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The Daugavet Property and Weak Neighborhoods in Banach Lattices

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The Daugavet property and the diameters of relatively weakly open subsets of unit balls in Banach lattices X on measure spaces are studied. It is shown that under mild assumptions the subspace X_a of order continuous elements inherits the Daugavet property from X. This is applied to prove that if X has the Daugavet property and the Köthe dual X' is strictly monotone (resp. order continuous) then X' contains a lattice isometric (resp. isomorphic) copy of $L_1(0,1)$. These results yield that a large class of r.i. spaces including several interpolation sums fail the Daugavet property and also that any r.i. space over a finite atomless measure space with the Daugavet property coincide to either L_1 or L_∞. Applications are shown for Orlicz, Lorentz, Marcinkiewicz spaces as well for Nakano spaces. It is established that in most cases these spaces do not enjoy the Daugavet property. However, it is proved that in a large class of Orlicz or Nakano spaces (variable exponent spaces), in particular those induced by fast growing Orlicz functions, all non-empty relatively weakly open subsets of their unit balls have diameter two.

Keywords: Daugavet property, weak neighborhoods, Banach lattices, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces, Nakano spaces, variable exponent spaces

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1. Introduction

Given a Banach space $(X, \|\cdot\|)$ and a bounded linear operator $T\colon X \to X$, the following equality

$$\|I + T\| = 1 + \|T\|,$$

is called the *Daugavet equation*, where I is the identity operator on X. A Banach space $(X, \|\cdot\|)$ is said to have the *Daugavet property* [21] if every rank one operator $T\colon X \to X$ satisfies the Daugavet equation. It is known that in a space with the Daugavet property, every weakly compact, even every operator not fixing a copy of ℓ_1 satisfies the Daugavet equation [21, 41].

In 1963, Daugavet [16] showed that the Daugavet equation holds for every compact operator T on $C(0,1)$, and in 1966, Lozanovskii [33] proved that the same equation is satisfied for compact operators on $L_1(0,1)$. These results were left without much attention approximately until the beginning of 1980's. Since then the Daugavet equation has been studied by many authors in various contexts. The textbook [3] and references therein provide a good source of results about this property. We also refer to a nice survey paper of Werner [44]. Here we will point out some results that are relevant to the studied topics.

The spaces $L_1(\mu)$ and $L_\infty(\mu)$ on any atomless measure space have the Daugavet property. Further such spaces as $C(K)$ where K is a compact Hausdorff space with no isolated points, the disk algebra, H^∞, non-atomic C^*-algebras or preduals of non-atomic von Neumann algebras have the Daugavet property [38]. A space with the Daugavet property does not have an unconditional basis [20] and does not even embed into a space with an unconditional basis [21]. In [20] Kadets used this result to give an elegant proof of the well known results that neither $C[0,1]$ nor $L_1(0,1)$ have unconditional bases. It is also known that a Banach space with the Daugavet property cannot have the Radon-Nikodým property [45] and that the Daugavet property can be lifted from components to the entire space in a finite direct sum of Banach spaces equipped with ℓ_∞ or ℓ_1 norm [2, 45, 21]. Some other examples of spaces with the Daugavet property are presented in [22, 11, 44].

In general the Daugavet property is not inherited by subspaces, even 1-complemented [44, 21]. On the other hand it is inherited by subspaces that are M-ideals or L-summands [21]. It should be mentioned that the Daugavet equation has found nice applications in approximation theory. Stechkin [42] used it to find the best constants in certain inequalities.

The main purpose of this paper is to investigate the Daugavet property in Köthe spaces that are the Banach lattices and ideals over measure spaces. So far with a few exceptions not much of such studies have been conducted.

The Daugavet property is quite restrictive, since it implies some severe isomorphic and isometric restrictions. So it is not surprising that many results presented here demonstrate that the Daugavet property in Banach lattices is rather unusual.

As we mentioned before, the Daugavet property is not generally inherited by subspaces, even the complemented ones. However our first result obtained here shows that in the case of Banach function lattices X, this property is inherited by the subspaces X_a of order continuous elements. This essential observation allows us to treat considerably larger family of spaces than has been studied so far. It also allows us to develop a special method for demonstrating that the space fails the Daugavet property.

In fact one starts with the assumption that the space X has the Daugavet property. It then yields that under mild assumptions its subspace X_a also has it. Hence X_a does contain an asymptotically isomorphic copy of ℓ_1 [21], which in turn is equivalent to the fact that the dual $(X_a)^*$ contains an isometric copy of $L_1(0,1)$ [18]. Further applying some additional assumptions on X or X' we get that this copy is order isometric [46], and then that $X(A) = L_1(A)$ as sets with equivalent norms [1], for every measurable set A with $\mu(A) < \infty$, which in fact often reduces X to L_∞ or leads to a contradiction. This method is a base for some general results on the Daugavet property in Banach function lattices or more specific results for such spaces as Orlicz, Nakano, Lorentz or Marcinkiewicz.

It is well known that if a Banach space has the Daugavet property, then every nonempty weakly open subset of its unit ball has diameter two [41, Lemma 3]. The class of Banach spaces having the latter property is considerably larger than the one with the Daugavet property. For instance in ℓ_∞ or c_0 every nonempty weakly open subset of their unit balls has diameter two, while they have no Daugavet property. It is also well known that every infinite-dimensional C^*-algebra satisfies that every weak neighborhood of the unit ball has diameter two [7] and just some of them satisfy the Daugavet property [38] (see also [8]). Analogous situation occurs in interpolation spaces $L_1 + L_\infty$ and $L_1 \cap L_\infty$ [5], and we will see similar phenomena in the class of Orlicz or Nakano spaces. Other results on spaces having the diameter two property can be found in [4, 9, 37]. A version of the Daugavet property for polynomials (instead of operators) has been studied in [15].

The paper consists of five sections. In the second section we investigate the Daugavet property for Köthe spaces, called further Banach function lattices. They are the ideals in the space of all measurable functions $L^0(\mu)$ over the measure space $(\Omega, \mathcal{S}, \mu)$. The first result, crucial for further studies, states that if a Banach function lattice X has the Daugavet property, then its subspace of all order continuous elements X_a inherits this property under mild assumptions (the support of X_a is the entire space Ω and X satisfies the weak Fatou property). It follows that for spaces X in a large class of Banach function lattices, both X and its Köthe dual X' contain an isometric copy of $L_1(0,1)$. We show further that a large class of r.i. spaces over atomless measure spaces fails the Daugavet property. In particular, if a r.i. space X over an atomless finite measure space $(\Omega, \mathcal{S}, \mu)$ has the Daugavet property, with the assumptions that the norm of X is order continuous and it satisfies the Fatou property, then X must coincide with $L_1(\mu)$ as sets and the norm of X is equivalent to the usual norm in $L_1(\mu)$. Under some additional assumptions we get that a r.i. space X with the Daugavet property must be isometric either to L_1 or to L_∞. We finish this section with the exact formulas of the norms in dual spaces of the interpolation spaces $X + L_\infty$ equipped with classical norms, and consequently we show that the interpolation spaces $L_1 \cap X$, $L_\infty \cap X$ or $X + L_\infty$ fail the Daugavet property for a wide class of r.i. spaces X.

In Section 3 we consider Orlicz spaces, an important class of r.i. Banach function lattices. In the case of the atomless measure space $(\Omega, \mathcal{S}, \mu)$, we show that many Orlicz spaces $L_\varphi := L_\varphi(\mu)$, equipped with either Luxemburg or Orlicz norm, fail the Daugavet property. In fact, if φ grows essentially faster at infinity than a linear function, that is, if $\lim_{t\to\infty} \varphi(t)/t = \infty$, then the Orlicz space L_φ has no Daugavet property. We show also that the graph of φ must be composed of straight segments whenever L_φ has the Daugavet property and $\mu(\Omega) < \infty$. We finish the section by showing that in function and sequence Orlicz spaces equipped with the Luxemburg norm, the diameter of any non-empty weakly open subset of the unit ball is equal to two whenever φ does not satisfy the appropriate condition Δ_2. Thus in the class of Orlicz spaces there exist many examples failing the Daugavet property but having the diameters of weak neighborhoods equal to two.

Another type of Köthe spaces, not symmetric in general, is studied in Section 4. Here we investigate Nakano spaces $L_{p(t)}$ with $1 \le p(t) \le \infty$, a direct generalization of the Lebesgue spaces L_p, $1 \le p \le \infty$, and a sub-family of much larger class of Musielak-Orlicz spaces [36]. In particular we show that whenever $1 < p(t) < \infty$ then $L_{p(t)}$ equipped with either of two standard norms fails the Daugavet property. Nakano spaces $L_{p(t)}$ are also called variable exponent spaces. However if $p(t)$ takes only two values, 1 or ∞, then the space $L_{p(t)}$ enjoys the Daugavet property. We finish the section by proving that whenever $\operatorname{esssup}_\Omega p(t) = \infty$, then the diameter of any weak neighborhood in the unit ball of this space is equal to two.

The last section is devoted to studies of the Daugavet property in the Lorentz and Marcinkiewicz spaces Λ_ψ and M_ψ, respectively. We prove that for large class of functions, both spaces fail the Daugavet property. As a consequence, we recover the analogous result for $L_1 + L_\infty$ and $L_1 \cap L_\infty$ equipped with one of the standard norms.

2. Banach lattices

Throughout this paper, we will consider only real Banach spaces. If $(X, \|\cdot\|)$ is a Banach space, then by B_X and S_X we denote the unit ball and the unit sphere of X, respectively. As usual by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ we denote the set of real, non-negative real and natural numbers, respectively. Let $(\Omega, \mathcal{S}, \mu)$ be a complete σ-finite measure space, and $L^0 = L^0(\mu)$ be the space of (equivalence classes of μ-a.e. equal) real valued measurable functions on Ω. We say that $(X, \|\cdot\|_X)$ is a *Banach function lattice Banach lattice* for short) on $(\Omega, \mathcal{S}, \mu)$ if X is an ideal in L^0 and whenever $x, y \in X$ and $|x| \le |y|$ a.e., then $\|x\|_X \le \|y\|_X$. If Ω is the set of natural numbers $\mathbb{N}$, and μ the counting measure on subsets of $\mathbb{N}$, then the elements of X are real sequences and in this case X is called a *Banach sequence lattice*.

Notice here that for any Banach lattice X on $(\Omega, \mathcal{S}, \mu)$ it is possible to construct (see [26, Corollary 1, p. 95] or [47, pp. 454–456]) a set $\Omega_X \in \mathcal{S}$ called the *support* of X such that every element of X vanishes μ-a.e. on $\Omega \setminus \Omega_X$ and every measurable set $E \subset \Omega_X$ with $\mu(E) > 0$ has a measurable subset F of finite positive measure with $\chi_F \in X$. Furthermore Ω_X is the union of an increasing sequence of measurable sets (E_n) such that $\chi_{E_n} \in X$ and $\mu(E_n) < \infty$ for each n. Notice that Ω_X is defined up to a null set. As usual the support Ω_X of a Banach function lattice X is denoted by $\operatorname{supp} X$.

Let X be a Banach function lattice on $(\Omega, \mathcal{S}, \mu)$. Given a measurable set $A \subset \Omega$, $X(A)$ denotes the space of all elements in X restricted to A, or $X(A) = \{x\chi_A : x \in X\}$. By $L_1 = L_1(\mu)$ and $L_\infty = L_\infty(\mu)$ we denote the spaces of integrable and μ-essentially bounded, real valued measurable functions on Ω, respectively. An element $x \in X$ is called *order continuous* if for every $0 \le x_n \le |x|$ such that $x_n \downarrow 0$ a.e. it holds $\|x_n\|_X \to 0$. By X_a we denote the set of all order continuous elements of X. X is said to have the *Fatou property* whenever for any $x_n \in X$ and $x \in L^0$ such that $x_n \to x$ a.e. and $\sup_n \|x_n\|_X < \infty$ we have that $x \in X$ and $\|x\|_X \le \liminf_n \|x_n\|_X$. X is said to have the *weak Fatou property* whenever $x_n, x \in X$ and $x_n \to x$ a.e. we have that $\|x\|_X \le \liminf_n \|x_n\|_X$.

The Köthe dual space X' of X is the subset of all elements $y \in L^0$ such that

$$\|y\|_{X'} = \sup\left\{ \int_\Omega |xy|\, d\mu : \ \|x\|_X \le 1\right\} < \infty.$$

It is well known that X' equipped with the norm $\|\cdot\|_{X'}$ is a Banach function lattice on $\operatorname{supp} X$. Furthermore $X'' = X$ with equality of norms if and only if X has the Fatou property (see [26, Theorem 6, p. 190] or [47, Theorems 3 and 4, p. 472]).

Let X be a Banach lattice. The order in its dual space X^* is given by $F \le G$ if and only if $F(x) \le G(x)$ for all $0 \le x \in X$. A functional $F \in X^*$ is called *order continuous* (resp., *σ-order continuous*) if $x_\alpha \downarrow 0$ (resp., $x_n \downarrow 0$) implies $F(x_\alpha) \to 0$ (resp., $F(x_n) \to 0$). The sets of order continuous functionals (resp., σ-order continuous) is denoted by X_n^* (resp., by X_c^*). It is well known that both sets form bands in X^*. The disjoint complement $(X_n^*)^d$ of X_n^* (in X^*) is denoted by X_s^* and is called the set of *singular* functionals. Thus $X^* = X_n^* \oplus X_s^*$ is a direct sum of X_n^* and X_s^*.

If X is a Banach function lattice over $(\Omega, \mathcal{S}, \mu)$, then $X_n^* = X_c^*$ and $F \in X_c^*$ if and only if there exists a unique $y \in X'$ such that $F(x) = \int_\Omega xy\, d\mu$ for all $x \in X$, with $\|F\|_{X^*} = \|y\|_{X'}$, and this gives the isometric equality $X_c^* \simeq X'$. Thanks to this representation, X_c^* is called the space of *regular functionals* and is denoted by X_r^*. Note that $\operatorname{supp} X_a = \Omega$ implies $X_s^* = (X_a)^\perp$, so every $F_s \in X_s^*$ is identically zero on the subspace X_a. Thus in this case $X^* \simeq X' \oplus (X_a)^\perp$. For details and more information on Banach lattices we refer to [6, 26, 31, 35].

We say that a Banach lattice X is *strictly monotone* if for any $x, y \in X$, we have $\|x\| < \|y\|$ whenever $0 \le x \le y$ and $x \ne y$.

Given $x \in L^0(\mu)$, its *distribution function* is defined by $d_x(\lambda) = \mu(\{t \in \Omega : |x|(t) > \lambda\})$, $\lambda \ge 0$, and its *decreasing rearrangement* by $x^*(t) = \inf\{s > 0 : d_x(s) \le t\}$, $t > 0$. A Banach lattice X is called a *rearrangement invariant space* (in short *r.i. space*) if $\|x\| = \|y\|$ whenever $d_x = d_y$ and $x \in X$. It is well known that for any r.i. space X we have $L_1 \cap L_\infty \subset X \subset L_1 + L_\infty$. If μ is an atomless measure, let $\phi_X(t) = \|\chi_A\|_X$, where $\mu(A) = t$ and $0 \le t \le \mu(\Omega)$, be the *fundamental* function of a r.i. space X. The fundamental function of the Köthe dual X' is given by $\phi_{X'}(t) = t/\phi_X(t)$ for any $0 \le t \le \mu(\Omega)$. The function $\phi_X(t)/t$ is decreasing on $(0,\mu(\Omega))$. Let further $\phi_X(0+) = \lim_{t\to 0^+}\phi_X(t)$. For r.i. spaces we refer to [10, 27, 31].

Given two Banach lattices X, Y we will write $X = Y$ whenever the sets coincide and the norms are equivalent. Two expressions U, V are said to be *equivalent* if for some constants $a, b > 0$ we have $aU \le V \le bU$. In this case we write $U \approx V$. The symbol $X \simeq Y$ means that X and Y are isometrically isomorphic.

