[$ satisfying $s := \|y - x\| < \bar{s}$ and consider the ball $B(x + sh, \lambda s)$. For $z \in B(x + sh, \lambda s)$, by Fact 3.2, one has

$$\mathrm{Lev}(A, z, \lambda s) \subset \mathrm{Lens}(x, y, d(x, A), 3\lambda s).$$

It follows from Lemma 3.3 and the choices of λ and $\bar{s}$ that

$$\begin{aligned} \alpha(\mathrm{Lev}(A, z, \lambda s)) &\leq (d(x, A) - s)\alpha(\mathrm{Cap}[h, 3\lambda/2]) + 6\lambda s \\ &< d(x, A)\frac{1}{2nd(x, A)} + \frac{1}{2n} = \frac{1}{n}, \end{aligned}$$

i.e., $z \in K_n$ as $\lim_{\varepsilon \downarrow 0} \alpha(\mathrm{Lev}(A, z, \varepsilon)) \leq \alpha(\mathrm{Lev}(A, z, \lambda s)) < 1/n$. Therefore,

$$\cup_{s \in (0, \bar{s})} B(x + sh, \lambda s) \subset K_n \subset \Phi_A \cup K_n.$$

One easily verifies for the cone $C(h, \lambda) = \cup_{s > 0} sB(h, \lambda)$ that

$$[x + C(h, \lambda)] \cap B(x, \bar{s}(1 - \lambda)) \subset \cup_{s \in (0, \bar{s})} B(x + sh, \lambda s)$$

whence $X\backslash(\Phi_A \cup K_n)$ is cone supported at x. And thus, the latter set is cone supported. □

The following corollaries are immediate. Their counterparts (for locally uniformly rotund Banach spaces) when uniqueness of the best approximation is considered, can be found in [14] .

Corollary 4.2. *Let A be a proximinal subset of a CLUR Banach space X. Then the set K_A of all points for which the corresponding best approximation problem is approximatively compact has a σ-cone supported complement in X.*

Corollary 4.3. *In a separable CLUR Banach space, for every closed nonempty set A, the set $X\backslash V_A$ can be covered by countably many Lipschitz surfaces of codimension 1.*

5. Best approximations in compactly uniformly rotund Banach spaces

In this section we will investigate to what extent the main result from the previous section can be improved if we restrict ourselves to the class of compactly uniformly rotund Banach spaces. As it is the case when the classical rotundity is considered (cf. [1] and [14]) we will see that if we want the set K_A to have a small complement in X, then the smallness can be obtained with respect to the notion of σ-porosity, not with respect to the notion of σ-cone supported set.

Before giving the main result in this section, let us recall some results. Stečkin [16] was the first to prove that in a uniformly rotund Banach space X the set of points for which the best approximation exists and is unique contains a dense G_δ-subset of X (thus its complement is of the first Baire category in X). Later, Lau [12] proved that in a reflexive Banach space with norm which satisfies the Kadec-Klee property the set K_A of the points x at which A is approximatively compact contains a dense G_δ-subset of X. Konyagin [9] showed that, in fact, the reflexivity and Kadec-Klee property are also necessary, namely: if for any closed nonempty set $A \subset X$ the set of points $\mathrm{Dom}(P_A) = \{x \in X : P_A(x) \neq \emptyset\}$ for which the best approximation to A exists is dense in X, then X must be reflexive with Kadec-Klee norm. According to the facts that we saw in the first section, namely, that a compactly uniformly rotund Banach space is reflexive and satisfies the Kadec-Klee property, it follows that the result of Lau holds in this class of spaces.

We have the following strengthening of the Lau's theorem in the case of compactly uniformly rotund spaces. An analogous result, in which also uniqueness of the best approximation is considered, is due to De Blasi, Myjak and Papini [1, Theorem 2.2].

