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Differential Variational Inequalities with Locally Prox-Regular Sets

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The paper studies local existence and uniqueness of absolutely continuous solution of differential variational inequality involving a locally prox-regular set of an infinite dimensional Hilbert space. The study requires a quantified concept of local prox-regularity termed as (r, α)-prox-regularity.

Keywords: Differential inclusions, differential variational inequalities, proximal normal cone, locally prox-regular set, (r, α)-prox-regular set, hypomonotonicity, sweeping process, perturbation

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1. Introduction

Sweeping processes have been introduced and thoroughly studied by J. J. Moreau in a series of seminal papers [26, 27, 28, 29] for a moving closed convex set $C(t)$ included in an infinite dimensional Hilbert space H. The point $x(t)$ which is swept by the process at time t is the state at time t of the solution of the differential inclusion given by

$$\begin{cases} -\dot{x}(t) \in N(C(t); x(t)) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0 \in C(0), \end{cases} \tag{P}$$

where $N(C(t); \cdot)$ denotes the usual normal cone of the closed convex set $C(t)$. According to the definition of the normal cone of convex sets, the above differential inclusion with the constraint $x(t) \in C(t)$ can be stated for almost all $t \in [0, T]$ in the variational form

$$\begin{cases} \langle -\dot{x}(t), y - x(t) \rangle \leq 0 \quad \text{for all } y \in C(t) \\ x(0) = x_0 \in C(0). \end{cases} \tag{1}$$

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Differential inclusion (P) modelizes several fundamental mechanical problems (see [29, 24]). Moreau [29] studied the problem in the case when the convex sets $C(t)$ move in an absolutely continuous way as well as in the more general case when they move with a bounded variation. In [4], C. Castaing dealt with sweeping processes associated with sets $C(t) = S + v(t)$, where S is a fixed nonconvex closed set, and v is a mapping with finite variation. Then, M. Valadier extended the study of the absolutely continuous case to a more general situation including sets in the form $C(t) = \mathbb{R}^n \setminus \operatorname{int}(K(t))$, where $K(t)$ are closed and convex sets. He obtained in [37] existence of solution for (P) in the finite dimensional setting. Afterwards, H. Benabdellah [1] and G. Colombo and V. V. Goncharov [10] independently proved the existence of a solution of (P) for general nonconvex sets $C(t)$ in $\mathbb{R}^n$ moving in a Lipschitz continuous way. Extensions of existence and uniqueness of solution of (P) when $C(t)$ is a uniformly prox-regular set of the Hilbert space H moving in a Lipschitz continuous way and in an absolutely continuous way have been obtained by G. Colombo and V. V. Goncharov [10] and M. Bounkhel and L. Thibault [3] respectively (see also the paper [36] by L. Thibault for a regularization of the process in such a context). For the case where the uniformly prox-regular sets $C(t)$ move with a bounded variation we refer the reader to the paper [14] by J. F. Edmond and L. Thibault.

The study of differential inclusions which are pertubations of sweeping processes began in some sense with C. Henry's paper [16]. In this paper, Henry introduced for the study of planning procedures in mathematical economy the differential inclusion

$$\begin{cases} -\dot{x}(t) \in P_{T_C(x(t))}(F(x(t))) \ \text{ a.e. } t \in [0,T] \\ x(0) = x_0 \in C, \end{cases} \tag{2}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous set-valued mapping with nonempty compact convex values, C is a fixed closed convex set, $T_C(\cdot)$ is the tangent cone to C, and $P_{T_C(x(t))}$ denotes the metric projection mapping onto the closed convex set $T_C(x(t))$. This differential inclusion has been also considered by B. Cornet [12] with a Clarke tangentially regular set C of $\mathbb{R}^n$, reducing the problem as in [16] to the existence of a solution of the differential inclusion

$$\begin{cases} -\dot{x}(t) \in N(C; x(t)) + F(x(t)) \ \text{ a.e. } t \in [0,T] \\ x(0) = x_0 \in C. \end{cases} \tag{P'}$$

The case of moving sets $C(t)$ (which is the true case of perturbation of sweeping process) in place of the fixed set C has then been developed. We refer the reader, e.g., to C. Castaing, T. X. Duc Ha and M. Valadier [5] and C. Castaing and M. D. P. Montero Marques [6] for the study of pertubed sweeping processes in the form

$$\begin{cases} -\dot{x}(t) \in N(C(t), x(t)) + F(t, x(t)) \ \text{ a.e. } t \in [0,T] \\ x(0) = x_0, \end{cases}$$

in the cases where all the sets $C(t)$ are either convex or complements of open convex sets. For the case of general nonconvex closed sets of $\mathbb{R}^n$ we refer to L. Thibault

[35]. Several other papers dealt later in the Hilbert setting with perturbed sweeping processes under uniform prox-regularity assumptions, as the works of M. Bounkhel and L. Thibault [3], J. F. Edmond and L. Thibault in [14, 15].

Coming back to the case of a fixed set C, it is worth pointing out that differential inclusion (P') has been used by O. Serea to modelize optimal control problems of Mayer type with controlled dynamical systems subject to reflection on the boundary of a fixed closed uniform r-prox-regular subset C of $\mathbb{R}^n$. The problem is reduced to the minimization over all measurable controls $u(\cdot) : [0,T] \to U$ of $g\left(x(T;t,x,u(\cdot)\right)$, where $x\left(\cdot;t,x,u(\cdot)\right)$ is, for each control $u(\cdot)$ and each initial state $x \in C$ at initial time $t \in [0,T]$, the solution of the differential variational inclusion

$$\begin{cases} \dot{x}(\tau) \in -N(C;x(\tau)) + \phi(x(\tau),u(\tau)) & \text{a.e. } \tau \in [t,T] \\ x(t) = x. \end{cases} \tag{3}$$

Therein, C is a fixed uniformly prox-regular set of $\mathbb{R}^n$, $g : C \to \mathbb{R}$ is a continuous real-valued function, U is a compact subset of $\mathbb{R}^m$, and $\phi : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous mapping which is Lipschitz in the first variable, i.e., for some real $\gamma \geq 0$

$$\|\phi(x_1,u) - \phi(x_2,u)\| \leq \gamma\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in \mathbb{R}^n,\ u \in U.$$

By [35, 33] the differential inclusion in (3) has one and only one absolutely continuous solution on $[t,T]$ for each $t \in [0,T[$. Through this model, O. Serea [33] showed (under some additional assumptions) that the value function $V_g(\cdot,\cdot)$ with

$$V_g(t,x) := \sup_{u(\cdot)\in\mathcal{U}(t)} g\left(x(T;t,x,u(\cdot))\right),$$

is the unique viscosity solution (in a suitable sense) of the Hamilton-Jacobi differential inclusion

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + H(x, \frac{\partial V}{\partial x}(t,x)) - \langle \frac{\partial V}{\partial x}(t,x), N(C;x)\rangle \ni 0 & \text{for } (t,x) \in [0,T[\times C \\ V(T,x) = g(x) & \text{for all } x \in C. \end{cases}$$

In the latter, the Hamiltonian H is defined by

$$H(x,p) := \min_{u\in U}\langle p, \phi(x,u)\rangle \quad \text{for all } (x,p) \in C \times \mathbb{R}^n$$

and $\mathcal{U}(t)$ denotes the set of all measurable mappings (that is, controls) from $[t,T]$ into the compact set U.

In a series of papers B. Maury and J. Venel [20, 21, 22, 38] also made use of differential inclusion (P') to provide an efficient mathematical model for handling of contacts in crowd motion situations. The problem considers n persons who are identified to planar discs with the same radius $\rho > 0$. Following those papers, the center of the ith disk is a point in $\mathbb{R}^2$ denoted by x_i. Overlapping being forbidden between the n persons, the vector of positions $x := (x_1, \cdots, x_n) \in \mathbb{R}^{2n}$ is required to be in the set of feasible configurations

$$C := \{x \in \mathbb{R}^{2n} :\ D_{i,j}(x) := \|x_i - x_j\| - 2\rho \geq 0,\ \forall i \neq j\}.$$

Let $V_0(x_i)$ be the spontaneous velocity that each person x_i would wish in the absence of other people, so that $V(x) := (V_0(x_1), \cdots, V_0(x_n)) \in \mathbb{R}^{2n}$ is the spontaneous velocity of the n-tuple of persons. The vector field $V(\cdot)$ is assumed to be locally Lipschitz continuous on $\mathbb{R}^{2n}$. The non-overlapping again says that the set of feasible velocities of x is

$$G(x) := \{v \in \mathbb{R}^{2n} : \forall i < j \; D_{i,j}(x) = 0 \Rightarrow \langle \nabla D_{i,j}(x), v \rangle \geq 0\}.$$

The actual velocity field at time $t \in [0, T]$ is then the closest to $V(x(t))$, that is, $\dot{x}(t) = P_{G(x(t))}(V(x(t)))$, where $P_{G(x(t))}$ denotes as above the metric projection onto the closed convex set $G(x(t))$ (which depends on $x(t)$). Like for Henry's model (2) above (but with different arguments) the authors reduce the problem to the differential inclusion

$$\begin{cases} \dot{x}(t) \in -N(C; x(t)) + V(x(t)) \\ x(0) = x^0 \in C \end{cases}$$

which is in the form of (P'). The authors also showed that the closed nonconvex set of configurations C is uniformly r-prox-regular for some real $r > 0$ and hence the latter differential inclusion has, according to [14, 15], one and only one absolutely continuous solution on $[0, T]$. An efficient algorithm along with its convergence is also provided in the papers [20, 21, 22, 38].

Our aim in this article is also to deal with differential inclusion (P') but in the infinite dimensional setting and with a weaker assumption than uniform prox-regularity, namely a quantified viewpoint of local prox-regularity. Consider a fixed closed set C in the Hilbert space H which is (r, α)-prox-regular at a point $x_0 \in C$ (a quantified concept of prox-regularity of C at x_0, see the next section for the definition and relevance) and a mapping $f : [0, T] \times H \to H$ which is measurable with respect to t and such that $f(t, \cdot)$ satisfies the Lipschitz property

$$\|f(t, x_1) - f(t, x_2)\| \leq k(t)\|x_1 - x_2\|$$

for some nonnegative Lebesgue integrable function $k(\cdot)$ on $[0, T]$ and all x_1, x_2 near x_0. We study the existence and uniqueness of a solution of the differential inclusion

$$\begin{cases} -\dot{x}(t) + f(t, x(t)) \in N(C; x(t)) \quad \text{a.e. } t \\ x(0) = x_0. \end{cases}$$

Otherwise stated, we provide some interval $[0, \tau] \subset [0, T]$ over which the latter differential inclusion has a unique absolutely continuous solution. Taking (11) into account, we then proved the existence and uniqueness of solution $x(\cdot)$, with $x(t) \in C$ on $[0, \tau]$ of the differential variational inequality written for almost all $t \in [0, \tau]$ in the form

$$\begin{cases} \langle -\dot{x}(t), y - x(t) \rangle \leq \langle -f(t, x(t)), y - x(t) \rangle \\ \qquad + \frac{1}{2r}\| -\dot{x}(t) + f(t, x(t))\| \, \|y - x(t)\|^2 \quad \forall y \in C \\ x(0) = x_0. \end{cases} \tag{4}$$

The paper is organized as follows. The next section recalls several concepts of nonsmooth and variational analysis which are involved throughout the paper and it also presents the definition of our quantified viewpoint of local prox-regularity for a subset C (of the Hilbert space H) at a point $x_0 \in C$. This quantification is termed as the (r, α)-prox-regularity of the set C at x_0 for two positive constants r, α. This property is also characterized in some sense through a quantified hypomonotonicity of the truncated normal cone $N(C; \cdot) \cap \mathbb{B}$, where $\mathbb{B}$ denotes the closed unit ball of H centered at the origin. The third section establishes (in a quantified way) the local existence and uniqueness result of solution of the differential variational inequality stated in (4). The approach yielding to this result uses in a crucial way both positive constants r and α defining the quantified local prox-regularity of the set C. The case where $\alpha = +\infty$ ensures the existence and uniqueness of a global solution on the whole interval $[0, T]$ for (4).

2. Preliminaries and locally prox-regular sets

Throughout, unless otherwise stated, $I := [T_0, T]$ is an interval of $\mathbb{R}$ and H is a real Hilbert space whose inner product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$.

