Applications of Convex Analysis to the Smallest Intersecting Ball Problem

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The smallest enclosing circle problem asks for the circle of smallest radius enclosing a given set of finite points on the plane. This problem was introduced in the 19th century by Sylvester [17]. After more than a century, the problem remains very active. This paper is the continuation of our effort in shedding new light on classical geometry problems using advanced tools of convex analysis and optimization. We propose and study the following generalized version of the smallest enclosing circle problem: given a finite number of nonempty closed convex sets in a reflexive Banach space, find a ball with the smallest radius that intersects all of the sets.

1. Introduction and Problem Formulation

A more general form of the smallest enclosing circle problem is called the *smallest enclosing ball problem*. Given a set $P = \{p_1, \ldots, p_n\}$, $n > 1$, on a Banach space X, it is always possible to find a ball $\mathbb{B}(a; r)$ such that

$$P \subset \mathbb{B}(a; r).$$

The smallest enclosing ball problem asks for such a ball with the smallest radius.

Consider the following optimization problem

$$\text{minimize } f(x), \quad x \in X, \tag{1}$$

where the function f therein is defined by

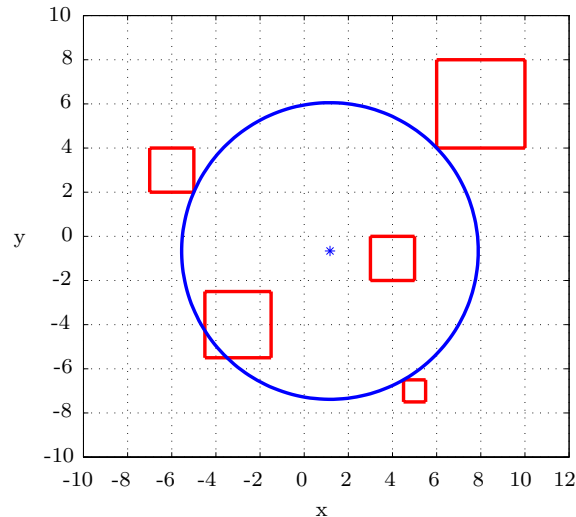$$f(x) = \max\{||x - p_i|| : i = 1, \ldots, n\}.$$

Figure 1.1: A smallest intersecting ball problem

The smallest enclosing ball can be found by solving (1).

Numerous articles have been written to study the smallest enclosing ball problem as well as its generalizations from both numerical and theoretical viewpoints. The reader are referred to [6, 7, 16, 18] and the reference therein for recent developments as well as the history of the problem.

In this paper we propose and study a new problem called *the smallest intersecting ball problem* as follows: given a finite number of nonempty closed convex sets in a Banach space, find a ball with the smallest radius that intersects all of the sets. It is obvious that when the sets under consideration are singletons, the problem reduces to the smallest enclosing ball problem. This is a continuation of our previous work from [15, 14] in an effort to shed new light to classical geometry problems using new tools of nonsmooth analysis.

Let $\Omega_i, i = 1, \ldots, n$, $n > 1$, be nonempty closed convex sets in a Banach space X. Let x be any point in X. Then there always exists $r > 0$ such that

$$\mathbb{B}(x; r) \cap \Omega_i \neq \emptyset \ \text{ for all } i = 1, \ldots, n. \tag{2}$$

We are looking for a ball with the smallest radius $r > 0$ (if exists) such that property (2) holds. Define the function

$$D(x) = \max\{d(x; \Omega_i) : \ i = 1, \ldots, n\}, \tag{3}$$

where the distance function generated by a set Ω is given by

$$d(x; \Omega) = \inf\{||x - \omega|| : \omega \in \Omega\}. \tag{4}$$

It is not hard to see that the function D is convex since each function $d(x; \Omega_i)$, $i = 1, \ldots, n$, is convex. Consider the problem of minimizing the function D on X below

$$\text{minimize } D(x) \ \text{ subject to } x \in X. \tag{5}$$

As we will see in Section 3, if X is a reflexive space and Ω_i, $i = 1, \ldots, n$, have no point in common, such a smallest intersecting ball can be found by solving problem (5). For this reason, we are going to use the following standing assumptions throughout the paper unless otherwise specified:

X is a reflexive Banach space and Ω_i, $i = 1, \ldots, n$, $n > 1$, are nonempty closed convex sets in X with

$$\cap_{i=1}^n \Omega_i = \emptyset.$$

Notice that if the Ω_i, $i = 1, \ldots, n$, have a common point, then the smallest intersecting ball problem has no solution (unless balls of radius 0 are allowed).

The smallest intersecting ball problem is an example of facility location problems. In contrast to most of the existing facility location problems which deal with locations of negligible sizes (points), this new problem deals with those that involve locations of non-negligible sizes (sets). The problem is mathematically interesting with promising applications to location models in which it is possible to *access* the entire of each location from a point in it. The difficulty when dealing with the smallest intersecting ball problem comes from the nonsmooth nature of the cost function D in (5), especially when the norm in X is Non-Euclidean. Our approach in this paper is to study the problem from both theoretical and numerical viewpoints using new tools from convex analysis and optimization. The results we are going to present in this paper provide improvements and generalizations of many results in [6, 7].

We organize the paper as follows. Section 2 provides necessary tools from convex analysis and optimization for solving the smallest intersecting ball problem. In Section 3, we study the problem from theoretical aspects. Section 4 is devoted to developing an algorithm of subgradient type to solve the smallest intersecting problem in finite dimensions. The MATLAB implementations of the algorithm are also presented.

2. Tools of Convex Analysis

This section provides important constructions and results from convex analysis that will be used in the next sections. Most of the material presented here can be found in [2, 9, 19].

Let X be a normed space with the dual space X^*. A function $f : X \to \mathbb{R}$ is called convex if

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \text{for all } x, y \in X \text{ and } \lambda \in (0, 1). \quad (6)$$

If the inequality in (6) becomes strict for $x \neq y$, the function f is called strictly convex.

A subset Ω of X is called convex if

$$\lambda x + (1 - \lambda)y \in \Omega \quad \text{for all } x, y \in \Omega \text{ and } \lambda \in (0, 1).$$

It is not hard to prove that a nonempty closed subset Ω of X is convex if and only if the corresponding distance function (4) is convex. Note that the distance function

$f(x) = d(x; \Omega)$ is Lipschitz continuous on X with modulus one, i.e.,

$$|f(x) - f(y)| \leq ||x - y|| \quad \text{for all } x, y \in X.$$

Let $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*. An element $x^* \in X^*$ is called a *subgradient* of a convex function f at $\bar{x}$ if the following holds

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \quad \text{for all } x \in X.$$

The set of all subgradients of f at $\bar{x}$ is called the *subdifferential* of f at $\bar{x}$ denoted by $\partial f(\bar{x})$.

Convex functions and subdifferentials have several important properties as far as optimization is concerned. For instance, a convex function f has a local minimum at $\bar{x}$ if and only if it has an absolute minimum at $\bar{x}$. Furthermore, the following generalized version of the Fermat rule holds:

$$\bar{x} \text{ is a minimizer of } f \quad \text{if and only if } 0 \in \partial f(\bar{x}). \tag{7}$$

It is well known that the subdifferential of the distance function (4) has a close connection to the *normal cone* of the generating set Ω. Recall that the normal cone of a convex set Ω at $\bar{x} \in \Omega$ is defined by

$$N(\bar{x}; \Omega) = \big\{ v \in X^* : \ \langle v, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega \big\}. \tag{8}$$

The *projection* from a point $\bar{x} \in X$ to a set Ω is

$$\Pi(\bar{x}; \Omega) = \big\{ \omega \in \Omega \ : \ ||\bar{x} - \omega|| = d(\bar{x}; \Omega) \big\}. \tag{9}$$

The following representation of the subdifferential for the distance function (4) will play an important role in the subsequent sections of the paper. The proof of the formulas can be found in [5], while their various extensions are presented in [13].

Proposition 2.1. *Let Ω be a nonempty closed convex set of a Banach space X and let $\bar{x} \in X$. Suppose that $\Pi(\bar{x}; \Omega) \neq \emptyset$ (which is always the case when X is reflexive). Then*

$$\partial d(\bar{x}; \Omega) = \begin{cases} \partial p(\bar{x} - \bar{\omega}) \cap N(\bar{\omega}; \Omega) & \text{if } \bar{x} \notin \Omega, \\ N(\bar{x}; \Omega) \cap I\!\!B^* & \text{if } \bar{x} \in \Omega, \end{cases}$$

where $I\!\!B^*$ is the closed unit ball of X^* and $\bar{\omega}$ is any element of $\Pi(\bar{x}; \Omega)$.

