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Lexicographical Representation of Convex Sets

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We introduce two new families of properties on convex sets of $\mathbb{R}^n$, in order to establish new theorems regarding open and closed separation of a convex set from any outside point by linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$, in the sense of the lexicographical order of $\mathbb{R}^m$, for each $m \in \{1, \dots, n\}$. We thus obtain lexicographical extensions of well known separation theorems for convex sets as well as characterizations of the solution sets of lexicographical (weak and strict) inequality systems defined by matrices of a given rank.

Keywords: Convex sets, open lexicographical separation, closed lexicographical separation

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1. Introduction

Separation theorems for convex sets are essential in the theory of convexity and in other fields of mathematics like optimization theory. The usual separation theorem for two nonempty disjoint convex subsets of $\mathbb{R}^n$ by a linear functional (or, equivalently, by a hyperplane) involves a very weak type of separation. Other well known separation theorems have appeared in the literature, but most of them deal with rather strong types of separation. In particular, Klee [3] obtained several maximal separation theorems involving four different types of separation, including the so called open and closed types of separation that we consider in this paper. On the other hand, the various known extensions of the usual separation theorem to separation by linear operators, in the sense of the usual order of $\mathbb{R}^n$, require rather strong assumptions as it has been remarked in [8]. A new theorem concerning the open separation of an arbitrary convex set in $\mathbb{R}^n$ and any outside point by orthogonal square matrices, in the sense of the lexicographical order of $\mathbb{R}^n$, has been given

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in [4]; we shall recall it in Theorem 2.7 below. Some applications of this separation theorem have been given in [4] (applications to quasiconvex conjugation theory), [7] (linear and convex inequality systems and optimization) and [6] (duality for vector optimization).

The motivation of this work is in extending the separation of a convex set and an outside point, in the above sense, to the case of open separation of a convex set in $\mathbb{R}^n$ and an outside point by non square matrices or, equivalently, by linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$, in the sense of the lexicographical order of $\mathbb{R}^m$, for each $m \in \{1, \ldots, n\}$ (the extreme cases $m = 1$ and $m = n$ are already well known). Furthermore, we shall characterize those convex sets that are openly lexicographically separated from any outside point. Analogously, we shall characterize those convex sets that are closedly lexicographically separated from any outside point by linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$, for each $m \in \{1, \ldots, n\}$. Thus, we shall obtain an analogue to Theorem 2.11 for this type of separation.

The layout of the paper is as follows. In Section 2 we recall the basic definitions and present some fundamental results on open and closed separation, which are crucial for developing new separation theorems. In Section 3 we define new properties on convex sets in order to derive new theorems concerning open and closed separation of a convex set from any outside point by linear operators.

2. Preliminaries

Let us recall some notions, notation and results, which we shall use in the sequel.

The elements of $\mathbb{R}^n$ will be considered column vectors, and the superscript T will mean transpose. Therefore, $u^T v$ corresponds to the standard inner product for vectors $u, v \in \mathbb{R}^n$. The convex hull and the closure of any subset C of $\mathbb{R}^n$ will be denoted by $\operatorname{co} C$ and $\operatorname{cl} C$, respectively. We also introduce the symbol $\mathbb{N}_p := \{1, \ldots, p\}$ for any $p \in \mathbb{N}$.

We recall from [8, 9] that $x := (x^1, \ldots, x^n)^T \in \mathbb{R}^n$ is said to be *lexicographically less than* $y := (y^1, \ldots, y^n)^T \in \mathbb{R}^n$ (in symbols, $x <_L y$) if $x \neq y$ and if for $k := \min\{i \in \mathbb{N}_n : x^i \neq y^i\}$ we have $x^k < y^k$. We write $x \leq_L y$ if $x <_L y$ or $x = y$. The notation $y >_L x$ and $y \geq_L x$, respectively, will also be used.

For $m \in \mathbb{N}_n$, we shall denote by $\mathbb{M}_{m,n}$ the family of all linear operators $A : \mathbb{R}^n \to \mathbb{R}^m$ and we shall identify each $A \in \mathbb{M}_{m,n}$ with its $m \times n$ matrix with respect to the unit vector bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. In the case $m = n$, we just write $\mathbb{M}_n$. We shall denote by $\mathbb{U}_{m,n}$ and $\mathbb{O}_{m,n}$ the families of all linear operators $A \in \mathbb{M}_{m,n}$ with $\operatorname{rank} A = m$, and all linear operators $A \in \mathbb{M}_{m,n}$ with $AA^T = I$ (I being the identity matrix of the appropriate size), respectively. In short, for $m = n$, we write $\mathbb{U}_n$ and $\mathbb{O}_n$, respectively. Given $A \in \mathbb{M}_{m,n}$ and $k \in \mathbb{N}_m$, the symbol $A^{(k)}$ stands for the submatrix consisting of the first k rows of A whereas the symbol a_k stands for the k-th row of A as a column vector. Analogously, if $z \in \mathbb{R}^m$ and $k \in \mathbb{N}_m$, we denote by $z^{(k)}$ and z^k the vector consisting of the first k components of z and the k-th component of z, respectively.

