Journal of Convex Analysis
Volume 19 (2012), No. 3, 865–874

Approximation of Bodies of Constant Width and Reduced Bodies in a Normed Plane

Marek Lassak
Institute of Mathematics and Physics, University of Technology and Life Sciences, al. Kaliskiego 7, Bydgoszcz 85–789, Poland
lassak@utp.edu.pl

Received: May 15, 2011
Revised manuscript received: January 22, 2012

We prove that for every $\varepsilon > 0$ and for every convex body of constant width in a normed plane there exists a convex body of the same constant width whose boundary consists only of arcs of circles in the sense of the norm such that the Hausdorff distance between the two bodies is at most ε. This generalizes the Euclidean case proved by Blaschke. We also present a more general theorem about approximation of reduced bodies.

Keywords: Reduced convex body, body of constant width, normed plane, Hausdorff distance, approximation

2010 Mathematics Subject Classification: 52A10, 52A21, 52A27, 46B25

1. Introduction

A theorem of Blaschke says that for every convex body W of constant width in the Euclidean plane E^2 we can find a convex body of constant width whose boundary consists only of arcs of circles and which can be as close to W as we wish (see [2] and also §65 of [3]). The proximity here is taken in the sense of Hausdorff distance. Our Theorem 2.1 presents a generalization of this theorem for bodies of constant width in any real normed plane M^2 (also called a two-dimensional Banach space or a Minkowski plane), and Theorem 3.3 gives a further generalization for reduced bodies in M^2 and, in particular, in E^2.

Let $C \subset M^2$ be a convex body. The symbol $\mathrm{bd}(C)$ denotes the boundary of C. If H_1 and H_2 are different parallel lines in M^2, then $S = \mathrm{conv}(H_1 \cup H_2)$ is called a *strip*, where the symbol conv stands for convex hull. We call H_1 and H_2 *bounding lines of* S. If these lines are perpendicular (in Euclidean sense) to a direction m, then S is called a *strip of direction* m. If both H_1 and H_2 are supporting lines of a convex body C, then we say that S is a C*-strip*. The C*-strip of direction* m is denoted by $S(C, m)$. If $p \in \mathrm{bd}(C)$ belongs to a bounding line of a C-strip S, we say that S *supports* C *at* p. By the *first strip supporting* C at a boundary point p we mean the first supporting strip of this body which contains p when we go on the

ISSN 0944-6532 / \$ 2.50

boundary counterclockwise. By ab we denote the segment with endpoints a and b, and by $|ab|$ we denote the distance $||b-a||$ of them. The set of points whose distance from a point c equals (respectively, is at most) d is called the *circle in the sense of the norm* or, for shortness, *circle* (respectively, *ball*) *of radius* d *and center* c.

2. Approximation of bodies of constant width

By the *M^2-width of a strip* S we mean the double radius of any largest ball contained in S. Let a and b be points of this ball in the two lines bounding S. Then we say that ab is *normal* to S (in the Euclidean case, a and b are unique, but uniqueness does not hold in general). Moreover, we say that ab is a *normal to a line* or *to a segment*, if ab is normal to a parallel strip.

By the *M^2-width* $w(C,m)$ of C in direction m we mean the M^2-width of $S(C,m)$. The number $\Delta(C) = \min_m w(C,m)$ is called the *M^2-thickness* of C. If a chord ab of $C \subset M^2$ connects the two lines bounding a C-strip of M^2-thickness $\Delta(C)$ and if $|ab| = \Delta(C)$, it is called a *thickness chord of* C. Recall that the union of all thickness chords of C which connect the straight lines bounding $S(C,m)$ is a trapezium whose bases are in these lines. We denote these bases by $A(C,m)$ and $B(C,m)$, such that $A(C,m)$ is in this line bounding $S(C,m)$ for which the outer normal of $S(C,m)$ has direction m. A convex body $W \subset M^2$ is said to be of *constant M^2-width* if $w(W,m)$ is the same for every direction m. For basic results on bodies of constant M^2-width see [6], and for a survey of results see Part 2 of [4] and Part 2 of [12]. For the Euclidean case see [3] and [4].

We omit the proof of the following theorem since it is a special case of Theorem 3.3 proved in Section 3 (see the short explanation after the proof of Theorem 3.3).