In particular, if X is a Hilbert space, then $\Pi(\bar{x}; \Omega)$ is singleton and

$$\partial d(\bar{x}; \Omega) = \begin{cases} \left\{ \dfrac{\bar{x} - \Pi(\bar{x}; \Omega)}{d(\bar{x}; \Omega)} \right\} & \text{if } \bar{x} \notin \Omega, \\ N(\bar{x}; \Omega) \cap I\!\!B & \text{if } \bar{x} \in \Omega, \end{cases}$$

In this proposition we also observe that when $\bar{x} \in \Omega$, one has $\Pi(\bar{x}; \Omega) = \{\bar{x}\}$. Since $\partial p(0) = I\!\!B^*$, the formula

$$\partial d(\bar{x}; \Omega) = \partial p(\bar{x} - \bar{\omega}) \cap N(\bar{\omega}; \Omega)$$

holds for any $\bar{x} \in X$.

Finally, we present the following well-known subdifferential rule that involves "max" functions.

Theorem 2.2. *Let X be a Banach space and let $f_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, be continuous convex functions. Define*

$$f(x) = \max\{f_i(x) : i = 1, \ldots, n\}.$$

Then

$$\partial f(\bar{x}) = \mathrm{co}\,\{\partial f_i(\bar{x}) : i \in I(\bar{x})\},$$

where $I(\bar{x}) = \{i = 1, \ldots, n : f(\bar{x}) = f_i(\bar{x})\}$.

3.   The Smallest Intersecting Ball Problem: Theoretical Aspects

This section is devoted to theoretical analysis of the smallest intersecting ball problem. We are able to provide improvements and generalizations of many results in [6, 7]. Our approach is based mostly on tools of convex analysis and optimization.

The following proposition allows us to reduce the smallest intersecting ball problem to a nonsmooth convex optimization problem in the reflexive space setting. For this reason, we will identify the smallest intersecting ball problem with problem (5).

Proposition 3.1. *Consider the minimization problem* (5). *Then $\bar{x} \in X$ is an optimal solution of this problem with $r = D(\bar{x})$ if and only if $\mathbb{B}(\bar{x}; r)$ is a smallest ball that satisfies* (2).

Proof. Suppose that $\bar{x}$ is an optimal solution of (5) with $r = D(\bar{x})$. Since Ω_i, $i = 1, \ldots, n$, have no point in common as in the standing assumptions, one has

$$D(\bar{x}) = \inf\{D(x) : x \in X\} = r > 0.$$

This implies

$$d(\bar{x}; \Omega_i) \leq r \ \text{ for all } i = 1, \ldots, n.$$

Since X is reflexive, there exist $\bar{\omega}_i \in \Omega_i, i = 1, \ldots, n$, satisfying

$$||\bar{x} - \bar{\omega}_i|| \leq r.$$

It follows that $\bar{\omega}_i \in \mathbb{B}(\bar{x}; r) \cap \Omega_i$, and hence

$$\mathbb{B}(\bar{x}; r) \cap \Omega_i \neq \emptyset \ \text{ for all } i = 1, \ldots, n.$$

Suppose there exists $r' < r$ and $\bar{x}' \in X$ with

$$\mathbb{B}(\bar{x}'; r') \cap \Omega_i \neq \emptyset \ \text{ for all } i = 1, \ldots, n.$$

Then

$$d(\bar{x}'; \Omega_i) \leq r' < r \ \text{ for all } i = 1, \ldots, n.$$

This implies $D(\bar{x}') \leq r' < r = D(\bar{x})$, which is a contradiction. Thus $\mathbb{B}(\bar{x}; r)$ is a smallest ball we are looking for.

We are now going to justify the converse. Let us first prove that $r = D(\bar{x})$. Since $\mathbb{B}(\bar{x}; r) \cap \Omega_i \neq \emptyset$, one has

$$d(\bar{x}; \Omega_i) \leq r \ \text{ for all } i = 1, \ldots, n.$$

This implies $D(\bar{x}) \leq r$. Assume by contradiction that $D(\bar{x}) < r$. Let r' satisfy $D(\bar{x}) < r' < r$. Then

$$\mathbb{B}(\bar{x}; r') \cap \Omega_i \neq \emptyset \quad \text{for all } i = 1, \ldots, n.$$

This contradicts the minimal property of r. Thus $r = D(\bar{x})$. Let x be any point in X and let $r' = D(x)$. Then

$$\mathbb{B}(x; r') \cap \Omega_i \neq \emptyset \quad \text{for all } i = 1, \ldots, n.$$

This implies $r \leq r'$ or $D(\bar{x}) \leq D(x')$. Therefore, $\bar{x}$ is an optimal solution of (5). $\quad\square$

In what follows, we will prove that under natural assumptions on the sets Ω_i, $i = 1, \ldots, n$, such a smallest intersecting ball does exist. We are going to use the fact that on a Banach space, any convex lower semicontinuous function is weakly lower semicontinuous.

Proposition 3.2. *Suppose that there exists $i = 1, \ldots, n$, such that Ω_i is bounded. Then the smallest intersecting ball problem* (5) *has a solution.*

Proof. Without loss of generality, suppose that Ω_1 is bounded. Define

$$r = \inf\{D(x) : x \in X\}.$$

Let (x_n) be a minimizing sequence for problem (5). That means

$$D(x_n) \to r \quad \text{as } n \to \infty.$$

Let $N \in \mathbb{N}$ with

$$d(x_n; \Omega_1) \leq D(x_n) < r + 1 \quad \text{for all } n \geq N.$$

Then there exists a sequence (ω_n) in Ω_1 such that

$$||x_n - \omega_n|| < r + 1 \quad \text{for all } n \geq N.$$

Since (ω_n) is a bounded sequence, (x_n) is also bounded. As X is reflexive, there exists a subsequence (x_{n_k}) that converges weakly to $\bar{x}$. This implies

$$D(\bar{x}) \leq \liminf D(x_{n_k}) \leq r$$

because D is weakly lower semicontinuous. Therefore, $\bar{x}$ is a solution of problem (5). $\quad\square$

Proposition 3.2 implies that the smallest enclosing ball problem (1) always has a solution because each $\Omega_i = \{\omega_i\}$, $i = 1, \ldots, n$, is obviously bounded. However, in general, the smallest intersecting ball problem (5) may not have any solution.

Example 3.3. Let $X = \mathbb{R}^2$ with the Euclidean norm. Consider $\Omega_1 = \{0\} \times \mathbb{R}$ and

$$\Omega_2 = \left\{ (x, y) \in \mathbb{R}^2 : y \geq \frac{1}{x}, \ x > 0 \right\}.$$

Then the smallest intersecting ball problem (5) generated by Ω_1 and Ω_2 does not have any solution.

In the case where the smallest intersecting ball (5) has a solution, the solution may not be unique as shown in the example below.

Example 3.4. Let $X = I\!\!R^2$ with the Euclidean norm. Consider $\Omega_1 = \{(x, y) \in I\!\!R^2 : y \geq 1\}$ and $\Omega_2 = \{(x, y) \in I\!\!R^2 : y \leq -1\}$. Then any $x \in I\!\!R \times \{0\}$ is a solution of the smallest intersecting ball problem (5) generated by Ω_1 and Ω_2.

Example 3.5. Consider $X = I\!\!R^2$ with the "max" norm $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$. Then the ball $I\!\!B(x; r)$ in X, where $x = (x_1, x_2)$ and $r > 0$, is the square

$$S(x; r) = [x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r].$$

Problem (5) can be equivalently interpreted as follows: find a smallest square $S(x; r)$ that intersects Ω_i for all $i = 1, \ldots, n$. Using different norms on X, we obtain different intersecting ball problems.

Lemma 3.6. *Let X be a Hilbert space and let $\omega_i \in X$, $i = 1, \ldots, n$. Then the function*

$$s(x) = \max\{||x - \omega_i||^2 : i = 1, \ldots, n\}, \quad n \geq 1,$$

is strictly convex.

Proof. We are going to prove that for $x \neq y$ and $t \in (0, 1)$, one has

$$s(tx + (1 - t)y) < ts(x) + (1 - t)s(y).$$

By induction, we only need to show that the function $p(x) = ||x||^2$ is strictly convex and the function $g(x) = \max\{g_1(x), g_2(x)\}$ is strictly convex if both g_1 and g_2 are strictly convex functions. Indeed, for $t \in (0, 1)$ and $x, y \in X$, one has

$$\begin{aligned}
p(tx + (1 - t)y) &= ||tx + (1 - t)y||^2 \\
&= t^2||x||^2 + 2t(1 - t)\langle x, y\rangle + (1 - t)^2||y||^2 \\
&\leq t^2||x||^2 + 2t(1 - t)||x||.||y|| + (1 - t)^2||y||^2 \\
&\leq t^2||x||^2 + t(1 - t)(||x||^2 + |y|^2) + (1 - t)^2||y||^2 \\
&= t||x||^2 + (1 - t)||y||^2 = tp(x) + (1 - t)p(y).
\end{aligned}$$

Notice that the equality holds if and only if $||x|| = ||y||$ and $\langle x, y\rangle = ||x||||y||$. This implies $||x - y||^2 = 0$, and hence $x = y$. Therefore, p is strictly convex.