The closed unit ball of H will be denoted by $\mathbb{B}$ and $B[x, \eta]$ (resp. $B(x, \eta)$) is the closed (resp. open) ball of radius $\eta > 0$ centered at the point x of H. For any subset C of H, $\overline{\text{co}}\, C$ stands for the closed convex hull of C, and $\sigma(C, \cdot)$ represents the support function of C, that is, for all $v \in H$,

$$\sigma(C, v) := \sup_{x \in C} \langle v, x \rangle.$$

The real vector space $\mathcal{C}(I, H)$ of all continuous mappings from I into H will be endowed with the norm of uniform convergence $\| \cdot \|_\infty$, where

$$\|g(\cdot)\|_\infty = \sup_{t \in I} \|g(t)\| \quad \text{for all } g(\cdot) \in \mathcal{C}(I, H).$$

The Lebesgue σ-field of I is denoted by $\mathcal{L}(I)$ and λ denotes the Lebesgue measure.

For $p \in [1, +\infty]$, the real vector space $L^p(I, H)$ is the (quotient) space of all *Bochner λ-measurable mappings* $g(\cdot) : I \to H$ such that the real-valued function $\|g(\cdot)\|$ belongs to $L^p(I, \mathbb{R})$. So, $L^1(I, H)$ is the (quotient) space of all *Bochner λ-integrable* mappings.

Let C be a nonempty closed subset of H and $y \in H$. The distance of y to C, denoted by $d_C(y)$ or $d(y, C)$ is given by

$$d_C(y) := \inf\{ \|x - y\| : x \in C \}.$$

One defines the (possibly empty) set of nearest points of y in C by

$$\text{Proj}_C(y) := \{ x \in C : \ d_C(y) = \|y - x\| \}.$$

When $\text{Proj}_C(y)$ is a singleton, we will write $P_C(y)$ in place of $\text{Proj}_C(y)$ to emphasize this singleton property. If $x \in \text{Proj}_C(y)$, and $s \geq 0$, then the vector $s(y - x)$ is called (see, e.g., [8]) a *proximal normal* to C at x. Sometimes, it will be convenient to write $\mathrm{Proj}\,(C,x)$ in place of $\mathrm{Proj}_C(x)$. The set of all vectors of this form is a cone which is termed the *proximal normal cone* of C at x. It is denoted by $N^P(C;x)$ or $N^P_C(x)$, and $N^P(C;x)=\emptyset$ whenever $x\notin C$. Observing that, for $x\in C$, a nonzero vector $v\in N^P(C;x)$ if and only if that for some $\rho>0$ one has $x\in \mathrm{Proj}_C(x+\frac{\rho}{\|v\|}v)$, and translating this as $\rho^2\le \|x+\frac{\rho}{\|v\|}v-x'\|^2$ for all $x'\in C$, we obtain that the inclusion $v\in N^P(C;x)$ is equivalent to the existence of some real $\sigma\ge 0$ such that

$$\langle v, x'-x\rangle \le \sigma\|x'-x\|^2 \ \text{ for all } x'\in C. \tag{5}$$

For $x\in C$, inequality (5) can be also localized in the sense that it holds for some $\sigma>0$ (i.e., $v\in N^P(C;x)$) if and only if there exist some $\gamma\ge 0$ and $\eta>0$ such that

$$\langle v, x'-x\rangle \le \gamma\|x'-x\|^2 \ \text{ for all } x'\in C\cap B(x,\eta). \tag{6}$$

One also defines the *Mordukhovich limiting normal cone* and the *Clarke normal cone* respectively by

$$N^L(C;x) := \left\{ v\in H:\, \exists v_n \to^w v, v_n\in N^P(C;x_n), x_n \underset{C}{\to} x \right\}$$

and

$$N^C(C;x) := \overline{\mathrm{co}} N^L(C;x),$$

where $v_n \to^w v$ means that the sequence $(v_n)_n$ converges weakly to v and $x_n \underset{C}{\to} x$ means that $x_n\to x$ and $x_n\in C$ for all $n\in\mathbb{N}$. It clearly appears in the definition above that $N^L(C;x)$ is the Painlevé-Kuratowski weak sequential outer (or superior) limit of $N^P(C;x')$ as $x'\to x$, where for a set-valued mapping $M:U\rightrightarrows H$ from a topological space U into H the Painlevé-Kuratowski weak sequential outer limit of M at $x\in U$ is the set

$$^{\mathrm{seq}}\operatorname*{Lim\,sup}_{x'\to x} M(x') := \{\, v\in H:\exists v_n \to^w v,\ v_n\in M(x_n),\ x_n\to x\,\}.$$

Like for $N^P(C;x)$, sometimes one writes $N^L_C(x)$ and $N^C_C(x)$ instead of $N^L(C;x)$ and $N^C(C;x)$. It is worth pointing out that for x outside the closed set C one has $N^L(C;x)=N^C(C;x)=\emptyset$, hence (since $0\in N^P(C;x)$ for all $x\in C$)

$$\mathrm{Dom}\, N^P(C;\cdot) = \mathrm{Dom}\, N^L(C;\cdot) = \mathrm{Dom}\, N^C(C;\cdot) = C,$$

where for a set valued-mapping $M:U\rightrightarrows H$ we denote by $\mathrm{Dom}\,M$ its (effective) domain, that is, $\mathrm{Dom}\,M := \{x\in U:\, M(x)\neq\emptyset\}$.

The elements in the Mordukhovich limiting normal cone can also be obtained as weak limits of sequences of Fréchet normal vectors. A vector $v\in H$ is a *Fréchet normal* of C at $x\in C$ whenever for any real $\varepsilon>0$ there exists some real $\eta>0$ such that

$$\langle v, x'-x\rangle \le \varepsilon\|x'-x\| \ \text{ for all } x'\in C\cap B(x,\eta).$$

Denoting by $N^F(C;x)$ or $N^F_C(x)$ the *cone of all Fréchet normals* to C at $x \in C$ and putting $N^F(C;x) = \emptyset$ for $x \notin C$, it is also known that

$$N^L(C;x) = {}^{\text{seq}}\operatorname*{Lim\,sup}_{x' \to x} N^F(C;x') \quad \text{for all } x \in C.$$

Taking (5) into account, we always have

$$N^P(C;x) \subset N^F(C;x) \subset N^L(C;x) \subset N^C(C;x) \quad \text{for all } x \in C. \tag{7}$$

After those preliminaries concerning normal cones, we can now present a view of the local prox-regularity property for sets. For a large development of the concept of local prox regularity of sets, the reader is referred to [31]. In this paper, our analysis needs a quantification of this concept through the constants that it involves in its definition.

Definition 2.1. For positive real numbers r and α, the closed set C is said to be (r,α)-prox regular at a point $\bar{x} \in C$ provided that for any $x \in C \cap B(\bar{x},\alpha)$, any $v \in N^P_C(x)$ such that $\|v\| < r$, one has

$$x = P_C(x+v). \tag{8}$$

The set C is r-prox-regular (resp. prox-regular) at $\bar{x}$ when it is (r,α)-prox-regular at $\bar{x}$ for some real $\alpha > 0$ (resp. for some numbers $r > 0$ and $\alpha > 0$).

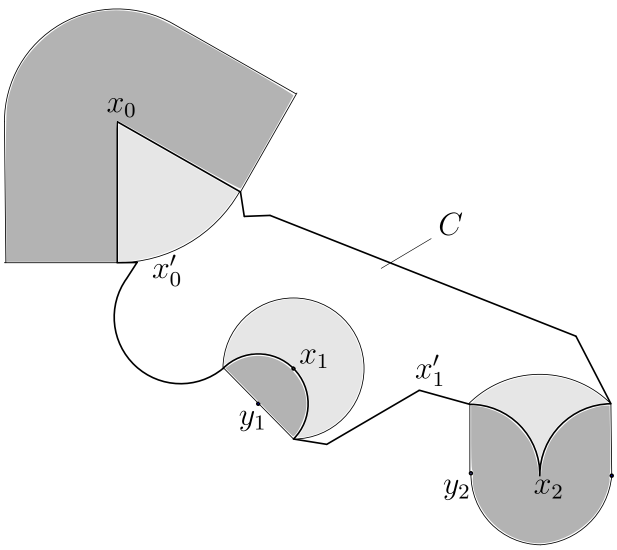

Figure 2.1: Example of an (r,α)-prox-regular set.

On Figure 2.1, the set C is (r_i,α_i)-prox-regular at x_i for $i = 0,1,2$. More precisely, it is (r_0,α_0)-prox-regular at x_0 with $r_0 = +\infty$, and (r_i,α_i)-prox-regular at x_i with $r_i = \|y_i - x_i\|$ for $i = 1,2$. The lightly shaded area around x_i is $C \cap B(x_i,\alpha_i)$ and the external shaded area around x_i describes all $x+v$ satisfying (8) with $v \neq 0$. Observe also that C is not prox-regular at x'_i for $i = 0,1$.

It is not difficult to see that the latter (r,α)-prox-regularity property of C at $\bar{x} \in C$ is equivalent to requiring that

$$x \in \mathrm{Proj}_C(x + rv) \quad \text{for all } x \in C \cap B(\bar{x},\alpha) \text{ and } v \in N^P(C;x) \cap \mathbb{B}. \tag{9}$$

We first observe that inclusion (9) means that for each positive real number $t < r$ and each $x \in C \cap B(\bar{x},\alpha)$

$$C \cap B[x + tv,\, t] = \{x\} \quad \text{for any } v \in N^P(C;x) \text{ with } \|v\| = 1, \tag{10}$$

that is, any unit normal vector in $N^P(C;x)$ can be realized (see, e.g., [9], [11] and [31]) by a t-ball for any positive real number $t < r$. Further, translating (like for (5)) the same inclusion $x \in \mathrm{Proj}_C(x+rv)$ as $\|rv\|^2 \le \|x+rv-x'\|^2$ for all $x' \in C$, it is easily seen that C is (r,α)-prox-regular at $\bar{x}$ if and only if for any $x \in C \cap B(\bar{x},\alpha)$ one has

$$\langle v, x' - x\rangle \le \frac{1}{2r}\|x' - x\|^2 \quad \text{for all } v \in N^P(C;x) \cap \mathbb{B} \text{ and } x' \in C. \tag{11}$$

Moreover we have the equalities

$$N^P(C;x) = N^F(C;x) = N^L(C;x) = N^C(C;x) \quad \text{for all } x \in C \cap B(\bar{x},\alpha),$$

whenever the set C is (r,α)-prox-regular at $\bar{x} \in C$. Indeed, fix $x \in C \cap B(\bar{x},\alpha)$ and $v \in N^L(C;x)$. There are sequences $(x_n)_n$ in C with $\|x_n - x\| \to 0$ and $(v_n)_n$ converging weakly to v with $v_n \in N^P(C;x_n)$ for all n. Choose $\beta > 0$ such that $\|v_n\| \le \beta$ for all n. For n large enough, say $n \ge N$, we have $x_n \in B(\bar{x},\alpha)$ hence $x_n \in C \cap B(\bar{x},\alpha)$. This and (11) yields $\langle v_n, x' - x_n\rangle \le \frac{\beta}{2r}\|x' - x_n\|^2$ for each $x' \in C$ which entails $\langle v, x' - x\rangle \le \frac{\beta}{2r}\|x' - x\|^2$. This translates that $v \in N^P(C;x)$ hence $N^L(C;x) \subset N^P(C;x)$. The convexity and closedness of $N^F(C;x)$ entails that $N^L(C;x)$ is closed and convex and hence it is also equal to $N^{Cl}(C;x)$ according to the equality $N^{Cl}(C;x) := \overline{\mathrm{co}} N^L(C;x)$.

So, for such an (r,α)-prox-regular set and for $x \in C \cap B(\bar{x},\alpha)$ we will sometimes write $N(C;x)$ in place of anyone of the four normal cones above.

In [31], the local prox-regularity is related to an open neighborhood O of $\bar{x}$. The above quantification of the concept with the positive numbers r and α will be crucial in several parts of the present paper.

The following proposition examines the quantified prox-regularity at points $x \in C \cap B(\bar{x},\alpha)$, and also the (r,α)-prox-regularity of translated sets of C.

Proposition 2.2. *Let C be (r,α)-prox-regular at $\bar{x}$ and $y \in B(\bar{x},\alpha) \cap C$. Then the following hold:*

(a) *C is $(r, \alpha - \|y - \bar{x}\|)$-prox-regular at y.*

(b) *For any $w \in H$ the set $C + w$ is (r,α)-prox-regular at $\bar{x} + w$.*

Proof. To prove (a), observe that for $y \in B(\bar{x},\alpha) \cap C$ we have

$$B(y, \alpha - \|y - \bar{x}\|) \cap C \subset B(\bar{x},\alpha) \cap C,$$

hence (9) still holds for all $x \in B(y, \alpha - \|y - \bar{x}\|) \cap C$ and all $v \in N^P(C; x) \cap \mathbb{B}$.

