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Convex Solids with Hyperplanar Shadow-Boundaries

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Extending the well-known characteristic property of solid ellipsoids as convex bodies with hyperplanar shadow-boundaries, we describe all n-dimensional closed convex sets in $\mathbb{R}^n$ whose shadow-boundaries satisfy certain conditions of hyperplanarity.

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1. Introduction

Various characterizations of solid ellipsoids among convex bodies in $\mathbb{R}^n$ form an established topic of convex geometry (see Bonnesen and Fenchel [5, § 70], Heil and Martini [8], Petty [15]). One of the best known, due to various applications (see, e.g., Amir [2]), is a characterization of solid ellipsoids as convex bodies with hyperplanar shadow-boundaries. We need some definitions to describe the existing results in this field.

In what follows, by a *convex solid* we mean a closed convex set $K \subset \mathbb{R}^n$, $n \geq 2$, with nonempty interior which is distinct from the entire space (*convex bodies* are compact convex solids). As usual, $\operatorname{bd} K$ and $\operatorname{int} K$ denote, respectively, the boundary and interior of K. Similarly, $\operatorname{rbd} M$ and $\operatorname{rint} M$ denote, respectively, the relative boundary and relative interior of a closed convex set $M \subset \mathbb{R}^n$ of some intermediate dimension.

If $l \subset \mathbb{R}^n$ is a line, then the *shadow-boundary* of K with respect to l, denoted $S_l(K)$, is the set of points in $\operatorname{bd} K$ at which the lines parallel to l support K. This terminology comes from the concept of illumination of K by a family of rays which are parallel to a given direction (see, e.g., the surveys of Martini [13] and Martini and Soltan [14]). Since any two parallel lines determine the same shadow-boundary of K, we consider, in what follows, the shadow-boundaries generated by 1-dimensional subspaces of $\mathbb{R}^n$. If l is a 1-dimensional subspace of $\mathbb{R}^n$, then

$$S_l(K) = \operatorname{bd} K \cap \operatorname{bd} (K + l), \tag{1}$$

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where $K+l$ is the vector sum of K and l (equivalently, $K+l$ is the union of all translates of l that intersect K).

Blaschke ([3] and [4, p. 157–159]) proved that a strictly convex body $K \subset \mathbb{R}^3$ with regular boundary is a solid ellipsoid if any shadow-boundary of K is a plane curve. Alexandrov [1] obtained a far-reaching local version of Blaschke's assertion, which characterizes pieces of convex quadrics or conic surfaces in $\mathbb{R}^3$ (see Lemma 4.1 below). Refining Blaschke's argument, Busemann [6, p. 93] showed that a convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is a solid ellipsoid if for any shadow-boundary $S_l(K)$ of K there is a hyperplane $H \subset \mathbb{R}^n$ such that

$$S_l(K) \subset H. \tag{2}$$

Marchaud [12] (for $n = 3$) and Gruber [7] (for all $n \geq 3$) proved (with some additional restrictions on l) that a convex body $K \subset \mathbb{R}^n$ is a solid ellipsoid provided for any 1-dimensional subspace $l \subset \mathbb{R}^n$ there is a hyperplane H which is not parallel to l and satisfies the inclusion

$$H \cap \operatorname{bd}(K+l) \subset S_l(K). \tag{3}$$

We observe here that a similar condition, with $H \cap \operatorname{bd} K \subset S_l(K)$ instead of (3), does not characterize ellipsoids: for example, if $K \subset \mathbb{R}^n$ is a convex polytope, then for any edge e of K that lies in $S_l(K)$ there is a hyperplane H which is not parallel to l and has the property $H \cap \operatorname{bd} K = e$. (In this regard, Marchaud's condition on page 36 of [12] which involves plane sections of $\operatorname{bd} K$, and not of $\operatorname{bd}(K+l)$, should be treated with caution.)

Our goal here is to describe all convex solids in $\mathbb{R}^n$ whose shadow-boundaries satisfy one of the conditions (2) and (3). The motivation for this comes from the fact that ellipsoids are particular cases of convex quadric hypersurfaces in $\mathbb{R}^n$, which satisfy both conditions (2) and (3) (see Lemma 3.6 below).

We recall that a *convex hypersurface* (*surface* if $n = 3$ or *curve* if $n = 2$) is the boundary of a convex solid in $\mathbb{R}^n$. This definition includes a hyperplane and a pair of parallel hyperplanes. In a standard way, a *quadric* (or a *second degree hypersurface*) in $\mathbb{R}^n$ is the locus of points $x = (\xi_1, \ldots, \xi_n)$ that satisfy a quadratic equation

$$F(\xi_1, \ldots, \xi_n) \equiv \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k + 2\sum_{i=1}^{n} b_i\xi_i + c = 0, \tag{4}$$

where not all scalars a_{ik} are zero. We say that a convex hypersurface $C \subset \mathbb{R}^n$ is a *convex quadric* provided there is a real quadric $Q \subset \mathbb{R}^n$ and a connected component U of $\mathbb{R}^n \setminus Q$ such that U is convex and $C = \operatorname{bd} U$.