Let $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ be Banach lattices over the measure space $(\Omega, \mathcal{S}, \mu)$. Then $\overline{X} = (X_0, X_1)$ denotes a Banach couple in the sense of interpolation [10, 27]. The spaces $\Sigma(\overline{X}) = X_0 + X_1$ and $\Delta(\overline{X}) = X_0 \cap X_1$ are often equipped with the following norms:

$$\|x\|_\Sigma = \|x\|_{\Sigma(\overline{X})} = \inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\},$$
$$\|x\|_\Delta = \|x\|_{\Delta(\overline{X})} = \max\{\|x\|_0, \|x\|_1\},$$
$$|||x|||_\Sigma = |||x|||_{\Sigma(\overline{X})} = \inf\{\max\{\|x_0\|_0, \|x_1\|_1\} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\},$$
$$|||x|||_\Delta = |||x|||_{\Delta(\overline{X})} = \|x\|_0 + \|x\|_1.$$

It is well known that for any couple $\overline{X}$ of Banach lattices and the couple $\overline{X'} = (X_0', X_1')$ the following Köthe duality formulas hold with equality of norms:

$$(\Delta(\overline{X}), \|\cdot\|_\Delta)' = (\Sigma(\overline{X'}), \|\cdot\|_\Sigma), \qquad (\Sigma(\overline{X}), \|\cdot\|_\Sigma)' = (\Delta(\overline{X'}), \|\cdot\|_\Delta),$$
$$(\Delta(\overline{X}), |||\cdot|||_\Delta)' = (\Sigma(\overline{X'}), |||\cdot|||_\Sigma), \qquad (\Sigma(\overline{X}), |||\cdot|||_\Sigma)' = (\Delta(\overline{X'}), |||\cdot|||_\Delta).$$

If $X_0 = L_1$ and $X_1 = L_\infty$ are equipped with their usual norms ($\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively), then we denote briefly by $\Sigma = L_1 + L_\infty$ and $\Delta = L_1 \cap L_\infty$, and their appropriate norms by $\|\cdot\|_\Sigma, |||\cdot|||_\Sigma$ and $\|\cdot\|_\Delta, |||\cdot|||_\Delta$. It is clear that if $(\Omega, \mathcal{S}, \mu)$ is a finite measure space, then $\Delta = L_\infty$ and $\Sigma = L_1$ up to equivalent norms.

We start with a first important observation that the Daugavet property is inherited by the subspace of order continuous elements. This basic fact is an essential ingredient in the proofs of several further results.

Theorem 2.1. *Let X be a Banach lattice on $(\Omega, \mathcal{S}, \mu)$ with the Daugavet property. If X has the weak Fatou property and* $\operatorname{supp} X_a = \Omega$ *then the Daugavet property is inherited by X_a.*

Proof. Notice first that if E is a Banach function lattice on $(\Omega, \mathcal{S}, \mu)$ with $\operatorname{supp} X = \Omega$ then E is an order dense ideal in L^0, i.e. for every $0 \le x \in L^0$ there exists a non-negative sequence (x_n) in E such that $x_n \uparrow x$ a.e. [26, Lemma 1, p. 95]. Thus our hypothesis $\operatorname{supp} X_a = \Omega$ implies that X_a is an order dense ideal in X. Then using Lebesgue's Monotone Convergence Theorem it is easily seen that the following formula holds with equality of norms

$$(X_a)' = X'.$$

Let $T = F \otimes x_0$ be a rank-one operator on X_a, that is $Tx = F(x)x_0$ for $x \in X_a$. Since the Banach lattice X_a has an order continuous norm, $(X_a)^*$ is order isometrically isomorphic to $(X_a)'$. Thus by the above equality, there exists $y \in X'$ such that $\|F\|_{X^*} = \|y\|_{X'}$ and

$$F(u) = \int_\Omega uy\, d\mu, \quad u \in X_a.$$

We define a rank-one operator $S\colon X \to X$ by $S = \tilde{F} \otimes x_0$, where

$$\tilde{F}(x) = \int_\Omega xy\, d\mu, \quad x \in X.$$

Since $\tilde{F}|_{X_a} = F$ and $\|\tilde{F}\|_{X^*} = \|y\|_{X'} = \|F\|_{(X_a)^*}$, we have $\|S\|_{X\to X} = \|T\|_{X_a\to X_a}$.

We fix $x \in B_X$. By order density of X_a in X, it follows that there exists a sequence (u_n) in X_a such that $u_n \to x$ a.e. and $|u_n| \le |x|$ for all $n \in \mathbb{N}$. Since $|u_n y| \le |xy|$ for all $n \in \mathbb{N}$ and $xy \in L_1(\mu)$, Lebesgue's Dominated Convergence Theorem reveals that $\tilde{F}(u_n) \to \tilde{F}(x)$. Consequently,

$$|u_n + \tilde{F}(u_n)x_0| \longrightarrow |x + \tilde{F}(x)x_0| = |x + S(x)| \ \text{ a.e.}$$

Combining now the weak Fatou property of X with $\tilde{F}|_{X_a} = F$ and $u_n \in B_X$, we obtain

$$\begin{aligned}\|x + S(x)\|_X &\le \liminf_{n\to\infty} \|u_n + \tilde{F}(u_n)x_0\|_X \\ &\le \sup_{\|u\|_{X_a}\le 1} \|u + F(u)x_0\|_{X_a} = \|I + T\|_{X_a\to X_a}.\end{aligned}$$

Hence $\|I + S\|_{X\to X} \le \|I + T\|_{X_a\to X_a}$. Since $\|S\|_{X\to X} = \|T\|_{X_a\to X_a}$, the Daugavet property of X yields the required inequality

$$1 + \|T\|_{X_a\to X_a} \le \|I + T\|_{X_a\to X_a},$$

and this completes the proof. □

The next two results state conditions under which the Daugavet property of X implies that X or X' contains an (lattice) isomorphic or isometric copy of $L_1(0,1)$.

Corollary 2.2. *Let X be a Banach lattice on $(\Omega, \mathcal{S}, \mu)$ with the weak Fatou property and let* $\operatorname{supp} X_a = \Omega$. *Suppose X has the Daugavet property. Then we have*:

(*i*) *The Köthe dual space X' contains an isometric copy of $L_1(0,1)$.*

(*ii*) *If X' is strictly monotone, then it contains a lattice isometric copy of $L_1(0,1)$.*

(*iii*) *If X' is order continuous, then X' contains a lattice isomorphic copy of $L_1(0,1)$.*

Proof. (*i*) It was shown in [18] that a Banach space contains asymptotically isometric copy of ℓ_1 if and only if its dual space contains an isometric copy of $L_1(0,1)$. It is also known that if a Banach space has the Daugavet property, then it contains asymptotically isometric copy of ℓ_1 (see the proof of [21, Theorem 2.9]).

If X is a Banach lattice which satisfies our conditions, it follows by Theorem 2.1 that X_a has the Daugavet property. Since $(X_a)^*$ is isometrically isomorphic to X', the proof is complete by the above mentioned results.

(*ii*) Since by (*i*), X' contains an isometric copy of $L_1(0,1)$, it follows by [46] that X' contains also a lattice isometric copy of $L_1(0,1)$.

(*iii*) It was shown in [23, Theorem 3.1] that if a Banach lattice Z contains no isomorphic copy of c_0 and has a subspace isomorphic to $L_1(0,1)$, then Z also has a sublattice which is order isomorphic to $L_1(0,1)$. Since X' has the Fatou property, thus our hypothesis implies that X' does not contain a copy of c_0. Combining the above mentioned result we obtain similarly as in (*i*) the required assertion. □

Proposition 2.3. *Let X be a Banach lattice on a measure space $(\Omega, \mathcal{S}, \mu)$ with the Fatou property. If X has the Daugavet property, then we have*:

(*i*) *If* $\operatorname{supp}(X')_a = \Omega$, *then X contains an isometric copy of $L_1(0,1)$.*

(*ii*) *If* $\operatorname{supp} X_a = \operatorname{supp}(X')_a = \Omega$, *then both X and X' contain isometric copies of $L_1(0,1)$.*

Proof. (*i*) It is obvious that a Banach space Y has the Daugavet property once its dual Y^* has it. Our hypotheses imply

$$((X')_a)^* \simeq ((X')_a)' \simeq X'' \simeq X.$$

This yields that $(X')_a$ has the Daugavet property. Combining the above formulas with the proof of Corollary 2.2(*i*), we conclude the result.

(*ii*) The statement follows by (*i*) and Corollary 2.2. □

In the remaining part of this section we shall investigate the Daugavet property of r.i. spaces and its relationship to the behavior of the fundamental functions of these spaces. We start with a result which will be applied to Lorentz and Marcinkiewicz spaces in the last section (see Theorem 5.3).

Proposition 2.4. *Let X be a r.i. space with the Fatou property on an atomless measure space $(\Omega, \mathcal{S}, \mu)$ and let μ be separable in case of infinite measure. If $\phi_{X'}(0+) > 0$, $X_a \neq \{0\}$ and X' is strictly monotone or $L_1(0,1)$ is order isometrically embedded into X', then X does not possess the Daugavet property.*

Proof. Notice that the assumption $X_a \neq \{0\}$ for r.i. space X is equivalent to $\operatorname{supp} X_a = \Omega$. Without loss of generality we can assume that $\mu(\Omega) = 1$ or $\mu(\Omega) = \infty$. Assume also that X' is strictly monotone. If instead we assume that X' contains an order isometric copy of $L_1(0,1)$ then the proof goes in a similar mode.

Assume that X has the Daugavet property. Then by Theorem 2.1 the space X_a inherits this property. We also have that the dual space $(X_a)^* \simeq X'$, and then in view of the assumption that X' is strictly monotone and by Corollary 2.2(*ii*), X' contains an order isometric copy of $L_1(0,1)$.

If $\mu(\Omega) = 1$ then $X' = L_\infty$. Indeed, since the inclusion $L_\infty \hookrightarrow X'$ holds, we need to show that $X' \hookrightarrow L_\infty$. Assuming on the contrary that there exists $x \in X'$ which is not bounded, we can find a sequence $(A_n) \subset \Omega$ such that $\mu(A_n) > 0$, $|x(t)| \geq n$ for $t \in A_n$ and $n \in \mathbb{N}$. Hence

$$\|x\|_{X'} \geq \|x\chi_{A_n}\|_{X'} \geq n\phi_{X'}(0+) \to \infty,$$

a contradiction. Let $T\colon L_1(0,1) \to X' = L_\infty$ be an order isometry. Letting $x_k \in L_1(0,1)$, $\|x_k\|_1 = 1$, $k = 1,\dots,n$, and $|x_k| \wedge |x_j| = 0$ for $k \neq j$, we have that $\|\sum_{k=1}^n x_k\|_1 = \sum_{k=1}^n \|x_k\|_1 = n$. Hence

$$n = \Big\| \sum_{k=1}^n x_k \Big\|_1 = \Big\| T\Big(\sum_{k=1}^n x_k\Big)\Big\|_{X'} \approx \Big\| T\Big(\sum_{k=1}^n x_k\Big)\Big\|_\infty$$
$$= \max_{1\le k\le n} \|Tx_k\|_\infty \approx \max_{1\le k\le n} \|Tx_k\|_{X'} = \max_{1\le k\le n} \|x_k\|_1 = 1,$$

and we get a contradiction.

Assume now that $\mu(\Omega) = \infty$. Applying the same reasoning as above we get that $X'(A) = L_\infty(A)$ for any measurable set A with $\mu(A) < \infty$.

By the assumption that μ is separable and the Caratheodory Theorem (see, e.g. Corollary on p. 128 in [29]), $L_1(\mu)$ is isometrically isomorphic to $L_1(0,1)$ which in turn is order isometrically embedded into X'. Hence $L_1 = L_1(\mu)$ is order isometrically embedded into $X' = X'(\Omega)$. Now by Corollary 9 in [1] and its proof, we have that $L_1(A) \subset X'(A)$ as sets for every A with $\mu(A) < \infty$, which contradicts the fact that $X'(A) = L_\infty(A)$. □

Before the proof of the next result let us recall that a Banach space X is said to be a weakly compactly generated Banach space, or a WCG-space, if X contains a linearly dense weakly compact subset K, i.e., $X = \overline{\mathrm{span}}(K)$. Separable Banach spaces and reflexive Banach spaces are trivial examples of WCG-spaces. It is well known that in every Banach lattice with order continuous norm, order intervals are weakly compact [6, Theorem 12.9]. This implies that if X is a Banach lattice with a weak unit e, then $X = \overline{\mathrm{span}}[-e, e]$ (by [34, Theorems 40.2 and 40.3]) and so X is a WCG-space.

We will use the following result [17, Corollary 7, p. 83]: *If X is a Banach space whose dual X^* is a subspace of a weakly compactly generated Banach space Y, then X^* has the Radon-Nikodým property.*

Proposition 2.5. *Let X be a r.i. space with the Fatou property on an atomless measure space $(\Omega, \mathcal{S}, \mu)$. If X is order continuous and has the Daugavet property then $L_1 \hookrightarrow X$. In particular, if $\mu(\Omega) < \infty$ then $X = L_1$.*

Proof. Let X be order continuous and have the Daugavet property. We claim that $\phi_{X'}(0+) > 0$. Assuming on the contrary that $\phi_{X'}(0+) = 0$. Then $(X')_a \neq \{0\}$, and so $\mathrm{supp}(X')_a = \Omega$. Since

$$((X')_a)^* \simeq ((X')_a)' \simeq X'' \simeq X$$

and X is order continuous with a weak unit (by $\mathrm{supp}\, X = \Omega$), we conclude by the above mentioned result that X has the Radon-Nikodým property. However it contradicts the Daugavet property of X by [45], in view of the well known fact that the unit ball of a space with the Radon-Nikodým property has a strongly exposed point. This proves the claim. We can now proceed analogously to the proof of

Proposition 2.3 and show that $X' \hookrightarrow L_\infty$. As a consequence $L_1 \hookrightarrow X$ by the Köthe duality and $X = X''$ isometrically.

We conclude recalling that any r.i. space over a finite atomless measure space is embedded into L_1 [10, 27] and so $X \hookrightarrow L_1$. By the previous opposite embedding we get that $X = L_1$. □

Proposition 2.6. *Let X be a r.i. space with the Fatou property on a finite separable atomless measure space $(\Omega, \mathcal{S}, \mu)$. Assume that X has the Daugavet property. If X' is strictly monotone or X' is order continuous, then $X = L_\infty(\mu)$.*

Proof. Let X' be strictly monotone and let X have the Daugavet property. Suppose that $\phi_X(0+) = 0$. It follows that $\operatorname{supp} X_a = \Omega$, and thus by Corollary 2.2(*ii*), X' contains a lattice isometric copy of $L_1(0,1)$. Now by the assumption that μ is separable, $L_1(0,1)$ is isometric to $L_1(\mu)$.

Applying in turn [1], we get that $L_1(\mu) \subset X'$. However $X' \subset L_1(\mu)$, so $X' = L_1(\mu)$. Thus $X = L_\infty(\mu)$, which contradicts $\phi_X(0+) = 0$. Hence $\phi_X(0+) > 0$, and consequently in view of finiteness of μ, $X = L_\infty(\mu)$.

If X' is order continuous, then by Corollary 2.2(*iii*) and the similar arguments as above, we obtain that $X = L_\infty(\mu)$. □

The following geometrical characterization of the Daugavet property will be used later.

