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Fixed Point Theorems for Monotone Generalized Contractions in Partially Ordered Metric Spaces and Applications to Integral Equations*

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The purpose of this paper is to present some fixed point theorems for monotone generalized contractions in a complete metric space endowed with a partial order. Some results appearing in [9] and in [12] can be obtained as particular cases of our theorems. An application to integral equations is presented in order to illustrate our results.

Keywords: Monotone generalized mapping, partially ordered set, altering distance function

1991 Mathematics Subject Classification: 47H10

1. Introduction and background

The Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in the vast applicability in a number of branches of mathematics. Generalization of the above principle has been a heavily investigated branch of research. In particular there has been a number of works involving altering distance functions.

There are control functions which alter the distance between two points in a metric space. Such functions were introduced by Khan et al. in [1], where they present some fixed point theorems with the help of such functions.

Recently, the authors presented in [2] some fixed point theorems in partially ordered metric spaces using altering distance functions.

Existence of fixed point in partially ordered sets has been considered recently in [2–16]. The main idea in these works involves a combination between iterative

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technique ideas of the contraction principle with the monotone iteration approach [17].

Previously, we recall some definitions and results.

Definition 1.1. An altering distance function is a function $\psi\colon [0,\infty) \to [0,\infty)$ satisfying

a) ψ is continuous and nondecreasing.
b) $\psi(t) = 0$ if and only if $t = 0$.

If $(X, \leq)$ is a partially ordered set and $f\colon X \to X$ then f is said to be nondecreasing if for any $x, y \in X$ and $x \leq y$ then $f(x) \leq f(y)$.

The main result in [2] is the following fixed point theorem.

Theorem 1.2 (see [2], Theorem 2.1). *Let $(X, \leq)$ be a partially ordered set and suppose that there exist a metric d in X such that (X, d) is a complete metric space. Let $f\colon X \to X$ be a nondecreasing mapping such that*

$$\psi\left(d\left(f(x), f(y)\right)\right) \leq \psi\left(d(x,y)\right) - \phi\left(d(x,y)\right), \quad \text{for } x \geq y,$$

where ψ and ϕ are altering distance functions.
Suppose either

(*i*) *f is continuous or*
(*ii*) *if (x_n) is a nondecreasing sequence such that $x_n \to x$ then $x_n \leq x$ for all $n \in \mathbb{N}$.*

If there exist $x_0 \in X$ with $x_0 \leq f(x_0)$ then f has a fixed point.

Recently, in [18] the author proves the equivalence between the contractive condition appearing in Theorem 1.2 with classical ones.
On the other hand, in [14] the authors prove some common fixed point theorems in partially ordered metric spaces under different assumptions that the ones used in this paper.
The purpose of this paper is to present a generalization of Theorem 1.2.
Therefore, in view of [18] our results generalize recent results (see [9, 12], for example).
We will use the concept of g-monotone mapping which was introduced by L. Ćirić et al. in [19].

2. Fixed point theorems

We start this section with the following definition.

Definition 2.1. Suppose $(X, \leq)$ is a partially ordered set and $f, g\colon X \to X$. Then f is said to be g-nondecreasing if for any $x, y \in X$

$$g(x) \leq g(y) \Rightarrow f(x) \leq f(y).$$

Remark 2.2. Notice that if $g = Id_X$ is the identity on X then the definition of g-nondecreasing function coincides with the classical definition of nondecreasing function.

Remark 2.3. There exist g-nondecreasing functions and not nondecreasing.
For example, we can consider $(\mathbb{R}, \leq)$ with the usual order in $\mathbb{R}$ and $f, g\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$ and $g(x) = x^2$. Obviously, f is a g-nondecreasing function and it is not nondecreasing.

Definition 2.4. Suppose $f, g\colon X \to X$. An element $x \in X$ is a coincidence point (common fixed point) of f and g if $f(x) = g(x)$ $(f(x) = g(x) = x)$.

In what follows we present the main result of the paper.