As shown in [18], a convex hypersurface $C \subset \mathbb{R}^n$ is a convex quadric if and only if there is an orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$ such that C is the locus of points

$x = (\xi_1, \dots, \xi_n)$ given by one of the conditions

$$a_1\xi_1^2 + \dots + a_k\xi_k^2 = 1, \qquad 1 \le k \le n, \tag{5}$$
$$a_1\xi_1^2 - a_2\xi_2^2 - \dots - a_k\xi_k^2 = 1,\ \xi_1 \ge 0, \qquad 2 \le k \le n, \tag{6}$$
$$a_1\xi_1^2 = 0, \tag{7}$$
$$a_1\xi_1^2 - a_2\xi_2^2 - \dots - a_k\xi_k^2 = 0,\ \xi_1 \ge 0, \qquad 2 \le k \le n, \tag{8}$$
$$a_1\xi_1^2 + \dots + a_{k-1}\xi_{k-1}^2 = \xi_k, \qquad 2 \le k \le n, \tag{9}$$

where all scalars a_i involved are positive. Various characteristic properties of convex quadrics are given in [16]–[19].

We recall that the *recession cone* of a convex solid $K \subset \mathbb{R}^n$ is given by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n : x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \ge 0\}.$$

It is known that $\operatorname{rec} K$ is a closed convex cone with apex o (the origin of $\mathbb{R}^n$), and $\operatorname{rec} K \ne \{o\}$ if and only if K is unbounded (see, e.g., [20] for general properties of convex sets). Furthermore, $\operatorname{rec} K$ is the union of all halflines h with apex o such that $x + h \subset K$ for any given $x \in K$. The subspace $\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K)$ is called the *linearity space* of K. Clearly, (i) K is line-free (that is, does not contain a line) if and only if $\operatorname{lin} K = \{o\}$, (ii) $\operatorname{lin} K$ is an $(n-1)$-dimensional subspace if and only if K is either a halfspace of a slab (that is, a closed solid bounded by a pair of distinct parallel hyperplanes). Given a subspace $L \subset \mathbb{R}^n$ complementary to $\operatorname{lin} K$ (put $L = \mathbb{R}^n$ if $\operatorname{lin} K = \{o\}$), the solid K can be expressed as the direct sum $K = \operatorname{lin} K \oplus (K \cap L)$, where $K \cap L$ is a line-free closed convex set. (The sum of sets K and M is called *direct*, denoted $K \oplus M$, provided K and M lie, respectively, in complementary subspaces of $\mathbb{R}^n$.)

We distinguish two special types of directions and respective shadow-boundaries: *non-recessional* and *sharp*. While non-recessional directions are useful in describing non-empty shadow-boundaries, sharp directions allow refinements of the existing results even for the case of convex bodies. A 1-dimensional subspace l is called *recessional* for the convex solid K if $l \subset \operatorname{rec} K \cup (-\operatorname{rec} K)$; otherwise, l is called *non-recessional*, as well as the respective shadow-boundary $S_l(K)$. The solid K has non-recessional subspaces if and only if K is not a halfspace (see Lemma 3.1). The 1-dimensional non-recessional for K subspace l is called *sharp* if every line which is parallel to l and supports K has precisely one point in common with K; the respective shadow-boundary $S_l(K)$ is also called *sharp* (see [11] for the number of sharp shadow-boundaries of a convex solid).

2. Main Results

Theorem 2.1. *Given a convex solid $K \subset \mathbb{R}^n$, $n \ge 2$, the following conditions are equivalent.*

1) *For any 1-dimensional subspace $l \subset \mathbb{R}^n$, there is a hyperplane $H \subset \mathbb{R}^n$ which intersects $K + l$ such that the inclusion (3) holds.*
2) *For any 1-dimensional non-recessional for K subspace $l \subset \mathbb{R}^n$, there is a hyperplane $H \subset \mathbb{R}^n$ which intersects $K + l$ such that the inclusion (3) holds.*

3) *For any* 1*-dimensional sharp for* K *subspace* $l \subset \mathbb{R}^n$*, there is a hyperplane* $H \subset \mathbb{R}^n$ *such that*

$$H \cap \operatorname{bd}(K+l) = S_l(K). \tag{10}$$

4) K *has one of the following shapes:*
a) $\operatorname{bd} K$ *is a convex quadric,*
b) $\dim(\operatorname{lin} K) = n-2$*, and* K *is the direct sum of* $\operatorname{lin} K$ *and a* 2*-dimensional line-free closed convex set,*
c) $\dim(\operatorname{lin} K) = n-3$*, and* K *is the direct sum of* $\operatorname{lin} K$ *and a* 3*-dimensional line-free closed convex cone.*

Remark 2.2. As follows from the proof of Theorem 2.1, condition *3)* can be slightly relaxed by replacing the family of all sharp for K subspaces with a dense subset of this family. Furthermore, the shapes *a)*–*c)* in condition *4)* are not mutually exclusive: a cylinder based on a 2-dimensional line-free convex quadric is a particular case of *b*), and a cylinder based on a sheet of a 3-dimensional elliptic cone is a particular case of *c*). Obviously, the shape *c*) occurs only if $n \geq 3$.

Corollary 2.3. *Given a line-free convex solid* $K \subset \mathbb{R}^n$*,* $n \geq 3$*, the following conditions are equivalent.*

1) *Each sharp shadow-boundary of* K *lies in a hyperplane* H*.*
2) $\operatorname{bd} K$ *is a convex quadric* (*additionally,* K *may be a convex cone if* $n = 3$).