Lemma 2.7 ([21, Lemma 2.2]). *For a Banach space X, the following conditions are equivalent*:

(*i*) *X has the Daugavet property.*
(*ii*) *For every $x \in S_X$ and $y^* \in S_{X^*}$ and every $\varepsilon > 0$ there is $x^* \in S_{X^*}$ such that $x^*(x) > 1 - \varepsilon$ and $\|x^* + y^*\| > 2 - \varepsilon$.*
(*iii*) *For every $x \in S_X$ and $x^* \in S_{X^*}$ and every $\varepsilon > 0$ there is $y \in S_X$ such that $x^*(y) > 1 - \varepsilon$ and $\|x + y\| > 2 - \varepsilon$.*

In what follows we also need the following technical result.

Lemma 2.8. *Assume that $f, g \in L_1(\mu)$ and D is a measurable set such that for a.e. $t \in D$,*

$$0 \le f(t) \quad \text{and} \quad g(t) = -c,$$

where $c > 0$. Let $A = \{t \in D : f(t) \le c\}$ and $B = \{t \in D : f(t) > c\}$. Then

$$\|(f+g)\chi_D\|_1 \le \|f\chi_D\|_1 + \|g\chi_D\|_1 - 2c\mu(B).$$

Proof. In view of $A \cup B = D$ and $A \cap B = \emptyset$, we have that

$$\|f\chi_D\|_1 + \|g\chi_D\|_1 = \|f\chi_A\|_1 + \|f\chi_B\|_1 + c(\mu(A) + \mu(B)),$$

and

$$\begin{aligned}\|(f+g)\chi_D\|_1 &= \int_A |f - c|\, d\mu + \int_B |f - c|\, d\mu = \int_A (c - f)\, d\mu + \int_B (f - c)\, d\mu \\ &= c(\mu(A) - \mu(B)) - \|f\chi_A\|_1 + \|f\chi_B\|_1.\end{aligned}$$

Hence

$$\|f\chi_D\|_1 + \|g\chi_D\|_1 - \|(f+g)\chi_D\|_1 = 2\|f\chi_A\|_1 + 2c\mu(B) \geq 2c\mu(B),$$

and the proof is done. □

The next statement extends [5, Proposition 4.3, c] from L_∞ to a wider class of r.i. spaces.

Theorem 2.9. *Let X be a r.i. space on an atomless infinite measure space $(\Omega, \mathcal{S}, \mu)$. Assume that one of the following two conditions is satisfied:*

(i) The fundamental function ϕ_X of X satisfies that $\phi_X(T) > T$ for some $0 < T < 1$, $\phi_X(1) = 1$ and $\phi_X(t_0) < t_0$ for some real number $t_0 > 1$.

(ii) $X \subset L_\infty(\mu)$ and the fundamental function ϕ_X satisfies that $\phi_X(1) = 1$ and $\phi_X(t_0) < t_0$ for some $t_0 > 1$.

Then the space $(L_1(\mu) \cap X, \|\cdot\|_\Delta)$ does not have the Daugavet property.

Proof. Let $(E, \|\cdot\|_E) = (L_1(\mu) \cap X, \|\cdot\|_\Delta)$.

(i) By the assumptions we can choose a real number a such that $a > \max\{\frac{\phi_X(T)}{\phi_X(T)-T}, t_0\}$. Now we choose measurable subsets Ω_1 and Ω_a of Ω such that $\Omega_1 \subset \Omega_a$, $\mu(\Omega_1) = 1$ and $\mu(\Omega_a) = a$. Take $x_0 = \frac{1}{a}\chi_{\Omega_a}$ and the functional $F_0 \in E^*$ generated by $-\chi_{\Omega_1}$. Since the function $t \mapsto \frac{\phi_X(t)}{t}$ is non-increasing, by the choice of a we have

$$\|x_0\|_E = \max\{\|x_0\|_1, \|x_0\|_X\} = \max\left\{1, \frac{\phi_X(a)}{a}\right\} = 1.$$

Since $B_E \subset B_{L_1}$, $\|F_0\|_{E'} \leq \|\chi_{\Omega_1}\|_\infty = 1$ and so $F_0 \in S_{E'}$ by $\|\chi_{\Omega_1}\|_E = 1$ and $F_0(\chi_{\Omega_1}) = -1$.

We shall show that for every $0 < \varepsilon < \min\left\{1 - \frac{T}{\phi_X(T)} - \frac{1}{a}, \frac{2T}{a}, 1 - \frac{\phi_X(a)}{a}\right\}$, if $y \in S_E$ satisfies $F_0(y) > 1 - \varepsilon$, then $\|x_0 + y\|_E < 2 - \varepsilon$, which will prove that E lacks the Daugavet property by Lemma 2.7.

So assume that $y \in S_E$ and $F_0(y) > 1 - \varepsilon$. Let us write

$$C = \{t \in \Omega_1 : -y(t) \geq 0\},$$

and

$$A = \{t \in C : -y(t) \leq a^{-1}\}, \qquad B = \{t \in C : a^{-1} < -y(t)\}.$$

We clearly have

$$\begin{aligned}
1-\varepsilon < F_0(y) &= \int_{\Omega_1} (-y)\, d\mu \le \int_C (-y)\, d\mu \\
&= \int_A (-y)\, d\mu + \int_B (-y)\, d\mu \\
&\le \frac{1}{a}\mu(A) + \|\chi_B\|_{X'} \|y\chi_B\|_X \\
&\le \frac{1}{a}\mu(A) + \frac{\mu(B)}{\phi_X(\mu(B))}\|y\|_X \quad (\text{since } A \subset \Omega_1) \\
&\le \frac{1}{a} + \frac{\mu(B)}{\phi_X(\mu(B))}.
\end{aligned}$$

The function $t \mapsto \frac{t}{\phi_X(t)}$ is non-decreasing and by the choice of ε we know that $\frac{1}{a} + \frac{T}{\phi_X(T)} < 1-\varepsilon$, hence we deduce from the above inequality that $\mu(B) \ge T$.

On the other hand, from the definitions of the subsets C, A and B, by applying Lemma 2.8 we obtain

$$\begin{aligned}
\|(x_0+y)\chi_C\|_1 &\le \|(x_0+y)\chi_C\|_1 + \|(x_0+y)\chi_C\|_1 - 2\mu(B)/a \\
&\le \|(x_0+y)\chi_C\|_1 + \|(x_0+y)\chi_C\|_1 - 2T/a.
\end{aligned}$$

Combining the above inequality with $\|x_0\|_E = \|y\|_E = 1$ and $\varepsilon < 2T/a$ we obtain

$$\begin{aligned}
\|x_0+y\|_1 &= \|(x_0+y)\chi_C\|_1 + \|(x_0+y)\chi_{\Omega\setminus C}\|_1 \\
&\le \|x_0\chi_C\|_1 + \|y\chi_C\|_1 - 2T/a + \|x_0\chi_{\Omega\setminus C}\|_1 + \|y\chi_{\Omega\setminus C}\|_1 \\
&= \|x_0\|_1 + \|y\|_1 - 2T/a \\
&\le 2 - 2T/a < 2-\varepsilon.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
\|x_0+y\|_E = &\ \max\left\{\|x_0+y\|_1, \|x_0+y\|_X\right\} \le \max\{\|x_0+y\|_1, \|x_0\|_X + \|y\|_X\} \\
\le &\ \max\left\{\|x_0+y\|_1, \frac{\phi_X(a)}{a} + 1\right\} < 2-\varepsilon
\end{aligned}$$

and so E does not have the Daugavet property.

(ii) By the assumptions we can choose measurable subsets Ω_1 and Ω_0 of Ω such that $\Omega_1 \subset \Omega_0$, $\mu(\Omega_1) = 1$ and $\mu(\Omega_0) = t_0$. Take $x_0 = a\chi_{\Omega_0}$, where $a = 1/t_0$ and the regular functional $F_0 \in E^*$ generated by $-\chi_{\Omega_1}$. We have $\|x_0\|_E = \max\{\|x_0\|_1, \|x_0\|_X\} = \max\{1, \frac{\phi_X(t_0)}{t_0}\} = 1$. Since $B_E \subset B_{L_1}$ one has that $\|F_0\|_{E'} \le \|\chi_{\Omega_1}\|_\infty = 1$. Indeed $F_0 \in S_{E'}$ since $\|\chi_{\Omega_1}\|_E = 1$ and $F_0(\chi_{\Omega_1}) = -1$.

We shall show that for every $\varepsilon > 0$ small enough, if $y \in S_E$ satisfies $F_0(y) > 1-\varepsilon$, then $\|x_0+y\|_E < 2-\varepsilon$, which will prove that E does not have the Daugavet property by Lemma 2.7.

Our hypothesis $X \subset L_\infty$ implies by the closed graph theorem that $B_X \subset cB_{L_\infty}$ for some $c > 0$. Without loss of generality we can assume that $c > a = 1/t_0$.

We choose $0 < \varepsilon < \min\{1 - \frac{\phi_X(t_0)}{t_0}, \frac{2(a-a^2)}{c+a}\}$. Let us notice that ϕ_X is increasing and so $\varepsilon < 1 - \frac{1}{t_0} = 1 - a$. For every $y \in S_X$ we clearly have that

$$\|x_0 + y\|_X \le \|x_0\|_X + \|y\|_X \le \frac{\phi_X(t_0)}{t_0} + 1 < 2 - \varepsilon.$$

We fix $y \in S_E$ satisfying $F_0(y) > 1 - \varepsilon$. In order to prove the desired inequality we will use the measurable sets given by

$$A = \{t \in \Omega_1 : -y(t) \le a\}, \qquad B = \{t \in \Omega_1 : -y(t) > a\}.$$

We clearly have

$$\begin{aligned} 1 - \varepsilon < F_0(y) = \int_{\Omega_1} (-y)\, d\mu &= \int_A (-y)\, d\mu + \int_B (-y)\, d\mu \quad (\text{since } \|y\|_\infty \le c) \\ &\le a\mu(A) + c\mu(B) \quad (\text{since } A, B \subset \Omega_1, A \cap B = \emptyset) \\ &\le a(1 - \mu(B)) + c\mu(B) = a + (c-a)\mu(B), \end{aligned}$$

and hence

$$\mu(B) > \frac{1 - \varepsilon - a}{c - a}.$$

We write $D = \Omega \backslash \Omega_1$. Then we have

$$\int_D |x_0|\, d\mu = \int_{\Omega_0 \backslash \Omega_1} |x_0|\, d\mu = \frac{1}{t_0}\big(\mu(\Omega_0) - \mu(\Omega_1)\big) = 1 - a.$$

Since A, B and D are pairwise disjoint sets, we know that

$$\int_A y\, d\mu - \int_B y\, d\mu + \|y\chi_D\|_1 \le \int_A |y|\, d\mu + \int_B |y|\, d\mu + \|y\chi_D\|_1 \le \|y\|_1.$$

Combining this estimate with $\|x_0\|_E = \|y\|_E = 1$ and the choice of ε we get

$$\begin{aligned} \|x_0 + y\|_1 &= \|(x_0 + y)\chi_{\Omega_1}\|_1 + \|(x_0 + y)\chi_D\|_1 \\ &\le \int_A |x_0 + y|\, d\mu + \int_B |x_0 + y|\, d\mu + \int_D |x_0|\, d\mu + \|y\chi_D\|_1 \\ &\le \int_A (a + y)\, d\mu - \int_B (a + y)\, d\mu + \int_D |x_0|\, d\mu + \|y\chi_D\|_1 \\ &= a\big(\mu(A) - \mu(B)\big) + \int_A y\, d\mu - \int_B y\, d\mu + 1 - a + \|y\chi_D\|_1 \\ &\le a(1 - 2\mu(B)) + \|y\|_1 + 1 - a \\ &< a\Big(1 + \frac{2\varepsilon + 2a - 2}{c - a}\Big) + 2 - a \\ &= 2 + a\frac{2\varepsilon + 2a - 2}{c - a} < 2 - \varepsilon. \end{aligned}$$

As a consequence we obtain

$$\|x_0 + y\|_E = \max\left\{\|x+y\|_1, \|x+y\|_X\right\} < 2 - \varepsilon,$$

and so E does not have the Daugavet property. $\square$

Corollary 2.10. *Let X be a r.i. space on $(0,\infty)$. If the fundamental function ϕ_X of X is concave and non-constant on (T, t_0) and constant on $[t_0,\infty)$ for some $0 < T < t_0$ and $\phi_{X'}(t) = t/\phi_X(t)$ is concave and non-constant on (T, t_0), then the space $(L_1 \cap X, \|\cdot\|_\Delta)$ does not have the Daugavet property.*

Proof. To see this take $s > 0$ such that $\phi(s) = s/\phi_X(s) = 1$ and define a r.i. space $X_0 = (X, \|\cdot\|_0)$ equipped with the norm $\|x\|_0 = \frac{1}{\phi_X(s)}\|D_s x\|_X$ for $x \in L_1 \cap \Delta$. Then we have $\phi_{X_0}(t) = \phi_X(st)/\phi_X(s)$ for every $t > 0$ and so $\phi_{X_0}(1) = 1$. Since $T = \frac{1}{\phi_X(s)} D_s$ is an isometrical isomorphism between the spaces $L_1 \cap X$ and $L_1 \cap X_0$, Theorem 2.9 can be applied. $\square$

It was proved in [5] that $(L_\infty \cap L_1, \|\cdot\|_\Delta)$ does not have the Daugavet property. The following result is a far-reaching extension of the mentioned result to a large class of r.i. spaces $(L_\infty \cap X, \|\cdot\|_\Delta)$.

Theorem 2.11. *Let X be a r.i. space on an atomless infinite measure space $(\Omega,\mathcal{S},\mu)$. Assume that the fundamental function ϕ_X is concave, continuous at 1, satisfies $\phi_X(1) = 1$, and it is non-constant on $[1,\infty[$. Assume also that there are $0 < u_0 < 1$ and $\varepsilon_0 > 0$ such that the following condition is satisfied*

$$\big(G \in \mathcal{S},\, x \in B_X,\, \|x\|_\infty \le 1, \|x\chi_G\|_X > 1 - \varepsilon_0\big) \;\Rightarrow\; \|x\chi_{\Omega\setminus G}\|_X < u_0.$$

Then the space $(L_\infty \cap X, \|\cdot\|_\Delta)$ does not have the Daugavet property.

Proof. We will prove that $Y = (L_\infty \cap X, \|\cdot\|_\Delta)$ does not satisfy the third condition of Lemma 2.7.

Let us choose a measurable set $\Omega_1 \subset \Omega$ such that $\mu(\Omega_1) = 1$ and let F_0 be the element of Y' represented by χ_{Ω_1}. Since $B_Y \subset B_{L_\infty}$ it is clear that $\|F_0\|_{Y'} \le \|\chi_{\Omega_1}\|_1 = 1$. By assumption $\phi_X(1) = 1$, so the element $y_0 = \chi_{\Omega_1} \in S_Y$ and $F_0^*(y_0) = 1$. Hence $F_0 \in S_{Y^*}$. Since ϕ_X is concave and non-constant on $[1,+\infty)$, $\phi_X(t) > \phi_X(1) = 1$ for every $t > 1$. We choose now a measurable set $\Omega_2 \subset \Omega\backslash\Omega_1$ with $\mu(\Omega_2) = 1/2$.

Let $s = \frac{3}{2}$. Consider the element $x_0 = \frac{-1}{\phi_X(s)}\,(\chi_{\Omega_1} + \chi_{\Omega_2}) \in Y$. It is clear that

$$\|x_0\|_\infty = \frac{1}{\phi_X(s)} < 1, \qquad \|x_0\|_X = \frac{1}{\phi_X(s)}\phi_X(\mu(\Omega_1 \cup \Omega_2)) = 1,$$

and so $x_0 \in S_Y$.

We shall show that for every $\varepsilon > 0$ small enough, if $x \in S_Y$ satisfies $F_0(x) > 1 - \varepsilon$, then $\|x_0 + x\|_Y < 2 - \varepsilon$, which will prove that Y does not have the Daugavet property by Lemma 2.7.