Now let $t \in (0, 1)$ and $x \neq y$. Then

$$g_i(tx + (1 - t)y) < tg_i(x) + (1 - t)g_i(y) \leq tg(x) + (1 - t)g(y) \quad \text{for } i = 1, 2.$$

This implies

$$g(tx + (1 - t)y) < tg(x) + (1 - t)g(y).$$

The proof is now complete. $\qquad\square$

The following proposition gives an example of a smallest intersecting ball problem which has a unique solution. We will use a natural convention that $I\!\!B(c; 0) = \{c\}$ for any $c \in X$.

Proposition 3.7. *Let X be a Hilbert space and let $r \geq 0$. Suppose that $\Omega_i = I\!B(\omega_i; r)$, $i = 1, \ldots, n$, are closed balls in a Hilbert space X. Then the smallest intersecting ball problem (5) generated by Ω_i, $i = 1, \ldots, n$, has a unique solution. Moreover, this unique solution coincides with the unique solution of the smallest enclosing ball problem (1) generated by the centers of the balls $\{\omega_i\}$, $i = 1, \ldots, n$.*

Proof. Let us first show that in this case the function D in (3) has the following representation

$$D(x) = \max\{||x - \omega_i|| : i = 1, \ldots, n\} - r. \tag{10}$$

Indeed, let

$$J(x) = \{i \in 1, \ldots, n : x \notin \Omega_i\}.$$

Since Ω_i, $i = 1, \ldots, n$, have no point in common by the standing assumptions, $J(x) \neq \emptyset$. For any $i \in \{1, \ldots, n\} \setminus J(x)$ and for any $j \in J(x)$, one has

$$||x - \omega_i|| \leq r \leq ||x - \omega_j||.$$

It follows that

$$
\begin{aligned}
D(x) &= \max\{d(x; \Omega_i) : i = 1, \ldots, n\} \\
&= \max\{d(x; \Omega_i) : i \in J(x)\} \\
&= \max\{||x - \omega_i|| : i \in J(x)\} - r \\
&= \max\{||x - \omega_i|| : i = 1, \ldots, n\} - r.
\end{aligned}
$$

Thus (10) has been justified. Using representation (10), we see that $\bar{x}$ is a solution of problem (5) if and only if it is a solution of the minimization problem

$$\text{minimize } s(x) = \max\{||x - \omega_i||^2 : i = 1, \ldots, n\}, \quad x \in X. \tag{11}$$

Since s is strictly convex by Lemma 3.6, problem (11) has a unique solution. Therefore, problem (5) also has a unique solution. Notice that $\bar{x}$ is a solution of problem (11) if and only if it is the solution of the smallest enclosing ball problem (1) generated by $\{\omega_i\}$, $i = 1, \ldots, n$. The proof is now complete. □

Example 3.8. In $I\!R^2$ with the Euclidean norm, consider the balls $\Omega_1 = I\!B((0, 3); 3)$, $\Omega_2 = I\!B((-2, 0); 1)$, and $\Omega_3 = \{B((2, 0); 1)\}$. Then $\bar{x} = (0, 0)$ is the solution of problem (5) generated by these balls, but this solution is the solution of the smallest enclosing ball problem (1) generated by the centers of the balls.

In what follows we are going to prove that in the Hilbert space setting, the smallest intersecting ball problem (5) generated by closed balls with different radii also has a unique solution although the solution may not coincide with the solution of the smallest enclosing ball problem generated by their centers.

Proposition 3.9. *Let X be a Hilbert space and let $\Omega_i = I\!B(\omega_i; r_i)$, $r_i \geq 0$, $i = 1, \ldots, n$, be closed balls in a Hilbert space X. Then the smallest intersecting ball problem (5) generated by Ω_i, $i = 1, \ldots, n$, has a unique solution.*

Proof. Similar to the proof of Proposition 3.7, one has

$$D(x) = \max\{\|x - \omega_i\| - r_i : i = 1, \ldots, n\} \quad \text{for all } x \in X.$$

Define

$$p_i(x) = \|x - \omega_i\| - r_i, \quad i = 1, \ldots, n.$$

Let

$$\ell = -\max\{r_i : i = 1, \ldots, n\}.$$

Then $p_i(x) \geq \ell$ for all $i = 1, \ldots, n$ and for all $x \in X$. Consider the optimization problem

$$\text{minimize } h(x) = \max\{(p_i(x) - \ell)^2 : i = 1, \ldots, n\}. \tag{12}$$

Notice that $p_i(x) - \ell \geq 0$ for all $i = 1, \ldots, n$ and for all $x \in X$. Then it is not hard to see that $\bar{x}$ is a solution of the smallest intersecting ball problem (5) if and only if it is also a solution of problem (12). Similar to the proof of Proposition 3.7, one sees that the function h in (12) is strictly convex and hence problem (12) has a unique solution. The proof is complete. $\qquad\square$

For each $x \in X$, the set of active indices for D at x is defined by

$$I(x) = \{i \in \{1, \ldots, n\} : D(x) = d(x; \Omega_i)\},$$

and let

$$A_i(x) = \partial p(x - \omega_i) \cap N(\omega_i; \Omega_i).$$

where $\omega_i \in \Pi(x; \Omega_i)$. Notice that the definition of $A_i(x)$ does not depend on the choice of ω_i by Proposition 2.1. It is also clear from the definition that $I(x) \neq \emptyset$ for any $x \in X$. Moreover, if $i \in I(x)$, then $D(x) > 0$ because $\cap_{i=1}^n \Omega_i = \emptyset$ as in the standing assumptions, and hence

$$d(x; \Omega_i) = D(x) > 0.$$

This implies $x \notin \Omega_i$.

Proposition 3.10. *Consider the smallest intersecting ball problem* (5). *Then $\bar{x} \in X$ is an optimal solution of the problem if and only if*

$$0 \in \text{co}\,\{A_i(\bar{x}) : i \in I(\bar{x})\}.$$

Proof. It follows from Theorem 2.2 that

$$\partial D(\bar{x}) = \text{co}\,\{\partial d(\bar{x}; \Omega_i) : i \in I(\bar{x})\}$$
$$= \text{co}\,\{A_i(\bar{x}) : i \in I(\bar{x})\}.$$

The result then follows from the subdifferential Fermat rule (7). $\qquad\square$

Corollary 3.11. *Let X be a Hilbert space. Consider the smallest intersecting ball problem* (5). *Then $\bar{x}$ is a solution of the problem if and only if*

$$\bar{x} \in \text{co}\,\{\bar{\omega}_i : i \in I(\bar{x})\}, \tag{13}$$

where $\bar{\omega}_i = \Pi(\bar{x}; \Omega_i)$.

In particular, if $\Omega_i = \{a_i\}$, $i = 1, \ldots, n$, then $\bar{x}$ is the solution of the smallest enclosing ball problem (1) generated by a_i, $i = 1, \ldots, n$, if and only if

$$\bar{x} \in \text{co}\,\{a_i : i \in I(\bar{x})\}. \tag{14}$$

Proof. According to Proposition 3.10, the element $\bar{x}$ is a solution of the smallest intersecting ball problem (5) if and only if

$$0 \in \text{co}\,\{A_i(\bar{x}) : i \in I(\bar{x})\}.$$

For each $i \in I(\bar{x})$, one has $\bar{x} \notin \Omega_i$, and hence

$$A_i(\bar{x}) = \left\{ \frac{\bar{x} - \bar{\omega}_i}{d(\bar{x}; \Omega_i)} \right\} = \left\{ \frac{\bar{x} - \bar{\omega}_i}{D(\bar{x})} \right\}.$$

It follows that

$$0 \in \text{co}\,\{A_i(\bar{x}) : i \in I(\bar{x})\}$$

if and only if there exists $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that $\sum_{i \in I(\bar{x})} \lambda_i = 1$ and

$$0 = \sum_{i \in I(\bar{x})} \lambda_i \frac{\bar{x} - \bar{\omega}_i}{D(\bar{x})}.$$

This equation is equivalent to

$$0 = \sum_{i \in I(\bar{x})} \lambda_i(\bar{x} - \bar{\omega}_i) \quad \text{or} \quad \bar{x} = \sum_{i \in I(\bar{x})} \lambda_i \bar{\omega}_i,$$

which is equivalent to (13).