Theorem 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f, g\colon X \to X$ are such that $f(X) \subset g(X)$, f is a g-nondecreasing mapping, $g(X)$ is closed and, if $(g(x_n)) \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X)$ then $g(x_n) \leq g(z)$ for all $n \in \mathbb{N}$ and $g(z) \leq g(g(z))$.*
Moreover, suppose that

$$\psi\left(d\left(f(x), f(y)\right)\right) \leq \psi\left(d\left(g(x), g(y)\right)\right) - \phi\left(d\left(g(x), g(y)\right)\right), \quad \textit{for } g(x) \geq g(y),$$

where ψ and ϕ are altering distance functions.
If there exists $x_0 \in X$ with $g(x_0) \leq f(x_0)$, then f and g have a coincidence point. Further, if f and g conmute at their coincidence points then f and g have a common fixed point.

Proof. Let $x_0 \in X$ be such that $g(x_0) \leq f(x_0)$. Since $f(X) \subset g(X)$ we can choose $x_1 \in X$ so that $g(x_1) = f(x_0)$. Again, from $f(X) \subset g(X)$ we can find $x_2 \in X$ so that $g(x_2) = f(x_1)$.
Continuing this process we find a sequence (x_n) in X such that

$$g(x_{n+1}) = f(x_n) \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

For better readability we divide the proof in several steps.

Step 1: $(f(x_n))$ is a nondecreasing sequence. We will use the mathematical induction.
Since $g(x_0) \leq f(x_0)$ and $f(x_0) = g(x_1)$ we have $g(x_0) \leq g(x_1)$. Using the fact that f is g-nondecreasing

$$f(x_0) \leq f(x_1)$$

and, consequently, our claim is satisfied for $n = 0$.
Suppose that $f(x_{n-1}) \leq f(x_n)$ for $n \in \mathbb{N}^*$, then, by (1) this means that

$$g(x_n) \leq g(x_{n+1})$$

and, again using the fact that f is g-nondecreasing

$$f(x_n) \leq f(x_{n+1}).$$

This proves *Step 1*.

Step 2: $\lim_{n\to\infty} d\left(f(x_n), f(x_{n+1})\right) = 0$. In fact, if $d\left(f(x_n), f(x_{n+1})\right) = 0$ for some $n \in \mathbb{N}$, then by (1), we have

$$g(x_{n+1}) = f(x_n) = f(x_{n+1})$$

and f and g have a coincidence at $x = x_{n+1}$ and this finishes the proof of the theorem.
Suppose that $d\left(f(x_n), f(x_{n+1})\right) > 0$ for all $n \in \mathbb{N}$. As, by *Step 1* $f(x_{n-1}) = g(x_n) \leq f(x_n) = g(x_{n+1})$, using the contractive condition of our theorem we have

$$\begin{aligned}\psi\left(d\left(f(x_n), f(x_{n+1})\right)\right) &\leq \psi\left(d\left(g(x_n), g(x_{n+1})\right)\right) - \phi\left(d\left(g(x_n), g(x_{n+1})\right)\right)\\ &= \psi\left(d\left(f(x_{n-1}), f(x_n)\right)\right) - \phi\left(d\left(f(x_{n-1}), f(x_n)\right)\right)\\ &\leq \psi\left(d\left(f(x_{n-1}), f(x_n)\right)\right).\end{aligned} \tag{2}$$

Using the fact that ψ is nondecreasing, the last inequality gives us

$$d\left(f(x_n), f(x_{n+1})\right) \leq d\left(f(x_{n-1}), f(x_n)\right).$$

Therefore $\left(d\left(f(x_n), f(x_{n+1})\right)\right)$ is nondecreasing and nonnegative sequence and, thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d\left(f(x_n), f(x_{n+1})\right) = r. \tag{3}$$

In the sequel, we will prove that $r = 0$.
In fact, by (3) and as ψ and ϕ are continuous and nonnegative functions, letting $n \to \infty$ in (2) we get

$$\psi(r) \leq \psi(r) - \phi(r) \leq \psi(r)$$

and the last inequality gives us $\phi(r) = 0$. Finally, as ϕ is an altering distance function $r = 0$.
This proves *Step 2*.