On the other hand, a set H in $\mathbb{R}^n$ is called a *hemispace* if both H and its complement

$\mathbb{R}^n \backslash H$ are convex. Clearly, $\emptyset$ and $\mathbb{R}^n$ are hemispaces. Open or closed halfspaces are also hemispaces. Moreover, every *semispace* (a maximal convex cone excluding its vertex) is also a hemispace [2, Theorem 2 (a)]. There is no need to justify again the importance of this definition, because this is done in [8, 9] and in other places. Several characterizations for a hemispace can be found in [9]. We point out the following one taken from [9, Theorem 1.1]: H is a hemispace if and only if there exist $A \in \mathbb{M}_n$ and $z \in \mathbb{R}^n$ such that either

$$H = \{x \in \mathbb{R}^n : Ax <_L z\} \tag{1}$$

or

$$H = \{x \in \mathbb{R}^n : Ax \leq_L z\}. \tag{2}$$

Definition 2.1. We will say that a hemispace $H \subset \mathbb{R}^n$ has *rank* $m \in \mathbb{N}_n \cup \{0\}$ if H can be written as in (1) or (2) with $A \in \mathbb{U}_{m,n}$ and $z \in \mathbb{R}^m$.

Following the conventions employed by [9], $\mathbb{M}_{0,n}$ contains the null operator as its unique element and $\mathbb{R}^0 := \{0\}$. As a consequence, $\emptyset$ and $\mathbb{R}^n$ are the unique hemispaces of rank 0.

Proposition 2.2 ([9, Theorem 2.1]). *Every hemispace $H \subset \mathbb{R}^n$ cannot simultaneously be represented as in* (1) *and* (2) *with matrices $A \in \mathbb{U}_{m,n}$, and its rank is uniquely determined.*

We shall use the following characterization theorem of faces of convex sets.

Theorem 2.3 ([5, Theorem 2]). *Let C be a convex subset of $\mathbb{R}^n$ and $\emptyset \neq F \subset C$. Then F is a face of C if and only if there exists $A \in \mathbb{M}_{k,n}$, for some $k \in \mathbb{N}_n \cup \{0\}$, such that*

$$F = \{y \in C : Ay = \max\nolimits_L \{Ax : x \in C\}\}. \tag{3}$$

According to [5, Definition 3], given a convex set $C \subset \mathbb{R}^n$ and a nonempty face F of C, the *degree of non-exposedness* of F with respect to C is defined by

$$d_C(F) := \min\{k : \exists\, A \in \mathbb{M}_{k,n} \text{ satisfying (3)}\} - 1.$$

Proposition 2.4 ([5, Proposition 4]). *Let $C \subset \mathbb{R}^n$ be a convex set and let F be a nonempty face of C. Then*

$$d_C(F) \leq \dim C - \dim F - 1,$$

where dim *denotes dimension of the affine hull.*

2.1. Open Separation

Consider C and D two subsets of $\mathbb{R}^n$. If $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the set C is *openly separated* [3] from D by the *open halfspace* $H := \left\{x \in \mathbb{R}^n : a^T x < \alpha\right\}$ provided that $C \subset H$ and $D \subset \mathbb{R}^n \backslash H$ or, equivalently, that $a^T x < \alpha \leq a^T y$ for all $x \in C$ and $y \in D$. This definition can be extended in a natural way to the lexicographical setting as follows: if $m \in \mathbb{N}_n$, $A \in \mathbb{M}_{m,n}$ and $z \in \mathbb{R}^m$, the set C is *openly lexicographically* *separated* from D by the *hemispace* $H := \{x \in \mathbb{R}^n : Ax <_L z\}$ provided that $C \subset H$ and $D \subset \mathbb{R}^n \backslash H$ or, equivalently, that $Ax <_L z \leq_L Ay$ for all $x \in C$ and $y \in D$.

On the other hand, a subset $C \subset \mathbb{R}^n$ is called *evenly convex* [1] provided that C is the intersection of a family of open halfspaces or, equivalently, that C is openly separated from every outside point by open halfspaces. It is well known that $\emptyset$ and $\mathbb{R}^n$ are evenly convex sets, and that any closed convex set is evenly convex. The evenly convex hull of a set $C \subset \mathbb{R}^n$, that is, the smallest evenly convex set that contains C, will be denoted by $\operatorname{eco} C$. From these definitions one easily gets the following two results:

Theorem 2.5. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *C is evenly convex.*
(*ii*) *For every $x_0 \in \mathbb{R}^n \backslash C$, there exists $a \in \mathbb{R}^n$ such that $a^T x < a^T x_0$ for all $x \in C$.*

Proposition 2.6. *Let $C \subset \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *$\bar{x} \notin \operatorname{eco} C$.*
(*ii*) *There exists $a \in \mathbb{R}^n$ such that $a^T x < a^T \bar{x}$ for all $x \in C$.*

Observe that in the previous results we can assume without loss of generality that the vector $a \in \mathbb{R}^n$ is nonzero, or even that $a \in \mathbb{R}^n$ is unitary.