Notice that (13) is equivalent to (14) when $\Omega_i = \{a_i\}$, $i = 1, \ldots, n$. The proof is now complete. $\qquad\square$

We say that the smallest ball $I\!B(\bar{x}; r)$ touches a target set Ω_i, $i = 1, \ldots, n$, if $\Omega_i \cap I\!B(\bar{x}; r)$ is singleton.

Corollary 3.12. *Let X be a Hilbert space. Consider the smallest intersecting ball problem (5). Then any smallest intersecting ball touches at least two sets among Ω_i, $i = 1, \ldots, n$.*

Proof. Let us first prove that $|I(\bar{x})| \geq 2$ if $\bar{x}$ is a solution of problem (5). Suppose by contradiction that $I(\bar{x}) = \{i_0\}$. Then by (13),

$$\bar{x} \in \Omega_{i_0} \quad \text{and} \quad d(\bar{x}; \Omega_{i_0}) = D(\bar{x}) > 0.$$

This is a contradiction. Let us now show if $i \in I(\bar{x})$, then $I\!B(\bar{x}; r)$ touches Ω_i, where $r = D(\bar{x})$. Indeed, in this case $d(\bar{x}; \Omega_i) = r$. If there are $u, v \in \Omega_i$ such that

$$u, v \in I\!B(\bar{x}; r) \cap \Omega_i, \quad u \neq v.$$

Then
$$||u - \bar{x}|| \le r = d(\bar{x}; \Omega_i) \quad \text{and} \quad ||v - \bar{x}|| \le r = d(\bar{x}; \Omega_i).$$

It follows that $u, v \in \Pi(\bar{x}; \Omega_i)$. This is a contradiction because $\Pi(\bar{x}; \Omega_i)$ is singleton. Thus $B(\bar{x}; r)$ touches Ω_i. The proof is now complete. $\qquad \square$

It is obvious that $\text{co}\{a_i : i \in I(\bar{x})\} \subset \text{co}\{a_i : i = 1, \ldots, n\}$. Thus our result in Corollary 3.11 covers [7, Theorem 3.6]. It is also possible to prove that the solution of the smallest intersecting ball problem (5) generated by closed balls in a Hilbert space belongs to the convex hull of their centers as in the next proposition.

Proposition 3.13. *Let X be a Hilbert space. Suppose that $\Omega_i = B(\omega_i; r_i)$, $r_i \ge 0$, $i = 1, \ldots, n$, are closed balls in X. Let $\bar{x}$ be the unique solution of problem (5). Then*
$$\bar{x} \in \text{co}\{\omega_i : i \in I(\bar{x})\}.$$

Proof. Let $\bar{x}$ be the solution of the smallest intersecting ball problem (5) generated by Ω_i, $i = 1, \ldots, n$. Since $\bar{x} \notin \Omega_i$ for all $i \in I(\bar{x})$, by (13) from Corollary 3.11, there exist $\lambda_i \ge 0$, $i \in I(\bar{x})$, such that $\sum_{i \in I(\bar{x})} \lambda_i = 1$ and

$$\bar{x} = \sum_{i \in I(\bar{x})} \lambda_i \left(\omega_i + r_i \frac{\bar{x} - \omega_i}{||\bar{x} - \omega_i||} \right).$$

For any $i \in I(\bar{x})$, one has $||\bar{x} - \omega_i|| = r + r_i$, where r is the radius of the smallest intersecting ball. It follows that

$$\bar{x} = \sum_{i \in I(\bar{x})} \lambda_i \bar{x} = \sum_{i \in I(\bar{x})} \lambda_i \left(1 - \frac{r_i}{r + r_i} \right) \omega_i + \sum_{i \in I(\bar{x})} \frac{\lambda_i r_i}{r + r_i} \bar{x}.$$

This implies

$$\bar{x} = \frac{1}{\sum_{i \in I(\bar{x})} \dfrac{\lambda_i}{r + r_i}} \sum_{i \in I(\bar{x})} \frac{\lambda_i}{r + r_i} \omega_i \in \text{co}\{\omega_i : i \in I(\bar{x})\}.$$

The proof is now complete. $\qquad \square$

Example 3.14. Let $a_i, i = 1, 2, 3$, be three points in $\mathbb{R}^2$ with the Euclidean norm and let $\bar{x}$ be the solution of problem (1) generated by a_i, $i = 1, 2, 3$. By Corollary 3.12, one has $|I(\bar{x})| = 2$ or $|I(\bar{x})| = 3$. If $|I(\bar{x})| = 2$, say $I(\bar{x}) = \{2, 3\}$, then $\bar{x} \in \text{co}\{a_2, a_3\}$ and $||\bar{x} - a_2|| = ||\bar{x} - a_3||$ by Corollary 3.11. In this case

$$\bar{x} = \frac{a_2 + a_3}{2}.$$

This also implies $\langle a_2 - a_1, a_3 - a_1 \rangle < 0$. Conversely, if $\langle a_2 - a_1, a_3 - a_2 \rangle < 0$, then the angle of the triangle formed by a_1, a_2, and a_3 at vertex a_1 is obtuse (we allow the case where a_i, $i = 1, 2, 3$, are on a straight line). One can easily see
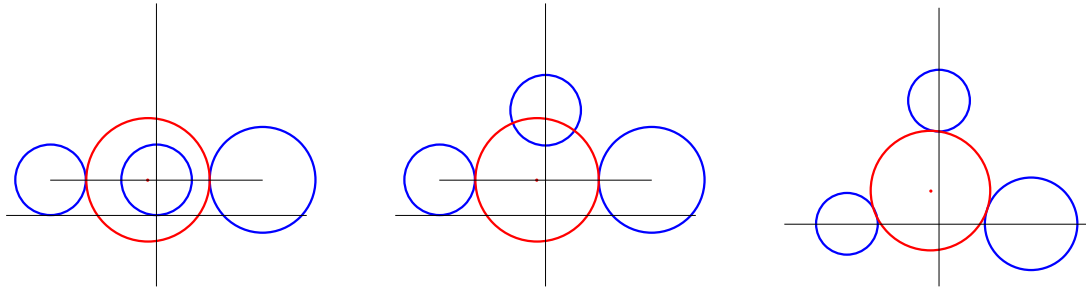
Figure 3.1: Smallest intersecting ball problems for three disks in $I\!\!R^2$

that $I(\frac{a_2+a_3}{2}) = \{2,3\}$, and $\frac{a_2+a_3}{2}$ satisfies the assumption of Corollary 3.11. Then $\bar{x} = \frac{a_2+a_3}{2}$ because of the uniqueness of the solution. In this case we have $|I(\bar{x})| = 2$.

Thus $|I(\bar{x})| = 2$ if and only if one of the angles of the triangle formed by a_1, a_2, and a_3 is obtuse. In this case the solution of problem (5) is the midpoint of the side opposite to the obtuse vertex.

If none of the angles of the triangle formed by a_1, a_2, a_3 is obtuse, then $|I(\bar{x})| = 3$. In this case, $\bar{x}$ is the unique point that satisfies

$$\bar{x} \in \text{co}\,\{a_1, a_2, a_3\}, \qquad ||\bar{x} - a_1|| = ||\bar{x} - a_2|| = ||\bar{x} - a_3||,$$

or $\bar{x}$ is the center of the circumscribing circle of the triangle.

Let us now consider the solution of problem (5) with the target sets being three disjoint disks in $I\!\!R^2$.

Example 3.15. Let $\Omega_i = I\!\!B(\omega_i, r_i)$, $r_i > 0$, $i = 1, 2, 3$, be disjoint disks in $I\!\!R^2$ with the Euclidean norm. We use $\text{bd}(\Omega_i)$ to denote the boundary of Ω_i, which is the circle of center ω_i and radius r_i, $i = 1, 2, 3$. Let $\bar{x}$ be the unique solution of the problem.

Let us consider the first case where one of the line segments connecting two of the centers intersects the other disks. For instance, the line segment connecting ω_2 and ω_3 intersects Ω_1. Let $u_2 = \overline{\omega_2\omega_3} \cap \text{bd}(\Omega_2)$ and $u_3 = \overline{\omega_2\omega_3} \cap \text{bd}(\Omega_3)$. Let $\bar{x}$ be the midpoint of $\overline{u_2 u_3}$. Then $I(\bar{x}) = \{2,3\}$ and we can apply Corollary 3.11 to see that $\bar{x}$ is the solution of the problem.