Step 3: $(f(x_n))$ is a Cauchy sequence. Otherwise, there exists $\epsilon > 0$ for which we can find subsequences $\left(f(x_{m(k)})\right)$ and $\left(f(x_{n(k)})\right)$ of $(f(x_n))$ with $n(k) > m(k) > k$ such that

$$d\left(f(x_{n(k)}), f(x_{m(k)})\right) \geq \epsilon. \tag{4}$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (4). Then

$$d\left(f(x_{n(k)-1}), f(x_{m(k)})\right) < \epsilon. \tag{5}$$

Using (4), (5) and the triangular inequality we have

$$\begin{aligned}\epsilon &\leq d\left(f(x_{n(k)}), f(x_{m(k)})\right)\\ &\leq d\left(f(x_{n(k)}), f(x_{n(k)-1})\right) + d\left(f(x_{n(k)-1}), f(x_{m(k)})\right)\\ &< d\left(f(x_{n(k)}), f(x_{n(k)-1})\right) + \epsilon.\end{aligned}$$

Using *Step 2* and letting $k \to \infty$ in the last inequality we obtain

$$\lim_{k\to\infty} d\left(f(x_{n(k)}), f(x_{m(k)}\right) = \epsilon. \tag{6}$$

Again, the triangular inequality gives us

$$\begin{aligned}
\epsilon &\leq d\left(f(x_{n(k)}), f(x_{m(k)})\right) \\
&\leq d\left(f(x_{n(k)}), f(x_{n(k)-1})\right) + d\left(f(x_{n(k)-1}), f(x_{m(k)-1})\right) + d\left(f(x_{m(k)-1}), f(x_{m(k)})\right) \\
&\leq d\left(f(x_{n(k)}), f(x_{n(k)-1})\right) + d\left(f(x_{n(k)-1}), f(x_{m(k)})\right) \\
&\qquad + d\left(f(x_{m(k)}), f(x_{m(k)-1})\right) + d\left(f(x_{m(k)-1}), f(x_{m(k)})\right) \\
&\leq d\left(f(x_{n(k)}), f(x_{n(k)-1})\right) + 2 \cdot d\left(f(x_{m(k)-1}), f(x_{m(k)})\right) + \epsilon.
\end{aligned}$$

Letting $k \to \infty$ in the last inequality and taking into account *Step 2* we get

$$\lim_{k\to\infty} d\left(f(x_{n(k)-1}), f(x_{m(k)-1})\right) = \epsilon. \tag{7}$$

Now, as $n(k) > m(k)$ and $(f(x_n))$ is a nondecreasing sequence (*Step 1*), we have $f(x_{n(k)+1}) = g(x_{n(k)}) \geq f(x_{m(k)+1}) = g(x_{m(k)})$ and using the contractive condition putting $x = x_{n(k)}$ and $y = x_{m(k)}$ we obtain

$$\begin{aligned}
&\psi\left(d\left(f(x_{n(k)}), f(x_{m(k)})\right)\right) \\
&\leq \psi\left(d\left(g(x_{n(k)}), g(x_{m(k)})\right)\right) - \phi\left(d\left(g(x_{n(k)}), g(x_{m(k)})\right)\right) \\
&= \psi\left(d\left(f(x_{n(k)-1}), f(x_{m(k)-1})\right)\right) - \phi\left(d\left(f(x_{n(k)-1}), f(x_{m(k)-1})\right)\right).
\end{aligned}$$

Taking into account (6), (7) and the fact that ψ and ϕ are continuous functions, letting $k \to \infty$ in the last inequality, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) \leq \psi(\epsilon)$$

and, consequently, $\phi(\epsilon) = 0$. This gives us $\epsilon = 0$ which is a contradiction.
This shows that $(f(x_n))$ is a Cauchy sequence.

Step 4: f and g have a coincidence. In fact, since $f(x_n) = g(x_{n+1}) \in g(X)$ for all $n \in \mathbb{N}$ and $g(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = g(z). \tag{8}$$