The next theorem characterizes convex sets as those sets that can be openly lexicographically separated from every outside point.

Theorem 2.7 ([4, Separation Theorem]). *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *C is convex.*
(*ii*) *For every $x_0 \in \mathbb{R}^n \backslash C$, there exists $A \in \mathbb{M}_n$ such that $Ax <_L Ax_0$ for all $x \in C$.*

In statement (ii) one can take $A \in \mathbb{U}_n$, or even $A \in \mathbb{O}_n$.

Corollary 2.8 ([4, Corollary 1.1]). *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *C is convex.*
(*ii*) *$C = \{x \in \mathbb{R}^n : A^s x <_L b^s, \ \forall\, s \in S\}$ for some index set S, $A^s \in \mathbb{M}_n$ and $b^s \in \mathbb{R}^n$.*

In statement (ii) one can take $A \in \mathbb{U}_n$, or even $A \in \mathbb{O}_n$.

An easy consequence of this result is that a subset of $\mathbb{R}^n$ is convex if and only if it is an intersection of semispaces (see [2, Corollary 4]). Theorem 2.7 has been extended to the case of lexicographical separation of two subsets of $\mathbb{R}^n$, with disjoint convex hulls, by linear operators, or isomorphisms, or orthonormal matrices, in the sense of the lexicographical order of $\mathbb{R}^n$.

Theorem 2.9 ([8, Theorem 2.1]). *Let C and D be subsets of $\mathbb{R}^n$. Then $\operatorname{co} C \cap \operatorname{co} D = \emptyset$ if and only if there exists $A \in \mathbb{M}_n$ such that $Ax <_L Ay$ for all $x \in C$ and $y \in D$.*

The matrix A can be taken in $\mathbb{U}_n$, or even in $\mathbb{O}_n$.

From Theorem 2.7 we obtain that if $C \subset \mathbb{R}^n$ is convex and $x_0 \in \mathbb{R}^n \backslash C$, then C is openly lexicographically separated from x_0 by a hemispace. The following result generalizes Theorem 2.7 by establishing a lexicographical separation theorem between a convex set and any disjoint affine manifold. In the particular case when the manifold has dimension 0, we recover Theorem 2.7.

Theorem 2.10. *Let $C \subset \mathbb{R}^n$ be a convex set and let M be an affine manifold of dimension $n-m$ with $m \in \mathbb{N}_n$. Then $C \cap M = \emptyset$ if and only if there exists $A \in \mathbb{M}_{m,n}$ such that $Ax <_L Ay$ for all $x \in C$ and $y \in M$.*

The matrix A can be taken in $\mathbb{U}_{m,n}$, or even in $\mathbb{O}_{m,n}$.

Proof. We will only prove the "only if" statement, since the converse is obvious. Let H be a maximal convex set containing C and not intersecting M. Then H is a maximal convex set not intersecting M. Hence, by [9, Theorem 3.2], H is a hemispace of type $<_L$ and M is the linear manifold associated to M. Therefore, by [9, Theorem 2.1], there exist $z \in \mathbb{R}^m$ and $A \in \mathbb{M}_{m,n}$ ($A \in \mathbb{U}_{m,n}$ or even $A \in \mathbb{O}_{m,n}$) such that $H = \{x \in \mathbb{R}^n : Ax <_L z\}$ and $M = \{y \in \mathbb{R}^n : Ay = z\}$, which ends the proof. □

2.2. Closed Separation

Consider again C and D two subsets of $\mathbb{R}^n$. If $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the set C is *closedly separated* [3] from D by the *closed halfspace* $H := \{x \in \mathbb{R}^n : a^T x \leq \alpha\}$ provided that $C \subset H$ and $D \subset \mathbb{R}^n \backslash H$ or, equivalently, that $a^T x \leq \alpha < a^T y$ for all $x \in C$ and $y \in D$. This definition can be also extended in a natural way to the lexicographical setting: if $m \in \mathbb{N}_n$, $A \in \mathbb{M}_{m,n}$ and $z \in \mathbb{R}^m$, the set C is *closedly lexicographically separated* from D by the *hemispace* $H := \{x \in \mathbb{R}^n : Ax \leq_L z\}$ provided that $C \subset H$ and $D \subset \mathbb{R}^n \backslash H$ or, equivalently, that $Ax \leq_L z <_L Ay$ for all $x \in C$ and $y \in D$.

It is well known that every closed convex set in $\mathbb{R}^n$ is characterized by the closed separation from any outside point by closed halfspaces. The following two results can be found in [10] and in other places.

Theorem 2.11. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *C is convex and closed.*

(*ii*) *For every $x_0 \in \mathbb{R}^n \backslash C$, there exist $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $a^T x \leq \alpha < a^T x_0$ for all $x \in C$.*

Proposition 2.12. *Let $C \subset \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. The following statements are equivalent:*

(*i*) *$\bar{x} \notin \operatorname{cl} \operatorname{co} C$.*

(*ii*) *There exist $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $a^T x \leq \alpha < a^T \bar{x}$ for all $x \in C$.*

Again, observe that in the previous results we can assume without loss of generality that the vector $a \in \mathbb{R}^n$ is nonzero, or even that $a \in \mathbb{R}^n$ is unitary. In addition, as a consequence of these results we obtain that a subset of $\mathbb{R}^n$ is closed and convex if and only if it is an intersection of closed halfspaces.