Now we only need to consider the case where any line segment connecting two centers of the disks does not intersect the remaining disk. Let

$$
\begin{aligned}
u_1 &= \overline{\omega_1\omega_2} \cap \text{bd}(\Omega_1), & v_1 &= \overline{\omega_1\omega_3} \cap \text{bd}(\Omega_1), \\
u_2 &= \overline{\omega_2\omega_3} \cap \text{bd}(\Omega_2), & v_2 &= \overline{\omega_2\omega_1} \cap \text{bd}(\Omega_2), \\
u_3 &= \overline{\omega_3\omega_1} \cap \text{bd}(\Omega_3), & v_3 &= \overline{\omega_3\omega_2} \cap \text{bd}(\Omega_3)
\end{aligned}
$$
$$a_1 = \overline{\omega_1 m_1} \cap \text{bd}(\Omega_1), \qquad a_2 = \overline{\omega_2 m_2} \cap \text{bd}(\Omega_2), \qquad a_3 = \overline{\omega_3 m_3} \cap \text{bd}(\Omega_3),$$

where m_1 is the midpoint of $\overline{u_2 v_3}$, m_2 is the midpoint of $\overline{u_3 v_1}$, and m_3 is the midpoint of $\overline{u_1 v_2}$. If one of the angles: $\widehat{u_2 a_1 v_3}$, $\widehat{u_3 a_2 v_1}$, $\widehat{u_1 a_3 v_2}$ is greater than $90°$. For instance, if $\widehat{u_2 a_1 v_3}$ is greater than $90°$. Then $I(m_1) = \{2,3\}$, and $\bar{x} = m_1$ is the unique solution

of the problem by Corollary 3.11. Now if all of the afore-mentioned angles are less than or equal to 90°, then $|I(\bar{x})| = 3$ and the smallest disk we are looking for is the unique disk that touches three other disks. The construction of this disk is the celebrated problem of *Apollonius*; see, e.g., [8].

We are going to prove that a smallest intersecting ball generated by n convex sets, $n > 1$, in $\mathbb{R}^m$ can be determined by at most $m + 1$ sets among them; see Figure 1.1 for the visualization of this property. The proof is based on a known result for points (see [18, Lemma 1 (iii)]), which can be easily proved by Corollary 14 and the Caratheodory theorem [10, Corollary 1, Sec. 3.5].

Lemma 3.16. *Let $P = \{p_1, \ldots, p_n\}$, $n > 1$, be a set of finite points in $\mathbb{R}^m$ with the Euclidean norm and let $\mathbb{B}(\bar{x}; r)$ be the smallest enclosing ball for problem* (1) *generated by points in P. Then there exists a subset $Q \subset P$ and $2 \leq |Q| \leq m + 1$ such that $\mathbb{B}(\bar{x}; r)$ is also the smallest enclosing ball of problem* (1) *generated by points in Q.*

Proof. By Corollary 3.11, one has

$$\bar{x} \in \text{co} \{p_i : i \in I(\bar{x})\}.$$

By the Caratheodory theorem, there exists an index set $J \subset I(\bar{x})$ with $|J| \leq m + 1$ and

$$\bar{x} \in \text{co} \{p_j : j \in J\}. \tag{15}$$

It is clear that $|J| \geq 2$ because $n > 1$. Let $Q = \{p_j : j \in J\}$. Then $2 \leq |Q| \leq m + 1$ and $||\bar{x} - q|| = r$ for all $q \in Q$. By converse of Corollary 3.11, one has that $\mathbb{B}(\bar{x}; r)$ is the smallest enclosing ball of problem (1) generated by points in Q. The proof is complete. $\square$

Proposition 3.17. *Let $X = \mathbb{R}^m$ with the Euclidean norm. Consider problem* (5) *in which Ω_i, $i = 1, \ldots, n$, are disjoint. Suppose that $\mathbb{B}(\bar{x}; r)$ is a smallest intersecting ball of the problem. Then there exists an index set J with $2 \leq |J| \leq m + 1$ such that $\mathbb{B}(\bar{x}; r)$ is also a smallest intersecting ball of problem* (5) *in which the target sets are Ω_j, $j \in J$.*

Proof. Let $\mathbb{B}(\bar{x}; r)$ be a smallest intersecting ball of problem (5) with target sets Ω_i, $i = 1, \ldots, n$. By Corollary 3.11 and Corollary 3.12, one has

$$\bar{x} \in \text{co} \{\omega_i : i \in I(\bar{x})\},$$

where $|I(\bar{x})| \geq 2$ and $\omega_i \in \Pi(\bar{x}; \Omega_i)$. Again, by Corollary 3.11, $\mathbb{B}(\bar{x}; r)$ is the solution of the smallest enclosing ball (1) generated by $\{\omega_i : i \in I(\bar{x})\}$.

Applying Lemma 3.16, one finds a subset $J \subset I(\bar{x})$, $2 \leq |J| \leq m + 1$ such that $\mathbb{B}(\bar{x}; r)$ is the solution of the smallest enclosing ball generated by $\{\omega_j : j \in J\}$. Then

$$\bar{x} \in \text{co} \{\omega_i : i \in J\}$$

and $||\bar{x} - \omega_j|| = r$ for all $j \in J$. Now consider problem (5) generated by $\{\Omega_j : j \in J\}$. Applying Corollary 3.11, we also see that $\bar{x}$ is a solution of this problem because

$$\bar{x} \in \text{co}\,\{\omega_j : j \in J\},$$

where $\omega_j \in \Pi(\bar{x}; \Omega_j)$ and $d(\bar{x}; \Omega_j) = r$ for each $j \in J$. Moreover, $I\!B(\bar{x}; r)$ is a smallest intersecting ball for the problem. The proof is now complete. □

Now we are going to give a generalization of [7, Theorem 4.4]. Our approach, which is based on the proof of [10, Proposition 1, Sec. 10.2], allows us to give an estimate of the radius of the smallest intersecting ball for problem (5) generated by closed balls in $I\!R^m$. Notice that [7, Theorem 4.4] holds in $I\!R^2$ for the classical smallest enclosing circle problem.

Theorem 3.18. *Let $X = I\!R^m$ with the Euclidean norm. Consider the smallest intersecting ball problem (5) generated by the closed balls $\Omega_i = I\!B(\omega_i; r_i)$, $i = 1, \ldots, n$. Let*

$$r_{\min} = \min\{r_i : i = 1, \ldots, n\}, \qquad r_{\max} := \max\{r_i : i = 1, \ldots, n\},$$

$$\ell = \min\{m + 1, n\}, \qquad P = \{\omega_i : i = 1, \ldots, n\},$$

and let $B_* = I\!B(\bar{x}; r)$ be the smallest enclosing ball. Then

$$\frac{1}{2}\,\text{diam}\,P - r_{\max} \leq r \leq \sqrt{\frac{\ell - 1}{2\ell}}\,\text{diam}\,P - r_{\min}. \qquad (16)$$

In particular,

$$\text{diam}\,P - \max\{\text{diam}\,\Omega_i : i = 1, \ldots, n\}$$

$$\leq \text{diam}\,B_* \leq \sqrt{\frac{2(\ell - 1)}{\ell}}\,\text{diam}\,P - \min\{\text{diam}\,\Omega_i : i = 1, \ldots, n\}.$$

Proof. For any $i, j = 1, \ldots, n$, one has

$$||\omega_i - \omega_j|| \leq ||\bar{x} - \omega_i|| + ||\bar{x} - \omega_j|| \leq 2r + r_i + r_j \leq 2r + 2r_{\max}.$$

Thus

$$\text{diam}\,P \leq 2r + 2r_{\max}$$

and the first inequality in (16) holds true.