In the sequel, we prove that z is a coincidence point of f and g.
In fact, by *Step 1*, $(f(x_n)) = (g(x_{n+1}))$ is a nondecreasing sequence, and, by (8), $g(x_n) \to g(z)$, consequently, by our assumption $g(x_n) \leq g(z)$ for all $n \in \mathbb{N}$. Applying the contractive assumption we have

$$\begin{aligned}
\psi\left(d\left(f(x_n), f(z)\right)\right) &\leq \psi\left(d\left(g(x_n), g(z)\right)\right) - \phi\left(d\left(g(x_n), g(z)\right)\right) \\
&= \psi\left(d\left(f(x_{n-1}), g(z)\right)\right) - \phi\left(d\left(f(x_{n-1}), g(z)\right)\right),
\end{aligned}$$

and, by (8) and letting $n \to \infty$ in the last inequality we have

$$\begin{aligned}\psi\left(d\left(g(z), f(z)\right)\right) &\leq \psi\left(d\left(g(z), g(z)\right)\right) - \phi\left(d\left(g(z), g(z)\right)\right)\\ &= \psi\left(0\right)) - \phi\left(0\right))\\ &= 0,\end{aligned}$$

and this gives us, $\psi\left(d\left(g(z), f(z)\right)\right) = 0$. Using the fact that ψ is an altering distance function, $d\left(g(z), f(z)\right) = 0$, or equivalently, $f(z) = g(z)$.
This proves that f and g have a coincidence point.

Step 5: If f and g conmute at coincidence points then f and g have a common fixed point. In fact, put $w = f(z) = g(z)$, where z is a coincidence point (whose existence is proved in *Step 4*).
Since f and g conmute in z we have

$$f(w) = f\left(g(z)\right) = g\left(f(z)\right) = g(w). \tag{9}$$

Since $g(z) \leq g\left(g(z)\right) = g(w)$ by our assumption, applying the contractive condition we obtain

$$\begin{aligned}\psi\left(d\left(f(w), f(z)\right)\right) &= \psi\left(d\left(f(w), w\right)\right)\\ &\leq \psi\left(d\left(g(w), g(z)\right)\right) - \phi\left(d\left(g(w), g(z)\right)\right)\\ &= \psi\left(d\left(f(w), w\right)\right) - \phi\left(d\left(f(w), w\right)\right)\\ &\leq \psi\left(d\left(f(w), w\right)\right),\end{aligned}$$

and this gives us, $d\left(f(w), w\right) = 0$, or equivalently,

$$f(w) = w. \tag{10}$$

Finally, by (9) and (10)

$$f(w) = g(w) = w.$$

This finishes the proof. □

Now, we present an example which shows that the hypotheses in Theorem 2.5 do not guarantee the uniqueness of the coincidence point.
Let $X = \{(1,0), (0,1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x, y) \leq (z, t) \Leftrightarrow x \leq z \text{ and } y \leq t.$$

Thus, $(X, \leq)$ is a partially ordered set whose different elements are not comparable. Besides, (X, d_2) is a complete metric space considering d_2 the Euclidean distance.
Put $f = g = Id_X$ (the identity mapping).
The contractive condition appearing in Theorem 2.5 is obviously satisfied since elements in X are only comparable to themselves. It is easily proved that f and g satisfy the others assumptions of Theorem 2.5 and $(1,0) = g\left((1,0)\right) \leq (1,0) = f(1,0)$. Notice that f and g have two coincidence points.

In the following result we present a sufficient condition for the uniqueness of the coincidence point.

Theorem 2.6. *Under the assumptions of Theorem* 2.5 *also suppose that* $g(X)$ *is a totally ordered set and* g *is an injective mapping. Then we obtain uniqueness of the coincidence point. Moreover, if* f *and* g *commute at the coincidence point then* f *and* g *have an unique common fixed point.*

Proof. Suppose that f and g have two coincidence points y and z. Then

$$\begin{aligned} f(y) &= g(y) = u, \\ f(z) &= g(z) = v. \end{aligned}$$

As $g(X)$ is totally ordered set and $g(y)$, $g(z) \in g(X)$ suppose that $g(y) \leq g(z)$. Applying the contractive condition we have

$$\begin{aligned} \psi\left(d\left(u,v\right)\right) &= \psi\left(d\left(f(y),f(z)\right)\right) \\ &\leq \psi\left(d\left(g(y),g(z)\right)\right) - \phi\left(d\left(g(y),g(z)\right)\right) \\ &= \psi\left(d\left(u,v\right)\right) - \phi\left(d\left(u,v\right)\right) \\ &\leq \psi\left(d\left(u,v\right)\right), \end{aligned}$$