3. Lexicographical Separation

In this section we will give lexicographical extensions of the classical separation theorems presented in the preceding section. More precisely, given a subset $C \subset \mathbb{R}^n$ and $m \in \mathbb{N}_n$, we determine the properties that C must satisfy in order to be characterized by open or by closed lexicographical separation from any outside point by a hemispace of rank m.

3.1. Open Lexicographical Separation

So as to answer the above question concerning open lexicographical separation, we shall consider the following definition:

Definition 3.1. Let $C \subset \mathbb{R}^n$ be a convex set and $m \in \mathbb{N}_n$. We will say that C satisfies property $\mathcal{O}(m)$ if for every face F of C with $\dim F \leq n-m+1$ and every $\bar{x} \in (\operatorname{eco} F) \setminus C$, there exists an affine manifold M with $\dim M = n-m+1$ satisfying the following conditions:

$$\bar{x} \in M, \tag{4a}$$

$$C \setminus M \quad \text{is convex}, \tag{4b}$$

$$\bar{x} \notin \operatorname{eco}(C \cap M). \tag{5}$$

Equivalently, C satisfies property $\mathcal{O}(m)$ if for every face F of C with $\dim F \leq n-m+1$ and every $\bar{x} \in \operatorname{eco} F$ such that there is no affine manifold M with $\dim M = n-m+1$ satisfying conditions (4a), (4b) and (5), one has $\bar{x} \in C$.

The following propositions characterize convex sets with property $\mathcal{O}(m)$ in the extreme cases $m = 1$ and $m = n$.

Proposition 3.2. *Let $C \subset \mathbb{R}^n$ be a convex set. The following statements are equivalent:*

(i) *C satisfies property $\mathcal{O}(1)$.*
(ii) *C is evenly convex.*

Proof. $(i) \Rightarrow (ii)$. Let $\bar{x} \in \operatorname{eco} C$. If $\bar{x} \notin C$, by property $\mathcal{O}(1)$ there exists an affine manifold M with $\dim M = n$ satisfying conditions (4a), (4b) and (5). Given that $M = \mathbb{R}^n$, we get a contradiction with $\bar{x} \in \operatorname{eco} C$. Thus $\operatorname{eco} C \subset C$ and, consequently, C is evenly convex.

$(ii) \Rightarrow (i)$. Let F be a face of C with $\dim F \leq n$. Given that $\operatorname{eco} F \subset \operatorname{eco} C = C$, we have $(\operatorname{eco} F) \setminus C = \emptyset$; hence property $\mathcal{O}(1)$ holds vacuously. □

Proposition 3.3. *Every convex set $C \subset \mathbb{R}^n$ satisfies property $\mathcal{O}(n)$.*

Proof. Let F be an edge of C and let $\bar{x} \in (\operatorname{eco} F) \setminus C$. Since F is unidimensional, it is evenly convex; hence $\operatorname{eco} F = F \subset C$ and, consequently, $(\operatorname{eco} F) \setminus C = \emptyset$. Therefore, property $\mathcal{O}(n)$ holds vacuously. □

The main result in this subsection extends Theorems 2.5 and 2.7 and characterizes convex sets with property $\mathcal{O}(m)$, for $m \in \mathbb{N}_n$, in terms of open lexicographical separation from any outside point by hemispaces of rank m. Its proof relies upon Theorem 2.7, which, in virtue of Proposition 3.3, corresponds to the particular case $m = n$.

Theorem 3.4. *Let $C \subset \mathbb{R}^n$ and $m \in \mathbb{N}_n$. The following statements are equivalent:*

(i) *C is convex and satisfies property $\mathcal{O}(m)$.*

(ii) *For every $x_0 \in \mathbb{R}^n \backslash C$, there exists $A \in \mathbb{U}_{m,n}$ such that $Ax <_L Ax_0$ for all $x \in C$.*

In statement (ii) one can take $A \in \mathbb{O}_{m,n}$.

Proof. In the case $m = 1$, by Proposition 3.2, the statement reduces to that of Theorem 2.5. Therefore we only need to consider the case $1 < m$.

$(i) \Rightarrow (ii)$ (with $A \in \mathbb{O}_{m,n}$). Pick any $x_0 \in \mathbb{R}^n \backslash C$. By Theorem 2.7, there exists $B \in \mathbb{O}_n$ such that $Bx <_L Bx_0$ for all $x \in C$, which implies $B^{(m-1)}x \leq_L B^{(m-1)}x_0$ for all $x \in C$. Let

$$F := \left\{x \in C : B^{(m-1)}x = B^{(m-1)}x_0\right\}.$$

Assume that F is nonempty, otherwise condition (ii) holds trivially. By Theorem 2.3, it is clear that F is a face of C with dimension not greater than $n - m + 1$.