Now we study assertion (b). Set $D := C + w$. Let $y \in D \cap B(\bar{x} + w, \alpha)$, that is, $y - w \in C \cap B(\bar{x}, \alpha)$. Let $v \in N^P(D; y)$. Then there exists $s > 0$ such that $y + sv \in \mathrm{Proj}(y, D)$ or equivalently $y - w + sv \in \mathrm{Proj}(y - w, C)$. So, $v \in N^P(C; y - w)$. Applying (11) to $y - w \in C \cap B(\bar{x}, \alpha)$ and $v \in N^P(C; y - w)$ we get

$$\langle v, y' - y \rangle = \langle v, y' - w - (y - w) \rangle \leq \frac{1}{2r} \|y' - y\|^2 \text{ for all } y' \in D$$

hence the set D is (r, α)-prox-regular at $\bar{x} + w$. □

Recall now that a set-valued mapping $T : H \rightrightarrows H$ is *hypomonotone* on a subset $O \subset H$ if there exists $\sigma > 0$ such that $T + \sigma I$ is monotone on O, where I denotes the identity mapping from H into H. This corresponds to having

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\sigma \|x_1 - x_2\|^2 \text{ whenever } v_i \in T(x_i), x_i \in O,\ i = 1, 2.$$

Concerning the quantified concept of (r, α)-prox-regularity of sets of the Hilbert space H, we have the following properties related to the hypomonotonicity of the truncated normal cone.

Proposition 2.3. *Let C be a closed subset of H, and $\bar{x} \in C$. The following hold.*

(a) *If there exist positive real numbers r and α such that C is (r, α)-prox-regular at $\bar{x}$, then the set-valued mapping $N^P(C, \cdot) \cap \mathbb{B}$ is $\frac{1}{r}$-hypomonotone on $B(\bar{x}, \alpha)$.*

(b) *Suppose that there exist $0 < r < \alpha$ such that the set-valued mapping $N^P(C, \cdot) \cap \mathbb{B}$ is $\frac{1}{r}$-hypomonotone on $B(\bar{x}, \alpha)$. Then for the real number $\sigma := \frac{1}{2}(\alpha - r) > 0$ the set C is $(\frac{r}{2}, \sigma)$-prox-regular at $\bar{x}$.*

Proof. Assuming that C is (r, α)-prox-regular at $\bar{x}$, then for all $x \in C \cap B(\bar{x}, \alpha)$, all $v \in N^P(C; x) \cap \mathbb{B}$, one has by (11)

$$\langle v, x' - x \rangle \leq \frac{1}{2r} \|x' - x\|^2 \text{ for all } x' \in C.$$

Writing this inequality for $x_i \in C \cap B(\bar{x}, \alpha)$ and $v_i \in N^P(C; x_i) \cap \mathbb{B}$, $i = 1, 2$, one obtains

$$\langle v_1, x_2 - x_1 \rangle \leq \frac{1}{2r} \|x_2 - x_1\|^2$$

and

$$\langle v_2, x_1 - x_2 \rangle \leq \frac{1}{2r} \|x_1 - x_2\|^2,$$

which yields

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\frac{1}{r} \|x_1 - x_2\|^2$$

for all $x_i \in B(\bar{x}, \alpha) \cap C$ and $v_i \in N^P(C; x_i) \cap \mathbb{B}$, $i = 1, 2$, and this translates assertion (a).

Now we study assertion (b). We will need the following claim:

Claim. Let C be a closed subset of H, $\bar{x} \in C$, and $0 < \alpha' \leq \frac{\alpha''}{2}$. Then for all $u \in B(\bar{x}, \alpha')$

$$\begin{cases} d(u, C) = d(u, C \cap B[\bar{x}, \alpha'']) = d(u, C \cap B(\bar{x}, \alpha'')) \\ \mathrm{Proj}(u, C) = \mathrm{Proj}(u, C \cap B[\bar{x}, \alpha'']) = \mathrm{Proj}(u, C \cap B(\bar{x}, \alpha'')). \end{cases}$$

Now let $0 < r < \alpha$ such that for all $x_i \in C \cap B(\bar{x}, \alpha)$, all $v_i \in N^P(C; x_i) \cap \mathbb{B}$,

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\frac{1}{r} \|x_1 - x_2\|^2.$$

Fix $x \in C \cap B(\bar{x}, \alpha)$ and $v \in N^P(C; x) \cap \mathbb{B}$. Computing the latter inequality, for $x_1 = x, x_2 = x'$ and $v_1 = v, v_2 = 0$, gives

$$\langle v, x' - x \rangle \leq \frac{1}{r} \|x' - x\|^2 \quad \text{for all } x' \in C \cap B(\bar{x}, \alpha).$$

So, for all $x' \in C \cap B(\bar{x}, \alpha)$ we have

$$0 \leq \|x' - x\|^2 - 2 \left\langle \frac{r}{2} v, x' - x \right\rangle$$
$$\left\| \frac{r}{2} v \right\|^2 \leq \|x' - x\|^2 - 2 \left\langle \frac{r}{2} v, x' - x \right\rangle + \left\| \frac{r}{2} v \right\|^2,$$

thus

$$\left\| \frac{r}{2} v \right\|^2 \leq \left\| x' - x - \frac{r}{2} v \right\|^2, \quad \text{for all } x' \in C \cap B(\bar{x}, \alpha),$$

that is, $x \in \mathrm{Proj}(x + \frac{r}{2} v, C \cap B(\bar{x}, \alpha))$.

For the real number $\sigma := \frac{1}{2}(\alpha - r) > 0$ (recall that $r < \alpha$), we see that for all $x \in B(\bar{x}, \sigma)$

$$\left\| x + \frac{r}{2} v - \bar{x} \right\| \leq \|x - \bar{x}\| + \frac{r}{2} < \frac{\alpha}{2} - \frac{r}{2} + \frac{r}{2} = \frac{\alpha}{2}.$$

Applying the claim for $\alpha'' = \alpha$ and $\alpha' = \alpha/2$, it follows that for all $x \in C \cap B(\bar{x}, \sigma)$ and all $v \in N^P(C; x) \cap \mathbb{B}$

$$x \in \mathrm{Proj}\left(x + \frac{r}{2} v, C \cap B(\bar{x}, \alpha) \right) = \mathrm{Proj}\left(x + \frac{r}{2} v, C \right),$$

which allows us to conclude by (9) that C is $(\frac{r}{2}, \sigma)$-prox-regular at $\bar{x}$.

Proof of Claim. Let $u \in B(\bar{x}, \alpha')$. Obviously, the inequalities

$$d(u, C) \leq d(u, C \cap B[\bar{x}, \alpha'']) \leq d(u, C \cap B(\bar{x}, \alpha'')) \tag{12}$$

are always true. Now consider $x \in C$. If $x \in B(\bar{x}, \alpha'')$, we get

$$\|u - x\| \geq d(u, C \cap B(\bar{x}, \alpha'')).$$

If $x \notin B(\bar{x}, \alpha'')$, we have

$$\|u - x\| \geq \|x - \bar{x}\| - \|u - \bar{x}\| > \alpha'' - \alpha',$$

because $\|u - \bar{x}\| < \alpha'$. As $\alpha'' - \alpha' \geq \alpha'$, this entails

$$\|u - x\| > \alpha' > \|u - \bar{x}\| \geq d(u, C \cap B(\bar{x}, \alpha'')).$$

Thus, for all $u \in B(\bar{x}, \alpha')$ we have $d(u, C) \geq d(u, C \cap B(\bar{x}, \alpha''))$ hence according to (12)

$$d(u, C) = d(u, C \cap B[\bar{x}, \alpha'']) = d(u, C \cap B(\bar{x}, \alpha'')). \tag{13}$$

Observe that our last equality ensures that

$$\mathrm{Proj}(u, C) \supset \mathrm{Proj}(u, C \cap B[\bar{x}, \alpha'']) \supset \mathrm{Proj}(u, C \cap B(\bar{x}, \alpha'')).$$

Now let $y \in \mathrm{Proj}(u, C)$. Then we have the estimation

$$\|y - \bar{x}\| \leq \|y - u\| + \|u - \bar{x}\| \leq 2\|u - \bar{x}\| < 2\alpha' \leq \alpha'' \text{ hence } y \in C \cap B(\bar{x}, \alpha'').$$

This combined with (13) ensures that $y \in \mathrm{Proj}(u, C \cap B[\bar{x}, \alpha''])$. Consequently, we conclude that

$$\mathrm{Proj}(u, C) = \mathrm{Proj}(u, C \cap B[\bar{x}, \alpha'']) = \mathrm{Proj}(u, C \cap B(\bar{x}, \alpha'')),$$

as desired. □

Now let us recall the definitions of subdifferentials of functions.
Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $x \in \mathrm{dom}\, f$, that is, $f(x) < +\infty$. Each one of the above normal cones (see, e.g., [25, 32]) leads to a subdifferential through the normal cone to the epigraph $\mathrm{epi}\, f$ of f, where

$$\mathrm{epi}\, f := \{\, (x, \rho) \in H \times \mathbb{R} : \, f(x) \leq \rho \,\}.$$

So, the *proximal subdifferential*, the *Fréchet subdifferential*, the *Mordukhovich limiting subdifferential*, and the *Clarke subdifferential* are the (possibly empty) subsets of H given by

$$\partial_? f(x) = \{\, v \in H : (v, -1) \in N^? \,(\mathrm{epi}\, f; (x, f(x))) \,\},$$

where "?" stands for P, F, L, C respectively. By convention anyone of the above subdifferentials of f at a point $x \notin \mathrm{dom}\, f$ is empty. The proximal and Fréchet subdifferentials have amenable analytical descriptions. Indeed, for $x \in \mathrm{dom}\, f$ it is known through (6) that $v \in \partial_P f(x)$ if and only if there exist $\gamma \geq 0$ and $\eta > 0$ such that

$$\langle v, x' - x \rangle \leq f(x') - f(x) + \gamma \|x' - x\|^2 \;\text{ for all } x' \in B(x, \eta).$$

Analogously, $v \in \partial_F f(x)$ if and only if for any real $\varepsilon > 0$ there exists some $\eta > 0$ such that

$$\langle v, x' - x \rangle \leq f(x') - f(x) + \varepsilon \|x' - x\| \;\text{ for all } x' \in B(x, \eta).$$

Conversely, considering the indicator function δ_C of the set C given by $\delta_C(x') = 0$ if $x' \in C$ and $\delta_C(x') = +\infty$ otherwise, the four normal cones to C at $x \in C$ can be recovered from the concept of subdifferential by the following equalities

$$N^?(C;x) = \partial_? \delta_C(x),$$

where as above ? stands for P, F, L, C respectively. The proximal and Fréchet normal cones can be also recovered through the distance function to the set C with the equalities

$$N^P(C;x) \cap \mathbb{B} = \partial_P d_C(x) \quad \text{and} \quad N^F(C;x) \cap \mathbb{B} = \partial_F d_C(x) \ \text{ for any } x \in C, \tag{14}$$

(see, e.g., [9, 2] for the first equality and [17, 18, 25] for the second one). Note also that for the Mordukhovich and Clarke normal cones one has

$$N_L(C;x) = \mathbb{R}_+ \partial_L d_C(x) \quad \text{and} \quad N_C(C;x) = \mathrm{cl}\,(\mathbb{R}_+ \partial_C d_C(x))$$

(see [7] for the second equality and [34] for the first one).

Like for the normal cone, the Mordukhovich limiting subdifferential (see [25, 32]) coincides with the graphical weak sequential outer limit of the proximal or Fréchet subdifferential, that is,

$$\partial_L f(x) = {}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{x' \to_f x} \partial_P f(x') = {}^{\mathrm{seq}}\operatorname*{Lim\,sup}_{x' \to_f x} \partial_F f(x'),$$

where $x' \to_f x$ means that $(x', f(x')) \to (x, f(x))$. So, $v \in \partial_L f(x)$ if and only if there are sequences $(x_n)_n$ with $(x_n, f(x_n)) \to (x, f(x))$, $(v_n)_n$ in H with $v_n \in \partial_P f(x_n)$ (resp. $v_n \in \partial_F f(x_n)$) and such that $v_n \to^w v$.