Theorem 2.1. *For every body $W \subset M^2$ of constant M^2-width and for arbitrary $\varepsilon > 0$ there exists a body $W_\varepsilon \subset M^2$ of constant M^2-width $\Delta(W_\varepsilon) = \Delta(W)$ whose boundary consists only of arcs of circles (in the sense of norm) of radius $\Delta(W)$, such that the Hausdorff distance between W and W_ε is at most ε.*

3. Approximation of reduced bodies

A convex body $R \subset M^2$ is *reduced* provided $\Delta(K) < \Delta(R)$ for every convex body $K \subset R$ different from R. For Euclidean space this notion was introduced in [8], and for finite dimensional real normed space it was extended in [11]. The class of reduced bodies is larger than the class of bodies of constant width. For basic properties of planar reduced bodies in E^2 see [10] (a larger context is given in [9]), and for analogues in M^2 see [7] and [11]. For many extremal problems concerning the thickness and, more generally, the M^2-thickness of convex bodies it is sufficient to consider only reduced bodies. So the subject is important.

For the proof of Theorem 3.3 we need a description of the structure of the boundary of a reduced body $R \subset M^2$. From [7] we see that it looks as follows.

The Theorem of [7] says that if $w(R,m_1) = \Delta(R) = w(R,m_2)$ and if $w(R,m) > \Delta(R)$ for every m strictly between m_1 and m_2, then the segments a_1a_2 and b_1b_2 are non-degenerate, and that they are in $\mathrm{bd}(R)$. Here by a_1 we denote the last point of

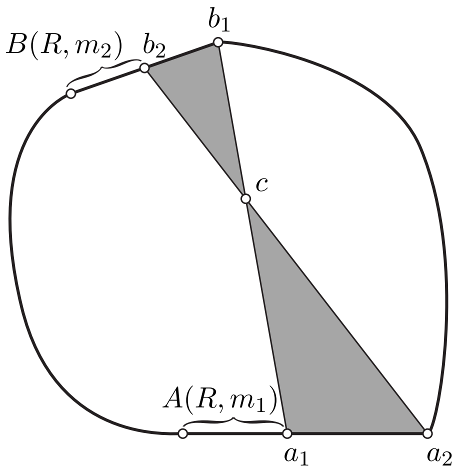

Figure 3.1: A butterfly of a reduced body

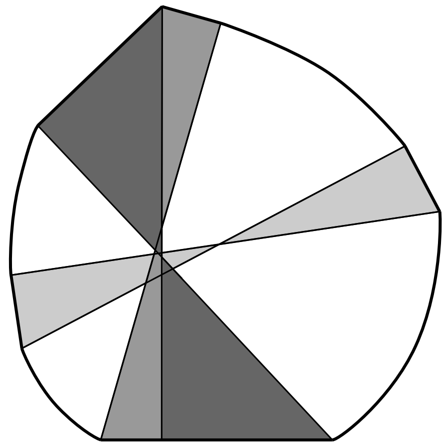

Figure 3.2: Reduced body with three butterflies

$A(R, m_1)$, by a_2 the first point of $A(R, m_2)$, by b_1 the last point of $B(R, m_1)$ and by b_2 the first point of $B(R, m_2)$. Denote by c the intersection point of the segments a_1a_2 and b_1b_2. The union of the triangles a_1a_2c and b_1b_2c is called a *butterfly*, a_1a_2 and b_1b_2 its *arms*, and a_1b_1 and a_2b_2 its *diagonals*. An illustration is given in Fig. 3.1; pay attention that here $B(R, m_1)$ consists of one point b_1, and $A(R, m_2)$ of one point a_2, which follows from Corollary 3 of [7]. Observe that ab, where a and b belong to the arms of a butterfly, is a thickness chord if and only if ab is a diagonal of this butterfly.

In Fig. 3.2 we see a reduced convex body in E^2 with three butterflies. Particular reduced bodies and some of their butterflies are also shown in Fig. 3–5 of [11]. Of course, a reduced body is a body of constant M^2-width if and only if there are no butterflies in it.