Since ϕ_X is continuous at 1 and $\phi_X(1) = 1$, there is $0 < t_0 < 1$ such that $t_0\phi_X(t_0) > 1 - \varepsilon_0$. So we choose

$$0 < \varepsilon < \min\left\{\frac{1}{4}, 1 - u_0, \frac{(\phi_X(s) - 1)^2}{(\phi_X(s))^2}, (1 - t_0)^2\right\}.$$

Assume that $x \in S_Y$ satisfies that $F_0(x) > 1 - \varepsilon$.

In view of $\|x_0\|_\infty = \frac{1}{\phi_X(s)} < \|x\|_X = 1$ and the choice of ε we obtain

$$\|x_0 + x\|_\infty \leq \|x_0\|_\infty + \|x\|_\infty \leq \frac{1}{\phi_X(s)} + 1 < 2 - \varepsilon.$$

Let $0 < r < 1$ be such that $\varepsilon = (1 - r)^2$ and let

$$G = \{t \in \Omega_1 : x(t) > r\}\ .$$

Since $x \in S_Y \subset B_{L_\infty}$ we have

$$\begin{aligned} 1 - \varepsilon < F_0(x) &= \int_{\Omega_1} x\ d\mu = \int_G x\ d\mu + \int_{\Omega_1\backslash G} x\ d\mu \\ &\leq \mu(G) + r\mu(\Omega_1\backslash G) = \mu(G) + r(1 - \mu(G)) = \mu(G)(1 - r) + r. \end{aligned}$$

That is,

$$\mu(G) \geq 1 - \frac{\varepsilon}{1 - r} = 1 - (1 - r) = r.$$

Hence by the choice of ε (and r), since ϕ_X is non-decreasing we have

$$\|x\chi_G\|_X \geq \|r\chi_G\|_X = r\phi_X(\mu(G)) \geq r\phi_X(r) \geq t_0\phi_X(t_0) > 1 - \varepsilon_0,$$

and so, by the assumption on X, we obtain

$$\|x\chi_{\Omega\backslash G}\|_X \leq u_0.$$

Observe that for every $t \in G$ we have

$$0 < r - \frac{1}{\phi_X(s)} < x(t) + x_0(t) \leq 1 - \frac{1}{\phi_X(s)}$$

and hence

$$\begin{aligned} \|(x_0 + x)\chi_G\|_X &\leq \Big(1 - \frac{1}{\phi_X(s)}\Big)\phi_X(\mu(G)) \\ &\leq \Big(1 - \frac{1}{\phi_X(s)}\Big)\phi_X(1) \\ &= \Big(1 - \frac{1}{\phi_X(s)}\Big). \end{aligned}$$

Combining the shown above inequalities we obtain

$$\begin{aligned}
\|x_0 + x\|_X &\le \|(x_0 + x)\chi_G\|_X + \|(x_0 + x)\chi_{\Omega\setminus G}\|_X \\
&\le \left(1 - \frac{1}{\phi_X(s)}\right) + \|x_0\chi_{\Omega\setminus G}\|_X + \|x\chi_{\Omega\setminus G}\|_X \\
&\le \left(1 - \frac{1}{\phi_X(s)}\right) + \frac{1}{\phi_X(s)}\phi_X(\mu((\Omega_1 \cup \Omega_2)\setminus G)) + u_0 \\
&= \left(1 - \frac{1}{\phi_X(s)}\right) + \frac{1}{\phi_X(s)}\phi_X(s - \mu(G)) + u_0 \\
&\le \left(1 - \frac{1}{\phi_X(s)}\right) + \frac{1}{\phi_X(s)}\phi_X(s - r) + u_0 \quad \left(\text{since } r \ge \frac{1}{2}\right) \\
&\le \left(1 - \frac{1}{\phi_X(s)}\right) + \frac{1}{\phi_X(s)}\phi_X(1) + u_0 \\
&\le \left(1 - \frac{1}{\phi_X(s)}\right) + \frac{1}{\phi_X(s)} + u_0 \\
&= 1 + u_0 < 2 - \varepsilon.
\end{aligned}$$

We have already shown that $\|x_0 + x\|_\infty < 2 - \varepsilon$ and so the above estimate yields

$$\|x_0 + x\|_Y = \max\{\|x_0 + x\|_\infty, \|x_0 + x\|_X\} < 2 - \varepsilon.$$

This shows that Y does not satisfy the Daugavet property and the proof is complete. □

The next two theorems below provide formulas of the norms in the dual spaces to $(X + L_\infty, |\!|\!|\cdot|\!|\!|_\Sigma)$ and to $(X + L_\infty, \|\cdot\|_\Sigma)$.

Theorem 2.12. *Let X be a Banach lattice on $(\Omega, \mathcal{S}, \mu)$. If $X \subset (X + L_\infty)_a$, in particular if X is order continuous, then*

$$(X + L_\infty, \|\cdot\|_\Sigma)^* \simeq (X' \cap L_1) \oplus (X + L_\infty)_a^\perp,$$

and for every $F \in (X + L_\infty, \|\cdot\|_\Sigma)^*$ we have $F = F_r + F_s$, where $F_s \in (X + L_\infty)_a^\perp$, and for some $y \in X' \cap L_1$,

$$F_r(x) = \int_\Omega xy\, d\mu, \quad x \in X + L_\infty,$$

and $\|F_r\| = \|y\|_{X' \cap L_1} = \max\{\|y\|_{X'}, \|y\|_1\}$. Moreover,

$$\|F\| = \max\{\|y\|_{X'}, \|y\|_1 + \|F_s\|\}.$$

Proof. By the assumption $X \subset (X + L_\infty)_a$ we have that $\operatorname{supp}(X + L_\infty)_a = \Omega$. Moreover by the Köthe duality formula we have $(X + L_\infty)' = X' \cap L_1$ with equality of norms. Combining with the general representation of the dual space of a Banach lattice described in the introduction we obtain the first statement of the theorem. In order to finish we need to show that the formula for the norm of the functional holds true.

Fix $x = u + v \in X + L_\infty$ with $u \in X, v \in L_\infty$. Then by Hölder's inequality and the fact that $F_s(u) = 0$ since u is order continuous, we get

$$\begin{aligned}|F(u+v)| &\le |F_r(u)| + |F_r(v)| + |F_s(v)| \\ &\le \int_\Omega |uy|\, d\mu + \int_\Omega |vy|\, d\mu + \|F_s\|\|v\|_{X+L_\infty} \\ &\le \|u\|_X\|y\|_{X'} + \|y\|_1\|v\|_\infty + \|F_s\|\|v\|_\infty \\ &\le \max\{\|y\|_{X'}, \|y\|_1 + \|F_s\|\}(\|u\|_X + \|v\|_\infty).\end{aligned}$$

Taking the infimum over all decompositions of x, we obtain

$$\|F\| \le \max\{\|y\|_{X'}, \|y\|_1 + \|F_s\|\}.$$

Notice that for every $0 \le x = u + v \in X + L_\infty$ with $0 \le u \in X$, $0 \le v \in L_\infty$ we have $|F_s|(x) = |F_s|(u) + |F_s|(v) = |F_s|(v)$, since $|F_s| \in X_s^*$ and so $|F_s|(u) = 0$. This implies that given $\varepsilon > 0$ there exist $0 \le u_0 \in X, 0 \le v_0 \in L_\infty$ such that

$$\|u_0\|_X + \|v_0\|_\infty < 1 + \varepsilon \quad \text{and} \quad \|F_s\| < |F_s|(v_0) + \varepsilon/2.$$

Since $y \in L_1$, there exists $0 \le v_1 \in L_\infty$ with $\|v_1\|_\infty \le 1$ such that

$$|F_r|(v) = \int_\Omega v_1|y|\, d\mu > \|y\|_1 - \varepsilon/2.$$

For $v = v_0 \vee v_1$ we have $0 \le v \in L_\infty \subset X + L_\infty$ and

$$\begin{aligned}|F|(v) = |F_r|(v) + |F_s|(v) &= \int_\Omega v|y|\, d\mu + |F_s|(v) \\ &\ge \int_\Omega v_1|y|\, d\mu + |F_s|(v_0) \ge (\|y\|_1 - \varepsilon/2) + (\|F_s\| - \varepsilon/2) \\ &= \|y\|_1 + \|F_s\| - \varepsilon.\end{aligned}$$

Since $\|v\|_\infty < 1 + \varepsilon$, we have $\|v\|_{X+L_\infty} < 1 + \varepsilon$ and hence $\|F\| \ge (\|y\|_1 + \|F_s\| - \varepsilon)/(1 + \varepsilon)$. Consequently,

$$\|F\| \ge \|y\|_1 + \|F_s\|.$$

To conclude the proof, it is enough to observe that

$$\begin{aligned}\|F\| &= \sup_{\|x\|_{X+L_\infty}\le 1} |F(x)| \ge \sup_{\|x\|_X \le 1} |F(x)| \\ &= \sup_{\|x\|_X\le 1} |F_r(x)| = \sup_{\|x\|_X \le 1} \left| \int_\Omega xy\, d\mu \right| = \|y\|_{X'}.\end{aligned}$$

□

Remark 2.13. The assumption $X \subset (X + L_\infty)_a$ is weaker than the order continuity of X. In fact there exist Banach function lattices X such that $X \neq X_a$ but $X \subset (X + L_\infty)_a$. Indeed let X be a r.i. space over $\mathbb{R}_+$, not order continuous, and such that $\phi_X(1) = 1$ and $X(0,1) = (X(0,1))_a$. Then we have $X \subset (X + L_\infty)_a$. It is standard to show that $\|x\|_{X+L_\infty} \approx \|x^*\chi_{(0,1)}\|_X$. Hence given $0 \le x \in X$, and $0 \le x_n \le x$, $x_n \downarrow 0$, we have $\|x_n\|_{X+L_\infty} = \|x_n^*\chi_{(0,1)}\|_X \to 0$ since $x_n^*\chi_{(0,1)} \in (X(0,1))_a$. In particular take $X = \Lambda_\psi$ with $\psi(0+) = 0$, $\psi(1) = 1$ and $\psi(\infty) < \infty$, which is defined in Section 5.

Theorem 2.14. *Let X be a Banach lattice on $(\Omega, \mathcal{S}, \mu)$. If $X \subset (X + L_\infty)_a$, in particular if X is order continuous, then*

$$(X + L_\infty, \|\!|\cdot|\!\|_\Sigma)^* \simeq (X' \cap L_1) \oplus (X + L_\infty)_a^\perp,$$

and for every $F \in (X + L_\infty, \|\!|\cdot|\!\|_\Sigma)^*$ we have $F = F_r + F_s$, where $F_s \in (X + L_\infty)_a^\perp$, and for some $y \in X' \cap L_1$,

$$F_r(x) = \int_\Omega xy\, d\mu, \quad x \in X + L_\infty,$$

and $\|F_r\| = \|\!|y|\!\|_{X' \cap L_1} = \|y\|_{X'} + \|y\|_1$. Moreover,

$$\|F\| = \|y\|_{X'} + \|y\|_1 + \|F_s\|.$$

Proof. The proof of the first part goes similarly as the corresponding part of the proof of Theorem 2.12. Let $x = u + v \in X + L_\infty$ with $u \in X$, $v \in L_\infty$. Then by $F_s(u) = 0$ and Hölder's inequality,

$$\begin{aligned} |F(u + v)| &\le \|u\|_X \|y\|_{X'} + \|y\|_1 \|v\|_\infty + \|F_s\| \|v\|_\infty \\ &\le \|u\|_X \|y\|_{X'} + \|v\|_\infty (\|y\|_1 + \|F_s\|) \\ &\le \max\{\|u\|_X, \|v\|_\infty\} (\|y\|_{X'} + \|y\|_1 + \|F_s\|). \end{aligned}$$

By taking the infimum over all decompositions of x we get

$$\|F\| \le \|y\|_{X'} + \|y\|_1 + \|F_s\|.$$

As before, given $\varepsilon > 0$ there exist $0 \le u_0 \in X$, $v_0 \in L_\infty$ such that

$$\max\{\|u_0\|_X, \|v_0\|_\infty\} < 1 + \varepsilon \quad \text{and} \quad \|F_s\| < |F_s|(v_0) + \varepsilon/3.$$

Since $y \in L_1 \cap X'$, there exists $0 \le v_1 \in L_\infty$ with $\|v_1\|_\infty \le 1$ such that

$$|F_r|(v_1) = \int_\Omega v_1 |y|\, d\mu > \|y\|_1 - \varepsilon/3,$$

and there exists $0 \le u \in X$ with $\|u\|_X \le 1$ such that

$$|F_r|(u) = \int_\Omega u|y|\, d\mu > \|y\|_{X'} - \varepsilon/3.$$

Letting $v = v_0 \vee v_1$ we have $v \in L_\infty$, $0 \le v \in X + L_\infty$ and so $x = u + v \in X + L_\infty$ with $0 \le u \in X$. Hence

$$\begin{aligned} |F|(x) &= |F_r|(x) + |F_s|(x) = |F_r|(u) + |F_r|(v) + |F_s|(v) \\ &= \int_\Omega u|y|\, d\mu + \int_\Omega v|y|\, d\mu + |F_s|(v) \\ &\ge (\|y\|_{X'} - \varepsilon/3) + \int_\Omega v_1 |y|\, d\mu + |F_s|(v_0) \\ &\ge (\|y\|_{X'} - \varepsilon/3) + (\|y\|_1 - \varepsilon/3) + (\|F_s\| - \varepsilon/3) \\ &= \|y\|_{X'} + \|y\|_1 + \|F_s\| - \varepsilon. \end{aligned}$$

Since $\|v\|_\infty \le \max\{\|v_0\|_\infty, \|v_1\|_\infty\} < 1+\varepsilon$, and $|\!|\!|x|\!|\!|_{X+L_\infty} \le \max\{\|u\|_X, \|v\|_\infty\} \le 1+\varepsilon$, we have $\|F\| \ge (\|y\|_{X'} + \|y\|_1 + \|F_s\| - \varepsilon)/(1+\varepsilon)$. Thus

$$\|F\| \ge \|y\|_{X'} + \|y\|_1 + \|F_s\|,$$

which completes the proof. □

Remark 2.15. In the case of $X = L_1$, Theorems 2.12 and 2.14 were proved in [25].

The concluding two theorems in this section claim that the large class of the spaces $(X + L_\infty, \|\cdot\|_\Sigma)$ and $(X + L_\infty, |\!|\!|\cdot|\!|\!|_\Sigma)$ do not have the Daugavet property.