If $x_0 \notin \operatorname{eco} F$, there exists a unitary vector $a \in \mathbb{R}^n$ such that $a^T x < a^T x_0$ for all $x \in F$. Without loss of generality, we can assume that a is orthogonal to the rows of $B^{(m-1)}$. Then (ii) holds with $A := \begin{pmatrix} B^{(m-1)} \\ a^T \end{pmatrix}$.

If $x_0 \in \operatorname{eco} F$, by property $\mathcal{O}(m)$, there exists an affine manifold M with $\dim M = n - m + 1$ such that $x_0 \in M$, $C \backslash M$ is convex and $x_0 \notin \operatorname{eco}(C \cap M)$. Given that $C \backslash M$ and M are disjoint convex sets, applying Theorem 2.10 we get the existence of $D \in \mathbb{O}_{m-1,n}$ such that $Dx <_L Dy = Dx_0$ for all $x \in C \backslash M$ and all $y \in M$. On the other hand, since $x_0 \notin \operatorname{eco}(C \cap M)$, by Proposition 2.6 there exists a unitary vector $a \in \mathbb{R}^n$ such that $a^T x < a^T x_0$ for all $x \in C \cap M$. There is no loss of generality in assuming that a is orthogonal to the rows of D. Then (ii) holds with $A := \begin{pmatrix} D \\ a^T \end{pmatrix}$.

$(ii) \Rightarrow (i)$. Statement (ii) means that C is an intersection of sets of the type

$$\{x \in \mathbb{R}^n : Ax <_L Ax_0\},$$

which are convex; therefore C is convex too. For proving that C satisfies property $\mathcal{O}(m)$, let F be a face of C with $\dim F \leq n - m + 1$ and let $\bar{x} \in (\operatorname{eco} F) \backslash C$. By (ii), there exists $A \in \mathbb{U}_{m,n}$ such that $Ax <_L A\bar{x}$ for all $x \in C$. If we consider the affine manifold $M := \left\{x \in \mathbb{R}^n : A^{(m-1)}x = A^{(m-1)}\bar{x}\right\}$, then we have $\dim M = n - m + 1$ and $\bar{x} \in M$. Since $a_m^T x < a_m^T \bar{x}$ for all $x \in C \cap M$, by Proposition 2.6 we have $\bar{x} \notin \operatorname{eco}(C \cap M)$. On the other hand, from $A^{(m-1)}x <_L A^{(m-1)}\bar{x}$ for all $x \in C \backslash M$ it follows that $C \backslash M = C \cap \left\{x \in \mathbb{R}^n : A^{(m-1)}x <_L t^{(m-1)}\right\}$, so $C \backslash M$ is an intersection of convex sets and is hence convex. We have thus proved property $\mathcal{O}(m)$. $\square$

Corollary 3.5. *Let $C \subset \mathbb{R}^n$ and $m \in \mathbb{N}_n$. The following statements are equivalent:*

(i) *C is convex and satisfies property $\mathcal{O}(m)$.*

(*ii*) $C = \{x \in \mathbb{R}^n : A^s x <_L b^s,\ \forall\, s \in S\}$ *for some index set* S, $A^s \in \mathbb{U}_{m,n}$ *and* $b^s \in \mathbb{R}^m$.

In statement (*ii*) *one can take* $A^s \in \mathbb{O}_{m,n}$.

Proof. It follows easily from Theorem 3.4 and Proposition 2.2. □

Corollary 3.6. *Let* $C \subset \mathbb{R}^n$ *be a convex set, and let* $m, q \in \mathbb{N}_n$. *If* C *satisfies property* $\mathcal{O}(m)$ *and* $q > m$, *then* C *also satisfies property* $\mathcal{O}(q)$.

Proof. It follows from Corollary 3.5, taking into account that each lexicographical inequality $A^s x <_L b^s$ with $A^s \in \mathbb{U}_{m,n}$ can be equivalently replaced by the system

$$\left\{ \begin{pmatrix} A^s \\ B \end{pmatrix} x <_L \begin{pmatrix} b^s \\ d \end{pmatrix},\ \forall\, d \in \mathbb{R}^{q-m} \right\}$$

with $B \in \mathbb{M}_{q-m,n}$ such that $\begin{pmatrix} A^s \\ B \end{pmatrix} \in \mathbb{U}_{q,n}$. □

3.2. Closed Lexicographical Separation

In this case, we shall consider the following definition:

Definition 3.7. Let $C \subset \mathbb{R}^n$ be a convex set and $m \in \mathbb{N}_n$. We will say that C satisfies property $\mathcal{C}(m)$ if for every face F of C with $\dim F \leq n - m + 1$ and every $\bar{x} \in (\operatorname{cl} F) \setminus C$, there exists an affine manifold M with $\dim M = n - m + 1$ satisfying the following conditions:

$$\bar{x} \in M, \tag{4a}$$

$$C \setminus M \ \text{ is convex}, \tag{4b}$$

$$\bar{x} \notin \operatorname{cl}(C \cap M). \tag{6}$$

Equivalently, C satisfies property $\mathcal{C}(m)$ if for every face F of C with $\dim F \leq n - m + 1$ and every $\bar{x} \in \operatorname{cl} F$ such that there is no affine manifold M with $\dim M = n - m + 1$ satisfying conditions (4a), (4b) and (6), one has $\bar{x} \in C$.