and the last inequality gives us $\phi\left(d\left(u,v\right)\right) = 0$ and, as ϕ is an altering distance function, $d\left(u,v\right) = 0$, and consequently, $u = v$.
Thus, $g(y) = u = v = g(z)$.
The injectivity of g gives us $y = z$.
Finally, suppose that z_1 and z_2 are two common fixed points of f and g.
Then, as

$$\begin{aligned} z_1 &= f(z_1) = g(z_1), \\ z_2 &= f(z_2) = g(z_2), \end{aligned}$$

z_1 and z_2 are two coincidence points of f and g and, by the uniqueness of the coincidence points above proved,

$$z_1 = z_2$$

This finishes the proof. □

In what follows, we present some corollaries of Theorem 2.5.

Corollary 2.7. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Suppose* $f, g \colon X \to X$ *are such that* $f(X) \subset g(X)$*,* f *is a* g*-nondecreasing mapping,* $g(X)$ *is closed and*

> *if* $(g(x_n)) \subset X$ *is a nondecreasing sequence with* $g(x_n) \to g(z)$ *in* $g(X)$ *then* $g(x_n) \leq g(z)$ *for all* $n \in \mathbb{N}$ *and* $g(z) \leq g\left(g(z)\right)$.

Moreover, if there exists $k \in [0, 1)$ *such that*

$$d\left(f(x), f(y)\right) \leq k\ d\left(g(x), g(y)\right) \quad \text{for } g(x) \geq g(y),$$

and there exists $x_0 \in X$ *with* $g(x_0) \leq f(x_0)$ *then* f *and* g *have a coincidence. Further, if* f *and* g *commute at their coincidence points then* f *and* g *have a common fixed point.*

Proof. Applying Theorem 2.5 for $\psi = Id_{\mathbb{R}}$ (identity mapping) and $\phi = c \cdot Id_{\mathbb{R}}$ with $c \in (0,1]$ and $1-c = k$ (which are obviously altering distance functions) we obtain the corollary. $\square$

Corollary 2.7 is a generalization of Theorem 2.2 of [9]. If in Corollary 2.7 we have as $g = Id_X$ we obtain Theorem 2.2 of [9].
Another consequence of Theorem 2.5 is the following fixed point theorem with a contractive condition of integral type.

Corollary 2.8. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f, g\colon X \to X$ are such that $f(X) \subset g(X)$, f is a g-nondecreasing mapping, $g(X)$ is closed and*

if $(g(x_n)) \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X)$ then $g(x_n) \leq g(z)$ for all $n \in \mathbb{N}$ and $g(z) \leq g(g(z))$.

Moreover, if there exists $k \in [0,1)$ such that

$$\int_0^{d(f(x),f(y))} \rho(t)\mathrm{d}t \leq k \cdot \int_0^{d(g(x),g(y))} \rho(t)\mathrm{d}t \quad \text{for } g(x) \geq g(y),$$

where $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping satisfying that $\int_0^{\epsilon} \rho(t)\mathrm{d}t > 0$ for $\epsilon > 0$, and there exists $x_0 \in X$ with $g(x_0) \leq f(x_0)$ then f and g have a coincidence. Further, if f and g conmute at their coincidence points then f and g have a common fixed point.

Proof. It is easily proven that the function $\rho\colon [0,\infty) \to [0,\infty)$ defined by

$$\int_0^t \psi(s)\mathrm{d}s > 0$$

is an altering distance function.
Applying Theorem 2.5 with the altering distance function above defined and $\phi = (1-k)\psi$ we obtain the corollary. $\square$

3. An application

In this section we present an example which illustrates the applicability of Theorem 2.5.
Previously, we recall some basic facts.
We consider the space $\mathcal{C}[0,1]$ of continuous functions defined on $[0,1]$. This space with the usual metric given by

$$d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|, \quad \text{for } x, y \in \mathcal{C}[0,1],$$

is a complete metric space. $\mathcal{C}[0,1]$ con also be equipped with a partial order given by

$$x, y \in \mathcal{C}[0,1], \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0,1].$$