Theorem 2.16. *Let X be a r.i. space over an atomless measure space $(\Omega, \mathcal{S}, \mu)$ satisfying $\mu(\Omega) \ge 2$. If $X \subset (X + L_\infty)_a$, in particular if X is order continuous, then the space $(X + L_\infty, \|\cdot\|_\Sigma)$ does not have the Daugavet property whenever ϕ_X is strictly increasing and $\phi_X(t) = t$ on $[0,1]$.*

Proof. Let $A \in \mathcal{S}$ be such that $\mu(A) = 1$. Choose $B \in \mathcal{S}$ such that $\mu(B) = 1$ and $A \cap B = \emptyset$. Let $x = \chi_A$. Then

$$\|x\|_\Sigma = \min\{\|\chi_A\|_X, 1\} = \min\{\phi_X(1), 1\}.$$

Define a regular functional $G \in (X + L_\infty)^*$ by

$$G(u) = \int_\Omega ug\, d\mu, \quad u \in X + L_\infty,$$

where

$$g = -\frac{1}{2}\chi_A + \frac{1}{2}\chi_B.$$

Then we have $\|g\|_1 = 1$, $\|g\|_{X'} = \frac{1}{2}\|\chi_{A\cup B}\|_{X'} = \frac{1}{2}\phi_{X'}(2) = \frac{1}{\phi_X(2)} < 1$, since $\phi_X(2) > \phi_X(1) = 1$. Thus, by Theorem 2.12,

$$\|G\| = \|g\|_{X' \cap L_1} = 1.$$

Let $0 < \varepsilon < \min\left\{\frac{1}{2\phi_X(2)}, 1 - \frac{1}{\phi_X(2)}\right\}$ and $F \in (X + L_\infty)^*$ with $\|F\| = 1$, $F(x) > 1 - \varepsilon$. Then, by Theorem 2.12, F can be represented as $F = F_r + F_s$, where F_r is a regular functional induced by $h \in X' \cap L_1$, and F_s is a singular functional. Set $C = \{t \in \Omega : h(t) \ge 0\}$. Since $x = \chi_A \in X$ it is order continuous and so $F_s(x) = 0$. Hence

$$1 - \varepsilon < F(x) = \int_A h\, d\mu \le \|h\chi_{A\cap C}\|_1 \le \|h\|_1.$$

By Theorem 2.12, $\|h\|_1 \le \max\{\|h\|_{X'}, \|h\|_1 + \|F_s\|\} = \|F\| = 1$, which yields $\|h\chi_{(A\cap C)'}\|_1 < \varepsilon$. Therefore

$$1 = \|F\| \ge \|h\|_1 + \|F_s\| > 1 - \varepsilon + \|F_s\|,$$

and so $\|F_s\| < \varepsilon$. Let

$$D = \{t \in C \cap A : h(t) \le 1/2\} \quad \text{and} \quad E = \{t \in C \cap A : h(t) > 1/2\}.$$

We will show that $\|F+G\|$ is far away from 2. To see this observe that $F+G = F_r + F_s + G$ and

$$\|g+h\|_{X'} \le \|g\|_{X'} + \|h\|_{X'} \le \|g\|_{X'} + \|F\| = 1/\phi_X(2) + 1.$$

On the other hand, if $(C\cap A)' = \Omega\backslash(C\cap A)$, then we have

$$\begin{aligned}\|g+h\|_1 &= \|(g+h)\chi_{C\cap A}\|_1 + \|(g+h)\chi_{(C\cap A)'}\|_1 \\ &= \int_D \Big(\frac{1}{2} - h\Big)\,d\mu + \int_E \Big(h - \frac{1}{2}\Big)\,d\mu + \|(g+h)\chi_{(C\cap A)'}\|_1 \\ &\le \frac{1}{2}\Big(\mu(D) - \mu(E)\Big) + \|h\chi_E\|_1 + \|g\chi_{(C\cap A)'}\|_1 + \|h\chi_{(C\cap A)'}\|_1.\end{aligned}$$

Since $\|h\|_{X'} \le \|F\| = 1$ and $\|h\chi_E\|_1 \le \|h\|_{X'}\|\chi_E\|_X \le \phi_X(\mu(E)) = \mu(E)$ by assumption and $\mu(E)\le 1$, we obtain

$$\begin{aligned}\|g+h\|_1 &\le \frac{1}{2}(\mu(D)+\mu(E)) + \|g\chi_{(C\cap A)'}\|_1 + \varepsilon \\ &\le \|g\chi_D\|_1 + \|g\chi_E\|_1 + \|g\chi_{(C\cap A)'}\|_1 + \varepsilon = \|g\|_1 + \varepsilon \le 1+\varepsilon.\end{aligned}$$

Combining the above estimates, by Theorem 2.12,

$$\begin{aligned}\|F+G\| = &\max\{\|g+h\|_{X'}, \|g+h\|_1 + \|F_s\|\} \le \max\{1+1/\phi_X(2), 1+2\varepsilon\} \\ &= 1 + 1/\phi_X(2) < 2-\varepsilon.\end{aligned}$$

Hence condition *(ii)* in Lemma 2.7 is not satisfied, and so the space under consideration does not possess the Daugavet property. □

Theorem 2.17. *Let X be a r.i. space on an infinite atomless measure space $(\Omega, \mathcal{S}, \mu)$. If the fundamental function ϕ_X is continuous on $[1,\infty)$, $\lim_{t\to 0+}\frac{\phi_X(t)}{t} < \infty$ and $X \subset (X+L_\infty)_a$, then the space $(X+L_\infty, \|\!|\cdot|\!\|_\Sigma)$ does not have the Daugavet property.*

Proof. We will denote by $Y = (X+L_\infty, \|\!|\cdot|\!\|_\Sigma)$. In this case we shall also show that the second condition of Lemma 2.7 is not satisfied. Put $c = \frac{1+\phi_X(1)}{\phi_X(1)}$ and choose a measurable subset $\Omega_1 \subset \Omega$ satisfying $\mu(\Omega_1) = 1$. Let $x_0 = c\chi_{\Omega_1}$, we have that $\|\!| x_0 |\!\|_\Sigma = 1$. Indeed, $x_0 = \chi_{\Omega_1} + \frac{1}{\phi_X(1)}\chi_{\Omega_1} \in B_{L^\infty} + B_X \subset B_Y$. Moreover, if F_0 is the functional induced by $f_0 = \frac{1}{c}\chi_{\Omega_1}$, then in view of Theorem 2.14, we have that

$$\|F_0\| = \|f_0\|_{X'} + \|f_0\|_1 = \frac{1}{c}\Big(\frac{1+\phi_X(1)}{\phi_X(1)}\Big) = 1$$

and $F_0(x_0) = 1$. Hence $\|F_0\| = 1$ and thus $x_0 \in S_Y$.

We choose a real number $K > c$. Since ϕ_X is continuous on $[1,\infty)$ and $c < K$, there is a real number $m_0 > 0$ such that $K = (1+m_0)\frac{1+\phi_X(1+m_0)}{\phi_X(1+m_0)}$. We choose a measurable subset $\Omega_2 \subset \Omega$ such that $\mu(\Omega_2) = m_0$ and $\Omega_2 \cap \Omega_1 = \varnothing$.

We will use the element $G_0 \in Y'$ associated to the function g_0 given by

$$g_0 = \frac{1}{K}\Big(-\chi_{\Omega_1} - \chi_{\Omega_2}\Big).$$

It satisfies

$$\|G_0\| = \|g_0\|_{X'} + \|g_0\|_1 = \frac{1}{K}\Big(\frac{1+m_0}{\phi_X(1+m_0)} + 1 + m_0\Big) = 1,$$

so $G_0 \in S_{Y^*}$.

By assumption there is $M \geq \max\Big\{1, \sup\Big\{\frac{\phi_X(t)}{t} : t \in (0,1)\Big\}\Big\}$ and choose $0 < \varepsilon < \frac{2(K-c)}{2K+(MK-1)Kc}$. Take now any element $F \in S_{Y^*}$ such that

$$1 - \varepsilon < F(x_0).$$

and by Theorem 2.14, decompose it as $F = F_r + F_s$, where F_r is induced by $h \in X' \cap L_1$, and F_s is a singular functional. Further Theorem 2.14 yields

$$\|F\| = \|h\|_{X'} + \|h\|_1 + \|F_s\| = 1$$

and so $\|h\|_{X'} \leq 1$. We will check that $\|h\|_\infty \leq M$. Observe that if $s > 0$ and $C \subset \Omega$ is a measurable set with $0 < \mu(C) < 1$ such that $h(t) \geq s$, then we have

$$\phi(\mu(C)) = \|\chi_C\|_X \geq \|h\|_{X'}\|\chi_C\|_X \geq \int_\Omega h\chi_C \, d\mu \geq s\mu(C).$$

Hence

$$s \leq \frac{\phi(\mu(C))}{\mu(C)} \leq \sup\Big\{\frac{\phi_X(t)}{t} : t \in (0,1)\Big\} \leq M$$

and so h is essentially bounded and $\|h\|_\infty \leq M$.

Now we apply Lemma 2.8 for the following sets

$$D = \Big\{t \in \Omega_1 : h(t) \geq 0\Big\}, \qquad A = \Big\{t \in D : h(t) \leq \frac{1}{K}\Big\},$$
$$B = \Big\{t \in D : h(t) > \frac{1}{K}\Big\},$$

and the elements g_0 and h that satisfy the required conditions, and so we obtain that

$$\|(h + g_0)\chi_D\|_1 \leq \|h\chi_D\|_1 + \|g_0\chi_D\|_1 - 2\mu(B)/K.$$

By the assumption $X \subset (X + L_\infty)_a$ and so $F_s(x_0) = 0$. Since $\|h\|_\infty \leq M$ we get a lower estimate of $\mu(B)$ as follows

$$\begin{aligned}
1 - \varepsilon < F(x_0) = F_r(x_0) &= c\int_{\Omega_1} h \, d\mu \leq c\int_D h \, d\mu \\
&= c\Big(\int_A h \, d\mu + \int_B h \, d\mu\Big) \leq \frac{c}{K}\mu(A) + cM\mu(B) \\
&\leq c\Big(\frac{1}{K}(1 - \mu(B)) + M\mu(B)\Big) = \mu(B)\Big(cM - \frac{c}{K}\Big) + \frac{c}{K}.
\end{aligned}$$

Hence, $\mu(B) > \frac{K-c-\varepsilon K}{cKM-c}$. Linking now this estimate of $\mu(B)$ to the previous one shown above we conclude that

$$\|(h+g_0)\chi_D\|_1 \leq \|h\chi_D\|_1 + \|g_0\chi_D\|_1 - \frac{2(K-c-\varepsilon K)}{K(cKM-c)}. \tag{1}$$

Combining this inequality with Theorem 2.14 we deduce that

$$\begin{aligned}
\|G_0+F\| &= \|g_0+h\|_{X'} + \|g_0+h\|_1 + \|F_s\| \\
&\leq \|g_0\|_{X'} + \|h\|_{X'} + \|(g_0+h)\chi_D\|_1 + \|(g_0+h)\chi_{\Omega\setminus D}\|_1 + \|F_s\| \\
&\leq \|g_0\|_{X'} + \|h\|_{X'} + \|g_0\chi_D\|_1 + \|h\chi_D\|_1 - \frac{2(K-c-\varepsilon K)}{K(cKM-c)} \\
&\qquad + \|g_0\chi_{\Omega\setminus D}\|_1 + \|h\chi_{\Omega\setminus D}\|_1 + \|F_s\| \\
&= \|g_0\|_{X'} + \|g_0\|_1 + \|h\|_{X'} + \|h\|_1 + \|F_s\| - \frac{2(K-c-\varepsilon K)}{K(cKM-c)} \\
&\leq 2 - \frac{2(K-c-\varepsilon K)}{K(cKM-c)}.
\end{aligned}$$

As a consequence we conclude by the choice of ε that $\|G_0+F\| < 2-\varepsilon$ and thus Lemma 2.7(ii) implies that Y does not have the Daugavet property. □

Remark 2.18. Theorems 2.16 and 2.17 recover the results from [5] proved for $X = L_1$.

3. Orlicz spaces

In this section we investigate the Daugavet property and the weak neighborhoods in Orlicz spaces. Let $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ be an Orlicz function, that is φ is even and convex, $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$. By L_φ denote the Orlicz space over $(\Omega, \mathcal{S}, \mu)$, that is, the set of all $x \in L^0$ such that $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$, where

$$I_\varphi(x) = \int_\Omega \varphi(x)\, d\mu.$$

The space L_φ is a Banach space when equipped with either the Luxemburg norm $\|\cdot\|_\varphi$ or the Orlicz norm $\|\cdot\|_\varphi^0$ defined as

$$\|x\|_\varphi = \inf\{\epsilon > 0 : I_\varphi(x/\epsilon) \leq 1\},$$

or

$$\|x\|_\varphi^0 = \inf_{k>0} \frac{1}{k}(1 + I_\varphi(kx)),$$

respectively. These norms are equivalent, and in fact $\|x\|_\varphi \leq \|x\|_\varphi^0 \leq 2\|x\|_\varphi$. The Orlicz space equipped with either norm is a r.i. space satisfying the Fatou property. By E_φ denote the space $(L_\varphi)_a$, which is the closure in L_φ of bounded functions supported on finite measure sets. In the sequence case, the Orlicz space is denoted by ℓ_φ, and $(\ell_\varphi)_a$ by h_φ. The sequence of unit vectors (e_n) is an unconditional basis in h_φ. Recall that φ satisfies condition Δ_2 (resp., Δ_2^∞, Δ_2^0) if

$$\limsup_{t\to\infty;\ t\to 0} \{\varphi(2t)/\varphi(t)\} < \infty,$$

$$(\text{resp. } \limsup_{t\to\infty}\{\varphi(2t)/\varphi(t)\} < \infty,\ \limsup_{t\to 0}\{\varphi(2t)/\varphi(t)\} < \infty).$$

The conditions Δ_2, Δ_2^∞, Δ_2^0 always correspond to the measure spaces $(\Omega, \mathcal{S}, \mu)$ that are atomless and infinite, atomless and finite, and purely atomic with $\Omega = \mathbb{N}$ and μ the counting measure, respectively. The Orlicz space L_φ is order continuous if and only if φ satisfies the corresponding condition Δ_2. It is also equivalent to $L_\varphi = (L_\varphi)_a$. Let

$$\varphi_*(t) = \sup_{u>0}\{ut - \varphi(u)\}$$

be a conjugate function to φ. By Köthe duality we have

$$(L_\varphi, \|\cdot\|_\varphi)' = (L_{\varphi_*}, \|\cdot\|_{\varphi_*}^0) \quad \text{and} \quad (L_\varphi, \|\cdot\|_\varphi^0)' = (L_{\varphi_*}, \|\cdot\|_{\varphi_*}).$$

Consequently, by general form of dual spaces of Banach lattices and by $\operatorname{supp} E_\varphi = \Omega$, it holds

$$L_\varphi^* \simeq L_{\varphi_*} \oplus E_\varphi^\perp.$$

It is well known that an Orlicz space L_φ is reflexive if and only if both φ and φ_* satisfy the appropriate condition Δ_2. For the theory of Orlicz spaces, we refer the reader to the monographs [13, 28, 39, 36] devoted entirely to Orlicz spaces and to [10, 30, 31] containing considerable parts on those spaces.

The next result provides a class of Orlicz spaces defined on an atomless measure space $(\Omega, \mathcal{S}, \mu)$ failing the Daugavet property. Since $L_1(\mu)$ has the Daugavet property for every atomless measure μ, some assumption on the function φ is needed in order to obtain such a result.

Theorem 3.1. *Assume that the Orlicz function φ satisfies N-condition at infinity, that is $\lim_{t\to\infty}\varphi(t)/t = \infty$. Then the Orlicz space L_φ on an atomless measure space $(\Omega, \mathcal{S}, \mu)$ and equipped with either Luxemburg or Orlicz norm has no Daugavet property.*

Proof. We notice first that $(L_\varphi, \|\cdot\|_\varphi^0)$ does not contain an isometric copy of ℓ_1. From Theorem 4 in [14] it follows that $(L_\varphi, \|\cdot\|_\varphi^0)$ contains an isometric copy of ℓ_1 if and only if $A := \sup_{u>0}\{u(\lim_{v\to\infty}\varphi(v)/v) - \varphi(u)\} < \infty$. Obviously, if an Orlicz function φ satisfies N-condition at infinity, then $A = \infty$ and $(L_\varphi, \|\cdot\|_\varphi^0)$ does not contain an isometric copy of ℓ_1.

It is well known that if an Orlicz function φ satisfies N-condition at infinity, then the Young function φ_* is an Orlicz function also satisfying N-condition at infinity [13]. It follows that φ_* assumes finite values, and so $\operatorname{supp} E_{\varphi_*} = \Omega$. Further, by the Köthe duality formula we have

$$\operatorname{supp}(L_\varphi')_a = \operatorname{supp}(L_{\varphi_*})_a = \operatorname{supp} E_{\varphi_*} = \Omega.$$

Since $(L_\varphi, \|\cdot\|_\varphi^0)$ does not contain an isometric copy of ℓ_1, it fails the Daugavet property in view of Proposition 2.3(i).