Assume that R is a reduced body which is not a body of constant M^2-width. Clearly, the endpoints of thickness chords of R form a family of pairs of "opposite" curves. We call them *pairs of opposite curves of M^2-width* $\Delta(R)$, or briefly *pairs of opposite curves*. Here and later we mean only such curves which are maximal by inclusion. Moreover we never treat a pair of points as a pair of opposite curves (e.g., endpoints of a diagonal of a butterfly do not form a pair of opposite curves). For instance, in Fig. 3.2 of this paper we see two pairs of opposite curves, and in Fig. 5 of [7] we see a pair of opposite curves of a reduced body R_2 such that one of them is a point and the other is a segment. Observe that the endpoints of each curve (from any pair of opposite curves of R) is always an endpoint of an arm of a butterfly or a limit of endpoints of butterflies. Thus the boundary of R is the union of at most countably many pairs of arms of butterflies and of at most countably many pairs of opposite curves (cf. Corollary 11 of [7]).

Angles in Lemmas 3.1 and 3.2 and in the proof of Theorem 3.3 are understood in the sense of E^2.

Lemma 3.1. *For every normed plane there exists a constant $\mu_{|| \, ||}$ such that for every non-degenerate triangle abd the distance between d and ab is at most $\mu_{|| \, ||} \cdot |ab| \cot(\frac{1}{2}\angle adb)$.*

Lemma 3.2. *For every normed plane there exists a positive number $\beta_{|| \, ||} < \pi$ such that for every strip and every normal to it both angles between them are at most $\beta_{|| \, ||}$.*

Proof. Here is a sketch of the proofs of Lemmas 3.1 and 3.2. Denote by o the center of the unit ball U of M^2 and by r the ratio of the radius of the smallest Euclidean disk centered at o which contains U to the radius of the largest Euclidean disk centered at o which is contained in U.

Since in E^2 Lemma 3.1 holds for $\mu_{|| \, ||_E} = \frac{1}{2}$, we pass to M^2 multiplying this value by r. Consequently, we may take $\mu_{|| \, ||} = \frac{1}{2} r$.

In order to see Lemma 3.2 take any strip S and any segment connecting opposite straight lines bounding S. Observe that the ratio of the Euclidean length of this segment to the Euclidean width of S is at most $\frac{1}{r}$. Hence the angles between this segment and S are between $\arcsin \frac{1}{r}$ and $\pi - \arcsin \frac{1}{r}$. So it is sufficient to take $\beta_{|| \, ||} = \pi - \arcsin \frac{1}{r}$ in M^2. □

Theorem 3.3. *For every reduced body $R \subset M^2$ and for arbitrary $\varepsilon > 0$ there exists a reduced body $R_\varepsilon \subset M^2$ such that the following hold:*

(*i*) *the boundary of R_ε consists only of arms of butterflies and arcs of circles (in the sense of norm) of radius $\Delta(R)$,*
(*ii*) *for every direction the M^2-widths of R and R_ε are equal,*
(*iii*) $\Delta(R_\varepsilon) = \Delta(R)$,
(*iv*) *the Hausdorff distance between $\Delta(R_\varepsilon)$ and R is at most ε.*

Proof. We omit the trivial case when the boundary of R consists only of arms of butterflies.

Of course, it is sufficient to consider any $\varepsilon > 0$ smaller than a constant. For our purpose, consider any $\varepsilon > 0$ smaller than $\pi - \beta_{|| \, ||}$. Here and below $\beta_{|| \, ||}$ and $\mu_{|| \, ||}$ are taken from Lemmas 3.1 and 3.2. Let $\rho_\varepsilon = \varepsilon \mu_{|| \, ||}^{-1} \cdot \tan \frac{1}{2}(\pi - \beta_{|| \, ||})$. In Part 2 of our proof we apply this ρ_ε when constructing R_ε, and in Part 6 we explain that such a choice of it guarantees that the Hausdorff distance between R and R_ε is at most ε.

Consider any pair of opposite curves F and G in the boundary of R. Exceptionally when R is a body of constant M^2-width, we divide $\mathrm{bd}(R)$ into a pair of curves F and G by an arbitrary thickness chord. Denote the endpoints of F by f' and f'', and the endpoints of G by g' and g'', in both cases according to positive orientation.

1. We show that the distance between any points $f \in F$ and $g \in G$ is at most $\Delta(R)$.

Assume the opposite, i.e., that the distance of some $f \in F$ and $g \in G$ is larger than $\Delta(R)$. Support R by a strip such that fg is normal to the lines bounding it. This strip is of M^2-width larger than $\Delta(R)$.