Let us prove the second inequality of (16). By the Caratheodory theorem [10, Corollary 1, Sec. 3.5] and the proof of Proposition 3.13, there exist $k \leq \min\{m + 1, I(\bar{x})\} \leq \min\{m + 1, n\}$ and $\lambda_i \geq 0$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} \lambda_i = 1$ (we reorder the indices if necessary) such that

$$\bar{x} = \sum_{i=1}^{k} \mu_i \omega_i, \quad \text{where } \mu_i = \frac{1}{\mu}\frac{\lambda_i}{r_i + r}, \ \mu = \sum_{i=1}^{k} \frac{\lambda_i}{r + r_i}$$

and

$$\|\bar{x} - \omega_i\| = r + r_i, \quad i = 1, \ldots, k.$$

One has

$$\begin{aligned}
\|\omega_i - \omega_j\|^2 &= \|\omega_i - \bar{x}\|^2 + \|\omega_j - \bar{x}\|^2 - 2\langle \omega_i - \bar{x}, \omega_j - \bar{x}\rangle \\
&= (r + r_i)^2 + (r + r_j)^2 - 2\langle \omega_i - \bar{x}, \omega_j - \bar{x}\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{i=1}^{k} \mu_i \|\omega_i - \omega_j\|^2 &= (r + r_j)^2 + \sum_{i=1}^{k} \mu_i (r + r_i)^2 - 2\left\langle \sum_{i=1}^{k} \mu_i \omega_i - \bar{x}, \omega_j - \bar{x}\right\rangle \\
&= (r + r_j)^2 + \sum_{i=1}^{k} \mu_i (r + r_i)^2.
\end{aligned}$$

It follows that

$$\sum_{i,j=1}^{k} \mu_i \mu_j \|\omega_i - \omega_j\|^2 = 2\sum_{i=1}^{k} \mu_i (r + r_i)^2.$$

We also have

$$\sum_{i,j=1,i\neq j}^{k} \mu_i \mu_j = \left(\sum_{i=1}^{k} \mu_i\right)^2 - \sum_{i=1}^{k} \mu_i^2 \leq 1 - \frac{1}{k} = \frac{k-1}{k} \leq \frac{\ell-1}{\ell}.$$

This implies

$$\sum_{i,j=1}^{k} \mu_i \mu_j \|\omega_i - \omega_j\|^2 = 2\sum_{i=1}^{k} \mu_i (r + r_i)^2 \leq (\operatorname{diam} P)^2 \frac{\ell-1}{\ell}.$$

Since

$$\mu_i (r + r_i)^2 = \frac{1}{\mu}\frac{\lambda_i}{r_i + r}(r + r_i)^2 = \frac{\lambda_i}{\mu}(r + r_i),$$

one has

$$\sum_{i=1}^{k} \frac{\lambda_i}{\mu}(r + r_i) \leq \frac{1}{2}\frac{\ell-1}{\ell}(\operatorname{diam} P)^2$$

Using the formula for μ, we arrive at

$$(r + r_{\min})^2 \leq \frac{1}{2}\frac{\ell-1}{\ell}(\operatorname{diam} P)^2.$$

This implies (16). The second estimates follow from (16). The proof is complete. □

4.   Subgradient Algorithm and Its Implementation

In this section let $(X, ||\cdot||)$ be a normed space where $X = I\!\!R^m$ and let $p(x) = ||x||$ be the norm function on X. We are going to present and justify an algorithm of subgradient type to solve problem (5) numerically and illustrate its implementations using MATLAB.

Theorem 4.1. *Let Ω_i, $i = 1, \ldots, n$, be nonempty closed convex subsets of X such that at least one of them is bounded. Picking a sequence $\{\alpha_k\}$ of positive numbers and a starting point $x_1 \in X$, consider the iterative algorithm:*

$$x_{k+1} = x_k - \alpha_k x_k^*, \quad k \in I\!\!N. \tag{17}$$

Let the vectors x_k^* in (17) be given by

$$x_k^* \in \partial p(x_k - \omega_k) \cap N(\omega_k; \Omega_i), \tag{18}$$

where $\omega_k \in \Pi(x_k; \Omega_i)$ and i is any index chosen from the following index set

$$I(x_k) = \{i = 1, \ldots, n : D(x_k) = d(x_k; \Omega_i)\}.$$

Assume that the given sequence $\{\alpha_k\}$ in (17) satisfies the conditions

$$\sum_{k=1}^{\infty} \alpha_k = \infty \quad and \quad \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \tag{19}$$

Then the iterative sequence $\{x_k\}$ in (18) converges to an optimal solution of the smallest intersecting ball problem (5) and the value sequence

$$V_k = \min \{D(x_j) : j = 1, \ldots, k\} \tag{20}$$

converges to the optimal value $\widehat{V}$ in this problem.

Proof. By Proposition 3.2 the smallest intersecting ball problem under consideration has a solution. Observe that the function D in (5) satisfies a Lipschitz condition with Lipschitz constant $\kappa = 1$. We have

$$\partial D(x_k) = \text{co}\, \{\partial d(x_k; \Omega_i) : i \in I(x_k)\}$$
$$= \text{co}\, \{\partial p(x_k - \omega_k) \cap N(\omega_k; \Omega_i) : i \in I(x_k)\},$$

where $\omega_k \in \Pi(x_k; \Omega_i)$. Notice that under the standing assumptions, $x_k \notin \Omega_i$ for $i \in I(x_k)$. It follows that for any $i \in I(x_k)$ one has

$$\partial p(x_k - \omega_k) \cap N(\omega_k; \Omega_i) \subset \partial D(x_k).$$

We also have that $\partial d(x_k; \Omega_i) = \partial p(x_k - \omega_k) \cap N(\omega_k; \Omega_i)$ is nonempty. Since all norms in X are equivalent, it suffices to show that

$$||x_k - \bar{x}||_2 \to 0 \quad \text{and} \quad V_k \to \widehat{V},$$

where $||\cdot||_2$ is the Euclidean norm in X and $\bar{x}$ is a solution of problem (5). However, these follow directly from the well-known results on the subgradient method for convex functions in the so-called "square summable but not summable case"; see, e.g., [1, 4]. □
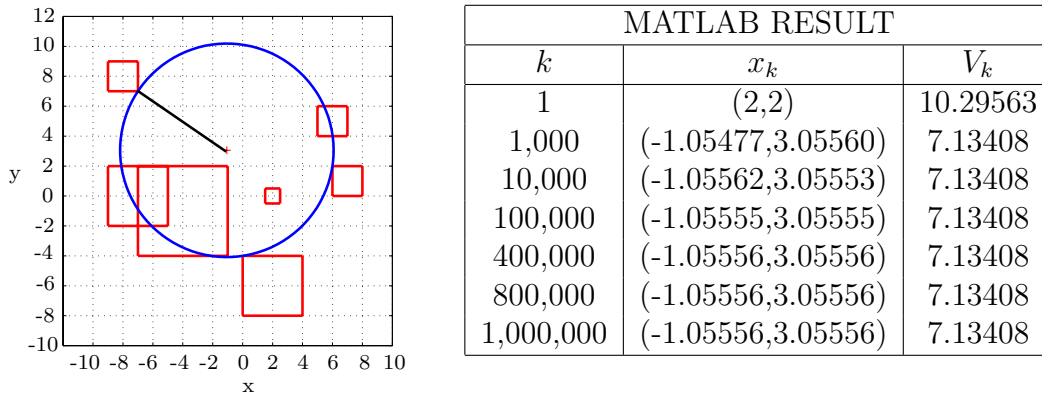
Figure 4.1: A smallest intersecting ball with Euclidean norm to square targets

MATLAB RESULT		
k	x_k	V_k
1	(2,2)	10.29563
1,000	(-1.05477,3.05560)	7.13408
10,000	(-1.05562,3.05553)	7.13408
100,000	(-1.05555,3.05555)	7.13408
400,000	(-1.05556,3.05556)	7.13408
800,000	(-1.05556,3.05556)	7.13408
1,000,000	(-1.05556,3.05556)	7.13408

One important feature of the subgradient method is that the subgradient x_k^* for each k is not uniquely defined. This also reflects in the following direct consequence of Theorem 4.1.

Corollary 4.2. *Let $X = \mathbb{R}^m$ with the Euclidean norm. Consider the smallest intersecting ball problem (5). For each $k \in \mathbb{N}$, the subgradient x_k^* in Theorem 4.1 is computed by*

$$x_k^* = \frac{x_k - \omega_k}{||x_k - \omega_k||},$$

where $\omega_k = \Pi(x_k; \Omega_i)$ and i is an index chosen from $I(x_k)$.