Since the closure of a convex set C contains the evenly convex hull of C, if C satisfies property $\mathcal{C}(m)$ then it also satisfies property $\mathcal{O}(m)$. To see that the converse statement is not true, consider the case $m = 1$ in the light of Proposition 3.8 below and Proposition 3.2, taking into account that not every evenly convex set is closed.

The following proposition characterizes convex sets with property $\mathcal{C}(m)$ in the extreme case $m = 1$.

Proposition 3.8. *Let* $C \subset \mathbb{R}^n$ *be a convex set. The following statements are equivalent:*

(*i*) C *satisfies property* $\mathcal{C}(1)$.
(*ii*) C *is closed.*

Proof. $(i) \Rightarrow (ii)$. Let $\bar{x} \in (\operatorname{cl} C) \backslash C$. Since C is a face of itself, by $\mathcal{C}(1)$ there exists a manifold M with $\dim M = n$ satisfying conditions (4a), (4b) and (6). Obviously, M is the affine manifold generated by C, which contradicts (6). This proves that $(\operatorname{cl} C) \backslash C$ is empty, so that C is closed.

$(ii) \Rightarrow (i)$. Let F be a face of C. Since $(\operatorname{cl} F) \backslash C = \emptyset$ because C is closed, property $\mathcal{C}(1)$ holds vacuously. □

The main result in this subsection extends Theorem 2.11 and characterizes convex sets with property $\mathcal{C}(m)$, for $m \in \mathbb{N}_n$, by means of the closed lexicographical separation from any outside point by hemispaces of rank m.

Theorem 3.9. *Let $C \subset \mathbb{R}^n$ and $m \in \mathbb{N}_n$. The following statements are equivalent:*

- (i) *C is convex and satisfies property $\mathcal{C}(m)$.*
- (ii) *For every $x_0 \in \mathbb{R}^n \backslash C$, there exist $A \in \mathbb{U}_{m,n}$ and $z \in \mathbb{R}^m$ such that $Ax \leq_L z <_L Ax_0$ for all $x \in C$.*

In statement (ii) one can take $A \in \mathbb{O}_{m,n}$.

Proof. In the case $m = 1$, the statement of this theorem coincides with that of Theorem 2.11, as a consequence of Proposition 3.8. Thus, we only need to consider the case $m > 1$.

$(i) \Rightarrow (ii)$ (with $A \in \mathbb{O}_{m,n}$). Pick any $x_0 \in \mathbb{R}^n \backslash C$. By Theorem 2.7, there exist $B \in \mathbb{O}_n$ such that $Bx <_L Bx_0$ for all $x \in C$, which implies $B^{(m-1)}x \leq_L B^{(m-1)}x_0$ for all $x \in C$. If the latter lexicographical inequality holds strictly for every $x \in C$, then (ii) holds with $A := B^{(m)}$ and any vector $z := \begin{pmatrix} B^{(m-1)}x_0 \\ b_m^T x_0 - \varepsilon \end{pmatrix}$ with $\varepsilon > 0$. Otherwise, the set

$$F := \left\{ x \in C : B^{(m-1)}x = B^{(m-1)}x_0 \right\}$$

is nonempty. Moreover, applying Theorem 2.3, we get that F is a face of C with dimension not greater than $n - m + 1$.

If $x_0 \notin \operatorname{cl} F$, by Proposition 2.12, there exist a unitary vector $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $a^T x \leq \alpha < a^T x_0$ for all $x \in F$. Without loss of generality, we can assume that a is orthogonal to the rows of $B^{(m-1)}$. Then (ii) holds with $A := \begin{pmatrix} B^{(m-1)} \\ a^T \end{pmatrix}$ and $z := \begin{pmatrix} B^{(m-1)}x_0 \\ \alpha \end{pmatrix}$.

On the contrary, if $x_0 \in \operatorname{cl} F$, by property $\mathcal{C}(m)$, there exists an affine manifold M with $\dim M = n - m + 1$ such that $x_0 \in M$, $C \backslash M$ is convex and $x_0 \notin \operatorname{cl}(C \cap M)$. Given that $C \backslash M$ and M are disjoint convex sets, applying Theorem 2.10 we get the existence of $D \in \mathbb{O}_{m-1,n}$ such that $Dx <_L Dy = Dx_0$ for all $x \in C \backslash M$ and all $y \in M$. On the other hand, since $x_0 \notin \operatorname{cl}(C \cap M)$, by Proposition 2.12 there exist a unitary vector $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $a^T x \leq \alpha < a^T x_0$ for all $x \in C \cap M$. There is no loss of generality in assuming that a is orthogonal to the rows of D. Then (ii) holds with $A := \begin{pmatrix} D \\ a^T \end{pmatrix}$ and $z := \begin{pmatrix} Dx_0 \\ \alpha \end{pmatrix}$.