On the other hand, since fg intersects $f'g'$ and $f''g''$, the strip supports R at a point of F and a point of G, a contradiction to the fact that F and G are opposite curves.

2. The aim of this part is to construct the set R_ε.

For each pair F, G of opposite curves in $\mathrm{bd}(R)$ we provide a number of different thickness chords $f_1g_1, \dots, f_ng_n$ of R such that $f_1, \dots, f_n \in F$ (with $f_1 = f'$ and $f_n = f''$), and $g_1, \dots, g_n \in G$ (with $g_1 = g'$ and $g_n = g''$), taking care that $|f_if_{i+1}| \leq \rho_\varepsilon$ and $|g_ig_{i+1}| \leq \rho_\varepsilon$, where $i = 1, \dots, n-1$, and that the positively oriented angle between every two successive of these chords is at most $\pi - \beta_{||\ ||}$. Observe that some of points $f_1, \dots, f_n$ (some of $g_1, \dots, g_n$) may coincide.

Denote by o_i the intersection point of f_ig_i and $f_{i+1}g_{i+1}$ for $i = 1, \dots, n-1$ (see Fig. 3.3). Moreover, denote by Φ_i (respectively, Γ_i) the angle between the rays from o_i through f_i and f_{i+1} (respectively, through g_i and g_{i+1}) for $i = 1, \dots, n-1$. Let $c_i \in \Gamma_i$ be a point of intersection of circles of radius $\Delta(R)$ with centers f_i and f_{i+1} (so c_i is in equal distances from f_i and f_{i+1}). Such c_i exists since, thanks to Part 1, we have $|f_ig_{i+1}| \leq \Delta(R)$ and $|f_{i+1}g_i| \leq \Delta(R)$. Moreover, by c_0 we mean g', and by c_n we mean g''.

For every $i \in \{1, \dots, n-1\}$ take the arc F_i of the circle F_i° of radius $\Delta(R)$ with center c_i and endpoints f_i and f_{i+1} which is in Φ_i. Moreover, for $i \in \{1, \dots, n\}$ take the arc G_i of the circle G_i° of radius $\Delta(R)$ with center f_i which begins at c_{i-1} and ends at c_i. Created arcs are marked by broken lines in Fig. 3.3. Clearly, $G_1 \subset \Gamma_1$, $G_i \subset \Gamma_{i-1} \cup \Gamma_i$ for $i = 2, \dots, n-1$, and $G_n \subset \Gamma_{n-1}$.

We constructed the pair of curves $F^* = F_1 \cup \dots \cup F_{n-1}$ and $G^* = G_1 \cup \dots \cup G_n$.

Denote by U_ε the closure of the union of all arms of the butterflies of R and of all pairs of curves of the form F^* and G^*. We see that U_ε is obtained from $\mathrm{bd}(R)$ by exchanging all pairs of opposite curves F and G by the constructed pairs of curves F^* and G^*.

We define R_ε as the set bounded by U_ε.

3. We intend to show that R_ε is a convex body and to conclude that $\Delta(R_\varepsilon) = \Delta(R)$.

Since we have at most countably many pairs of opposite curves in $\mathrm{bd}(R)$, imagine