It is also worth pointing out that $\partial_C f(x)$ is closed and convex since $N^C(C;x)$ is a closed convex cone of H by definition. Assume that f is locally Lipschitz continuous near x. Then it is known (see [8]) that

$$\partial_C f(x) = \{ v \in H : \langle v, y \rangle \leq f^0(x;y) \ \forall y \in H \},$$

where

$$f^0(x;y) := \limsup_{x' \to x;\, t \downarrow 0} t^{-1}[f(x'+ty) - f(x')].$$

Since $f^0(x;\cdot)$ is also convex, positively homogeneous and continuous (as easily seen from the latter equality), it is the support function $\sigma(\partial_C f(x);\cdot)$ of $\partial_C f(x)$. So, whenever f is locally Lipschitz continuous near x, then for each fixed $y \in H$ the function

$$\sigma(\partial_C f(\cdot);y) = f^0(\cdot;y) \ \text{ is upper semicontinuous,} \tag{15}$$

according to the above definition of $f^0(\cdot;y)$.

Coming back to the case of a general (non locally Lipschitz continuous) function, the inclusions between the four normal cones make clear that the inclusions

$$\partial_P f(\bar{x}) \subset \partial_F f(\bar{x}) \subset \partial_L f(\bar{x}) \subset \partial_C f(x)$$

always hold true.

A particular class of functions for which those four subdifferentials coincide is the class of primal lower nice functions. These functions will be involved in several parts of the next section.

Definition 2.4 (see [30, 19]). Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper function (i.e., $\operatorname{dom} f \neq \emptyset$) and consider $\bar{x} \in \operatorname{dom} f$. The function f is said to be primal lower nice (pln for short) at $\bar{x}$, if there exist positive constant real numbers s_0, c_0, q_0, such that for all $x \in B[\bar{x}, s_0]$, for all $q \geq q_0$ and all $v \in \partial_P f(x)$ with $\|v\| \leq c_0 q$, one has

$$f(y) \geq f(x) + \langle v, y - x \rangle - \frac{q}{2}\|y - x\|^2 \tag{16}$$

for each $y \in B[\bar{x}, s_0]$.

Remark 2.5. Clearly, if f is pln at $\bar{x}$ with constants s_0, c_0, q_0, one has

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -q\|x_1 - x_2\|^2 \tag{17}$$

for any $v_i \in \partial_P f(x_i)$ with $\|v_i\| \leq c_0 q$ whenever $q \geq q_0$ and $x_i \in B[\bar{x}, s_0]$, $i = 1, 2$.

A (nontrivial) result due to Poliquin [30] in $\mathbb{R}^n$ and Levy, Poliquin and Thibault [19] in the Hilbert setting ensures that if a function $f : H \to \mathbb{R} \cup \{+\infty\}$ is pln at $\bar{x} \in \operatorname{dom} f$ then for all x in a neighborhood of $\bar{x}$, the proximal subdifferential of f at x agrees with the Clarke subdifferential of f at x, hence

$$\partial_P f(x) = \partial_F f(x) = \partial_L f(x) = \partial_C f(x).$$

So, for such a pln function f we will write $\partial f(x)$ in place of the above subdifferentials for x close enough to $\bar{x}$. The latter equalities are related to another (nontrivial) result (proved in [30] for $\mathbb{R}^n$ and [19] for a Hilbert space) saying that (17) is in fact a characterization of the pln property of f at $\bar{x}$.

The next result recalled below makes the bridge between prox-regular sets and pln functions through the indicator functions of the concerned sets. For completeness we reproduce the main ideas of the proof in [31] in such a way that we obtain a quantified version.

Proposition 2.6 (see [31]). *A closed set C of the Hilbert space H is prox-regular at $\bar{x} \in C$ if and only if the indicator function δ_C of C is pln at the point $\bar{x}$. In general, the following quantified properties hold.*

(a) *If the set C is (r, α)-prox-regular at $\bar{x} \in C$, then for $c_0 = r$, any $q_0 > 0$, and any positive real $s_0 < \alpha$, the function δ_C is pln at $\bar{x}$ with the constants s_0, c_0, q_0.*

(b) *If δ_C is pln at $\bar{x}$ with the positive constants s_0, c_0, q_0, then for any positive real $\alpha < \frac{s_0}{2}$ and any $q \geq \max\{1, c_0 q_0, (\frac{s_0}{2} - \alpha)^{-1} c_0\}$, the set C is $(\frac{c_0}{q}, \alpha)$-prox-regular at $\bar{x}$.*

Proof. The (r, α)-prox-regularity of C leads to the pln property of δ_C as quantified in (a). Indeed, if C is (r, α)-prox-regular at $\bar{x}$, according to (11), for all $x \in$

$B(\bar{x}, \alpha) \cap C$,

$$\langle v, x' - x \rangle \leq \frac{1}{2r}\|x' - x\|^2 \quad \forall v \in N^P(C; x) \cap \mathbb{B}, \forall x' \in C.$$

Now if $v \in N^P(C, x)$, $\|v\| \leq rq$, with $q > 0$, we have $\|\frac{1}{rq}v\| \leq 1$. Putting this in the latter inequality, we have for all x such that $\|x - \bar{x}\| < \alpha$, all $q > 0$, and all $v \in N^P(C; x) = \partial_P \delta_C(x)$ with $\|v\| \leq rq$,

$$\delta_C(x') \geq \delta_C(x) + \langle v, x' - x \rangle - \frac{q}{2}\|x' - x\|^2$$

for all x' with $\|x' - x\| < \alpha$, hence for $c_0 = r$, any $q_0 > 0$ and any positive $s_0 < \alpha$, the function δ_C is pln at $\bar{x}$ with the constants s_0, c_0, q_0. So, assertion (a) is established.

To prove assertion (b), suppose that δ_C is pln at $\bar{x}$ with positive constants s_0, c_0 and q_0, that is, for all $x \in B[\bar{x}, s_0]$, all $q \geq q_0$ and all $v \in \partial_P \delta_C(x) = N^P(C; x)$ with $\|v\| \leq c_0 q$, the inequality

$$\delta_C(x') \geq \delta_C(x) + \langle v, x' - x \rangle - \frac{q}{2}\|x' - x\|^2$$

is valid for all $x' \in B[\bar{x}, s_0]$. Fix $\alpha < \frac{s_0}{2}$ and $q \geq \max\{1, c_0 q_0, (\frac{s_0}{2} - \alpha)^{-1} c_0\}$. Take any $x \in C \cap B(\bar{x}, \alpha)$ and any $v \in N^P(C; x) \cap \mathbb{B}$. We have $v \in N^P(C; x)$ and $\|v\| \leq 1 \leq c_0 \frac{q}{c_0}$ with $\frac{q}{c_0} \geq q_0$, then we get

$$\langle v, x' - x \rangle \leq \frac{q}{2c_0}\|x' - x\|^2 \quad \forall x' \in C \cap B(\bar{x}, s_0),$$

which ensures that

$$x \in \operatorname{Proj}\left(x + \frac{c_0}{q}v, C \cap B(\bar{x}, s_0)\right). \tag{18}$$

Further, according to the choice of q we have

$$\left\|x + \frac{c_0}{q}v - \bar{x}\right\| \leq \|x - \bar{x}\| + \frac{c_0}{q} < \alpha + \left(\frac{s_0}{2} - \alpha\right) = \frac{s_0}{2},$$

thus the claim in the proof of Proposition 2.3 ensures

$$x \in \operatorname{Proj}\left(x + \frac{c_0}{q}v, C\right).$$

So the set C is $(\frac{c_0}{q}, \alpha)$-prox-regular at $\bar{x}$. □

3. Local prox-regularity and variational inequality

This section is devoted to the study of a differential variational inequality involving prox-regular sets. Throughout, C is a nonempty closed set of the Hilbert space H and T_0, T are two real numbers with $T_0 < T$.

Lemma 3.1. *Let $u_0 \in C$ and let $u(\cdot)$ be a solution of the differential inclusion*

$$\begin{cases} -\dot{u}(t) + h(t) \in N^F(C; u(t)), & \text{a.e. } t \in [T_0, T] \\ u(T_0) = u_0. \end{cases} \tag{19}$$

Then the following inequality holds true

$$\|\dot{u}(t) - h(t)\| \le \|h(t)\|, \quad \text{a.e. } t \in [T_0, T].$$

Proof. Some of the arguments are similar to those in the proof of Propostion 2.1 in [35].

Consider the closed set $S(t) := C - \int_{T_0}^{t} h(s)\,ds$ of H and put $\psi(t) := -\int_{T_0}^{t} h(s)\,ds$ for all $t \in [T_0, T]$. If $u(\cdot)$ is a solution of (19), then clearly $y(\cdot) := u(\cdot) + \psi(\cdot)$ is a solution of the sweeping process

$$\begin{cases} -\dot{y}(t) \in N^F(S(t); (y(t)) & \text{a.e. } t \in [T_0, T] \\ y(T_0) = u_0. \end{cases}$$

For all $y \in H$ and $T_0 \le s < t \le T$, one has

$$\begin{aligned} |d(y, S(t)) - d(y, S(s))| &= |d(y - \psi(t), C) - d(y - \psi(s), C)| \\ &\le \|\psi(t) - \psi(s)\| \\ &\le \int_{s}^{t} \|h(s)\|\,ds \\ &\le v(t) - v(s) \quad \text{where } v(t) := \int_{T_0}^{t} \|h(s)\|\,ds, \end{aligned}$$

which means that the closed set $S(t)$ varies in an absolutely continuous way with respect to $t \in [T_0, T]$. Fix any $t \in [T_0, T]$ where $\dot{y}(t)$ and $\dot{v}(t)$ exist along with $\dot{y}(t) \neq 0$, and note that

$$\frac{-\dot{y}(t)}{\|\dot{y}(t)\|} \in N^F(S(t); y(t)).$$

But, (see (14)), one knows that for any closed set S and $x \in S$

$$\partial_F d_S(x) = N^F(S; x) \cap \{x^\star \in H, \|x^\star\| \le 1\}.$$

Therefore, one has

$$-\frac{\dot{y}(t)}{\|\dot{y}(t)\|} \in \partial_F d_{S(t)}(y(t)).$$

Now fix any $\varepsilon > 0$. For $s < t$ sufficiently closed to t we can write

$$\begin{aligned} \left\langle -\frac{\dot{y}(t)}{\|\dot{y}(t)\|}, y(s) - y(t) \right\rangle &\le d_{S(t)}(y(s)) + \varepsilon \|y(s) - y(t)\| \\ &= d_{S(t)}(y(s)) - d_{S(s)}(y(s)) + \varepsilon \|y(s) - y(t)\| \\ &\le v(t) - v(s) + \varepsilon \|y(s) - y(t)\| \end{aligned}$$

and hence

$$\left\langle -\frac{\dot{y}(t)}{\|\dot{y}(t)\|}, \frac{y(s)-y(t)}{t-s} \right\rangle \le \frac{1}{t-s}(v(t)-v(s)) + \varepsilon \frac{1}{t-s}\|y(s)-y(t)\|,$$

that is,

$$\left\langle \frac{\dot{y}(t)}{\|\dot{y}(t)\|}, \frac{y(s)-y(t)}{s-t} \right\rangle \le \frac{v(s)-v(t)}{s-t} + \varepsilon \left\| \frac{y(s)-y(t)}{s-t} \right\|.$$

Taking the limit as $s \uparrow t$ we get

$$\left\langle \frac{\dot{y}(t)}{\|\dot{y}(t)\|}, \dot{y}(t) \right\rangle \le \dot{v}(t) + \varepsilon \|\dot{y}(t)\|,$$

that is, $\|\dot{y}(t)\| \le \dot{v}(t) + \varepsilon\|\dot{y}(t)\|$. This inequality being also true for $\dot{y}(t) = 0$, it follows that $\|\dot{y}(t)\| \le \dot{v}(t)$ for a.e. $t \in [t_0, T]$. As $y(\cdot) = u(\cdot) + \psi(\cdot)$, we conclude that

$$\|\dot{u}(t) - h(t)\| \le \|h(t)\| \text{ a.e. } t \in [T_0, T]. \qquad \square$$

Before establishing the main result of this work, let us recall the following theorem from S. Marcellin and L. Thibault, which will be relevant in our proof.

Theorem 3.2 (Theorem 2.9 in [23]). *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Consider any real T_0 and let $u_0 \in \operatorname{dom} f$ be such that f is pln at u_0 with constants s_0, c_0, Q_0. Then, there exist a real number $T \in]T_0, +\infty[$ and a unique absolutely continuous mapping $u : [T_0, T] \to B(u_0, s_0)$ which is a solution of the problem*

$$\begin{cases} \dot{u}(t) + \partial_F f(u(t)) \ni 0 & \text{a.e. } t \in [T_0, T] \\ u(T_0) = u_0. \end{cases}$$

We also recall the Gronwall's lemma for absolutely continuous solutions of differential inequalities.