Theorem 5.1. *Let X be a compactly uniformly rotund Banach space and A be its nonempty closed subset. Then the set $X \setminus K_A$ of best approximation problems, which are not well-posed, is σ-porous.*

Proof. As we mentioned in the previous section, $K_A = \cap_n K_n$ where for every natural integer $n = 1, 2, \dots$

$$K_n := \left\{ x \in X : \lim_{\varepsilon \downarrow 0} \alpha(\mathrm{Lev}(A, x, \varepsilon)) < \frac{1}{n} \right\},$$

and thus $X \setminus K_A = \cup_n (X \setminus K_n)$. Therefore, it is enough to show that each $X \setminus K_n$ is porous.

To this end, fix any integer $n \geq 1$ and $x \in X \setminus K_n$. Let $\lambda_n \in (0, 1/(8n)]$ be such that

$$\alpha\left(\mathrm{Cap}[h, \lambda_n]\right) < \frac{1}{8nd(x, A)} \quad \text{for any } h \in S_X.$$

Such a choice of λ_n is possible because of Definition 2.4 and the obvious fact that $d(x, A) > 0$. Put further $r_n = 1/(6n)$. With the so chosen λ_n and r_n we will check the porosity of $X \setminus K_n$ at x.

Let $r \in (0, r_n]$. Since $x \notin K_n$ it is easily seen that $d(x, A) > 1/(3n)$. On the other hand, according to the above cited result of Lau, $\mathrm{Dom}(P_A) = \{x \in X : P_A(x) \neq \emptyset\}$

is dense in X. Therefore, there exists $x' \in \mathrm{Dom}(P_A)$ so that $\|x' - x\| < r/4$ and $1/(3n) < d(x', A) < 2d(x, A)$. Fix some $z \in P_A(x')$ and let us consider the point

$$y := x' + \frac{1}{2} r \frac{z - x'}{\|z - x'\|}.$$

We will show that $B(y, \lambda_n r) \subset B(x, r) \cap K_n$ and this will complete the proof.

The inclusion $B(y, \lambda_n r) \subset B(x, r)$ is clear because of the choice of y and since $\lambda_n r < r/4$. Further, take an arbitrary $y' \in B(y, \lambda_n r)$ and observe that $\|y - x'\| = r/2 < r_n \le 1/(6n) < (1/2)d(x', A)$. We have also $\lambda_n r < r = 2\|y - x'\|$. Hence, by Fact 3.2 we have

$$\mathrm{Lev}\left(A, y', \frac{1}{3}\lambda_n r\right) \subset \mathrm{Lens}(x', y, d(x', A), \lambda_n r). \tag{1}$$

But on the other hand, by Lemma 3.3 and the choice of λ_n and x' we obtain that

$$\begin{aligned} \alpha(\mathrm{Lens}(x', y, d(x', A), \lambda_n r) &\le 2\lambda_n r + 2d(x', A)\alpha\left(\mathrm{Cap}\left[\frac{y - x'}{\|y - x'\|}, \frac{\lambda_n r}{2\|y - x'\|}\right]\right) \\ &< 2\frac{1}{8n} + 4d(x, A)\alpha\left(\mathrm{Cap}\left[\frac{y - x'}{\|y - x'\|}, \lambda_n\right]\right) \\ &\le \frac{1}{4n} + \frac{1}{2n} = \frac{3}{4n}. \end{aligned}$$

This together with (1) implies that $\alpha(\mathrm{Lev}(A, y', (1/3)\lambda_n r)) \le 3/(4n)$ and consequently $y' \in K_n$. This completes the proof. □

Let us mention once again, that a counter example from [14, Example 4.2] shows that the conclusion of the above theorem cannot be strengthened to obtain that the complement of K_A in X is σ-cone supported in X.

Note Added in Proof. When the article has been submitted for publication we were notified by Denka Kutzarova [11] that the compact uniform rotund property that we consider in our article seems equivalent to the so-called β-property introduced by Rolewicz [15] (see also [10]). The proof and further discussions will appear elsewhere.

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