In particular, a convex body $K \subset \mathbb{R}^n$ *is a solid ellipsoid if and only if each sharp shadow-boundary of* K *lies in a hyperplane.* □

A convex solid $K \subset \mathbb{R}^n$ distinct from a cone is called *strictly convex* provided its boundary contains no line segment; an n-dimensional closed convex cone $K \subset \mathbb{R}^n$ is *strictly convex* provided it has a unique apex, say, p such that any line segment in $\operatorname{bd} K$ belongs to a line through p.

Corollary 2.4. *Given a convex solid* $K \subset \mathbb{R}^n$*,* $n \geq 2$*, the following conditions are equivalent.*

1) *For any* 1*-dimensional non-recessional for* K *subspace* $l \subset \mathbb{R}^n$*, there is a hyperplane* $H \subset \mathbb{R}^n$ *that satisfies the inclusion* (2).
2) K *has one of the following shapes:*
a) $\operatorname{bd} K$ *is a convex quadric,*
b′) $\dim(\operatorname{lin} K) = n-2$*, and* K *is the direct sum of* $\operatorname{lin} K$ *and a* 2*-dimensional line-free closed convex set which is either unbounded or bounded and strictly convex,*
c′) $\dim(\operatorname{lin} K) = n-3$*, and* K *is the direct sum of* $\operatorname{lin} K$ *and a* 3*-dimensional line-free closed strictly convex cone.* □

3. Auxiliary Lemmas

We say that a plane $P \subset \mathbb{R}^n$ (of certain dimension m) *properly supports* a closed convex set $M \subset \mathbb{R}^n$ provided P meets the relative boundary, $\operatorname{rbd} M$, of M and is disjoint from its relative interior, rint M. Furthermore, P is parallel to a line $l \subset \mathbb{R}^n$ if a translate of l lies in P.

Lemma 3.1. *A convex solid $K \subset \mathbb{R}^n$ has 1-dimensional non-recessional subspaces if and only if K is not a halfspace. If K is not a halfspace and $l \subset \mathbb{R}^n$ is a 1-dimensional non-recessional subspace for K, then $S_l(K) \neq \varnothing$ if and only if K is not a slab.*

Proof. The first assertion immediately follows from the fact that K is a closed halfspace of $\mathbb{R}^n$ if and only if $\operatorname{rec} K \cup (-\operatorname{rec} K) = \mathbb{R}^n$. Assume that K is not a halfspace and choose a 1-dimensional non-recessional for K subspace l. If K is a slab, then l is not parallel to the boundary hyperplanes of K, which gives $K+l = \mathbb{R}^n$ and

$$S_l(K) = \operatorname{bd} K \cap \operatorname{bd}(K+l) \subset \operatorname{bd}(K+l) = \operatorname{bd} \mathbb{R}^n = \varnothing.$$

Conversely, let K be neither a halfspace nor a slab. Then $\dim(\operatorname{lin} K) \leq n-2$ because halfspaces and slabs are the only convex solids in $\mathbb{R}^n$ with $(n-1)$-dimensional linearity spaces. Due to $l \cap \operatorname{rec} K = \{o\}$, there is a subspace $L \subset \mathbb{R}^n$ containing l and complementary to $\operatorname{lin} K$ (put $L = \mathbb{R}^n$ if $\operatorname{lin} K = \{o\}$). Then

$$\dim L = n - \dim(\operatorname{lin} K) \geq 2 \quad \text{and} \quad K = \operatorname{lin} K \oplus (K \cap L),$$

where $K \cap L$ is a line-free closed convex set. Obviously, L meets $\operatorname{int} K$. Choose a point $x \in (L \setminus l) \cap \operatorname{int} K$ and consider the 2-dimensional subspace $E = \operatorname{span}(\{x\} \cup l)$. Denote by l' the line through x which is parallel to l. Since $\operatorname{rec}(E \cap K) = E \cap \operatorname{rec} K$, the subspace l is non-recessional for $E \cap K$. Therefore $(E \cap K) \cap l'$ is a line segment, and at least one of the closed halfplanes of E determined by l' meets $E \cap K$ along a bounded set (otherwise $E \cap K$ would be a slab of E between a pair of parallel lines, which is impossible because $E \cap K$ is line-free as a section of $K \cap L$). Continuously translating l' within this halfplane, we find a line $l'' \subset E$ that is parallel to l' and properly supports $E \cap K$. Hence l'' properly supports $K \cap L$. The equalities

$$\operatorname{bd} K = \operatorname{lin} K \oplus \operatorname{rbd}(K \cap L) \quad \text{and} \quad \operatorname{int} K = \operatorname{lin} K \oplus \operatorname{rint}(K \cap L)$$

imply that l'' supports K, which gives $S_l(K) \neq \varnothing$. □

Given a convex solid $K \subset \mathbb{R}^n$, a hypersubspace $L \subset \mathbb{R}^n$ (that is, a subspace of dimension $n-1$) is called *ordinary* if there is a translate of L that supports K and no translate of L supports K along an $(n-1)$-dimensional set. From the standard facts of Convex Analysis it follows that $K \subset \mathbb{R}^n$ has ordinary hypersubspaces provided it is neither a halfspace nor a slab. Since any convex solid has at most countably many $(n-1)$-dimensional faces, the union of all ordinary for K hypersubspaces is dense in $\mathbb{R}^n \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K))$ provided K is neither a halfspace nor a slab.