Lemma 3.3 (Gronwall's lemma). *Let $\alpha, \beta, \zeta : [T_0, T] \to \mathbb{R}$ be three real-valued Lebesgue integrable functions. If the function $\zeta(\cdot)$ is absolutely continuous and if for almost all $t \in [T_0, T]$*

$$\dot{\zeta}(t) \le \alpha(t) + \beta(t)\zeta(t),$$

then for all $t \in [T_0, T]$

$$\zeta(t) \le \zeta(T_0) \exp\left(\int_{T_0}^t \beta(\theta)\, d\theta \right) + \int_{T_0}^t \alpha(s) \exp\left(\int_s^t \beta(\theta)\, d\theta \right) ds.$$

Now let us state and prove the local existence of a solution for a differential variational inequality using the preceding theorem in the special case when $f = \delta_C$.

Proposition 3.4. *Let C be an (r, α)-prox-regular set at the point $u_0 \in C$ and let any real number $\eta_0 \in]0, \alpha[$. Then for any $\bar{u} \in B(u_0, \alpha - \eta_0) \cap C$, any positive real* *number $\tau \leq T - T_0$, and any mapping $h \in L^1([T_0, T], H)$ such that $\int_{T_0}^{T_0+\tau} \|h(s)\| ds < \eta_0/2$, the differential variational inequality*

$$\begin{cases} -\dot{u}(t) + h(t) \in N(C; u(t)) \quad \text{a.e. } t \in [T_0, T_0 + \tau] \\ u(T_0) = \bar{u} \end{cases} \tag{20}$$

has an absolutely continuous solution $u : [T_0, T_0 + \tau] \to B(\bar{u}, \eta_0) \cap C$.

Proof. *Case A: $h(\cdot)$ is a step mapping.* Let $p \geq 2$ be an integer and $(t_i)_{i=1...p}$ be real numbers such that $T_0 = t_1 < ... < t_p = T$. Let also $v_1, \cdots, v_{p-1}$ be $p-1$ vectors in H and $h := \sum_{i=1}^{p-1} v_i 1_{[t_i;t_{i+1}[}$ be the associated step mapping which lies in $L^1([T_0, T], H)$. Let $\bar{u} \in B(u_0, \alpha - \eta_0) \cap C$. Fix any $i \in \{1, ..., p-1\}$ and consider on H the lsc function $\phi_i(\cdot) := \delta_C(\cdot) - \langle v_i, \cdot \rangle$ with $\operatorname{dom} \phi_i = \operatorname{dom} \delta_C = C$. We observe first that $\operatorname{Dom} \partial_P \phi_i = \operatorname{Dom} \partial_P \delta_C = C$ for any $i = 1, ..., p-1$ and $\partial_P \phi_i(y) = \partial_P \delta_C(y) - v_i$ for all $y \in B(\bar{u}, \alpha) \cap C$. Further, by Proposition 2.2 the set C is $(r, \alpha - \|\bar{u} - u_0\|)$-prox-regular at $\bar{u}$. Put $s_i := \eta_0$, $c_i := \frac{r}{2}$, and take any positive real $q_i \geq \frac{2\|v_i\|}{r}$. Consider any $x \in B[\bar{u}, \eta_0]$ and any $w \in \partial_P \phi_i(x)$ with $\|w\| \leq c_i q$ and $q \geq q_i$. This and what precedes yield that there exists $v \in \partial_P \delta_C(x) = N^P(C; x)$ such that $w = v - v_i$ with

$$\|v\| \leq \|w\| + \|v_i\| \leq \frac{r}{2} q + \frac{r}{2} q_i \leq rq.$$

On the other hand it is easily seen that $B[\bar{u}, \eta_0] \subset B(u_0, \alpha)$, hence $x \in B(u_0, \alpha) \cap C$. Therefore, the (r, α)-prox-regularity of C at u_0 ensures by (11) that

$$\langle v, y - x \rangle \leq rq \frac{1}{2r} \|y - x\|^2 = \frac{q}{2} \|y - x\|^2 \quad \text{for all } y \in C,$$

which entails

$$\langle w, y - x \rangle = \langle v - v_i, y - x \rangle \leq \phi_i(y) - \phi_i(x) + \frac{q}{2} \|y - x\|^2 \quad \text{for all } y \in H.$$

Consequently, ϕ_i is pln at $\bar{u}$ with the constant $s_i = \eta_0$, and some positive constants c_i and q_i.

Thus, for $i = 1$, according to Theorem 3.2, we can find a real number $T_1 > T_0$ and a unique mapping $x_1 : [T_0; T_1] \to B(\bar{u}, \eta_0)$ which is absolutely continuous and satisfies

$$\begin{cases} -\dot{x}_1(t) \in \partial \phi_1(x_1(t)) \quad \text{a.e. } t \in [T_0, T_1] \\ x_1(T_0) = \bar{u}. \end{cases}$$

The inclusion $-\dot{x}_1(t) \in \partial \phi_1(x_1(t))$ a.e. $t \in [T_0, T_1]$ implies that $x_1(t) \in \operatorname{dom} \phi_1 = C$ a.e. $t \in [T_0, T_1]$ and hence $x_1(t) \in C$ for all t according to the closedness of C and the continuity of $x_1(\cdot)$ on $[T_0, T_1]$. So $x_1(t) \in B(\bar{u}, \eta_0) \cap C$ for all $t \in [T_0, T_1]$.

Put $\theta_1 = \min\{T_1, t_2\}$. Then the mapping $x_1 : [T_0, \theta_1] \to B(\bar{u}, \eta_0) \cap C$ is an absolutely continuous solution of

$$\begin{cases} -\dot{x}_1(t) + h(t) \in N(C; x_1(t)) \quad \text{a.e. } t \in [T_0, \theta_1] \\ x_1(T_0) = \bar{u}. \end{cases} \tag{21}$$

Denote by Λ_0 the collection of pairs $(\hat{T}, x(\cdot))$ such that $\hat{T} \in]T_0, T]$, and $x(\cdot)$ is an absolutely continuous solution of (21) on each segment $[T_0, \beta] \subset [T_0, \hat{T}[$ with $x(t) \in B(\bar{u}, \eta_0) \cap C$ for all $t \in [T_0, \hat{T}[$. The preceding analysis guarantees the nonemptyness of the set Λ_0. Endow Λ_0 with the partial ordering $\preceq$ defined by

$$(T_1, x_1) \preceq (T_2, x_2) \Leftrightarrow T_1 \leq T_2 \quad \text{and} \quad x_1(t) = x_2(t) \text{ for all } t \in [T_0, T_1[,$$

for $(T_i, x_i) \in \Lambda_0, i = 1, 2$. Let $\{(T_\alpha, x_\alpha(\cdot)) : \alpha \in A\}$ be a totally ordered family in Λ_0. If $\tilde{T} := \sup_{\alpha \in A} T_\alpha$, then the mapping $\tilde{x} : [T_0, \tilde{T}[\to H$ given by $\tilde{x}(t) := x_\alpha(t)$ if $t \in [T_0, T_\alpha[$ is well defined, and on each interval $[T_0, \beta] \subset [T_0, \tilde{T}[$ it is absolutely continuous and it is a solution of (21) with $\tilde{x}(t) \in B(\bar{u}, \eta_0) \cap C$ for all $t \in [T_0, \tilde{T}[$. To see this, let $(\theta_n)_n \subset]T_0, \tilde{T}[$ converging increasingly to $\tilde{T}$ hence

$$[T_0, \tilde{T}[= \bigcup_{n \geq 1} [T_0, \theta_n].$$

Since $\tilde{T}$ is a supremum, there exists $\alpha_n \in A$ such that $\theta_n < T_{\alpha_n}$. So $[T_0, \theta_n] \subset [T_0, T_{\alpha_n}[$ and for each $t \in [T_0, \theta_n]$ one has $\tilde{x}(t) = x_{\alpha_n}(t)$. Moreover, x_{α_n} is an absolutely continuous solution of (21) on $[T_0, \theta_n]$ and so does $\tilde{x}$. Now observe that for any $\sigma \in]T_0, \tilde{T}[$, there exists some integer n_σ such that $[T_0, \sigma] \subset [T_0, \theta_{n_\sigma}[$, so $\tilde{x}$ is absolutely continuous on $[T_0, \sigma]$. For any $n \in \mathbb{N}$, there exists some Lebesgue-null set $N_n \subset [T_0, \theta_n]$ such that the absolutely continuous mapping $\tilde{x}$ on $[T_0, \theta_n]$ verifies the inclusion of (21) for all $t \in [T_0, \theta_n] \backslash N_n$. Hence $\tilde{x}$ verifies the inclusion of (21) for all $t \in [T_0, \tilde{T}[\backslash \left(\bigcup_{n \geq 1} N_n\right)$. Thus $(\tilde{T}, \tilde{x}) \in \Lambda_0$, and $(T_\alpha, x_\alpha) \preceq (\tilde{T}, \tilde{x})$ for all $\alpha \in A$, which means that $(\tilde{T}, \tilde{x})$ is an upper bound for the family $\{(T_\alpha, x_\alpha(\cdot)) : \alpha \in A\}$. According to Zorn's lemma, there exists a maximal element in Λ_0 that we call (θ, u). The mapping $u(\cdot)$ satisfies $u(t) \in B(\bar{u}, \eta_0) \cap C$ for all $t \in [T_0, \theta[$ and it is an absolutely continuous solution of (21) on each interval $[T_0, \sigma] \subset [T_0, \theta[$, and it cannot be extended on the right hand side of θ.

Case B: General case. Fix $h \in L^1([T_0, T], H)$ with $\int_{T_0}^{T_0+\tau} \|h(s)\| \, ds < \eta_0/2$. Then there exists a sequence of step mappings $(h_n)_{n \geq 1}$ which converges to h almost everywhere on $[T_0, T]$ and also with respect to the strong topology of $L^1([T_0, T], H)$. Since $\lim_{n \to \infty} \int_{T_0}^{T_0+\tau} \|h_n(t)\| \, dt = \int_{T0}^{T_0+\tau} \|h(t)\| \, dt < \eta_0/2$, we may without loss of generality suppose that

$$\int_{T_0}^{T_0+\tau} \|h_n(t)\| \, dt < \eta_0/2 \text{ for all } n \in \mathbb{N}. \tag{22}$$

By the preceding analysis in Case A, for each $n \in \mathbb{N}$, there exist $\theta_n \in]T_0, T]$ and a mapping $u_n : [T_0, \theta_n[\to B(\bar{u}, \eta_0) \cap C$ defined on its maximal interval of existence, that is locally absolutely continuous on $[T_0, \theta_n[$ and for which

$$\begin{cases} -\dot{u}_n(t) + h_n(t) \in N(C; u_n(t)) \text{ a.e. } t \in [T_0, \theta_n[\\ u_n(T_0) = \bar{u}. \end{cases} \tag{23}$$

Besides, for any $n \geq 1$, $T_0 \leq s < t < \theta_n$, one has by Lemma 3.1

$$\|u_n(t) - u_n(s)\| = \left\| \int_s^t \dot{u}_n(\sigma) \, d\sigma \right\| \leq \int_s^t \|\dot{u}_n(\sigma)\| \, d\sigma \leq 2 \int_s^t \|h_n(\sigma)\| \, d\sigma. \tag{24}$$

Fix an arbitrary $n \in \mathbb{N}$. The last estimation in (24) combined with the Cauchy's criterion shows that $\lim_{t\uparrow\theta_n} u_n(t) = \overline{u}_n$ exists in $(H, \|\cdot\|)$. This limit point clearly belongs to $B[\bar{u}, \eta_0]$, and since C is closed, it also lies in C. Extending $u_n(\cdot)$ on the closed interval $[T_0, \theta_n]$ by setting $u_n(\theta_n) := \bar{u}_n$, the inequalities in (24) still hold for all $T_0 \le s \le t \le \theta_n$, hence the so extended mapping $u_n(\cdot)$ is absolutely continuous on $[T_0, \theta_n]$ with values in $B[\bar{u}, \eta_0] \cap C$ and it is a solution on the closed interval $[T_0, \theta_n]$ of (23).