If L_φ is equipped with the Luxemburg norm and would have the Daugavet property, then we would conclude by the Köthe duality and Corollary 2.2(i) that the Orlicz space L_{φ_*} equipped with an Orlicz norm would contain an isometric copy of ℓ_1, which is impossible since φ_* is N-function at infinity. This completes the proof. $\square$

Before presenting the next result, let us recall some further concepts from the theory of Orlicz spaces. Let φ be an Orlicz function. An interval $[a,b] \subset \mathbb{R}$ is called affine interval of φ provided that φ is affine on $[a,b]$ and it is not affine on either $[a-\varepsilon, b]$ or $[a, b+\varepsilon]$ for any $\varepsilon > 0$. By $\mathrm{Ext}(\varphi)$, we define the set of all strictly convex points of φ, i.e. such $u \in \mathbb{R}$ that there are no $s,t \in \mathbb{R}$, $s \neq t$, satisfying

$$u = \frac{s+t}{2} \quad \text{and} \quad \varphi\Big(\frac{s+t}{2}\Big) = \frac{\varphi(s)+\varphi(t)}{2}.$$

Let X be a Banach space. A point $x \in B_X$ is said to be a *locally uniformly rotund point* (in short LUR point) if for any sequence $(x_n) \subset B_X$, the condition $\|x_n + x\| \to 2$ implies $x_n \to x$ in X. Recall also that a subset S of X is a *slice of* B_X if for some $F \in S_{X^*}$ and $0 < \alpha < 1$, S coincides with

$$S(F,\alpha) = \{x \in B_X : F(x) > 1 - \alpha\}.$$

We say that $x \in B_X$ is *strongly exposed* if there exists a functional $F \in S_{X^*}$ such that $F(x) = 1$ and for any sequence $(x_n) \subset B_X$, if $F(x_n) \to 1$ then $\|x - x_n\| \to 0$. It is well-known that if $x \in S_X$ is a strongly exposed point then $\inf_{\alpha>0} \operatorname{diam} S(F,\alpha) = 0$ for the functional F exposing x. So any space with strongly exposed points has slices of arbitrarily small diameters. It is easy to show that every LUR point is strongly exposed.

The criteria for LUR points in Orlicz spaces equipped with the Luxemburg norm and generated by N-functions are given in [13, Theorem 2.22]. Following its proof we have the following lemma.

Lemma 3.2. *Let φ satisfy Δ_2 (resp. Δ_2^∞) condition. Let L_φ be the Orlicz space on an atomless measure space $(\Omega, \mathcal{S}, \mu)$ with $\mu(\Omega) = \infty$ (resp. $\mu(\Omega) < \infty$), equipped with the Luxemburg norm. Assume that $x \in S_{L_\varphi}$ satisfies the following conditions.*

(i) $x(t) \in \mathrm{Ext}(\varphi)$ *for μ-a.e. $t \in \Omega$,*

(ii) $\mu(\{t \in \Omega : |x(t)| = a_i\}) = 0$, $i = 1,2$, *for all affine intervals $[a_1, a_2]$ of φ.*

Then x is a LUR point of B_{L_φ}.

Proof. Since L_φ is order continuous, it follows from [19, Corollary 1] that L_φ has the Kadec-Klee property with respect to measure, i.e., if for any $x \in B_X$ and any sequence $(x_n) \subset B_X$, the conditions $\|x_n\| \to \|x\|$ and $x_n \to x$ in $L^0(\mu)$ imply $x_n \to x$ in X.

Combining the above remark with a careful analysis of the proof of the implication (iii) $\Rightarrow$ (i) in Theorem 2.22 in [13] shows that the proof works also under the assumption that φ is not N-function, and yields the desired result. $\square$

Since every LUR point is strongly exposed, the next result is an immediate consequence of the above lemma.

Corollary 3.3. *If φ satisfies condition Δ_2 (resp. Δ_2^∞) when $\mu(\Omega) = \infty$ (resp. $\mu(\Omega) < \infty$) and φ is strictly convex on some interval, then L_φ has slices of arbitrarily small diameters.*

Theorem 3.1 and the following corollary show that the Daugavet property in the class of Orlicz spaces is rare.

Theorem 3.4. *Let L_φ be an Orlicz space on an atomless measure space equipped with the Luxemburg norm. Assume that L_φ satisfies the Daugavet property.*

(*i*) *If $\mu(\Omega) < \infty$, then $L_\varphi = L_1$ up to equivalence of norms and φ is not strictly convex on any open interval in $\mathbb{R}$ (the graph of φ is composed of straight segments).*

(*ii*) *If $\mu(\Omega) = \infty$ and φ satisfies condition Δ_2 then φ is not strictly convex on any open interval.*

Proof. It is obvious that $L_\varphi \hookrightarrow L_1$ for any Orlicz space on a finite measure space. Assume that L_φ has the Daugavet property. From Theorem 3.1, it follows that φ is not N-function at infinity. This implies that φ is equivalent to the identity function $s(t) = t$ for large enough arguments, and whence $L_1 \hookrightarrow L_\varphi$. Combining both continuous inclusions, we conclude that $L_\varphi = L_1$ in the case $\mu(\Omega) < \infty$.

It follows from the above that φ satisfies Δ_2^∞ condition whenever $\mu(\Omega) < \infty$. Let's also assume that φ satisfies condition Δ_2 if $\mu(\Omega) = \infty$. Now by applying Corollary 3.3 we obtain the above statement. □

Despite that for a large class of Orlicz spaces the Daugavet property fails, substantially more Orlicz spaces have the weaker property that every nonempty weakly open subset of their unit ball has diameter two. Of course, all the spaces having the last property fail the Radon-Nikodým property. For the Orlicz spaces L_φ the Radon-Nikodým property is characterized in terms of φ (see e.g. [13, Theorem 3.32]). The assumption of the next result implies non-reflexivity of the Orlicz space L_φ.

Theorem 3.5. *Let the measure μ be atomless. If $\mu(\Omega) = \infty$ and φ does not satisfy condition Δ_2, or $\mu(\Omega) < \infty$ and φ does not satisfy condition Δ_2^∞, then the diameter of any relatively weakly open subset of the unit ball in Orlicz space L_φ equipped with the Luxemburg norm is equal to 2.*

Proof. Let $W \neq \varnothing$ be a weakly open subset of the unit ball in L_φ. Since μ is atomless and μ is σ-finite by the initial assumption, it follows that L_φ is infinite-dimensional. Then there exists $x \in W$ with $\|x\|_\varphi = 1$. Choose $c > 0$ and $E \in \mathcal{S}$ such that $\mu(E) > 0$ and $|x(t)| \leq c$ on E. Suppose that φ does not satisfy Δ_2 condition in case when $\mu(\Omega) = \infty$, and φ does not satisfy Δ_2^∞ when $\mu(\Omega) < \infty$. Then there exists a sequence (t_n) of positive real numbers such that

$$\varphi((1 + 1/n)t_n) > 2^n \varphi(t_n).$$

We observe that in the case of finite measure we can always assume that $t_n \to \infty$, and in the case when the measure is infinite we can suppose that either $t_n \to \infty$ or $t_n \to 0$. In the case when $t_n \to \infty$, without loss of generality we choose a sequence (E_n) of measurable disjoint sets satisfying for each $n \in \mathbb{N}$,

$$\mu(E_n) = 1/(2^n \varphi(t_n)).$$

In the case when the measure is infinite and $t_n \to 0$, we choose (E_n) satisfying the above equation and such that $E_n \subset \Omega \setminus \Omega_n$ where (Ω_n) is an increasing sequence of sets of finite measure and such that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. Define

$$x'_n = x\chi_{\Omega \setminus E_n} + t_n \chi_{E_n}, \qquad x''_n = x\chi_{\Omega \setminus E_n} - t_n \chi_{E_n}.$$

It is clear that $x'_n \to x$ and $x''_n \to x$ a.e., and thus by the Fatou property $1 = \|x\|_\varphi \leq \liminf \|x'_n\|_\varphi$ and $1 = \|x\|_\varphi \leq \liminf \|x''_n\|_\varphi$. On the other hand $I_\varphi(x'_n) \leq I_\varphi(x) + 1/2^n \to I_\varphi(x) \leq 1$. Hence $\limsup I_\varphi(x'_n) \leq 1$ and so $\limsup \|x'_n\|_\varphi \leq 1$. Since the similar inequalities hold also for x''_n, we get that

$$\lim_n \|x'_n\|_\varphi = \lim_n \|x''_n\|_\varphi = 1.$$

Let now F be a bounded linear functional on L_φ. Then $F = H + S$, where H is the integral functional associated to a function $h \in L_{\varphi_*}$, and S a singular functional identically equal to zero on E_φ. Notice that $x - x'_n \in E_\varphi$ since x is bounded on each E_n, and so

$$F(x - x'_n) = \int_{E_n} xh \, d\mu - \int_{E_n} t_n h \, d\mu.$$

Since $h \in L_{\varphi_*}$ we have $I_{\varphi_*}(\lambda h) < \infty$ for some $\lambda > 0$. Applying Young's inequality we obtain

$$\begin{aligned}
|F(x - x'_n)| &= \left| \lambda^{-1} \int_{E_n} x\lambda h \, d\mu - \lambda^{-1} \int_{E_n} t_n \lambda h \, d\mu \right| \\
&\leq \int_{E_n} \lambda^{-1} \big(\varphi(x) + \varphi_*(\lambda h)\big) \, d\mu + \int_{E_n} \lambda^{-1} \big(\varphi(t_n) + \varphi_*(\lambda h)\big) \, d\mu \\
&= \lambda^{-1} \int_{E_n} \varphi(x) \, d\mu + 2\lambda^{-1} \int_{E_n} \varphi_*(\lambda h) \, d\mu + \lambda^{-1} \varphi(t_n) \mu(E_n).
\end{aligned}$$

By the choice of (E_n), $\int_\Omega \varphi(x) \, d\mu \leq 1$ and $\int_\Omega \varphi_*(\lambda h) \, d\mu < \infty$, the right side approaches zero. Thus $x'_n \to x$ and $x''_n \to x$ weakly.

We notice also that

$$\|x'_n - x''_n\|_\varphi = 2\|t_n \chi_{E_n}\|_\varphi = 2t_n / \varphi^{-1}(1/\mu(E_n)),$$

and so by $\varphi((1 + 1/n)t_n) > 2^n \varphi(t_n)$ and $\mu(E_n) = 1/(2^n \varphi(t_n))$ we get

$$t_n / \varphi^{-1}(1/\mu(E_n)) \geq n/(n+1).$$

Hence

$$\|x'_n - x''_n\|_\varphi \to 2.$$

Finally taking $f'_n = x'_n / \|x'_n\|_\varphi$ and $f''_n = x''_n / \|x''_n\|_\varphi$, we have $f'_n, f''_n \in B_{L_\varphi}$, $f'_n \to x$ and $f''_n \to x$ weakly, and $\|f'_n - x''_n\| \to 2$. This concludes the proof showing that the diameter of W is equal to two. $\square$

The proof of the next theorem is similar as of Theorem 3.5, however it requires some modifications. We include the proof for the sake of completeness.

Theorem 3.6. *If an Orlicz function φ does not satisfy condition Δ_2^0, then the diameter of any non-empty relatively weakly open subset of the unit ball in Orlicz space ℓ_φ equipped with the Luxemburg norm is equal to* 2.

Proof. Let $W \neq \emptyset$ be a weakly open subset of the unit ball in ℓ_φ. Then there exists $x \in W$ with $\|x\|_\varphi = 1$. By the Fatou property we have $\lim_{n\to\infty} \|x\chi_{\{1,\dots,n\}}\|_\varphi \to \|x\|_\varphi$. Since φ does not satisfy Δ_2^0, there exists a positive decreasing sequence (t_n) such that $t_n \to 0$ and

$$\varphi((1+1/n)t_n) > 2^n \varphi(t_n).$$

By passing to a subsequence we can assume without loss of generality that there is a sequence (E_n) of finite subsets of $\mathbb{N} \setminus \{1,\dots,n\}$ such that

$$2^{-n} \leq \varphi(t_n)|E_n| \leq 2^{-n+1},$$

where $|A| = \operatorname{card}(A)$ for any $A \subset \mathbb{N}$. Define the sequences (x_n) and (y_n) by

$$x_n = x\chi_{\mathbb{N}\setminus E_n} + t_n\chi_{E_n}, \qquad y_n = x\chi_{\mathbb{N}\setminus E_n} - t_n\chi_{E_n}.$$

We have

$$I_\varphi(x_n) = I_\varphi(x\chi_{\mathbb{N}\setminus E_n}) + \varphi(t_n)|E_n| \leq 1 + 2^{-n+1}.$$

This implies, $\|x_n\|_\varphi \leq 1 + 2^{-n+1}$. Since $\|x\chi_{\{1,\dots,n\}}\|_\varphi \to \|x\|_\varphi$ and

$$\|x_n\|_\varphi \geq \|x\chi_{\mathbb{N}\setminus E_n}\|_\varphi \geq \|x\chi_{\{1,\dots,n\}}\|_\varphi,$$

it follows $\|x_n\|_\varphi \to 1$. Similarly we get that $\|y_n\|_\varphi \to 1$.

We show that $x_n \to x$ weakly. Since $x - x_n = x\chi_{E_n} + t_n\chi_{E_n} \in h_\varphi$, it is enough to show that $H(x - x_n) \to 0$ for any order continuous functional H on ℓ_φ. Let H be order continuous functional, generated by $\eta = (\eta_n) \in \ell_{\varphi^*}$, i.e., $H(\xi) = \sum_{n=1}^\infty \xi_n\eta_n$ for any $\xi = (\xi_n) \in \ell_\varphi$.

Since $\eta \in \ell_{\varphi_*}$, there exists $\lambda > 0$ such that $I_{\varphi_*}(\lambda\eta) < \infty$. Applying the Young's inequality we obtain

$$\begin{aligned}|H(x - x_n)| &= \Big|\sum_{j\in E_n} t_n\eta_j\Big| + \Big|\sum_{j\in E_n} x_j\eta_j\Big| \\ &\leq \lambda^{-1}(I_\varphi(t_n\chi_{E_n}) + I_{\varphi_*}(\lambda\eta\chi_{E_n})) + \lambda^{-1}(I_\varphi(x\chi_{E_n}) + I_{\varphi_*}(\lambda\eta\chi_{E_n}).\end{aligned}$$

The right side approaches zero since $I_\varphi(x\chi_{E_n}) \leq 2^{-n+1} \to 0$, $\sum_{j=1}^\infty \varphi_*(\lambda\eta_j) < \infty$ and $E_n \subset \mathbb{N} \setminus \{1,\dots,n\}$. Similarly we get that $y_n \to x$ weakly in ℓ_φ.