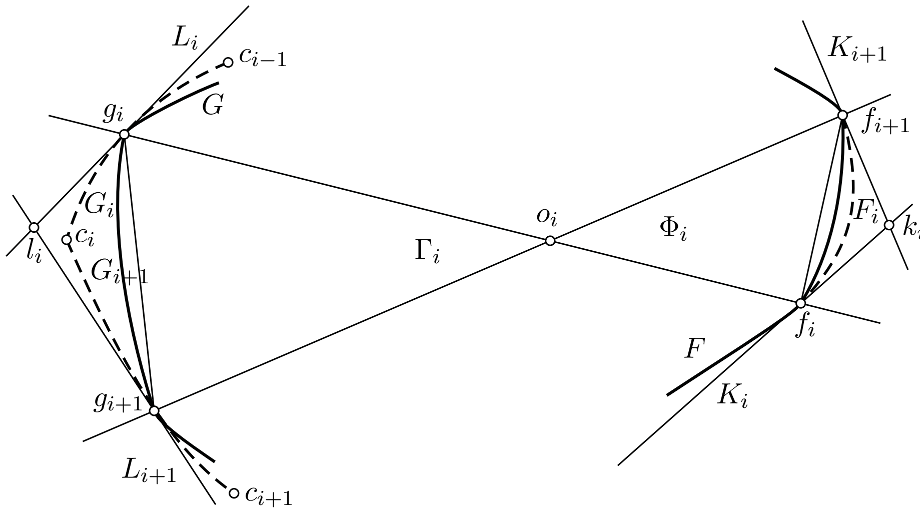

Figure 3.3: Illustration of the proof of Theorem 3.3

that we provide the exchange which gives U_ε step by step, each time exchanging exactly one more pair of opposite curves. Put $R^0 = R$, and the sets bounded by the successively obtained curves are denoted by $R^1, R^2, \dots$.

If we obtain R_ε, we stop creating successive sets. In the opposite case we have $R_\varepsilon = \lim_{j\to\infty} R^j$.

In order to show that each of the sets $R^0, R^1, \dots$ is a convex body, we proceed by induction. Of course, R^0 is a convex body. We assume that R^{j-1} is a convex body, and our aim is to show that R^j is a convex body

We intend to apply Theorem 9 of [5], p. 21, which says that if at every boundary point of a set $X \subset E^2$ with nonempty interior there is a straight line which supports X, then X is a convex body. Recall that we say that a line *supports a set* $X \subset E^2$ if it intersects X but does not cut X (see again [5]). So we consider now R^j in the part of X. Our aim is to show that for every boundary point p of R^j there is a line L_p through p which supports R^j.

The boundary of R^j consists of those two pieces of $\mathrm{bd}(R^{j-1})$ which remain after we remove F and G (in particular, when we create R^1 for a body R of constant width, those two pieces are empty) together with the pair F^* and G^* of curves added when constructing R^j. Observe that these four pieces are disjoint. The reason is that they are in four parts of the plane dissected by the lines containing $f'g'$ and $f''g''$.

First consider any point $p \in F^* \cup G^*$. Look for the first supporting $\mathrm{conv}(G_i^\circ)$-strip at every point g of every G_i (respectively, $\mathrm{conv}(F_i^\circ)$-strip at every point $f \in F_i$). Take the strip being the “half” of this strip with bounding lines passing through g and f_i (respectively, through f and c_i). Since the centers $f_1, \dots, f_n$ (respectively, $c_1, \dots, c_{n-1}$) of the arcs of circles are in the counterclockwise order on F^* (respectively, on G^*), we conclude that through every $p \in F^* \cup G^*$ passes a line L_p which does not cut $F^* \cup G^*$.

If $p \in \mathrm{bd}(R^j)$ does not belong to the curves F^* and G^*, then in the part of L_p we take any supporting line of R^{j-1} at p. Clearly, L_p does not cut the two fragments of the boundary of R^j taken from the boundary of R^{j-1}.

Take into account that for every point $p \in \mathrm{bd}(R^j)$ the straight line L_p through p taken in the two preceding paragraphs does not cut R^j (hint: first observe that this is true for points f', f'', g', g'' in the part of p, and next that this remains true if p moves on the boundary of R_j). Moreover, since $\mathrm{conv}(F^* \cup G^*)$ has nonempty interior, R^j also has nonempty interior. Consequently, R^j is a convex body (comp. Theorem 9 of [5], p. 21).

If after a finite number of steps we obtain R_ε, we are done. In the opposite case, $R_\varepsilon = \lim_{j\to\infty} R^j$. So it is also a convex body (see [14], p. 94).

From the construction of R_ε we conclude that for every direction the M^2-widths of R and R_ε are equal and thus that $\Delta(R_\varepsilon) = \Delta(R)$.

4. Now we prove that R_ε is a reduced body.

We must show that for any convex body $K \subset R_\varepsilon$ different from R_ε the inequality $\Delta(K) < \Delta(R_\varepsilon)$ holds true. The body K does not contain an extreme point e of R_ε (see §18 of [13]). So K is disjoint with an open disk D centered at e.

By the definition of R_ε in Part 2, every point of $\mathrm{bd}(R_\varepsilon)$ belongs to the closure of the union of all arms of the butterflies of R, or to the union of all curves of the form F^* and G^*.