In particular, if $\Omega_i = \mathbb{B}(c_i; r_i)$, $i = 1, \dots, n$, are closed balls in X. Then the subgradient x_k^* has the following explicit representation

$$x_k^* = \frac{x_k - c_i}{||x_k - c_i||} \quad \text{for an index } i \in I(x_k).$$

Example 4.3. Consider $X = \mathbb{R}^2$ with the Euclidean norm. The target sets are the squares $\Omega_i = S(\omega_i; r_i)$, $\omega_i = (\omega_{1i}, \omega_{2i})$, $i = 1, \dots, n$, where

$$S(\omega_i; r_i) = [\omega_{1i} - r_i, \omega_{1i} + r_i] \times [\omega_{2i} - r_i, \omega_{2i} + r_i].$$

Let the vertices of the ith square be denoted by $v_{1i} = (\omega_{1i} + r_i, \omega_{2i} + r_i)$, $v_{2i} = (\omega_{1i} - r_i, \omega_{2i} + r_i)$, $v_{3i} = (\omega_{1i} - r_i, \omega_{2i} - r_i)$, $v_{4i} = (\omega_{1i} + r_i, \omega_{2i} - r_i)$ and let $x_k = (x_{1k}, x_{2k})$. Fix an index $i \in I(x_k)$. Then the vectors x_k^* in Theorem 4.1 are given by

$$x_k^* = \begin{cases} \dfrac{x_k - v_{1i}}{\|x_k - v_{1i}\|} & \text{if } x_{1k} - \omega_{1i} > r_i \text{ and } x_{2k} - \omega_{2i} > r_i, \\[2mm] \dfrac{x_k - v_{2i}}{\|x_k - v_{2i}\|} & \text{if } x_{1k} - \omega_{1i} < -r_i \text{ and } x_{2k} - \omega_{2i} > r_i, \\[2mm] \dfrac{x_k - v_{3i}}{\|x_k - v_{3i}\|} & \text{if } x_{1k} - \omega_{1i} < -r_i \text{ and } x_{2k} - \omega_{2i} < -r_i, \\[2mm] \dfrac{x_k - v_{4i}}{\|x_k - v_{4i}\|} & \text{if } x_{1k} - \omega_{1i} > r_i \text{ and } x_{2k} - \omega_{2i} < -r_i, \\[2mm] (0,1) & \text{if } |x_{1k} - \omega_{1i}| \le r_i \text{ and } x_{2k} - \omega_{2i} > r_i, \\[1mm] (0,-1) & \text{if } |x_{1k} - \omega_{1i}| \le r_i \text{ and } x_{2k} - \omega_{2i} < -r_i, \\[1mm] (1,0) & \text{if } x_{1k} - \omega_{1i} > r_i \text{ and } |x_{2k} - \omega_{2i}| \le r_i, \\[1mm] (-1,0) & \text{if } x_{1k} - \omega_{1i} < -r_i \text{ and } |x_{2k} - \omega_{2i}| \le r_i. \end{cases}$$

It is also not hard to determine the index set $I(x_k)$ sequence and V_k for each $k \in \mathbb{N}$. Thus the algorithm is explicit.

Consider the target sets $\Omega_i, i = 1, \ldots, 7$, to be the squares with centers $(-8, 8)$, $(-7, 0), (-4, -1), (2, 0), (2, -6), (7, 1)$, and $(6, 5)$ and the radii $r_i = \{1, 2, 3, 0.5, 2, 1, 1\}$ for $i = 1, \ldots, 7$, respectively. A MATLAB program is performed for the sequence $\alpha_k = 1/k$ satisfying (19) and the starting point $x_1 = (2,2)$; see Figure 4.1.

Observe that the numerical results computed up to five decimal places yield an optimal solution $\bar{x} \approx (-1.05556, 3.05556)$ and the optimal value $\widehat{V} \approx 7.13408$.

When working with a norm in X that is different from the Euclidean norm, it may be difficult to find the distance functions, the projections to sets, as well as the subdifferential of the norm. The following remark allows us to have an intuitive way to find a subgradient x_k^*, $k \in \mathbb{N}$, in Theorem 4.1 in the case $X = \mathbb{R}^2$ with the "sum" norm.

Remark 4.4. Let $X = \mathbb{R}^2$ with the "sum" norm $p(x) = |x_1| + |x_2|$, $x = (x_1, x_2)$. The ball $\mathbb{B}(\bar{x}; t)$, $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $t > 0$, is the following diamond shape

$$\mathbb{B}(\bar{x}; t) = \{(x_1, x_2) \in X : |x_1 - \bar{x}_1| + |x_2 - \bar{x}_2| \le t\}.$$

The distance from $\bar{x}$ to a nonempty closed set Ω and the corresponding projection are given by

$$d(\bar{x}; \Omega) = \min\{t \ge 0 : \mathbb{B}(\bar{x}; t) \cap \Omega \neq \emptyset\} \tag{21}$$

and

$$\Pi(\bar{x}; \Omega) = \mathbb{B}(\bar{x}; t) \cap \Omega, \quad \text{where } t = d(\bar{x}; \Omega).$$

Moreover, the subdifferential $\partial p(\bar{x})$, $\bar{x} \in X$, has the following explicit representation

$$
\partial p(\bar{x}_1, \bar{x}_2) = \begin{cases}
[-1, 1] \times [-1, 1], & \text{if } (\bar{x}_1, \bar{x}_2) = (0, 0), \\
[-1, 1] \times \{1\}, & \text{if } \bar{x}_1 = 0, \bar{x}_2 > 0, \\
[-1, 1] \times \{-1\}, & \text{if } \bar{x}_1 = 0, \bar{x}_2 < 0, \\
\{1\} \times [-1, 1], & \text{if } \bar{x}_1 > 0, \bar{x}_2 = 0, \\
\{-1\} \times [-1, 1], & \text{if } \bar{x}_1 < 0, \bar{x}_2 = 0, \\
\{1\} \times \{1\}, & \text{if } \bar{x}_1 > 0, \bar{x}_2 > 0, \\
\{1\} \times \{-1\}, & \text{if } \bar{x}_1 > 0, \bar{x}_2 < 0, \\
\{-1\} \times \{1\}, & \text{if } \bar{x}_1 < 0, \bar{x}_2 > 0, \\
\{-1\} \times \{-1\}, & \text{if } \bar{x}_1 < 0, \bar{x}_2 < 0.
\end{cases}
$$

By considering the "sum" norm in X, we are able to introduce a new smallest intersecting ball problem in which a "ball" has a diamond shape. The algorithm is going to be implemented in the following example.

Example 4.5. Let us consider an example when $X = I\!R^2$ with the "sum" norm. Let Ω_i be the squares $S(\omega_i; r_i)$, $i = 1, \ldots, n$, given in Example 4.3. Notice that a ball $I\!B(\bar{x}; r)$, $\bar{x} = (\bar{x}_1, \bar{x}_2)$, in X is the diamond shape

$$
I\!B(\bar{x}; r) = \{(x_1, x_2) \in I\!R^2 : |x_1 - \bar{x}_1| + |x_2 - \bar{x}_2| \leq r\}.
$$

Therefore, the smallest intersecting ball problem (5) can be interpreted as follows: find a diamond shape of the smallest radius $I\!R^2$ that intersects all n given squares. Using the same notation for the vertices of the target set Ω_i as in Example 4.3, one can see that the vectors x_k^* in Theorem 4.1 are given by

$$
x_k^* = \begin{cases}
(1, 1), & \text{if } x_{1k} - \omega_{1i} > r_i \text{ and } x_{2k} - \omega_{2i} > r_i, \\
(-1, 1), & \text{if } x_{1k} - \omega_{1i} < -r_i \text{ and } x_{2k} - \omega_{2i} > r_i, \\
(-1, -1), & \text{if } x_{1k} - \omega_{1i} < -r_i \text{ and } x_{2k} - \omega_{2i} < -r_i, \\
(1, -1), & \text{if } x_{1k} - \omega_{1i} > r_i \text{ and } x_{2k} - \omega_{2i} < -r_i, \\
(0, 1), & \text{if } |x_{1k} - \omega_{1i}| \leq r_i \text{ and } x_{2k} - \omega_{2i} > r_i, \\
(0, -1), & \text{if } |x_{1k} - \omega_{1i}| \leq r_i \text{ and } x_{2k} - \omega_{2i} < -r_i, \\
(1, 0), & \text{if } x_{1k} - \omega_{1i} > r_i \text{ and } |x_{2k} - \omega_{2i}| \leq r_i, \\
(-1, 0), & \text{if } x_{1k} - \omega_{1i} < -r_i \text{ and } |x_{2k} - \omega_{2i}| \leq r_i.
\end{cases}
$$

Consider the target sets $\Omega_i, i = 1, \ldots, 7$, to be the squares with centers $(-5, 3)$, $(-3, 0)$, $(-2, -3)$, $(0, -8)$, $(4, -3)$, $(3, 0)$, and $(5, 4)$ and the radii $r_i = 1$ for $i = 1, \ldots, 7$, respectively. A MATLAB program is performed for the sequence $\alpha_k = 1/k$ satisfying (19) and the starting point $x_1 = (2, 0)$; see Figure 4.2.

Observe that the numerical results computed up to five decimal places yield optimal solution $\bar{x} \approx (0.50000, -0.25000)$ and the optimal value $\widehat{V} \approx 6.75000$.