Now observe that

$$\|x_n - y_n\|_\varphi = 2\|t_n\chi_{E_n}\|_\varphi = 2t_n/\varphi^{-1}(1/|E_n|).$$

From the inequalities at the beginning of the proof we have

$$t_n/\varphi^{-1}(1/|E_n|) \geq n/(n+1).$$

Hence

$$\|x_n - y_n\|_\varphi \to 2.$$

Finally taking $u_n = x_n/\|x_n\|_\varphi$ and $v_n = y_n/\|y_n\|_\varphi$, we have $u_n, v_n \in B_{\ell_\varphi}$, $u_n \to x$ and $v_n \to x$ weakly, and $\|u_n - v_n\|_\varphi \to 2$. This shows that the diameter of W is equal to two, and the proof is complete. □

4. Nakano spaces

Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let $p \in L^0$ with $1 \le p(t) \le \infty$, for $t \in \Omega$. Then for a.e. $t \in \Omega$ and $u \in \mathbb{R}_+$, define

$$\Phi_p(u,t) = \begin{cases} \frac{u^{p(t)}}{p(t)} & \text{if } t \in A, \\ \alpha(u) & \text{if } t \in A^c, \end{cases}$$

where $A = \{t : 1 \le p(t) < \infty\}$, $A^c = \{t : p(t) = \infty\}$, and $\alpha(u) = 0$ for $0 \le u \le 1$ and $\alpha(u) = \infty$ for $u > 1$. For every $t \in \Omega$ the function $u \to \Phi_p(u,t)$ is an extended real valued Orlicz function. Denote by $\Phi_q(u,t)$ the Young conjugate function to $\Phi_p(u,t)$. For any $x \in L^0$ denote by $I(x)$ the modular

$$I(x) = I_p(x) = \int_\Omega \Phi_p(|x(t)|, t)\, d\mu.$$

By $I_q(x)$ we denote the modular determined by the function $\Phi_q(u,t)$. The collection of all measurable functions $x \in L^0$ satisfying

$$\|x\| = \|x\|_{p(t)} = \inf\Big\{\lambda > 0 : I(x/\lambda) \le 1\Big\} < \infty$$

is the Nakano space $L_{p(t)}$. It is equipped with either the Luxemburg norm $\|\cdot\|_{p(t)}$ or the Amemyia norm

$$\|x\|^0 = \|x\|^0_{p(t)} = \inf_{k>0} \frac{1}{k}\Big(1 + I(kx)\Big),$$

which is identical with the Orlicz norm $\|x\|^0_{p(t)} = \sup\{\int_\Omega |xy| : I_{q(t)}(y) \le 1\}$ where $1/p(t) + 1/q(t) = 1$ a.e. with the usual convention that if $p(t) = 1$ then $q(t) = \infty$. The latter fact follows by the same techniques as for Orlicz spaces [28, 39]. By $L_{q(t)}$ we denote the Nakano space associated to the modular I_q. We have $\|x\| \le \|x\|^0 \le 2\|x\|$ for every $x \in L_{p(t)}$. The Nakano space equipped with either norm is a Banach function space with the Fatou property. It is also well known that if $p(t) < \infty$ a.e., then $\operatorname{supp}(L_{p(t)})_a = \Omega$ a.e. [24, p. 64]. A Nakano space is not rearrangement invariant unless the exponent $p(t)$ is a constant function. The Köthe dual spaces are described as follows

$$(L_{p(t)}, \|\cdot\|^0_{p(t)})' = (L_{q(t)}, \|\cdot\|_{q(t)}), \qquad (L_{p(t)}, \|\cdot\|_{p(t)})' = (L_{q(t)}, \|\cdot\|^0_{q(t)}).$$

Nakano spaces belong to the large family of Musielak-Orlicz spaces, and therefore many basic properties of these spaces follow from general results on Musielak-Orlicz spaces [36].

Theorem 4.1. *If $1 < p(t) < \infty$ a.e., then the Nakano space $L_{p(t)}$ equipped with either the Luxemburg or Orlicz norm has no Daugavet property.*

Proof. We start with $L_{p(t)}$ equipped with the Amemiya norm. Notice that since $1 < p(t) < \infty$, $(1/k)(1 + I(kx)) \to \infty$ as $k \to 0$ or $k \to \infty$, for any $x \in L_{p(t)} \backslash \{0\}$. Hence the infimum in the Amemyia norm is attained on $(0, \infty)$ for any $x \in L_{p(t)} \backslash \{0\}$. Moreover, if $x \neq 0$ and $I(x) < \infty$, then $I(\lambda x) < \lambda I(x)$ for any $0 < \lambda < 1$. Hence for any disjoint non-zero elements $x, y \in L_{p(t)}$ there exist $k_1, k_2 > 0$ such that $\|x\|^0 = (1/k_1)(1 + I(k_1 x))$ and $\|y\|^0 = (1/k_2)(1 + I(k_2 y))$, and

$$\begin{aligned}\|x+y\|^0 &\leq \frac{k_1+k_2}{k_1k_2}\Big(1 + I\Big(\frac{k_1k_2}{k_1+k_2}(x+y)\Big)\Big)\\ &= \frac{1}{k_1} + \frac{1}{k_2} + \frac{k_1+k_2}{k_1k_2} I\Big(\frac{k_1k_2}{k_1+k_2}x\Big) + \frac{k_1+k_2}{k_1k_2} I\Big(\frac{k_1k_2}{k_1+k_2}y\Big)\\ &< \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_1}I(k_1x) + \frac{1}{k_2}I(k_2y) = \|x\|^0 + \|y\|^0.\end{aligned}$$

Consequently, $L_{p(t)}$ equipped with the Amemiya norm cannot contain an order isometric copy of ℓ_1. We also observe that $L_{p(t)}$ with the Amemiya norm is strictly monotone. Indeed, letting $0 \leq x \leq y$, $x \neq y$, we have $\|y\|^0 = (1/k)(1 + I(ky))$ for some $k > 0$. Hence

$$\|x\|^0 \leq \frac{1}{k}\Big(1 + I(kx)\Big) < \frac{1}{k}\Big(1 + I(ky)\Big) = \|y\|^0.$$

Thus the space is strictly monotone and by [46] it cannot contain any isometric copy of ℓ_1 too.

Now assume that $(L_{p(t)}, \|\cdot\|^0_{p(t)})$ has the Daugavet property. Since $(L_{p(t)}, \|\cdot\|^0_{p(t)})' = (L_{q(t)}, \|\cdot\|_{q(t)})$ and $\operatorname{supp}(L_{q(t)})_a = \Omega$, the space $(L_{p(t)}, \|\cdot\|^0_{p(t)})$ should contain an isometric copy of ℓ_1 by Proposition 2.3(i), which is impossible.

If $(L_{p(t)}, \|\cdot\|_{p(t)})$ has the Daugavet property, then it follows from Corollary 2.2(i) that $(L_{p(t)}, \|\cdot\|_{p(t)})' = (L_{q(t)}, \|\cdot\|^0_{q(t)})$ contains ℓ_1 isometrically, which is again impossible and the proof is completed. □

The next lemma is surely well-known but we use it in the next result and include a proof for completeness.

Lemma 4.2. *Let $X = (X_1 \oplus \cdots \oplus X_n)_\infty$ be a finite direct sum of Banach spaces $(X_i, \|\cdot\|_i)$, $i = 1, \ldots, n$, equipped with the norm $\|x\| = \max_{1 \leq i \leq n} \|x_i\|_i$, where $x = (x_1, \ldots, x_n)$, $x_i \in X_i$. If X has the Daugavet property, then it is inherited by each component X_i, $i = 1, \ldots, n$.*

Proof. Without loss of generality assume that $n = 2$. Suppose that X has the Daugavet property. It is enough to show that X_1 has also this property. Let's take any normalized rank one operator T on X_1, i.e., $Tx_1 = x_1^*(x_1)y_1$ with $\|x_1^*\|_{X_1^*} = \|y_1\|_1 = 1$. Obviously $\|I + T\|_{X_1 \to X_1} \leq 2$, thus we need to show the opposite inequality. Define for any $x = (x_1, x_2) \in X$,

$$x^*(x) = x_1^*(x_1), \qquad y = (y_1, 0) \quad \text{and} \quad \widetilde{T}(x) = x^*(x)y = (x_1^*(x_1)y_1, 0).$$

Since $X^* \simeq (X_1^* \oplus X_2^*)_1$, $\|x^*\| = \|x_1^*\|_{X_1^*} = 1$. This implies by $\|y\| = \|y_1\|_1 = 1$, $\|\widetilde{T}\|_{X\to X} = \|x^*\|\|y\| = 1$. Since $\widetilde{T}$ is a rank-one operator on X, so by the Daugavet property of X, $\|I + \widetilde{T}\|_{X\to X} = 2$. In consequence,

$$\begin{aligned} 2 &= \sup_{\|x\|\le 1} \|x + x^*(x)y\| = \sup_{\|x_1\|_1\le 1, \|x_2\|_2\le 1} \max\{\|x_1 + x_1^*(x_1)y_1\|_1, \|x_2\|_2\} \\ &= \sup_{\|x_1\|_1\le 1, \|x_2\|_2\le 1} \|x_1 + x_1^*(x_1)y_1\|_1 = \|I + T\|_{X_1\to X_1}. \end{aligned}$$

This shows the Daugavet property of X_1 and finishes the proof. □

Theorem 4.3. *Let $L_{p(t)}$ be a Nakano space on $(\Omega, \mathcal{S}, \mu)$. Then for any $x \in L_{p(t)}$ we have*

$$\|x\| = \max\{\|x\chi_A\|, \|x\chi_{A^c}\|_\infty\},$$

and thus

$$L_{p(t)} \simeq (L_{p(t)}(A) \oplus L_\infty(A^c))_\infty,$$

where $A = \{t \in \Omega : 1 \le p(t) < \infty\}$ and $A^c = \Omega\backslash A$. Consequently, if $1 < p(t) \le \infty$ a.e. and $(L_{p(t)}, \|\cdot\|)$ has the Daugavet property then $L_{p(t)} = L_\infty$ with the equality of norms.

Proof. By $\|x\chi_{A^c}\| = \|x\chi_{A^c}\|_\infty$ and monotonicity of the norm, $\max\{\|x\chi_A\|, \|x\chi_{A^c}\|_\infty\} \le \|x\|$. Now, if $\|x\chi_A\| > \|x\chi_{A^c}\|_\infty$ then

$$I(x/\|x\chi_A\|) = I(x\chi_A/\|x\chi_A\|) \le 1,$$

and so $\|x\| \le \|x\chi_A\|$. In the opposite case when $\|x\chi_A\| \le \|x\chi_{A^c}\|_\infty$ we have

$$I(x/\|x\chi_{A^c}\|_\infty) = I(x\chi_A/\|x\chi_{A^c}\|_\infty) \le I(x\chi_A/\|x\chi_A\|) \le 1,$$

and thus $\|x\| \le \|x\chi_{A^c}\|_\infty$ which proves the desired equality $\|x\| = \max\{\|x\chi_A\|, \|x\chi_{A^c}\|_\infty\}$.

Applying now Lemma 4.2, both $L_{p(t)}(A)$ and $L_\infty(A)$ must have the Daugavet property. However in view of Theorem 4.1, it is only possible if $p(t) = \infty$ a.e. □

Remark 4.4. If $p(t) = 1$ for $t \in A$ and $p(t) = \infty$ for $t \in A^c$, then

$$L_{p(t)} = L_1(A) \oplus L_\infty(A^c),$$

and for all $x \in L_{p(t)}$,

$$\|x\| = \max\{\|x\chi_A\|_1, \|x\chi_{A^c}\|_\infty\} \quad \text{and} \quad \|x\|^0 = \|x\chi_A\|_1 + \|x\chi_{A^c}\|_\infty.$$

In this case $L_{p(t)}$ equipped with either Luxemburg or Orlicz norm has the Daugavet property if the measure μ is atomless [45].

If μ is atomless, it is known that every slice of the unit ball of $L_\infty(\mu)$ has diameter two. Hence, in the case when $\mu(A^c) > 0$ and μ is atomless, in view of Theorem 4.3, it is easy to check that $L_{p(t)}$ satisfies the same property. Below we state and prove a more general result.

Theorem 4.5. *Let $(\Omega, \mathcal{S}, \mu)$ be an atomless measure space. If $1 \le p(t) < \infty$ for a.a. $t \in \Omega$ and $\operatorname{esssup}_\Omega p(t) = \infty$, then the diameter of any nonempty relatively weakly open subset V of the unit ball in $L_{p(t)}$ equipped with the Luxemburg norm is equal to 2.*

Proof. Let $x \in V$ with $\|x\| = 1$, and assume that $\operatorname{esssup}_\Omega p(t) = \infty$. Let (Ω_m) be an increasing sequence of measurable sets such that $\bigcup_m \Omega_m = \Omega$, $0 < \mu(\Omega_m) < \infty$ and for every $m \in \mathbb{N}$, $u \ge 0$,

$$\operatorname{esssup}_{t\in\Omega_m} u^{p(t)}/p(t) < \infty.$$

A construction of such a sequence is given e.g. in [24, p. 64]. Let (B_i) be a sequence of measurable sets which covers Ω and $|x|$ be bounded on every B_i. By the assumption on p, there is a sequence (C_j) of measurable sets in Ω satisfying that for every j, $0 < \mu(C_j)$, $p(t) > 2^{2j}$ for $t \in C_j$ and $\mu(C_j) \to 0$. Since $C_j = \cup_{m,i}(C_j \cap \Omega_m \cap B_i)$ for each j, we can find a subsequence (A_n) of elements in $\{C_j \cap \Omega_m \cap B_i : j \in \mathbb{N}, m \in \mathbb{N}, i \in \mathbb{N}\}$ such that $|x|$ is bounded on every A_n, $0 < \mu(A_n)$, $(\mu(A_n)) \to 0$ and for every $t \in A_n$,

$$p(t) > 2^{2n}.$$

If $t \in A_n$ then for each $n \in \mathbb{N}$,

$$\Big(1 + \frac{1}{n}\Big)^{p(t)} \ge \Big(1 + \frac{1}{n}\Big)^{2^{2n}} = \Big[\Big(1 + \frac{1}{n}\Big)^{n}\Big]^{2^{2n}/n} \ge 2^n.$$

Since for each n, the function $u \mapsto \int_{A_n} u^{p(t)}/p(t)\, d\mu$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ is continuous and surjective, there exist $u_n > 0$ such that

$$I_p(u_n \chi_{A_n}) = 1/2^n.$$

We have for all n,

$$I_p\Big(\Big(1 + \frac{1}{n}\Big) u_n \chi_{A_n}\Big) = \int_{A_n} \frac{(1 + \frac{1}{n})^{p(t)} u_n^{p(t)}}{p(t)}\, d\mu \ge 2^n \int_{A_n} \frac{u_n^{p(t)}}{p(t)} = 1.$$

and on the other hand

$$1 > I_p(u_n \chi_{A_n}) \to 0.$$

By the definition of the Luxemburg norm this yields

$$1 \ge \|u_n \chi_{A_n}\| \ge n/(n+1).$$

Define now

$$y_n' = x\chi_{A_n^c} + u_n \chi_{A_n} \quad \text{and} \quad y_n'' = x\chi_{A_n^c} - u_n \chi_{A_n}.$$

Since $\mu(A_n) \to 0$, so $y_n' \to x$ and $y_n'' \to x$ a.e.. Thus by the Fatou property we have that $1 = \|x\| \le \liminf \|y_n'\|$ and $1 = \|x\| \le \liminf \|y_n''\|$. On the other hand $I_p(y_n') \le I_p(x) + I_p(u_n \chi_{A_n}) \le 1 + 1/n$. This yields $\limsup I_p(y_n') \le 1$, which in turn implies that $\limsup \|y_n'\| \le 1$. Similarly $\limsup \|y_n''\| \le 1$. Finally we have

$$\lim \|y_n'\| = \lim \|y_n''\| = 1.$$

By the choice of A_n, $I_p(\beta x\chi_{A_n}) < \infty$ and $I_p(\beta u_n\chi_{A_n}) < \infty$ for every $\beta > 0$ and all n. Hence $x - y_n' = x\chi_{A_n} - u_n\chi_{A_n} \in (L_{p(t)})_a$ and $x - y_n'' = x\chi_{A_n} + u_n\chi_{A_n} \in (L_{p(t)})_a$.