Lemma 3.2. *If $K \subset \mathbb{R}^n$ is a convex solid, which is neither a halfspace nor a slab, and L is an ordinary for K hypersubspace, then the union of all 1-dimensional sharp for K subspaces $l \subset L$ is dense in $L \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K))$.*

Proof. Translating K on a suitable vector, we may suppose that L supports K such that $o \in K \cap L$. By the assumption, $\dim(K \cap L) \leq n-2$. The solid K

can be expressed as the union of an increasing sequence of convex bodies $K \cap B_r$, $r = 1, 2, \ldots$, where B_r is the closed ball of radius r centered at o. Clearly, L is ordinary for each body $K \cap B_r$. As proved in [10], the set E_r of unit vectors in L which span all 1-dimensional non-sharp for $K \cap B_r$ subspaces has zero $(n-2)$-dimensional Hausdorff measure. Since $\operatorname{rec} K \cup (-\operatorname{rec} K)$ is the union of two closed convex cones with common apex o, each set $E_r \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K))$, $r \geq 1$, also has zero $(n-2)$-dimensional Hausdorff measure. Put

$$E = (E_1 \cup E_2 \cup \ldots) \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K)), \qquad F = L \cap S^{n-1},$$

where S^{n-1} is the unit sphere of $\mathbb{R}^n$. Then E is the set of unit vectors in $F \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K))$ which span all 1-dimensional non-recessional and non-sharp for K subspaces. By the above, E has zero $(n-2)$-dimensional Hausdorff measure. Hence the complement of E in $F \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K))$, which is the set of unit vectors that span all 1-dimensional sharp for K spaces, is dense in $F \setminus (\operatorname{rec} K \cup (-\operatorname{rec} K))$. $\square$

The following lemma describes the case when a 1-dimensional subspace l and a hyperplane H that satisfy one of the conditions (2) and (10) are parallel.

Lemma 3.3. *Given a convex solid $K \subset \mathbb{R}^n$ which is neither a halfspace nor a slab, the following conditions are equivalent.*

1) *There is a 1-dimensional sharp for K subspace $l \subset \mathbb{R}^n$ and a hyperplane $H \subset \mathbb{R}^n$ such that l and H are parallel and satisfy the equality* (10).

2) *There is a 1-dimensional non-recessional for K subspace $l \subset \mathbb{R}^n$ and a hyperplane $H \subset \mathbb{R}^n$ such that l and H are parallel and satisfy the inclusion* (2).

3) $\dim(\operatorname{lin} K) = n - 2$, *and K is the direct sum of $\operatorname{lin} K$ and a 2-dimensional unbounded line-free closed convex set.*

Proof. Since *1)* trivially implies *2)*, we proceed with *2)* $\Rightarrow$ *3)*. By Lemma 3.1, $S_l(K) \neq \varnothing$ and $\dim(\operatorname{lin} K) \leq n - 2$. Write $K = \operatorname{lin} K \oplus (K \cap L)$, where L is a subspace containing l and complementary to $\operatorname{lin} K$. Choose a proper translate H_1 of H which meets $\operatorname{int} K$. We state that $K \cap H_1$ is a slab of H_1. Indeed, we observe that l is non-recessional for $K \cap H_1$ because of

$$l \cap \operatorname{rec}(K \cap H_1) \subset l \cap \operatorname{rec} K = \{o\},$$

which implies that $K \cap H_1$ is not a halfplane of H_1. Since H_1 contains no line which is parallel to l and supports K (due to the assumption, all such lines are in H), Lemma 3.1 (with H_1 instead of $\mathbb{R}^n$) gives that $K \cap H_1$ must be a slab of H_1. Therefore $K \cap H_1$ contains an $(n-2)$-dimensional plane. Hence $\dim(\operatorname{lin}(K \cap H_1)) = n - 2$. This equality and $\dim(\operatorname{lin} K) \leq n - 2$ imply that $\dim(\operatorname{lin} K) = n - 2$, and whence $\dim L = n - \dim(\operatorname{lin} K) = 2$. Furthermore, a suitable translate of $\operatorname{lin} K$ lies in H.

Repeating the argument above for H (instead of H_1), we obtain that H supports K, since otherwise $K \cap H$ would be a slab of H and each line in H which is parallel to l would meet $\operatorname{int} K$, contradicting the assumption $\varnothing \neq S_l(K) \subset H$. Clearly, $H = \operatorname{lin} K \oplus (H \cap L)$, and the line $H \cap L$ (which is a translate of l) properly supports $K \cap L$. We state that $K \cap L$ is unbounded. Indeed, if $K \cap L$ were bounded, $L \setminus H$ would contain another translate of l properly supporting $K \cap L$ (and thus supporting K), in contradiction with $S_l(K) \subset H$.

3) $\Rightarrow$ *1*). Choose any 1-dimensional sharp for K subspace l. As above, let $K = \operatorname{lin} K \oplus (K \cap L)$, where L is a subspace containing l and complementary to $\operatorname{lin} K$. We have $\dim L = n - \dim(\operatorname{lin} K) = 2$. Because l is sharp for the line-free unbounded set $K \cap L$, there is a translate $l' \subset L$ of l that properly supports $K \cap L$ along a singleton (see the proof of Lemma 3.1). Furthermore, $l' \cap \operatorname{rbd}(K \cap L + l) = S_l(K \cap L)$, since otherwise $K \cap L$ would be bounded and poses another support line which is parallel to l. Put $H = \operatorname{lin} K \oplus l'$. Then H is a hyperplane supporting K such that

$$\begin{aligned} H \cap \operatorname{bd}(K + l) &= (\operatorname{lin} K \oplus l') \cap (\operatorname{lin} K \oplus \operatorname{rbd}(K \cap L + l)) \\ &= \operatorname{lin} K \oplus (l' \cap \operatorname{rbd}(K \cap L + l)) \\ &= \operatorname{lin} K \oplus S_l(K \cap L) = S_l(K). \end{aligned}$$

□

The next lemma describes some relations between the conditions (2), (3), and (10); obviously, (10) $\Rightarrow$ (2) and (10) $\Rightarrow$ (3).