If D has nonempty intersection with a curve of the form F^* or G^*, then from the construction of these curves we deduce that $\Delta(\mathrm{conv}(R_\varepsilon \setminus D)) < \Delta(R_\varepsilon)$. Since $K \subset \mathrm{conv}(R_\varepsilon \setminus D)$, we obtain $\Delta(K) < \Delta(R_\varepsilon)$.

In the opposite case, the construction of R_ε implies that there is an open disk $D' \subset D$ centered at e such that $D' \cap \mathrm{bd}(R_\varepsilon) = D' \cap \mathrm{bd}(R)$ and that this set has empty intersection with all curves of the form F^* and G^* in $\mathrm{bd}(R_\varepsilon)$, and with all curves of the form F and G in $\mathrm{bd}(R)$. Thus from the construction of R_ε and $\Delta(R_\varepsilon) = \Delta(R)$ we see that $\Delta(\mathrm{conv}(R_\varepsilon \setminus D')) = \Delta(\mathrm{conv}(R \setminus D'))$. Thus the inequality $\Delta(\mathrm{conv}(R \setminus D')) < \Delta(R)$ (resulting by the reducedness of R), and $\Delta(R_\varepsilon) = \Delta(R)$ imply that $\Delta(\mathrm{conv}(R_\varepsilon \setminus D')) < \Delta(R_\varepsilon)$. So from $K \subset \mathrm{conv}(R_\varepsilon \setminus D')$ we get $\Delta(K) < \Delta(R_\varepsilon)$.

Since $\Delta(K) < \Delta(R_\varepsilon)$ in both cases, we see that R_ε is a reduced body.

5. Consider any pair of curves F^* and G^* constructed in Part 3. Let N_i be the first R_ε-strip supporting R_ε at f_i and g_i. Its bounding line through f_i is denoted by K_i, and that through g_i by L_i (again see Fig. 3.3). For $i \in \{1, \dots, n-1\}$, by k_i we mean the point of intersection of K_i with K_{i+1} (if these lines coincide, then take k_i as the midpoint of $f_i f_{i+1}$). By l_i we mean the point of intersection of L_i with L_{i+1} (if these lines coincide, take l_i as the midpoint of $g_i g_{i+1}$).

For every $c \in G_i$ take the first R_ε-strip such that cf_i is normal to it, and denote by $T(c)$ its bounding line through f_i. In particular, $T(g_i) = K_i$. When we move $c \in G_i$ counterclockwise from g_i to c_i, the strip "rotates" counterclockwise (with some possible breaks in the movement). Thus $T(c)$ also rotates counterclockwise. This and the fact that the distance between c_i and any point of $T(c_i)$ is at least

$\Delta(R)$ imply that the distance from c_i to every point of the ray from f_i through k_i is at least $\Delta(R)$. Analogously, the distance between c_i and any point of the ray from f_{i+1} through k_i is at least $\Delta(R)$. So since every point of F_i is at distance $\Delta(R)$ from c_i, we see that $F_i \subset f_i k_i f_{i+1}$. Thus F^* is contained in the union of triangles $f_i k_i f_{i+1}$, where $i = 1, \dots, n$. Similarly one can show that G^* is in the union of triangles $g_i l_i g_{i+1}$, where $i = 1, \dots, n$.

6. It remains to show that the Hausdorff distance between R and R_ε is at most ε.

Denote by P the closure of the convex hull of all points f_i and g_i taken from all curves of the form F and G and of endpoints of all arms of butterflies of R. Denote by Q the intersection of all strips N_i (defined in Part 5) for all pieces of curves F, G and of all half-planes containing R whose bounding lines contain the arms of butterflies of R. Part 5 implies inclusions $P \subset R \subset Q$ and $P \subset R_\varepsilon \subset Q$. So in order to estimate the Hausdorff distance between R and R_ε it is sufficient to estimate the Hausdorff distance between P and Q. Taking into account the fact that the closure of $Q \setminus P$ is the union of all triangles of the form $f_i k_i f_{i+1}$ and of the form $g_i l_i g_{i+1}$, it is sufficient to show that all (i.e., for all pairs F, G and all i) distances between k_i and $f_i f_{i+1}$, and also between l_i and $g_i g_{i+1}$, are at most ε.