Moreover, in [7] it is proved that if (x_n) is a nondecreasing sequence in $\mathcal{C}[0,1]$ and $x_n \to x$ then $x_n \leq x$ for all $n \in \mathbb{N}$.
Now, we consider $X = \mathcal{C}^+[0,1] = \{x \in \mathcal{C}[0,1] \colon x \geq 0\}$. Obviously, X is a closed set of $\mathcal{C}[0,1]$ and, consequently, (X,d), where d is the above mentioned metric, is a complete metric space.
Now, we consider the mapping $g \colon X \to X$ given by

$$(gx)(t) = e^{x(t)} - 1, \quad \text{for } x \in X.$$

It is easily seen that $g(X) = X$ is closed, g is a nondecreasing mapping and $g(x) \leq g(g(x))$ for $x \in X$ (notice that using elemental calculus it can be proven that $x \leq g(x)$ for $x \in X$).
Now, we claim that if $(g(x_n)) \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X) = X$ then $g(x_n) \leq g(z)$ for all $n \in \mathbb{N}$ and $g(z) \leq g(g(z))$.
In fact, if $(g(x_n)) \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X) = X$, then for every $t \in [0,1]$ we get

$$g(x_1)(t) \leq g(x_2)(t) \leq g(x_3)(t) \leq \cdots \leq g(x_n)(t) \leq \cdots$$

and the convergence of this nondecreasing sequence of real numbers to $g(z)(t)$ implies

$$g(x_n)(t) \leq g(z)(t), \quad \text{for all } n \in \mathbb{N}$$

and, therefore, $g(x_n) \leq g(z)$ for all $n \in \mathbb{N}$.
On the other hand, as $g(x) \leq g(g(x))$ for every $x \in X$, particularly, $g(z) \leq g(g(z))$. This proves the claim.
Now, we consider the following operator defined on X by

$$(fx)(t) = \int_0^1 K(t, s, (gx)(s))\, \mathrm{d}s, \quad \text{for } x \in X \text{ and } t \in [0,1]. \tag{11}$$

We assume that

(i) $K \colon [0,1] \times [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.
(ii) $K(t,s,\cdot) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function for any $t, s \in [0,1]$.

Under assumptions (i) and (ii) it is easily proved that the operator f applies X into itself. Moreover, assumption (ii) gives us that f is a g-nondecreasing mapping.
Now, we consider the following assumption:

(iii) There exists a continuous function $\rho \colon [0,1] \times [0,1] \to \mathbb{R}_+$

$$|K(t,s,u) - K(t,s,v)| \leq \rho(t,s)\sqrt{\ln\left[(u-v)^2 + 1\right]}$$

for any $t, s \in [0,1]$ and $u, v \in \mathbb{R}_+$ and, moreover, this function satisfies that

$$\sup_{t \in [0,1]} \left(\int_0^1 \rho(t,s)^2 \mathrm{d}s \right)^{\frac{1}{2}} \leq 1$$

.

Theorem 3.1. *Under the assumptions* (*i*), (*ii*) *and* (*iii*) *the integral equation* (11) *has at least one nonnegative solution in* $\mathcal{C}[0,1]$.

Proof. Firstly, we check that the contractive condition appearing in Theorem 2.5 is satisfied.
In fact, we take $x, y \in X$ with $g(x) \geq g(y)$.
Since f is a g-nondecreasing mapping and taking into account our assumptions we get

$$\begin{aligned} d\left(f(x), f(y)\right) &= \sup_{t\in[0,1]} |f(x)(t) - f(y)(t)| \\ &= \sup_{t\in[0,1]} \left(f(x)(t) - f(y)(t)\right) \\ &= \sup_{t\in[0,1]} \int_0^1 (K(t,s,g\left(x\right)(s)) - K(t,s,g\left(y\right)(s))\,\mathrm{d}s \\ &\leq \sup_{t\in[0,1]} \int_0^1 \rho(t,s)\sqrt{\ln\left[\left(g\left(x\right)(s) - g\left(y\right)(s)\right)^2 + 1\right]}\mathrm{d}s. \end{aligned}$$