Lemma 3.4. *Let $K \subset \mathbb{R}^n$ be a convex solid which is neither a halfspace nor a slab, $l \subset \mathbb{R}^n$ a 1-dimensional non-recessional for K subspace, and $H \subset \mathbb{R}^n$ a hyperplane. The following assertions hold.*

1) *If H intersects $\operatorname{bd}(K + l)$ such that (3) holds, then l and H are not parallel.*

2) *If l and H are not parallel, then (2) $\Leftrightarrow$ (10). Furthermore, if (2) holds, then l is sharp for K.*

3) *If l and H are not parallel and l is sharp for K, then (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (10).*

Proof. *1*). Let H intersect $\operatorname{bd}(K + l)$ such that (3) holds. Assume for a moment that H is parallel to l. Then $H \cap \operatorname{bd}(K + l)$ and, subsequently, $S_l(K)$ contains a line l' which is a translate of l. From (1) we obtain $l' \subset \operatorname{bd} K$. In this case, $l \subset \operatorname{lin} K$, contradicting the assumption on l. Hence H cannot be parallel to l.

2). Since (10) $\Rightarrow$ (2) trivially holds, it remains to show that (2) $\Rightarrow$ (10). Because of

$$S_l(K) = S_l(K) \cap \operatorname{bd}(K + l) \subset H \cap \operatorname{bd}(K + l),$$

one has to prove the opposite inclusion. Choose a point $x \in H \cap \operatorname{bd}(K + l)$. We state that the line $x + l$ intersects $\operatorname{bd} K$. Indeed, suppose for a moment that $x + l$ and $\operatorname{bd} K$ are disjoint. From $x + l \subset \operatorname{bd}(K + l)$ it follows that $x + l$ is asymptotic for K. In this case, l is recessional for K, contradicting the assumption on l. Hence $\operatorname{bd} K \cap (x + l) \neq \varnothing$. If $z \in \operatorname{bd} K \cap (x + l)$, then

$$z \in \operatorname{bd} K \cap \operatorname{bd}(K + l) = S_l(K) \subset H.$$

Hence $z \in H \cap (x + l) = \{x\}$, giving $x = z \in S_l(K)$. Therefore (10) holds. Furthermore, since any line l' that supports K and is parallel to l can be expressed as $l' = u + l$, where $u \in l' \cap H$, the argument above shows that $l' \cap \operatorname{bd} K$ is a singleton. Hence l is sharp for K.

3). By the facts proved above, it suffices to show that (3) $\Rightarrow$ (2). Let x be a point in $S_l(K)$. Then the line $x+l$ supports K and lies in $\mathrm{bd}\,(K+l)$. Since H and l are not parallel, $x+l$ intersects H at a single point u. Due to (3),

$$u \in H \cap (x+l) \subset H \cap \mathrm{bd}\,(K+l) \subset S_l(K).$$

Since l is sharp for K, we have $x = u \in H$, which implies (2). $\square$

Remark 3.5. We observe that (3) $\not\Rightarrow$ (2) and (3) $\not\Rightarrow$ (10) if l is not sharp for K. Indeed, if K is a square in the coordinate plane, whose base lies on the x-axis and l is the y-axis, then $S_l(K)$ is the union of two vertical sides of K, while the intersection of $\mathrm{bd}\,(K+l)$ with any non-vertical line H consists of two points.

Lemma 3.6. *Let $K \subset \mathbb{R}^n$ be a convex solid distinct from a halfspace or a slab such that* $\mathrm{bd}\,K$ *a convex quadric, and $l \subset \mathbb{R}^n$ a* 1*-dimensional non-recessional for K subspace. Then l is sharp for K and there is a hyperplane $H \subset \mathbb{R}^n$ which is not parallel to l and satisfies the equality* (10).

Proof. Let $\mathrm{bd}\,K$ be as described by (4). By Lemma 3.1, there is a line l' which is parallel to l and supports $\mathrm{bd}\,K$. Write $l' = \{u + tv \in \mathbb{R}^n : t \in \mathbb{R}\}$, where $u \in l' \cap \mathrm{bd}\,K$ and v is a unit vector of l. Equivalently, $x = (\xi_1, \dots, \xi_n)$ belongs to l' if and only if

$$\xi_i = u_i + tv_i, \quad t \in \mathbb{R}, \ i = 1, \dots, n,$$

where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. To determine the values of t for which $x \in l' \cap \mathrm{bd}\,K$, we substitute $\xi_i = u_i + tv_i$ into (4). This results in a quadratic equation

$$A(v)\,t^2 + 2B(u,v)\,t + C(u) = 0, \tag{11}$$

where

$$A(v) = \sum_{i,k=1}^{n} a_{ik} v_i v_k, \qquad B(u,v) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} u_k + b_i \right) v_i,$$
$$C(u) = F(u_1, \dots, u_n).$$

Because (11) has at most two solutions, which correspond to the points of $l' \cap \mathrm{bd}\,K$, the convex set $l' \cap \mathrm{bd}\,K$ must be a singleton, $\{u\}$. Hence l is sharp for K.