We claim that for any integer n satisfying $\theta_n < T$ we have $T_0 + \tau < \theta_n$. Indeed suppose that for some integer k we have $\theta_k < T$ and $\theta_k \le T_0 + \tau$. Then, in view of (24)

$$\|u_k(t) - \bar{u}\| = \|u_k(t) - u_k(T_0)\| \le \int_{T_0}^{t} \|\dot{u}_k(\sigma)\|\, d\sigma \le 2\int_{T_0}^{t} \|h_k(\sigma)\|\, d\sigma,$$

hence with (22)

$$\|\overline{u}_k - \bar{u}\| = \lim_{t\uparrow\theta_k} \|u_k(t) - \bar{u}\| \le 2\int_{T_0}^{\theta_k} \|h_k(\sigma)\|\, d\sigma \le 2\int_{T_0}^{T_0+\tau} \|h_k(\sigma)\|\, d\sigma < \eta_0.$$

which implies in particular that $u_k(t) \in B(\bar{u}, \eta_0) \cap C$ for all $t \in [T_0, \theta_k]$. Since $\bar{u} \in B(u_0, \alpha - \eta_0) \cap C$, according to Proposition 2.2, the set C is (r, η_0)-prox-regular at $\bar{u}$, hence C is $(r, \eta_0 - \|\overline{u}_k - u_0\|)$-prox-regular at $\overline{u}_k$ thanks to the same Proposition and to the inequality $\|\overline{u}_k - \bar{u}\| < \eta_0$ above. Therefore, since h_k restricted to $[\theta_k, T[$ is a step mapping, the previous case A applied with the initial data $(\theta_k, \overline{u}_k)$ and $h_k \mathbb{1}_{[\theta_k, T]}$ enables us to find some $\tilde{\theta}_k \in]\theta_k, T]$ and a locally absolutely continuous mapping

$$v_k : [\theta_k, \tilde{\theta}_k[\to B(\overline{u}_k, \eta_0 - \|\overline{u}_k - \bar{u}\|) \cap C \subset B(\bar{u}, \eta_0) \cap C$$

such that

$$\begin{cases} -\dot{v}_k(t) + h_k(t) \in N(C, v_k(t)) & \text{a.e. } t \in [\theta_k, \tilde{\theta}_k[\\ v_k(\theta_k) = \overline{u}_k. \end{cases}$$

Now the continuous extension of u_k to $[T_0, \tilde{\theta}_k[$ given by

$$\tilde{u}_k(\cdot) := \begin{cases} u_k(\cdot) & \text{on } [T_0, \theta_k[\\ v_k(\cdot) & \text{on } [\theta_k, \tilde{\theta}_k[\end{cases}$$

extends $u_k(\cdot)$ beyond θ_k, and with values staying in $B(\bar{u}, \eta_0) \cap C$. This contradicts the definition of θ_k and proves the claim.

Consequently, for all integers $n \ge 1$ the inequality $T_0 + \tau \le \theta_n$ holds and the restriction to $[T_0, T_0 + \tau]$ of the mapping $u_n(\cdot)$ as extended above takes on values in $B[\bar{u}, \eta_0] \cap C$. Further, it is an absolutely continuous solution of (23) on the closed interval $[T_0, T_0 + \tau]$.

We show that the restrictions of the mappings $(u_n)_n$ to $[T_0, T_0 + \tau]$ converge uniformly on this segment to some mapping which will turn out to be the expected local solution of (20).

Let $n, p \geq 1$. Since $\{u_n(t), u_p(t)\} \subset B[\bar{u}, \eta_0] \subset B(u_0, \alpha)$ and $-\dot{u}_i(t) + h_i(t) \in N(C; u_i(t))$ for $i = n$ and p, the hypomonotonicity property of $N(C; \cdot) \cap \mathbb{B}$ (see Proposition 2.3) entails

$$\begin{aligned} &\langle -\dot{u}_n(t) + h_n(t) + \dot{u}_p(t) - h_p(t), u_n(t) - u_p(t)\rangle \\ \geq\ & -\frac{1}{r}(\| - \dot{u}_n(t) + h_n(t)\| + \| - \dot{u}_p(t) + h_p(t)\|)\|u_n(t) - u_p(t)\|^2. \end{aligned}$$

This and the inequality $\| - \dot{u}_n(t) + h_n(t)\| \leq \|h_n(t)\|$ for a.e. $t \in [T_0, T_0 + \tau]$ from Lemma 3.1 yield

$$\begin{aligned} &\frac{d}{dt}\|u_n(t) - u_p(t)\|^2 \\ \leq\ & \frac{2}{r}(\|h_n(t)\| + \|h_p(t)\|)\|u_n(t) - u_p(t)\|^2 + 2\langle h_n(t) - h_p(t), u_n(t) - u_p(t)\rangle. \end{aligned}$$

Then, since for almost $t \in [T_0, T_0 + \tau]$,

$$\frac{d}{dt}\|u_n(t) - u_p(t)\|^2 = 2\|u_n(t) - u_p(t)\|\frac{d}{dt}\|u_n(t) - u_p(t)\|,$$

we obtain

$$\frac{d}{dt}\|u_n(t) - u_p(t)\| \leq \frac{2}{r}(\|h_n(t)\| + \|h_p(t)\|)\|u_n(t) - u_p(t)\| + \|h_n(t) - h_p(t)\|.$$

Since $u_n(T_0) - u_p(T_0) = 0$, it follows from Gronwall's lemma (see Lemma 3.3) that

$$\begin{aligned} &\|u_n(t) - u_p(t)\| \\ \leq\ & \int_{T_0}^{t} \|h_n(s) - h_p(s)\| \exp\left\{\int_s^t \left(\frac{2}{r}(\|h_n(\sigma)\| + \|h_p(\sigma)\|\right) d\sigma\right\} ds \\ \leq\ & \exp\left\{\int_{T_0}^{T_0+\tau} \left(\frac{2}{r}(\|h_n(\sigma)\| + \|h_p(\sigma)\|)\right) d\sigma\right\} \int_{T_0}^{T_0+\tau} \|h_n(s) - h_p(s)\|\, ds. \end{aligned}$$

Next, due to the boundedness of (h_n) in $L^1([T_0, T]; H)$, for any $n, p \geq 1$, we deduce that

$$\sup_{t\in[T_0,T_0+\tau]} \|u_n(t) - u_p(t)\| \leq \|h_n - h_p\|_{L^1([T_0,T],H)} \exp\left\{\frac{4}{r} \sup_{n\in\mathbb{N}} \|h_n\|_{L^1([T_0,T],H)}\right\}.$$

By virtue of the strong convergence of $(h_n)_n$ to h in $L^1([T_0, T], H)$, it follows that

$$\lim_{n,p\to\infty} \sup_{t\in[T_0,T_0+\tau]} \|u_n(t) - u_p(t)\| = 0,$$

and the uniform Cauchy's criterion implies that $(u_n(\cdot))_n$ converges uniformly on $[T_0, T_0 + \tau]$ to some continuous mapping $u : [T_0, T_0 + \tau] \to H$ such that $u(T_0) = \bar{u}$. In addition,

$$\{u(t) : t \in [T_0, T_0 + \tau]\} \subset B[\bar{u}, \eta_0] \cap C,$$

since the set C is closed.

Referring to Lemma 3.1, observe as above that for almost every $t \in [T_0, T_0 + \tau]$, we have

$$\|\dot{u}_n(t) - h_n(t)\| \le \|h_n(t)\|. \tag{25}$$

Putting $w_n(t) := \dot{u}_n(t) - h_n(t)$ we see by (25) that the set $\{w_n(\cdot) : n \in \mathbb{N}\}$ is bounded in $L^1([T_0, T_0 + \tau], H)$ since it is the case of the restrictions of $h_n(\cdot)$ to $[T_0, T_0 + \tau]$, and this also ensures that for each Lebesgue measurable subset A of $[T_0, T_0 + \tau]$ the set $\{\int_A w_n(t)\, dt : n \in \mathbb{N}\}$ is bounded in H and hence relatively weakly compact in H. We also observe that the set $\{w_n(\cdot) : n \in \mathbb{N}\}$ is uniformly integrable. Indeed let any $\varepsilon > 0$. Choose an integer n_0 such that

$$\int_{T_0}^{T_0+\tau} \|h_n(t) - h(t)\|\, dt < \varepsilon/2 \ \text{ for all } n \ge n_0 + 1,$$

and choose a real $\delta > 0$ such that for any Lebesgue measurable subset A of $[T_0, T_0+\tau]$ with $\lambda(A) < \delta$ we have

$$\int_A \|h(t)\|\, dt < \varepsilon/2 \quad \text{and} \quad \int_A \|h_n(t)\|\, dt < \varepsilon \ \text{ for all } n \le n_0.$$

Then, taking any Lebesgue measurable subset A of $[T_0, T_0 + \tau]$ with $\lambda(A) < \delta$, we obtain on one hand for all $n \ge n_0 + 1$

$$\int_A \|w_n\|\, dt \le \int_A \|h_n(t)\|\, dt \le \int_A \|h(t)\|\, dt + \int_{T_0}^{T_0+\tau} \|h_n(t) - h(t)\|\, dt < \varepsilon$$

and on the other hand for all $n \le n_0$

$$\int_A \|w_n(t)\|\, dt \le \int_A \|h_n(t)\|\, dt < \varepsilon,$$

that is, $\int_A \|w_n(t)\|\, dt < \varepsilon$ for all $n \in \mathbb{N}$, which translates the desired uniform integrability. Consequently, the three above properties along with the reflexivity of H entails according to the Dunford theorem (see, e.g., [13, Theorem IV.2.1, p. 101]) that the set $\{w_n(\cdot) : n \in \mathbb{N}\}$ is relatively weakly compact in $L^1([T_0, T_0 + \tau], H)$. Extracting a weakly convergent subsequence by the Eberlein-Šmulian theorem, we may suppose that $(\dot{u}_n(\cdot))$ converges weakly in $L^1([T_0, T_0 + \tau], H)$ to some map $g(\cdot) \in L^1([T_0, T_0 + \tau], H)$. So for any $t \in [T_0, T_0 + \tau]$, fixing any $z \in H$ and writing

$$\left\langle z, \int_{T_0}^{t} \dot{u}_n(s)\, ds \right\rangle = \int_{T_0}^{T_0+\tau} \langle z \mathbb{1}_{[T_0,t]}(s), \dot{u}_n(s) \rangle\, ds,$$

we see that

$$\int_{T_0}^{t} \dot{u}_n(s)\, ds \to \int_{T_0}^{t} g(s)\, ds \ \text{ weakly in } H. \tag{26}$$

Since $u_n(t)$ converges weakly to $u(t)$ (because of the uniform convergence of $(u_n(\cdot))_n$ to $u(\cdot)$), from the equality $u_n(t) = u_n(T_0) + \int_{T_0}^{t} \dot{u}_n(s)\, ds$, it results that $u(t) =$

$u_0 + \int_{T_0}^t g(s)\,ds$, and hence $u(\cdot)$ is absolutely continuous with $\dot{u}(t) = g(t)$ a.e. $t \in [T_0, T_0 + \tau]$. As a result,

$$\dot{u}_n(\cdot) \to \dot{u}(\cdot) \;\text{ weakly in } L^1([T_0, T_0 + \tau], H).$$

Clearly $h_n(\cdot) \to h(\cdot)$ weakly in $L^1([T_0, T_0 + \tau], H)$. Applying Mazur's lemma, there exists a sequence $(\zeta_n(\cdot))$ which converges strongly in $L^1([T_0, T_0+\tau], H)$ to $-\dot{u}(\cdot)+h(\cdot)$ with

$$\zeta_n(\cdot) \in \operatorname{co}\{-\dot{u}_i(\cdot) + h_i(\cdot) : i \geq n\}.$$

Extracting a subsequence, we may suppose that

$$\zeta_n(t) \to -\dot{u}(t) + h(t) \;\text{ a.e. } t \in [T_0, T_0 + \tau].$$

Consequently, for almost all $t \in [T_0, T_0 + \tau]$,

$$-\dot{u}(t) + h(t) \in \cap_n \overline{\operatorname{co}}\{-\dot{u}_i(t) + h_i(t) : i \geq n\}.$$

It follows for almost all $t \in [T_0, T_0 + \tau]$ that for any $\xi \in H$

$$\langle \xi, -\dot{u}(t) + h(t)\rangle \leq \inf_n \sup_{i \geq n} \langle \xi, -\dot{u}_i(t) + h_i(t)\rangle. \tag{27}$$

On the other hand, from the inclusion $-\dot{u}_n(t)+h_n(t) \in N(C; u_n(t))$ a.e. $t \in [T_0, T_0+\tau]$, and from the inequality in (25), we deduce according to (14) that

$$-\dot{u}_n(t) + h_n(t) \in \|h_n(t)\|\, \partial_P d_C(u_n(t)) \;\text{ a.e. } t \in [T_0, T_0 + \tau].$$

So, by (27) for almost all $t \in [T_0, T_0 + \tau]$ one has for any $\xi \in H$

$$\begin{aligned}
\langle \xi, -\dot{u}(t) + h(t)\rangle &\leq \limsup_n \sigma(\|h_n(t)\|\, \partial_P d_C(u_n(t)), \xi) \\
&= \|h(t)\| \limsup_n \sigma(\partial_P d_C(u_n(t)), \xi) \\
&\leq \|h(t)\| \limsup_n \sigma(\partial_C d_C(u_n(t)), \xi)
\end{aligned}$$

and hence by the upper semicontinuity of $\sigma(\partial_C d_C(\cdot), \xi)$ (see (15))

$$\langle \xi, -\dot{u}(t) + h(t)\rangle \leq \|h(t)\|\, \sigma(\partial_C d_C(u(t)), \xi).$$

Since the Clarke subdifferential always has closed and convex values, this last inequality yields for almost all $t \in [T_0, T_0 + \tau]$,

$$-\dot{u}(t) + h(t) \in \|h(t)\|\, \partial_C d_C(u(t)) \subset N(C; u(t)).$$

Moreover,

$$u(T_0) = \lim_n u_n(T_0) = \bar{u}.$$

Therefore $u(\cdot)$ is a solution of (20) on $[T_0, T_0 + \tau]$.