Let now F be an arbitrary functional on $L_{p(t)}$. By the equality $(L_{p(t)})^* \simeq L_{q(t)} \oplus [(L_{p(t)})_a]^\perp$, $F = H + S$, where H is a regular (integral) functional induced by $h \in L_{q(t)}$, and S is a singular functional vanishing on the subspace $(L_{p(t)})_a$. Hence $F(x - y_n') = H(x - y_n')$ and $F(x - y_n'') = H(x - y_n'')$. Let $\beta > 0$ be such that $I_q(\beta h) < \infty$. Then by Young's inequality $uv \le \Phi_p(u,t) + \Phi_q(v,t)$, $u, v \ge 0$, we have

$$\begin{aligned}
&|F(x - y_n')| \\
&= \Big|\beta^{-1}\int_{A_n} x\beta h - \beta^{-1}\int_{A_n} u_n\beta h\Big| \\
&\le \int_{A_n} \beta^{-1}(\Phi_p(|x(t)|,t) + \Phi_q(\beta|h(t)|,t))\,d\mu + \int_{A_n} \beta^{-1}(\Phi_p(u_n,t) + \Phi_q(\beta|h(t)|,t))\,d\mu \\
&= \beta^{-1}\int_{A_n} \Phi_p(|x(t)|,t)\,d\mu + 2\beta^{-1}\int_{A_n} \Phi_q(\beta|h(t)|,t)\,d\mu + \beta^{-1}\int_{A_n} \Phi_p(u_n,t)\,d\mu.
\end{aligned}$$

The right side of the above inequality approaches zero since $\mu(A_n) \to 0$, $I_p(x) \le 1 < \infty$ and $I_q(\beta h) < \infty$, and by the choice of u_n. Thus $F(x - y_n') \to 0$ and similarly $F(x - y_n'') \to 0$, which means that $y_n' \to x$ and $y_n'' \to x$ weakly. However

$$\|y_n' - y_n''\| = 2\|u_n\chi_{A_n}\| \to 2.$$

We finish by setting new functions $z_n' = y_n'/\|y_n'\|$, $z_n'' = y_n''/\|y_n''\|$ that belong to the intersection of V and the unit sphere of $L_{p(t)}$, and since $\|z_n' - z_n''\| \to 2$, the diameter of V is two. □

5. Lorentz and Marcinkiewicz spaces

Let $\mathcal{P}$ be a class of functions $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ that are concave, not trivially equal to zero and $\psi(0) = 0$. Denote $\psi(0+) = \lim_{t\to 0+}\psi(t)$ and $\psi(\infty) = \lim_{t\to\infty}\psi(t)$. Given $\psi \in \mathcal{P}$, the *Lorentz space* Λ_ψ over the measure space (Ω, μ) consists of all $x \in L^0$ such that

$$\|x\|_{\Lambda_\psi} = \int_0^\gamma x^*(t)d\psi(t) = \|x\|_\infty \cdot \psi(0+) + \int_0^\gamma x^*(t)\psi'(t)\,dt < \infty,$$

where $\gamma := \mu(\Omega)$ and ψ' is the derivative of ψ. Note that the derivative of a concave function exists except a countable set. The reason that in the definition of Λ_ψ we use a concave function is that the functional $\|\cdot\|_{\Lambda_\psi}$ induced by increasing $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a norm if and only if ψ is concave and $\psi(0) = 0$ [32].

Let $\mathcal{Q}$ be the class of functions $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ that are not trivially equal to zero, increasing, quasi-concave i.e. $t/\psi(t)$ is increasing, and $\psi(0+) = 0$. For $\psi \in \mathcal{Q}$, the *Marcinkiewicz space* M_ψ over $(\Omega, \mathcal{S}, \mu)$ is the set of all $x \in L^0$ such that

$$\|x\|_{M_\psi} = \sup_{0<t<\gamma} \frac{\int_0^t x^*}{\psi(t)} < \infty.$$

Observe that $\mathcal{P} \subset \mathcal{Q}$, and a quasi-concave function ψ is also subadditive, meaning $\psi(s+t) \leq \psi(s) + \psi(t)$.

Unless we say otherwise, any time we discuss the Lorentz space Λ_ψ we assume a priori that $\psi \in \mathcal{P}$, and in the case of Marcinkiewicz space M_ψ, that $\psi \in \mathcal{Q}$.

Both $(\Lambda_\psi, \|\cdot\|_{\Lambda_\psi})$ and $(M_\psi, \|\cdot\|_{M_\psi})$ are Banach function lattices satisfying the Fatou property.

We summarize below the basic facts on Λ_ψ and M_ψ [27]. We note that the proof of the Köthe duality between Lorentz and Marcinkiewicz spaces stated below can be obtained in a similar way as corresponding results on the description of the Banach duality between those spaces in [27]. The strict monotonicity is a direct consequence of the definition of the norm in Λ_ψ.

Proposition 5.1. *Let $(\Omega, \mathcal{S}, \mu)$ be an atomless measure space. Then the following conditions hold true.*

(*1*) *The space $(M_\psi)_a$ is not trivial, that is $\operatorname{supp}(M_\psi)_a = \Omega$, if and only if $\lim_{t\to 0+} t/\psi(t) = 0$.*

(*2*) *For any $\psi \in \mathcal{P}$ it holds*

$$(\Lambda_\psi)' = M_\psi \quad \textit{and} \quad (M_\psi)' = \Lambda_\psi$$

with equality of norms.

(*3*) *If $\psi \in \mathcal{P}$ and $\lim_{t\to 0+} t/\psi(t) = 0$, then*

$$((M_\psi)_a)^* \simeq ((M_\psi)_a)' = (M_\psi)' = \Lambda_\psi,$$

where the last equation holds with equality of norms.

(*4*) *If $\mu(\Omega) < \infty$, then Λ_ψ is order continuous (resp. separable) if and only if $\psi(0+) = 0$ (resp. $\psi(0+) = 0$ and μ is separable).*

(*5*) *If $\mu(\Omega) = \infty$, then Λ_ψ is order continuous (resp. separable) if and only if $\psi(0+) = 0$ and $\psi(\infty) = \infty$ (resp. $\psi(0+) = 0$, $\psi(\infty) = \infty$ and μ is separable).*

(*6*) *The space Λ_ψ is strictly monotone if and only if ψ is strictly monotone on $(0, \gamma)$ and $\psi(\infty) = \infty$ when $\mu(\Omega) = \infty$.*

We present results on the Daugavet property in Lorentz and Marcinkiewicz spaces which show that for a large class of concave or quasi-concave functions up to equivalence of norms the spaces are of special types.

Theorem 5.2. *Let $(\Omega, \mathcal{S}, \mu)$ be an atomless measure space. Assume that $\psi \in \mathcal{P}$. Then the Lorentz space Λ_ψ does not have the Daugavet property whenever one of the following conditions is satisfied:*

(*i*) *Let $\mu(\Omega) < \infty$, $\psi(0+) = 0$ and $\lim_{t\to 0+} t/\psi(t) = 0$.*

(*ii*) *Let $\mu(\Omega) = \infty$, $\psi(0+) = 0$ and $\lim_{t\to 0+} t/\psi(t) = 0$ and $\psi(\infty) = \infty$.*

Proof. If $\psi(0+) = 0$ and $\psi(\infty) = \infty$ provided $\mu(\Omega) = \infty$, then $X = \Lambda_\psi$ is order continuous Banach lattices. Since X has the Fatou property, our hypothesis $\lim_{t\to 0+} t/\psi(t) = 0$ implies by Proposition 2.5 the statement. $\square$

In the case of separable atomless measure space we have the following result.

Theorem 5.3. *Let $(\Omega, \mathcal{S}, \mu)$ be an atomless separable measure space. Assume that $\psi \in \mathcal{P}$. Then both Lorentz space Λ_ψ and Marcinkiewicz space M_ψ do not have the Daugavet property whenever one of the following conditions is satisfied*:

(*i*) *Let $\mu(\Omega) < \infty$, $\psi(0+) = 0$ and $\lim_{t\to 0+} t/\psi(t) = 0$.*
(*ii*) *Let $\mu(\Omega) = \infty$, $\psi(0+) = 0$ and $\lim_{t\to 0+} t/\psi(t) = 0$ and $\psi(\infty) = \infty$.*
(*iii*) $\lim_{t\to 0+} t/\psi(t) = 0$, *$\psi(\infty) = \infty$ in case $\mu(\Omega) = \infty$, and ψ is strictly increasing on $(0, \gamma)$.*

Proof. *Case* 1^0. Consider first the Lorentz space Λ_ψ. If the condition (*i*) or (*ii*) is satisfied then Theorem 5.2 applies.

(*iii*) If $\psi(0+) = 0$, then we have the case (*i*) or (*ii*) above. Assume now that $\psi(0+) > 0$. By Proposition 5.1, our hypotheses on ψ imply that $(M_\psi)_a$ is nontrivial and Λ_ψ is strictly monotone. We also have the identity

$$((M_\psi)_a)^* \simeq ((M_\psi)_a)' = \Lambda_\psi$$

isometrically. Let Λ_ψ have the Daugavet property. It follows that $(M_\psi)_a$ has also this property. Apply now Proposition 2.4 to $X = (M_\psi)_a$. We see that $\phi_{X'}(0+) = \psi(0+) > 0$, $\operatorname{supp} X_a = \Omega$ and $X' = \Lambda_\psi$ is strictly monotone, and so $(M_\psi)_a$ cannot have the Daugavet property, and this contradiction completes the proof.

Case 2^0. Now consider the Marcinkiewicz space M_ψ. If (*i*) or (*ii*) is fulfilled then by the isometric identity

$$(\Lambda_\psi)^* \simeq ((\Lambda_\psi)_a)' = M_\psi,$$

if M_ψ have had the Daugavet property then Λ_ψ would have this property, but it is impossible by case 1^0.

(*iii*) We can assume that $\psi(0+) > 0$. Then applying Proposition 2.4 to $X = (M_\psi)_a$ exactly as in (*iii*) of case 1^0, we get that $(M_\psi)_a$ cannot have the Daugavet property, and so M_ψ has no Daugavet property. □

The following result can be proved by careful modification of the proof in the case of $\Omega = (0, 1)$ due to Briskin and Semenov [12, Theorem 3].

Theorem 5.4. *Let $(\Omega, \mathcal{S}, \mu)$ be an atomless measure space and $\psi \in \mathcal{P}$. Let $\psi(\infty) = \infty$ if $\mu(\Omega) = \infty$, and let ψ do not coincide with a constant on $\mathbb{R}_+$. Then the following conditions are equivalent*:

(*i*) *Two dimensional space $\ell_1^{(2)}$ is isometrically embedded into Λ_ψ.*
(*ii*) *Two dimensional space $\ell_1^{(2)}$ is order isometrically embedded into Λ_ψ.*
(*iii*) *The function ψ is linear in some neighborhood of zero.*

If ψ is constant on $\mathbb{R}_+$, then $\Lambda_\psi = L_\infty$ with $\|\cdot\|_{\Lambda_\psi} = \psi(0+)\|\cdot\|_\infty$, and then $\ell_1^{(2)}$ is isometrically embedded into Λ_ψ.

In the next proposition, applying the above result, we improve condition (*iii*) of Theorem 5.3.

Proposition 5.5. *Let $(\Omega, \mathcal{S}, \mu)$ be an atomless measure space and $\psi \in \mathcal{P}$. Then both the Marcinkiewicz space M_ψ and the Lorentz space Λ_ψ do not have the Daugavet property whenever $\lim_{t\to 0+} t/\psi(t) = 0$, ψ is not constant on $\mathbb{R}_+$, and $\psi(\infty) = \infty$ if $\mu(\Omega) = \infty$.*

Proof. By the assumption $\lim_{t\to 0+} t/\psi(t) = 0$, $\operatorname{supp}(M_\psi)_a = \Omega$ and so $((M_\psi)_a)' = \Lambda_\psi$. If M_ψ has the Daugavet property then $(M_\psi)_a$ has this property and by Corollary 2.2(i), Λ_ψ contains an isometric copy of $L_1(0,1)$. Hence Λ_ψ contains an isometric copy of $\ell_1^{(2)}$, and by Theorem 5.4, ψ is linear around zero and so $\lim_{t\to 0+} t/\psi(t) > 0$ contradicting the initial assumption.

If Λ_ψ has the Daugavet property then by $((M_\psi)_a)^* \simeq \Lambda_\psi$, $(M_\psi)_a$ inherits this property, but it is impossible by the above considerations. □

A space without the Daugavet property may satisfy that all slices of the unit ball have diameter two. This is not the case of Λ_ψ under some restrictions on ψ.

Proposition 5.6. *Let $\psi(0+) = 0$, $\lim_{t\to 0+} t/\psi(t) = 0$, μ be separable and $\psi(\infty) = \infty$ if $\mu(\Omega) = \infty$. Then the function Lorentz space Λ_ψ over an atomless measure space $(\Omega, \mathcal{S}, \mu)$ has slices of arbitrarily small diameter.*

Proof. Since the space Λ_ψ has the Radon-Nikodým property by the assumptions and Proposition 5.1, so it has strongly exposed points. Hence there exist slices of arbitrarily small diameters. □

Observe that under suitable function ψ, $(\Delta, |\!|\!|\cdot|\!|\!|_\Delta)$ and $(\Sigma, |\!|\!|\cdot|\!|\!|_\Sigma)$ are Λ_ψ and M_ψ, respectively. Hence from Proposition 5.5 we will show that some results from [5] can be recovered.

Corollary 5.7. *Both Banach lattices $(\Delta, |\!|\!|\cdot|\!|\!|_\Delta)$ and $(\Sigma, |\!|\!|\cdot|\!|\!|_\Sigma)$ over any atomless measure space do not have the Daugavet property.*

Proof. Letting $\psi(t) = 1 + t$ for every $t \geq 0$, we have $\psi(0+) = 1$ and $\psi'(t) = 1$ for all $t > 0$. This implies that $\Lambda_\psi = (\Delta, |\!|\!|\cdot|\!|\!|_\Delta)$ with equality of norms. We have $t/\psi(t) \to 0$ as $t \to 0$ and ψ is strictly increasing on $(0, \infty)$. Hence $(\Delta, |\!|\!|\cdot|\!|\!|_\Delta)$ does not have the Daugavet property by Theorem 5.3(iii) applied to Lorentz space Λ_ψ.

To conclude, observe that $(\Delta, |\!|\!|\cdot|\!|\!|_\Delta)' = (\Sigma, |\!|\!|\cdot|\!|\!|_\Sigma)$ with equality of norms, and thus $(\Sigma, |\!|\!|\cdot|\!|\!|_\Sigma) = M_\psi$ with equality of norms, which in turn implies that it does not have the Daugavet property by Theorem 5.3(iii) applied to Marcinkiewicz space M_ψ. □

Corollary 5.8. *Let $\psi \in \mathcal{P}$ be a constant function on $[t_0, \infty)$ for some $t_0 > 0$. Then the Marcinkiewicz space M_ψ over $(0, \infty)$ has the Daugavet property if and only if ψ is constant on $(0, \infty)$.*

Proof. Without loss of generality we assume that $t_0 = 1$ and $\psi(1) = 1$ (see Corollary 2.10). Define $\psi_0(t) = \max\{t, \psi(t)\}$ for every $t \geq 0$ and let $X = M_{\psi_0}$ be the Marcinkiewicz space generated by a quasi-concave function ψ_0. It is easy to check that

$$M_\psi = L_1 \cap X$$

with equality of norms. Since $\phi_X(t) = t/\psi_0(t)$ for every $t > 0$, $\phi_{X'}(t) = t/\phi_X(t) = \psi_0(t)$ for every $t > 0$. Our hypothesis on concavity of ψ yields $\phi_{X'}(t) = \psi(t)$ on $(0, 1]$.

If we assume that M_ψ has the Daugavet property, Theorem 2.9 applies and concludes that ψ is a constant function on $(0, \infty)$.

If $\psi(t) = c$ for every $t > 0$, then obviously $M_\psi = L_1$ with $c\| \cdot \|_{M_\psi} = \| \cdot \|_1$, and the proof is complete by the well known fact that any L_1 space on an atomless measure space has the Daugavet property. □

In view of Theorem 2.9 and Corollary 2.10, we can state the analogous results as above replacing $(0, \infty)$ by an atomless measure space $(\Omega, \mathcal{S}, \mu)$ under the additional assumption that $t_0 = 1$.

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