Since l is non-recessional for K, there is a translate l_1 of l that meets $\mathrm{int}\,K$ such that $l_1 \cap K$ is a line segment $[a, c]$. Writing $l_1 = \{u_1 + tv \in \mathbb{R}^n : t \in \mathbb{R}\}$, where u_1 is any given point of l_1, we obtain that a and c correspond to two distinct solutions for the equation

$$A(v)\,t^2 + 2B(u_1, v)\,t + C(u_1) = 0,$$

which gives $A(v) \neq 0$. Because (11) has precisely one solution, $t = 0$, we have

$$B(u,v) \equiv \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} u_k + b_i \right) v_i = 0.$$

Equivalently,

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ik} v_i \right) u_k + \sum_{i=1}^{n} b_i v_i = 0. \tag{12}$$

Interpreted as an equation in $u_1, \dots, u_n$, (12) describes a hyperplane, H, because at least one of the scalars

$$c_k = \sum_{i=1}^{n} a_{ik} v_i, \quad k = 1, \dots, n,$$

is not zero. Indeed, assuming $c_1 = \cdots = c_n = 0$, we would obtain

$$A(v) = c_1 v_1 + \cdots + c_n v_n = 0.$$

If we fix v and vary u as the point of contact of a variable line l' supporting K, then (12) shows that u belongs to H. Hence $S_l(K) \subset H$. By Lemma 3.4, l and H are not parallel and the equality (10) holds. $\square$

4. Proof of Theorem 2.1

4) $\Rightarrow$ *1)*. Let l be a 1-dimensional subspace of $\mathbb{R}^n$. Write $K = \operatorname{lin} K \oplus (K \cap L)$, where L is a subspace of $\mathbb{R}^n$ complementary to $\operatorname{lin} K$ (put $L = \mathbb{R}^n$ if $\operatorname{lin} K = \{o\}$). In what follows, we consider separately the following cases:

$$l \subset \operatorname{lin} K, \qquad l \subset (\operatorname{rec} K \setminus \operatorname{lin} K) \cup \{o\}, \qquad l \cap \operatorname{rec} K = \{o\}.$$

(i). Let $l \subset \operatorname{lin} K$. Then $x + l \subset \operatorname{bd} K$ for any point $x \in \operatorname{bd} K$, which shows that $S_l(K) = \operatorname{bd} K = \operatorname{bd}(K + l)$. If G is a subspace complementary to l within $\operatorname{lin} K$ (put $G = \{o\}$ if $l = \operatorname{lin} K$), then $H = G \oplus L$ is a hyperplane which intersects $K + l$ and

$$H \cap \operatorname{bd}(K + l) \subset \operatorname{bd}(K + l) = S_l(K).$$

(ii). Let $l \subset (\operatorname{rec} K \setminus \operatorname{lin} K) \cup \{o\}$. (a). If $\operatorname{bd} K$ is a convex quadric, then the inclusion $l \subset (\operatorname{rec} K \setminus \operatorname{lin} K) \cup \{o\}$ implies that $\operatorname{bd} K$ has one of the types (6), (8), (9). In this case, $\operatorname{int}(K + l)$ is either the whole space, an open halfspace, or an open slab. Choosing a hyperplane $H \subset \operatorname{int}(K + l)$, we obtain

$$H \cap \operatorname{bd}(K + l) = \varnothing \subset S_l(K). \tag{13}$$

(b). If $\dim L = 2$ and $K \cap L$ is a line-free closed convex set, then $\operatorname{rint}(K \cap L) + l$ is either L, an open halfplane of L, or an open slab of L. If G is a line in $\operatorname{rint}(K \cap L) + l$, then the hyperplane $H = \operatorname{lin} K \oplus G$ satisfies (13).

(c). If $\dim L = 3$ and $K \cap L$ is a 3-dimensional line-free closed convex cone, then $\operatorname{rint}(K \cap L) + l$ is an open halfspace of L. If G is a 2-dimensional plane in $\operatorname{rint}(K \cap L) + l$, then the hyperplane $H = \operatorname{lin} K \oplus G$ satisfies (13).

(iii). Let $l \cap \operatorname{rec} K = \{o\}$. If $\operatorname{bd} K$ is a convex quadric, then (3) follows from Lemma 3.6. Assume that K has one of the shapes (b), (c). Then $2 \le \dim L \le 3$. From the standard properties of 2-dimensional convex sets (respectively, 3-dimensional closed convex cones) we conclude the existence of a line $G \subset L$ (respectively, of a 2-dimensional plane $G \subset L$) which is not parallel to l and satisfies the inclusion $G \cap \operatorname{rbd}(K \cap L + l) \subset S_l(K \cap L)$. Then $H = \operatorname{lin} K \oplus G$ is a hyperplane such that

$$\begin{aligned} H \cap \operatorname{bd}(K + l) &= (\operatorname{lin} K \oplus G) \cap (\operatorname{lin} K \oplus \operatorname{rbd}(K \cap L + l)) \\ &= \operatorname{lin} K \oplus (G \cap \operatorname{rbd}(K \cap L + l)) \\ &\subset \operatorname{lin} K \oplus S_l(K \cap L) = S_l(K). \end{aligned}$$

Since *1)* $\Rightarrow$ *2)* trivially holds and *2)* $\Rightarrow$ *3)* due to Lemma 3.4, it remains to show that *3)* $\Rightarrow$ *4)*. This part is organized by induction on $n \geq 3$. The case $n = 3$ is considered in Proposition 4.2 below, which involves the following result of Alexandrov [1].