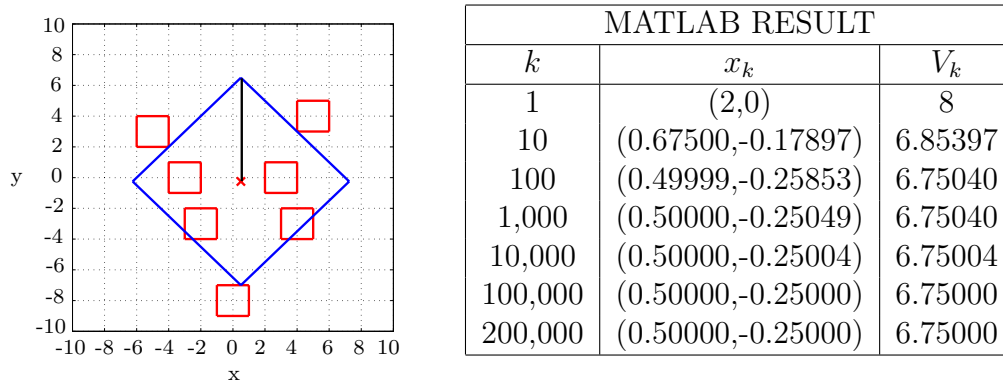Similar observations for $X = I\!R^2$ with the "max" norm can be easily seen:

MATLAB RESULT		
k	x_k	V_k
1	(2,0)	8
10	(0.67500,-0.17897)	6.85397
100	(0.49999,-0.25853)	6.75040
1,000	(0.50000,-0.25049)	6.75040
10,000	(0.50000,-0.25004)	6.75004
100,000	(0.50000,-0.25000)	6.75000
200,000	(0.50000,-0.25000)	6.75000

Figure 4.2: A smallest intersecting ball with sum norm to square targets

Remark 4.6. Let $X = I\!R^2$ with the "max" norm $p(x) = \max\{|x_1|, |x_2|\}$, $x = (x_1, x_2)$. The ball $I\!B(\bar{x}; t)$, $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $t > 0$, is the following square

$$I\!B(\bar{x}; t) = [\bar{x}_1 - t, \bar{x}_1 + t] \times [\bar{x}_2 - t, \bar{x}_2 + t].$$

The distance from $\bar{x}$ to a nonempty closed set Ω and the corresponding projection are given by

$$d(\bar{x}; \Omega) = \min\{t \geq 0 : I\!B(\bar{x}; t) \cap \Omega \neq \emptyset\}$$

and

$$\Pi(\bar{x}; \Omega) = I\!B(\bar{x}; t) \cap \Omega, \quad \text{where } t = d(\bar{x}; \Omega).$$

Moreover, the subdifferential $\partial p(\bar{x})$, $\bar{x} \in X$, has the following explicit representation

$$\partial p(\bar{x}_1, \bar{x}_2) = \begin{cases} \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| \leq 1\} & \text{if } (\bar{x}_1, \bar{x}_2) = (0, 0), \\ \{(0, 1)\} & \text{if } |\bar{x}_1| < \bar{x}_2, \\ \{(0, -1)\} & \text{if } \bar{x}_2 < -|\bar{x}_1|, \\ \{(1, 0)\} & \text{if } x_1 > |\bar{x}_2|, \\ \{(-1, 0)\} & \text{if } \bar{x}_1 < -|\bar{x}_2|, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1, \ v_1 \geq 0, \ v_2 \geq 0\} & \text{if } \bar{x}_1 = \bar{x}_2 > 0, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1, \ v_1 \geq 0, \ v_2 \leq 0\} & \text{if } \bar{x}_1 = -\bar{x}_2 > 0, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1, \ v_1 \leq 0, \ v_2 \leq 0\} & \text{if } \bar{x}_1 = \bar{x}_2 < 0, \\ \{(v_1, v_2) \in I\!R^2 \mid |v_1| + |v_2| = 1, \ v_1 \leq 0, \ v_2 \geq 0\} & \text{if } \bar{x}_1 = -\bar{x}_2 < 0. \end{cases}$$

Example 4.7. Let us consider an example when $X = I\!R^2$ with the "max" norm. Let Ω_i be the squares given in Example 4.3. Notice that a ball $I\!B(\bar{x}; r)$, $\bar{x} = (\bar{x}_1, \bar{x}_2)$, in X is the square

$$I\!B(\bar{x}; r) = [\bar{x}_1 - r, \bar{x}_1 + r] \times [\bar{x}_2 - r, \bar{x}_2 + r].$$

Therefore, the smallest intersecting ball problem (5) can be interpreted as follows: find a smallest square in $I\!R^2$ that intersects all n given squares. Using the same
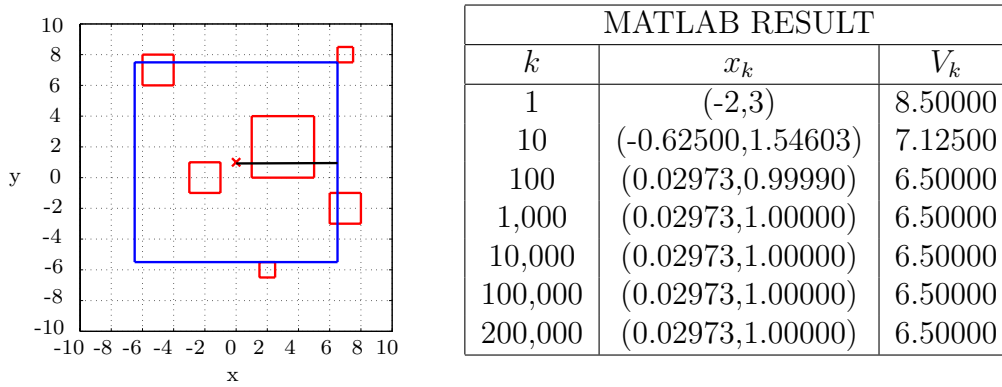
MATLAB RESULT		
k	x_k	V_k
1	(-2,3)	8.50000
10	(-0.62500,1.54603)	7.12500
100	(0.02973,0.99990)	6.50000
1,000	(0.02973,1.00000)	6.50000
10,000	(0.02973,1.00000)	6.50000
100,000	(0.02973,1.00000)	6.50000
200,000	(0.02973,1.00000)	6.50000

Figure 4.3: A smallest intersecting ball with max norm to square targets

notation for the vertices of the target set Ω_i as Example 4.3, one can see that the vectors x_k^* in Theorem 4.1 are given by

$$
x_k^* = \begin{cases}
(1,0), & \text{if } |x_{2k} - \omega_{2i}| \le x_{1k} - \omega_{1i} \text{ and } x_{1k} > \omega_{1i} + r_i, \\
(-1,0), & \text{if } |x_{2k} - \omega_{2i}| \le \omega_{1i} - x_{1k} \text{ and } x_{1k} < \omega_{1i} - r_i, \\
(0,1), & \text{if } |x_{1k} - \omega_{1i}| \le x_{2k} - \omega_{2i} \text{ and } x_{2k} > \omega_{2i} + r_i, \\
(0,-1), & \text{if } |x_{1k} - \omega_{1i}| \le \omega_{2i} - x_{2k} \text{ and } x_{2k} < \omega_{2i} - r_i,
\end{cases}
$$

where $i \in I(x_k)$. The sequence (V_k) is determined based on $D(x_k)$. Fix any $i \in I(x_k)$. Then

$$
D(x_{1k}, x_{2k}) = \begin{cases}
x_{1k} - (\omega_{1i} + r_i), & \text{if } |x_{2k} - \omega_{2i}| \le x_{1i} - \omega_{1i}, \ x_{1k} > \omega_{1i} + r_i, \\
(\omega_{1i} - r_i) - x_{1k}, & \text{if } |x_{2k} - \omega_{2i}| \le \omega_{1i} - x_{1k}, \ x_{1k} < \omega_{1i} - r_i, \\
x_{2k} - (\omega_{2i} + r_i), & \text{if } |x_{1k} - \omega_{1i}| \le x_{2k} - \omega_{2i}, \ x_{2k} > \omega_{2i} + r_i, \\
(\omega_{2i} - r_i) - x_{2k}, & \text{if } |x_{1k} - \omega_{1i}| \le \omega_{2i} - x_{2k}, \ x_{2k} < \omega 2i - r_i.
\end{cases}
$$

Consider the target sets $\Omega_i, i = 1, \ldots, 6$, to be the squares with centers $(-5,7)$, $(-2,0)$, $(2,-6)$, $(7,-2)$, $(3,2)$, and $(7,8)$ and the radii $r_i = \{1,1,0.5,1,2,0.5\}$ for $i = 1, \ldots, 6$, respectively. A MATLAB program is performed for the sequence $\alpha_k = 1/k$ satisfying (19) and the starting point $x_1 = (-2,3)$; see Figure 4.3. Observe that the numerical results computed up to five decimal places yield an optimal solution $\bar{x} \approx (0.02973, 1.00000)$ and the optimal value $\widehat{V} \approx 6.50000$.

The advantage of the algorithm comes from the fact that we are able to deal with the smallest intersecting ball problem generated by target sets of different types and different norms. Although a faster subgradient algorithm may be applied to this problem, we have chosen the simplest one for demonstrations.

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