Finally, for almost all $t \in [T_0, T_0 + \tau]$ we have by Lemma 3.1

$$\|\dot{u}(t)\| \leq \| - \dot{u}(t) + h(t)\| + \|h(t)\| \leq 2\|h(t)\|,$$

so for all $t \in [T_0, T_0 + \tau]$

$$\|u(t) - \bar{u}\| \leq 2 \int_{T_0}^{T_0+\tau} \|h(s)\|\, ds < \eta_0.$$

The proof of the proposition is then complete. □

Now we study the case of a mapping depending on t and x which motivated the study of the paper.

Theorem 3.5. *Let H be a Hilbert space. Assume that the closed set C is (r, α)-prox-regular at the point $u_0 \in C$ and let any real number $\eta_0 \in]0, \alpha[$. Let $f : [T_0, T] \times B(u_0, \alpha) \to H$ be a mapping which is Bochner measurable with respect to $t \in [T_0, T]$, i.e., $f(\cdot, x)$ is Bochner measurable for all $x \in B(u_0, \alpha)$. Assume that:*

(i) there exists a nonnegative function $\beta \in L^1([T_0, T], \mathbb{R})$ such that:

$$\|f(t, x)\| \leq \beta(t) \quad \text{for all } (t, x) \in [T_0, T] \times B(u_0, \alpha);$$

(ii) there exists a non-negative function $k(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for all $t \in [T_0, T]$ and for all $(x, y) \in B(u_0, \alpha) \times B(u_0, \alpha)$

$$\|f(t, x) - f(t, y)\| \leq k(t)\|x - y\|.$$

Then for all positive real $\tau \leq T - T_0$ such that $\int_{T_0}^{T_0+\tau} \beta(t)\, dt < \min\{\eta_0/2, (\alpha - \eta_0)/2\}$, the following variational differential inequality

$$\begin{cases} -\dot{u}(t) + f(t, u(t)) \in N(C; u(t)) \\ u(T_0) = u_0 \in C \end{cases} \tag{28}$$

has one and only one absolutely continuous solution $u(\cdot)$ on $[T_0, T_0 + \tau]$ satisfying $u([T_0, T_0 + \tau]) \subset B(u_0, \alpha)$. Further the solution even stays in $B(u_0, \eta_0)$, that is,

$$u(t) \in B(u_0, \eta_0) \quad \text{for all } t \in [T_0, T_0 + \tau].$$

Proof. Fix any positive real $\tau \leq T - T_0$ such that

$$\int_{T_0}^{T_0+\tau} \beta(s)\, ds < \min\left\{\frac{\eta_0}{2}, \frac{\alpha - \eta_0}{2}\right\}. \tag{29}$$

Step 1. Consider for each integer $n \geq 1$, a partition of $I := [T_0, T_0 + \tau]$ defined by $t_i^n = T_0 + i\frac{\tau}{n}$ for the integers $i = 0, \cdots, n$. Applying Proposition 3.4, we obtain that the problem

$$\begin{cases} -\dot{u}(t) + f(t, u_0) \in N(C; u(t)) \quad \text{a.e. } t \in [T_0, t_1^n] \\ u(T_0) = u_0 \in C \end{cases}$$

has an absolutely continuous solution $u_0^n(\cdot) : [T_0, t_1^n] \to B(u_0, \eta_0) \cap C$. Moreover Lemma 3.1 ensures that for almost all $t \in [T_0, t_1^n]$

$$\|\dot{u}_0^n(t)\| \le \|\dot{u}_0^n(t) - f(t, u_0)\| + \|f(t, u_0)\| \le 2\|f(t, u_0)\| \le 2\beta(t).$$

This yields for every $t \in [t_0^n, t_1^n]$ that

$$u_0^n(t) \in B(u_0, \alpha - \eta_0) \cap C,$$

because by the latter inequality and by (29)

$$\|u_0^n(t) - u_0\| \le \int_{T_0}^{t_1^n} \|\dot{u}_0(s)\|\, ds \le 2\int_{T_0}^{t_1^n} \beta(s)\, ds \le 2\int_{T_0}^{T_0+\tau} \beta(s)\, ds < \alpha - \eta_0.$$

Since $\|u_0^n(t_1^n) - u_0\| < \alpha - \eta_0$, we may apply Proposition 3.4 again with $u_0^n(t_1^n)$ as initial condition at time t_1^n, and this gives the existence of an absolutely continuous solution $u_1^n(\cdot) : [t_1^n, t_2^n] \to B(u_0^n(t_1^n), \eta_0) \cap C$ of the problem

$$\begin{cases} -\dot{u}(t) + f(t, u_0^n(t_1^n)) \in N(C; u(t)) & \text{a.e. } t \in [t_1^n, t_2^n] \\ u(t_1^n) = u_0^n(t_1^n) \in C. \end{cases}$$

For $k = 0, 1$, $u_k^n(\cdot)$ satisfies by Lemma 3.1 the inequalities (with $u_{-1}^n(t_0^n) = u_0$)

$$\|\dot{u}_k^n(t)\| \le \|\dot{u}_k^n(t) - f(t, u_{k-1}^n(t_k^n))\| + \|f(t, u_{k-1}^n(t_k^n))\| \le 2\beta(t), \quad \text{a.e. } t \in [t_k^n, t_{k+1}^n].$$

So, for any $t \in [t_1^n, t_2^n]$ one has

$$\begin{aligned} &\|u_1^n(t) - u_0\| \\ &\le \|u_1^n(t) - u_1^n(t_1^n)\| + \|u_1^n(t_1^n) - u_0\| \\ &= \|u_1^n(t) - u_1^n(t_1^n)\| + \|u_0^n(t_1^n) - u_0\| \\ &= \left\| \int_{t_1^n}^{t} \dot{u}_1^n(s)\, ds \right\| + \left\| \int_{T_0}^{t_1^n} \dot{u}_0^n(s)\, ds \right\| \\ &\le 2\int_{t_1^n}^{t} \beta(s)\, ds + 2\int_{T_0}^{t_1^n} \beta(s)\, ds = 2\int_{T_0}^{t} \beta(s)\, ds \le 2\int_{T_0}^{T_0+\tau} \beta(s) ds < \alpha - \eta_0. \end{aligned}$$

Then, on one hand

$$u_1^n(t) \in B(u_0, \eta_0) \cap C \quad \text{for all } t \in [t_1^n, t_2^n].$$

On the other hand, since $\|u_1^n(t_2^n) - u_0\| < \alpha - \eta_0$, applying Proposition 3.4 once more gives for the problem

$$\begin{cases} -\dot{u}(t) + f(t, u_1^n(t_2^n)) \in N(C; u(t)) & \text{a.e. } t \in [t_2^n, t_3^n] \\ u(t_2^n) = u_1^n(t_2^n) \in C \end{cases}$$

an absolutely continuous solution $u_2^n(\cdot) : [t_2^n, t_3^n] \to B(u_1^n(t_2^n), \eta_0) \cap C$.

Thus, iterating, for $2 \le k \le n-1$, and using Lemma 3.1 one has

$$\begin{aligned}\|u_{k-1}^n(t_k^n) - u_0\| &\le \sum_{i=0}^{k-1} \|u_i^n(t_{i+1}^n) - u_i^n(t_i^n)\| \\ &\le 2\sum_{i=0}^{k-1} \int_{t_i^n}^{t_{i+1}^n} \beta(s)ds \le 2\int_{T_0}^{T_0+\tau} \beta(s)ds < \alpha - \eta_0,\end{aligned}$$

hence there exists a finite sequence of absolutely continuous maps $u_k^n(\cdot) : [t_k^n, t_{k+1}^n] \to B(u_0, \eta_0) \cap C$ with $0 \le k \le n-1$, such that for each $k \in \{0, ..., n-1\}$, (with $u_{-1}^n(T_0) = u_0$),

$$\begin{cases} -\dot{u}_k^n(t) + f(t, u_{k-1}^n(t_k^n)) \in N(C; u_k^n(t)) & \text{a.e. } t \in [t_k^n, t_{k+1}^n] \\ u_k^n(t_k^n) = u_{k-1}^n(t_k^n) \end{cases}$$

and

$$\|\dot{u}_k^n(t) - f(t, u_{k-1}^n(t_k^n))\| \le \|f(t, u_{k-1}^n(t_k^n))\| \quad \text{a.e. } t \in [t_k^n; t_{k+1}^n].$$

Define $u_n(\cdot) : I \to H$ by

$$u_n(t) := u_k^n(t), \quad \text{if } t \in [t_k^n, t_{k+1}^n] \text{ for } k \in \{0, ..., n-1\}.$$

The mapping $u_n(\cdot)$ is absolutely continuous on $[T_0, T_0 + \tau]$ with

$$u_n(t) \in B(u_0, \eta_0) \cap C \quad \text{for all } t \in [T_0, T_0 + \tau]. \tag{30}$$

Further, putting

$$\begin{cases} \theta_n(T_0) := T_0 \\ \theta_n(t) := t_k^n \quad \text{if } t \in]t_k^n, t_{k+1}^n] \text{ for } k \in \{0, ..., n-1\} \end{cases} \tag{31}$$

we have

$$\begin{cases} -\dot{u}_n(t) + f(t, u_n(\theta_n(t)) \in N(C; u_n(t)) & \text{a.e. } t \in I \\ u_n(T_0) = u_0 \end{cases} \tag{32}$$

and

$$\|\dot{u}_n(t) - f(t, u_n(\theta_n(t))\| \le \|f(t, u_n(\theta_n(t)))\| \quad \text{a.e. } t \in I. \tag{33}$$

Step 2. Now we prove that the sequence $(u_n(\cdot))$ converges uniformly to a solution of (28). Let us first note that $(\dot{u}_n(\cdot))$ is uniformly dominated by an integrable function. Indeed, according to the definition of $\beta(\cdot)$ and to (33), we have for almost all t and for any n,

$$\|f(t, u_n(\theta_n(t)))\| \le \beta(t), \tag{34}$$

$$\|\dot{u}_n(t) - f(t, u_n(\theta_n(t)))\| \le \beta(t), \tag{35}$$

and thus,

$$\|\dot{u}_n(t)\| \le 2\beta(t). \tag{36}$$

Let us prove that $(u_n(\cdot))_n$ is a Cauchy sequence in $(\mathcal{C}(I,H), \|.\|_\infty)$. By (30) we know that

$$u_n(t) \in B(u_0, \eta_0) \cap C \quad \text{for all } t \in [T_0, T_0 + \tau].$$

Then we can use the hypomonotonicity property of the normal cone. Let $m, n \in \mathbb{N}$. For almost all $t \in I$, taking (32) and (35) into account yields

$$\begin{aligned} &\langle \dot{u}_n(t) - f(t, u_n(\theta_n(t))) - \dot{u}_m(t) + f(t, u_m(\theta_m(t))), u_n(t) - u_m(t)\rangle \\ \leq\ & \frac{\beta(t)}{r} \|u_n(t) - u_m(t)\|^2 \end{aligned}$$

hence

$$\begin{aligned} &\langle \dot{u}_n(t) - \dot{u}_m(t), u_n(t) - u_m(t)\rangle \\ \leq\ & \frac{\beta(t)}{r} \|u_n(t) - u_m(t)\|^2 + \langle f(t, u_n(\theta_n(t))) - f(t, u_m(\theta_m(t))), u_n(t) - u_m(t)\rangle. \end{aligned}$$