Lemma 4.1 ([1]). *Let $K \subset \mathbb{R}^3$ be a convex solid and T a non-planar, bounded, open, and simply connected piece of* $\operatorname{bd} K$. *If for any shadow-boundary $S_l(K)$ of K that meets T there is a plane H such that $S_l(K) \cap T \subset H$, then T is a piece of a line-free convex quadric or a piece of the boundary of a strictly convex cone.* □

We note that Lemma 4.1 deals with shadow-boundaries corresponding to all (possibly, non-sharp, or even recessional for K) 1-dimensional subspaces l, and the plane H is allowed to be parallel to l. Furthermore, Lemma 4.1 refines Alexandrov's original conclusion "T is a piece of a convex quadric or a piece of the boundary of a convex cone."

Proposition 4.2. *If a convex solid $K \subset \mathbb{R}^3$ satisfies condition 3) of Theorem* 2.1*, then it has one of the following shapes:*

a) $\operatorname{bd} K$ *is a convex quadric,*
b) K *is a cylinder based on a* 2*-dimensional line-free closed convex set,*
c) K *is a line-free closed convex cone.*

Proof. If K contains a line, then K is a cylinder based on a 2-dimensional closed convex set M. If M contains a line, then K is either a halfplane or a slab between two parallel planes, implying that $\operatorname{bd} K$ is a degenerate convex quadric. If M is line-free, then K has the shape *b)*.

Assuming that K is line-free, we divide the proof of Proposition 4.2 into a sequence of assertions.

Assertion 4.3. *If* $\operatorname{bd} K$ *contains an open strictly convex piece S, then the whole surface* $\operatorname{bd} K$ *is a strictly convex quadric.*

Proof. Choose a point $x \in S$ and a scalar $r > 0$ such that the set $T = U_r(x) \cap \operatorname{bd} K$ lies in S, where $U_r(x)$ is an open ball of radius $r > 0$ centered at x. Translating K on $-x$, we may assume that $x = o$. Denote by P a 2-dimensional subspace which supports K at o. Then $P \cap K = \{o\}$ because T is strictly convex. By Lemma 3.2, the family $\mathcal{F}$ of 1-dimensional sharp for K subspaces is dense in the family of all 1-dimensional subspaces of P. Due to condition *3)*, each sharp shadow-boundary $S_l(K)$, $l \in \mathcal{F}$, lies in a plane $H(l)$. Because T is strictly convex, the family of planar arcs $S_l(K) \cap T$, $l \in \mathcal{F}$, is dense in the family of all arcs of the form $S_l(K) \cap T$, $l \subset P$. Therefore each arc $S_l(K) \cap T$, $l \subset P$, lies in a plane. Since this argument holds for any point $z \in T$, Lemma 4.1 implies that T is a piece of a strictly convex quadric Q.

Because Q is strictly convex, it is either an ellipsoid, an elliptic paraboloid, or a sheet of hyperboloid on two sheets. In either case, there is a line m through o which is the axis of affine symmetry of Q. Applying a suitable linear transformation that keeps P fixed, we may assume that m is orthogonal to P (clearly, the image of K under this transformation satisfies condition *3)*). Therefore, m is the axis of symmetry of Q and each plane $H(l)$ contains m. Hence the family of planes $H(l)$, $l \in \mathcal{F}$, is dense in the family of all planes containing m.

Next, we state that K is strictly convex. For contradiction, assume for a moment the existence of a line segment $[u, v] \subset K$. Then $[u, v]$ should lie in a plane through m. Suppose that $[u, v]$ and m do not lie in a common plane. By the above, there is a 1-dimensional sharp for K subspace $l \in \mathcal{F}$ such that the respective plane $H(l)$ meets the open segment (u, v). This gives the inclusion $[u, v] \subset S_l(K)$, in contradiction with the condition $S_l(K) \subset H(l)$.

Denote by M the plane which contains $m \cup [u, v]$. Choosing a point $z \in T \setminus M$ and repeating the consideration above for z instead of x, we conclude that $[u, v]$ should lie in a common plane with the axis of affine symmetry of Q that contains z. Since this is impossible, we obtain a contradiction with the assumption $[u, v] \subset K$. Hence K is strictly convex.

Finally, cover $\operatorname{bd} K$ with countably many pieces of the form $T = U_r(x) \cap \operatorname{bd} K$. Since any two overlapping pieces of strictly convex quadrics belong to the same strictly convex quadric, the whole surface $\operatorname{bd} K$ is a strictly convex quadric. □

Our further goal is to show that K is a convex cone provided $\operatorname{bd} K$ is not a convex quadric. Let us recall that a subset F of $\operatorname{bd} K$ is an *exposed face* of K provided $F = K \cap P$ for a suitable plane P that supports K.