$(ii) \Rightarrow (i)$. Statement (ii) means that C is an intersection of sets of the type

$$\{x \in \mathbb{R}^n : Ax \leq_L Ax_0\},$$

which are convex; therefore C is convex too. For proving that C satisfies property $\mathcal{C}(m)$, let F be a face of C with $\dim F \leq n - m + 1$ and let $\bar{x} \in (\operatorname{cl} F) \backslash C$. By (ii), there exist $A \in \mathbb{U}_{m,n}$ and $z \in \mathbb{R}^m$ such that $Ax \leq_L z <_L Ax_0$ for all $x \in C$. If we consider the affine manifold $M := \left\{x \in \mathbb{R}^n : A^{(m-1)}x = A^{(m-1)}\bar{x}\right\}$, then we have $\dim M = n-m+1$ and $\bar{x} \in M$. Since $a_m^T x \leq z^m < a_m^T \bar{x}$ for all $x \in C \cap M$, by Proposition 2.12 we have $\bar{x} \notin \operatorname{cl}(C \cap M)$. On the other hand, from $A^{(m-1)}x <_L A^{(m-1)}\bar{x}$ for all $x \in C \backslash M$ it follows that $C \backslash M = C \cap \left\{x \in \mathbb{R}^n : A^{(m-1)}x <_L A^{(m-1)}\bar{x}\right\}$, so $C \backslash M$ is an intersection of convex sets and is hence convex. We have thus proved property $\mathcal{C}(m)$. □

Corollary 3.10. *Let $C \subset \mathbb{R}^n$ and $m \in \mathbb{N}_n$. The following statements are equivalent:*

- (i) *C is convex and satisfies property $\mathcal{C}(m)$.*
- (ii) *$C = \{x \in \mathbb{R}^n : A^s x \leq_L b^s,\ \forall\, s \in S\}$ for some index set S, $A^s \in \mathbb{U}_{m,n}$ and $b^s \in \mathbb{R}^m$.*

In statement (ii) one can take $A^s \in \mathbb{O}_{m,n}$.

Proof. It follows easily from Theorem 3.9 and Proposition 2.2. □

Corollary 3.11. *Let $C \subset \mathbb{R}^n$ be a convex set, and let $m, q \in \mathbb{N}_n$. If C satisfies property $\mathcal{C}(m)$ and $q > m$, then C also satisfies property $\mathcal{C}(q)$.*

Proof. It follows from Corollary 3.10, taking into account that each lexicographical inequality $A^s x \leq_L b^s$ with $A^s \in \mathbb{U}_{m,n}$ can be equivalently replaced by the system

$$\left\{\begin{pmatrix} A^s \\ B \end{pmatrix} x \leq_L \begin{pmatrix} b \\ d \end{pmatrix},\ \forall\, b >_L b^s\right\}$$

with $B \in \mathbb{M}_{q-m,n}$ such that $\begin{pmatrix} A^s \\ B \end{pmatrix} \in \mathbb{U}_{q,n}$ and d an arbitrary vector in $\mathbb{R}^{q-m}$. □

The next proposition provides a deeper insight on property $\mathcal{C}(n)$.

Proposition 3.12. *Let $C \subset \mathbb{R}^n$ be a convex set. The following statements are equivalent:*

- (i) *C satisfies property $\mathcal{C}(n)$.*
- (ii) *For every edge F of C and every $\bar{x} \in (\operatorname{cl} F) \backslash C$, there exists a straight line r such that $\bar{x} \in r$ and $C \cap r = \emptyset$.*

Proof. $(i) \Rightarrow (ii)$. Let F be an edge of C and let $\bar{x} \in (\operatorname{cl} F) \backslash C$. By Theorem 3.9, there exist $A \in \mathbb{O}_n$ and $z \in \mathbb{R}^n$ such that $Ax \leq_L z <_L A\bar{x}$ for all $x \in C$. Consider the set

$$s := \left\{x \in \mathbb{R}^n : A^{(n-1)}x = A^{(n-1)}\bar{x}\right\},$$

which contains $\bar{x}$. Given that the matrix $A^{(n-1)}$ has rank $n-1$, s is a straight line. Next, we distinguish the following two cases:

Firstly, suppose that $z^{(n-1)} <_L A^{(n-1)}\bar{x}$. Since $A^{(n-1)}x \leq_L z^{(n-1)} <_L A^{(n-1)}\bar{x}$ for all $x \in C$, we have $C \cap s = \emptyset$. Therefore, (ii) holds with $r := s$.