Using the Cauchy-Schwarz inequality in the last integral we have

$$\begin{aligned} & d\left(f(x), f(y)\right) \\ \leq & \sup_{t\in[0,1]} \left(\int_0^1 \rho(t,s)^2 \mathrm{d}s\right)^{\frac{1}{2}} \cdot \left(\int_0^1 \ln\left[\left(g\left(x\right)(s) - g\left(y\right)(s)\right)^2 + 1\right] \mathrm{d}s\right)^{\frac{1}{2}} \\ \leq & \left(\sup_{t\in[0,1]} \int_0^1 \rho(t,s)^2 \mathrm{d}s\right)^{\frac{1}{2}} \cdot \left(\ln\left[d\left(g(x), g(y)\right)^2 + 1\right]\right)^{\frac{1}{2}} \\ \leq & \left(\ln\left[d\left(g(x), g(y)\right)^2 + 1\right]\right)^{\frac{1}{2}}. \end{aligned}$$

Put $\psi(x) = x^2$ and $\phi(x) = x^2 - \ln\left(x^2 + 1\right)$. Obviously, ψ and ϕ are altering distance functions.
Therefore, from the last inequality we obtain

$$d\left(f(x), f(y)\right)^2 \leq d\left(g(x), g(y)\right)^2 - \left(d\left(g(x), g(y)\right)^2 - \ln\left[d\left(g(x), g(y)\right)^2 + 1\right]\right),$$

or, equivalently,

$$\psi\left(d\left(f(x), f(y)\right)\right) \leq \psi\left(d\left(g(x), g(y)\right)\right) - \phi\left(d\left(g(x), g(y)\right)\right).$$

This proves the contractive condition appearing in Theorem 2.5.
Finally, as $g(0) = 0 \leq f(0)$ (because $K(t,s,u) \geq 0$ for $t, s \in [0,1]$ and $u \in \mathbb{R}_+$), Theorem 2.5 gives us the existence of a coincidence point in X for f and g.
This means that there exists $x \in \mathcal{C}[0,1]$ with $x \geq 0$ such that

$$\int_0^1 K\left(t, s, e^{x(s)} - 1\right) \mathrm{d}s = e^{x(t)} - 1, \quad \text{for any } t \in [0,1].$$

□

In what follows, we refine Theorem 3.1 proving that the solution $x(t)$ is positive (this means that $0 < x(t)$ for $t \in (0,1)$).

Theorem 3.2. *Under assumptions of Theorem* 3.1 *if additionally the following condition holds:*

(iv) *For each $t \in [0,1]$ there exists a set $A_t \subset [0,1]$ with $\mu(A_t) > 0$, where μ is the Lebesgue measure and such that $K(t,\cdot,0)\colon [0,1] \to \mathbb{R}_+$ satisfies $K(t,s,0) \neq 0$ for $s \in A_t$,*

then the solution of the integral equation (11) *is positive.*

Proof. By Theorem 3.1 there exists a coincidence point x for f and g such that $x(t) \geq 0$.
In the sequel, we prove that $0 < x(t)$ for $t \in (0,1)$.
Otherwise, suppose that there exists $0 < t_0 < 1$ such that $x(t_0) = 0$.
By the definition of coincidence point we have

$$f(x)(t) = \int_0^1 K(t,s,g(x)(s))\,\mathrm{d}s = \int_0^1 K\left(t,s,e^{x(s)}-1\right)\mathrm{d}s = g(x)(t) = e^{x(t)} - 1,$$

and, particularly, for $t = t_0$ we have

$$\int_0^1 K\left(t_0,s,e^{x(s)}-1\right)\mathrm{d}s = e^{x(t_0)} - 1 = 0.$$

As $x \geq 0$ and $K\colon I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$, $K\left(t_0,s,e^{x(s)}-1\right) \geq 0$ for each $s \in [0,1]$ and the nulity of the last integral gives us

$$K\left(t_0,s,e^{x(s)}-1\right) = 0 \quad \text{a.e. } (s).$$

As, $e^{x(s)} - 1 \geq 0$ because $x \geq 0$, and (ii) we obtain

$$0 \leq K(t_0,s,0) \leq K\left(t_0,s,e^{x(s)}-1\right) = 0 \quad a.e(s).$$

Consequently, $K(t_0,s,0) = 0$ a.e. (s).
This contradicts assumption (iv) because there exists $A_{t_0} \subset [0,1]$ with $\mu(A_{t_0}) > 0$ and $K(t_0,s,0) \neq 0$ for $s \in A_{t_0}$.
Thus, $0 < x(t_0)$.
This finishes the proof. □

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