Further, we know that for each $t \in I$ the mapping $f(t, \cdot)$ is $k(t)$-Lipschitz continuous on $B(u_0, \alpha)$ and $k(\cdot) \in L^1(I, \mathbb{R})$. It follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_n(t) - u_m(t)\|^2) \\ \leq\ & \left(\frac{\beta(t)}{r} + k(t)\right) \|u_n(t) - u_m(t)\|^2 + k(t)(\|u_n(\theta_n(t)) - u_n(t)\| \\ & \qquad + \|u_m(t) - u_m(\theta_m(t))\|) \times \|u_n(t) - u_m(t)\|. \end{aligned} \tag{37}$$

Observing also, according to the inequality $\|\dot{u}_n(s)\| \leq 2\beta(s)$ for almost all s, that for any $t \in I$

$$\|u_n(t) - u_n(\theta_n(t))\| \leq 2 \int_{\theta_n(t)}^{t} \beta(s)\, ds,$$

it results from (37) that for $\gamma := \eta_0 + \|u_0\|$ it holds for almost all t,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_n(t) - u_m(t)\|^2) \\ \leq\ & \left(\frac{\beta(t)}{r} + k(t)\right) \|u_n(t) - u_m(t)\|^2 + 4\gamma k(t) \left(\int_{\theta_n(t)}^{t} \beta(s) ds + \int_{\theta_m(t)}^{t} \beta(s) ds\right). \end{aligned}$$

Put $G_{n,m}(t) := 4\gamma k(t)(\int_{\theta_n(t)}^{t} \beta(s) ds + \int_{\theta_m(t)}^{t} \beta(s) ds)$ and note, since $\theta_n(t) \to t$, that

$$\lim_{n\to\infty; m\to\infty} G_{n,m}(t) = 0, \quad \text{for all } t \in I.$$

Moreover, using the inequality $|G_{n,m}(t)| \leq 8\gamma(\int_{T_0}^{T_0+\tau} \beta(s)\, ds) k(t)$, for all $t \in I$ and the Lebesgue dominated convergence theorem we see that

$$\lim_{n\to\infty; m\to\infty} \int_{T_0}^{T_0+\tau} G_{n,m}(s) ds = 0. \tag{38}$$

Since $\|u_n(T_0) - u_m(T_0)\| = 0$, applying Gronwall lemma 3.3, we obtain

$$\begin{aligned}\|u_n(t) - u_m(t)\|^2 &\leq \int_{T_0}^{t} 2G_{n,m}(s) \exp\left(2\int_{s}^{t}\left(\frac{\beta(\theta)}{r} + k(\theta)\right)d\theta\right)ds \\ &\leq 2\exp\left(2\int_{T_0}^{T_0+\tau}\left(\frac{\beta(\theta)}{r} + k(\theta)\right)d\theta\right)\int_{T_0}^{t} G_{n,m}(s)\,ds.\end{aligned}$$

It then follows that

$$\|u_n(\cdot) - u_m(\cdot)\|_\infty^2 \leq 2\exp\left(2\int_{T_0}^{T_0+\tau}\left(\frac{\beta(\theta)}{r} + k(\theta)\right)d\theta\right)\int_{T_0}^{T_0+\tau} G_{n,m}(s)\,ds,$$

and this leads with (38) to

$$\lim_{n\to\infty;m\to\infty} \|u_n(\cdot) - u_m(\cdot)\|_\infty = 0.$$

Therefore, $(u_n(\cdot))$ is a Cauchy sequence in $(C(I,H), \|\cdot\|_\infty)$ and converges uniformly to some map $u(\cdot) \in C(I,H)$. According to (36), we may suppose without loss of generality that $(\dot{u}_n(\cdot))$ converges weakly in $L^1(I,H)$ to some map $g(\cdot) \in L^1(I,H)$(see, e.g., [13, Theorem IV.2.1, p. 101]). Thus, for any $t \in I$ like for (26)

$$\int_{T_0}^{t} \dot{u}_n(s)ds \to \int_{T_0}^{t} g(s)\,ds \quad \text{weakly in } H.$$

This combined with the strong convergence of $(u_n(t))_n$ in H to $u(t)$ ensures that

$$u(t) = u_0 + \int_{T_0}^{t} g(s)ds \quad \text{for all } t \in I.$$

Consequently, $u(\cdot)$ is absolutely continuous with $\dot{u}(t) = g(t)$ for almost t and

$$\dot{u}_n(\cdot) \to \dot{u}(\cdot) \quad \text{weakly in } L^1(I,H). \tag{39}$$

Step 3. Let us prove that $u(\cdot)$ is a solution of (28) on I.
Since $\theta_n(t) \to t$ for any $t \in I$, and $(u_n(\cdot))_n$ converges uniformly to $u(\cdot)$, we have $u_n(\theta_n(t)) \to u(t)$ for each $t \in I$ and $u(t) \in B[u_0, \eta_0] \cap C$. The continuity of the mapping $f(t,\cdot)$ on $B(u_0,\alpha)$ entails that for every $t \in I$,

$$f(t, u_n(\theta_n(t))) \to f(t, u(t))$$

and further

$$\|f(t, u(t))\| \leq \beta(t).$$

We deduce that the sequence $\left(f(\cdot, u_n(\theta_n(\cdot)))\right)_n$ converges strongly in $L^1(I,H)$ to $f(\cdot, u(\cdot))$ and hence $\left(\dot{u}_n(\cdot) - f(\cdot, u_n(\theta_n(\cdot)))\right)_n$ converges weakly in $L^1(I,H)$ to $\dot{u}(\cdot) - f(\cdot, u(\cdot))$. Using Mazur's lemma, there exists a sequence $(\zeta_n(\cdot))$ which converges strongly in $L^1(I,H)$ to $\dot{u}(\cdot) - f(\cdot, u(\cdot))$ with

$$\zeta_n(\cdot) \in \mathrm{co}\{\, \dot{u}_k(\cdot) - f(\cdot, u_k(\theta_n(\cdot))) : \, k \geq n \,\}$$

for each n and for all $t \in I$. Extracting a subsequence if necessary, we may also suppose that the convergence of $(\zeta_n(\cdot))_n$ to $\dot{u}(\cdot) - f(\cdot, u(\cdot))$ also holds almost everywhere on I. Consequently, for almost all $t \in I$

$$-\dot{u}(t) + f(t, u(t)) \in \bigcap_n \overline{\mathrm{co}}\{\, -\dot{u}_k(t) + f(t, u_k(\theta_k(t))) : \, k \geq n \,\}.$$

It follows like for (27) that for almost all $t \in I$, for any $\xi \in H$,

$$\langle \xi, -\dot{u}(t) + f(t, u(t)) \rangle \leq \inf_n \sup_{k \geq n} \langle \xi, -\dot{u}_k(t) + f(t, u_k(\theta_k(t))) \rangle,$$

then by the equality $\partial_P d_S(y) = N^P(S; y) \cap \mathbb{B}$ for $y \in S$ and by (32) and (35), we obtain

$$\begin{aligned}\langle \xi, -\dot{u}(t) + f(t, u(t)) \rangle &\leq \beta(t) \limsup_n \sigma(\partial_P d_C(u_n(t)), \xi) \\ &\leq \beta(t) \limsup_n \sigma(\partial_C d_C(u_n(t)), \xi).\end{aligned}$$

Since $\sigma(\partial_C d_C(\cdot), \xi)$ is upper semicontinuous on H (see (15)), we deduce for almost all $t \in I$ that

$$\langle \xi, -\dot{u}(t) + f(t, u(t)) \rangle \leq \beta(t) \sigma(\partial_C d_C(u(t)), \xi) \quad \text{for all } \xi \in H,$$

hence by the closedness and convexity of $\partial_C d_C(u(t))$ we see that

$$-\dot{u}(t) + f(t, u(t)) \in \beta(t) \partial_C d_C(u(t)) \subset N(C; u(t)).$$

Further, $u(T_0) = \lim_n u_n(T_0) = u_0$, so $u(\cdot)$ is an absolutely continuous solution of (28) on I, and Lemma 3.1 gives

$$\|\dot{u}(t)\| \leq 2\|f(t, u(t))\| \leq 2\beta(t),$$

hence $\|u(t) - u_0\| \leq \int_{T_0}^t \beta(s) ds < \eta_0$, which completes the proof of the statement concerning the existence of a solution with values in $B(u_0, \eta_0) \cap C$.

Uniqueness. We proceed to showing that the uniqueness follows from the hypomonotonicity property of the normal cone and the Lipschitz condition on f. Consider two solutions $u_1(\cdot)$ and $u_2(\cdot)$ with $u_i : I \to B(u_0, \alpha) \cap C$. The hypomonotonicity property of the normal cone on $C \cap B(u_0, \alpha)$ (see Proposition 2.3) and Lemma 3.1 yield for almost $t \in I$

$$\begin{aligned}&\langle \dot{u}_1(t) - f(t, u_1(t)) - \dot{u}_2(t) + f(t, u_2(t)), u_1(t) - u_2(t) \rangle \\ \leq\ & \frac{1}{r}(\|f(t, u_1(t))\| + \|f(t, u_2(t))\|) \|u_1(t) - u_2(t)\|^2.\end{aligned}$$

Since $\|f(t, u(t)\| \leq \beta(t)$, one obtains

$$\begin{aligned}&\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t) \rangle \\ \leq\ & \frac{2\beta(t)}{r} \|u_1(t) - u_2(t)\|^2 + \langle f(t, u_1(t)) - f(t, u_2(t)), u_1(t) - u_2(t) \rangle.\end{aligned}$$

By assumption, $f(t, \cdot)$ is $k(t)$-Lipschitz continuous on $B(u_0, \alpha)$, so

$$\frac{1}{2}\frac{d}{dt}(\|u_1(t) - u_2(t)\|^2) \le \frac{1}{r}\left(2\beta(t) + k(t)\right)\|u_1(t) - u_2(t)\|^2 .$$

According to Gronwall's lemma (see Lemma 3.3), it follows that $u_1(\cdot) = u_2(\cdot)$ on I. The proof of the theorem is then complete. □

Another view of Theorem 3.5 is the following:

Theorem 3.6. *Let H be a Hilbert space. Assume that the closed set C is (r, α)-prox-regular at the point $u_0 \in C$ and let any real number $\eta_0 \in]0, \alpha/2[$. Let $f : [T_0; T] \times B(u_0, \alpha) \to H$ be a mapping which is Bochner measurable in $t \in [T_0, T]$ with respect to the Lebesgue measure λ and such that:*

(*i*) *the mapping $f(\cdot, u_0)$ is Bochner integrable on $[T_0, T]$ with respect to the Lebesgue measure;*

(*ii*) *there exists a non-negative function $k(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for all $t \in [T_0, T]$ and for all $(x, y) \in B(u_0, \alpha) \times B(u_0, \alpha)$,*

$$\|f(t, x) - f(t, y)\| \le k(t)\|x - y\|.$$

Then for all positive real $\tau \le T - T_0$ such that

$$\int_{T_0}^{T_0+\tau} (2\alpha k(t) + \|f(t, u_0)\|)\, dt < \min\left(\eta_0/2, (\alpha - \eta_0)/2\right),$$

the following variational differential inequality

$$\begin{cases} -\dot{u}(t) + f(t, u(t)) \in N(C; u(t)) \\ u(T_0) = u_0 \in C \end{cases}$$

has one and only one absolutely continuous solution $u(\cdot)$ on $[T_0, T_0 + \tau]$ satisfying $u([T_0, T_0 + \tau]) \subset B(u_0, \alpha)$. Further the solution even stays in $B(u_0, \eta_0)$, that is,

$$u(t) \in B(u_0, \eta_0) \quad \textit{for all } t \in [T_0, T_0 + \tau].$$

Proof. Put

$$\beta(t) := 2\alpha k(t) + \|f(t, u_0)\| \quad \text{for all } t \in [T_0, T],$$

and observe that $\beta \in L^1([T_0, T], \mathbb{R})$ and

$$\|f(t, x)\| \le \beta(t) \quad \text{for all } t \in [T_0, T] \text{ and } x \in B(u_0, \alpha).$$

Then it is enough to apply Theorem 3.5. □

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