By Lemma 3.1 the distance between k_i and $f_i f_{i+1}$ is at most $\mu_{|| \, ||} \cdot |f_i f_{i+1}| \cot(\frac{1}{2}\angle f_i k_i f_{i+1})$, and the distance between l_i and $g_i g_{i+1}$ is at most $\mu_{|| \, ||} \cdot |g_i g_{i+1}| \cot(\frac{1}{2}\angle g_i l_i g_{i+1})$.

Look at the quadrangle $o_i f_i k_i f_{i+1}$. Since the sum of its angles is 2π, applying Lemma 3.2 for angles $\angle o_i f_i k_i$ and $\angle o_i f_{i+1} k_i$, and the fact that $\angle f_i o_i f_{i+1} \le \pi - \beta_{|| \, ||}$ (see the second paragraph of Part 2) we conclude that $\angle f_i k_i f_{i+1} \ge 2\pi - 2\beta_{|| \, ||} - (\pi - \beta_{|| \, ||}) = \pi - \beta_{|| \, ||}$. Similarly, always $\angle g_i l_i g_{i+1} \ge \pi - \beta_{|| \, ||}$.

From the above two paragraphs and having in mind that $|f_i f_{i+1}| \le \rho_\varepsilon$ and $|g_i g_{i+1}| \le \rho_\varepsilon$ (see Part 2) we get that all distances between k_i and $f_i f_{i+1}$, and between l_i and $g_i g_{i+1}$ (for all F, G) are at most $\rho_\varepsilon \cdot \mu_{|| \, ||} \cdot \cot \frac{1}{2}(\pi - \beta_{|| \, ||})$. Since $\rho_\varepsilon = \varepsilon \mu_{|| \, ||}^{-1} \cdot \tan \frac{1}{2}(\pi - \beta_{|| \, ||})$, we conclude that the Hausdorff distance between P and Q, and thus between R and R_ε, is at most ε. □

If R is a body of constant M^2-width, from Parts 2 and 3 of this proof we see that R_ε (which is R^1 there) is also a body of constant M^2-width. So Theorem 2.1 is a special case of Theorem 3.3.

4. Corollaries

From the paper [15], in particular we know that for every two convex bodies in M^2 the Hausdorff distance of them is equal to the Hausdorff distance of their boundaries. This and our Theorems 2.1 and 3.3 lead to the following corollary.

Corollary 4.1. *Every curve of constant M^2-width $\Delta(W)$ may be approximated by a curve of constant M^2-width $\Delta(W)$ whose boundary consists only of arcs of circles (in the sense of norm) of radius $\Delta(W)$, such that the Hausdorff distance between them is at most any given $\varepsilon > 0$.*

A more general corollary holds true for curves bounding reduced bodies in M^2.

Here are two corollaries which generalize the well-known theorem of Barbier [1] (see also [5], p. 127) that the perimeter of every convex body $W \subset E^2$ of constant width is $2\pi \cdot \Delta(W)$. The statements of these corollaries can also be obtained by applying central symmetrization.

Corollary 4.2. *All bodies of constant M^2-width of a normed plane have the same perimeter in the sense of the norm.*

Corollary 4.3. *For every planar reduced body R we have* $\mathrm{perim}(R) \geq 2\pi \cdot \Delta(R)$ *with equality if and only if R is a body of constant width. For every planar reduced polygon R we have* $\mathrm{perim}(R) > 2\pi \cdot \Delta(R)$*, and in general this inequality cannot be improved.*

These two corollaries result from the construction of R_ε in the proof of Theorem 3.3, by Corollary 4.1 (with its generalization for reduced bodies), and by the limit approach. Additionally, in order to obtain Corollary 4.3, we apply the fact that the sum of lengths of arms of butterflies in a butterfly is $2\tan\frac{\psi}{2} \cdot \Delta(R)$, where ψ denotes the angle between its diagonal (see Theorem 3 of [10]). Of course, this is more than the length $\psi \cdot \Delta(R)$ of the usual circular arc of radius $\Delta(R)$ and angle ψ. The fact that the second inequality in Corollary 4.3 cannot be improved follows by taking into account the sequence of regular n-gons, where n is odd and tends to infinity.

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