Now, suppose that $z^{(n-1)} = A^{(n-1)}\bar{x}$. In this case, we can also assume that there exists $\hat{x} \in C$ such that $A^{(n-1)}\hat{x} = z^{(n-1)}$, otherwise (ii) holds with $r := s$. Moreover, we claim that there exists $\tilde{x} \in F$ such that $A^{(n-1)}\tilde{x} <_L z^{(n-1)}$. On the contrary, if $A^{(n-1)}x = z^{(n-1)}$ for all $x \in F$, then $a_n^T x \leq z^n < a_n^T \bar{x}$ for all $x \in F$. Hence $F \subset \left\{x \in \mathbb{R}^n : a_n^T x \leq z^n\right\}$ and, consequently, $\operatorname{cl} F \subset \left\{x \in \mathbb{R}^n : a_n^T x \leq z^n\right\}$. Given that $\bar{x} \in \operatorname{cl} F$, it follows that $a_n^T \bar{x} \leq z^n$, which is a contradiction. Thus, there exists $\tilde{x} \in F$ such that $A^{(n-1)}\tilde{x} <_L z^{(n-1)}$. Next, we will prove that

$$A^{(n-1)}x <_L z^{(n-1)} \text{ for all } x \in F. \tag{7}$$

Indeed, for every $x \in F$ one has $A^{(n-1)}x \leq_L z^{(n-1)} = A^{(n-1)}\bar{x}$. Since $\bar{x} \in \operatorname{cl} F$ and $\tilde{x} \in F$ are different and F is an edge, we have $x - \bar{x} = \lambda\left(\tilde{x} - \bar{x}\right)$ for some $\lambda > 0$, from which we deduce that $A^{(n-1)}\left(x - \bar{x}\right) = \lambda A^{(n-1)}\left(\tilde{x} - \bar{x}\right) = \lambda\left(A^{(n-1)}\tilde{x} - z^{(n-1)}\right) <_L 0$. Therefore, $A^{(n-1)}x <_L A^{(n-1)}\bar{x}$ for all $x \in F$, and this confirms the claim (7). On the other hand, by Theorem 2.3 and Proposition 2.4 we can write

$$F = \left\{y \in C : By = \max_L \left\{Bx : x \in C\right\}\right\},$$

for some matrix $B \in \mathbb{M}_{n-1,n}$. Now, let us consider the affine manifold

$$M := \left\{x \in \mathbb{R}^n : \left(A^{(n-1)} + B\right)x = \left(A^{(n-1)} + B\right)\bar{x}\right\},$$

which contains $\bar{x}$. Given that the rank of $A^{(n-1)} + B$ is not greater than $n-1$, we have $\dim M \geq 1$ and, therefore, M contains a straight line r through $\bar{x}$. For proving that $C \cap r = \emptyset$, assume, ad absurdo, that there exists $x_0 \in C \cap r$. If $x_0 \in F$, then $A^{(n-1)}x_0 <_L A^{(n-1)}\bar{x}$ and, besides, $Bx_0 = B\bar{x}$ because of $\bar{x} \in \operatorname{cl} F$. Thus, it follows that $\left(A^{(n-1)} + B\right)x_0 <_L \left(A^{(n-1)} + B\right)\bar{x}$, and hence $x_0 \notin M$. Consequently, $x_0 \notin r$, which is a contradiction. Therefore $x_0 \in C \backslash F$, which implies that $Bx_0 <_L B\bar{x}$ and so, by $A^{(n-1)}x_0 \leq_L A^{(n-1)}\bar{x}$, we obtain $\left(A^{(n-1)} + B\right)x_0 <_L \left(A^{(n-1)} + B\right)\bar{x}$. As before, this inequality contradicts $x_0 \in r$. We thus conclude that $C \cap r = \emptyset$.

$(ii) \Rightarrow (i)$. It follows immediately from the definition of property $\mathcal{C}(n)$. □

Based on the preceding proposition, we obtain the following analogue to Theorem 2.11, characterizing those convex sets that are closedly lexicographically separated from any outside point by complements of semispaces. Let us recall that, according to [11, Lemma 1.1], a set $S \subset \mathbb{R}^n$ is a semispace if and only if there exist $A \in \mathbb{O}_n$ and $z \in \mathbb{R}^n$ such that $S = \left\{x \in \mathbb{R}^n : Ax >_L z\right\}$; the unique $x_0 \in \mathbb{R}^n$ such that $Ax_0 = z$ is the vertex of S.

Theorem 3.13. *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

(i) *C is convex and for every edge F of C and every $\bar{x} \in (\operatorname{cl} F) \backslash C$, there exists a straight line r such that $\bar{x} \in r$ and $C \cap r = \emptyset$.*

(*ii*) *For every* $x_0 \in \mathbb{R}^n \backslash C$, *there exist* $A \in \mathbb{U}_n$ *and* $z \in \mathbb{R}^n$ *such that* $Ax \leq_L z <_L Ax_0$ *for all* $x \in C$.

In statement (*ii*) *one can take* $A \in \mathbb{O}_n$.

Corollary 3.14. *Let* $C \subset \mathbb{R}^n$. *The following statements are equivalent:*

(*i*) C *is convex and for every edge* F *of* C *and every* $\bar{x} \in (\operatorname{cl} F) \backslash C$, *there exists a straight line* r *such that* $\bar{x} \in r$ *and* $C \cap r = \emptyset$.

(*ii*) $C = \{x \in \mathbb{R}^n : A^s x \leq_L b^s,\ \forall\, s \in S\}$ *for some index set* S, $A^s \in \mathbb{U}_n$ *and* $b^s \in \mathbb{R}^m$.

(*iii*) C *is an intersection of complements of semispaces.*

In statement (*ii*) *one can take* $A \in \mathbb{O}_n$.

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