Assertion 4.4. *Any 2-dimensional exposed face of K is a convex cone.*

Proof. Assume, for contradiction, that K has a 2-dimensional exposed face F which is distinct from a cone. Denote by P the plane containing F. Translating K on a suitable vector we may assume that P is a subspace. Choose a unit vector $u \in P$ such that the 1-dimensional subspace $l(u)$ spanned by u is sharp for F (u exists because the family of line segments in $\operatorname{rbd} F$ is at most countable). Then $\operatorname{rbd} F$ can be expressed as the union of two convex arcs γ and γ' such that $\gamma \cap \gamma' = \{p\}$ if F is unbounded (respectively, $\gamma \cap \gamma' = \{p, q\}$ if F is bounded) where $\{p\}$ (respectively, $\{p, q\}$) is the set of contact of F with the line(s) parallel to $l(u)$ and supporting F.

Because F is not a cone, at least one of the arcs γ, γ', say, γ does not belong to a halfline. Denote by Q a closed slab of P which is bounded by a pair of lines l_1, l_2 both parallel to $l(u)$ and intersecting $\operatorname{rint} F$ such that $\gamma \cap Q$ is not a line segment. We may assume that namely γ is the part of $\operatorname{rbd} F$ illuminated in the direction u (that is, the halfline $\{x + \lambda u : \lambda \geq 0\}$ intersects $\operatorname{rint} F$ for any point $x \in \gamma \cap Q$); otherwise replace u with $-u$.

Choose a plane N which is parallel to $l(u)$ but not to P and supports K such that $K \cap N$ is a bounded set with $\dim(K \cap N) \le 1$ (if, additionally, there is another plane N' which is parallel to N and supports K, then we also require that $\dim(K \cap N') \le 1$). As above, N exists because the family of 2-dimensional faces of K parallel to $l(u)$ is at most countable and K is line-free. Furthermore, choose a unit vector $v \in \mathbb{R}^3$ which is parallel to N but not to $l(u)$ such that P separates v from K. Let N_1 and N_2 be the planes through l_1 and l_2, respectively, both parallel to N. Denote by l'_j the line in N_j which is parallel to l_j and supports the 2-dimensional compact convex set $K \cap N_j$ from the opposite to l_j side, $j = 1, 2$. Also, let V be the closed slab of $\mathbb{R}^3$ bounded by N_1 and N_2.

By Lemma 3.2 and the choice of N, the open interval $(0, \infty) \subset \mathbb{R}$ contains a dense subset Λ such that every 1-dimensional subspace $l(u + \varepsilon v)$, $\varepsilon \in \Lambda$, is sharp for K. Choose in Λ a sequence $\varepsilon_1, \varepsilon_2, \ldots$ which converges to 0. Each set $S_{l(u+\varepsilon_i v)}(K) \cap V$ is a disjoint union of two curves; one of these curves tends to the non-line curve $\gamma \cap Q$ as $i \to \infty$, while the end-points of the second curve approach the sets $K \cap l'_1$ and $K \cap l'_2$, respectively. This argument shows that $S_{l(u+\varepsilon_i v)}(K) \cap V$ cannot lie in a plane for a sufficiently large i, in contradiction with condition *3*) of the theorem. Hence F must be a convex cone. □

Assertion 4.5. *If P_1 and P_2 are distinct parallel planes both supporting K such that $K \cap P_1$ is a cone, then $K \cap P_2$ is a translate of $K \cap P_1$.*

Proof. Let z_1 be the apex of $K \cap P_1$. Then $K \cap P_1 - z_1$ is a convex cone with apex o, which lies in $\operatorname{rec} K$. Choose a point $u \in K \cap P_2$. From $u + \operatorname{rec} K \subset K$ it follows that

$$(u - z_1) + K \cap P_1 \subset K \cap P_2. \tag{14}$$

First, assume that $\dim(K \cap P_1) = 2$. Then (14) shows that $\dim(K \cap P_2) = 2$. By Assertion 4.4, $K \cap P_2$ is a convex cone. Denote by z_2 the apex of $K \cap P_2$. By the argument above,

$$(z_2 - z_1) + K \cap P_1 \subset K \cap P_2 \quad \text{and} \quad (z_1 - z_2) + K \cap P_2 \subset K \cap P_1.$$

Hence $(z_2 - z_1) + K \cap P_1 = K \cap P_2$.

Now, assume that $\dim(K \cap P_1) = 1$. Then $K \cap P_1$ is a halfline, h_1, with endpoint z_1. If $K \cap P_2$ were 2-dimensional, then, by the facts proved above, $K \cap P_1$ would be a translate of $K \cap P_2$, which is impossible. Hence $\dim(K \cap P_2) = 1$. From (14) we obtain that $K \cap P_2$ is a translate of h_1. □

Assertion 4.6. *Any 1-dimensional exposed face of K is a halfline.*

Proof. Assume for a moment that K has a 1-dimensional exposed face F which is not a halfline. Since K is line-free, F is a line segment, $[x, z]$. We may suppose that $x = o$, so that the line l through o and z is a subspace. Choose a 2-dimensional subspace L with the property $L \cap K = [o, z]$ and a 2-dimensional subspace M through l that meets $\operatorname{int} K$. If there is another plane L' which is parallel to L and supports K, then, due to Assertions 4.4 and 4.5 above, $K \cap L'$ should be a point or a line segment. Denote by $\mathcal{F}$ the family of 1-dimensional subspaces from L which are sharp for K. From Lemma 3.2 it follows that $\mathcal{F}$ is dense in the family of all 1-dimensional subspaces of L.

Choose a plane N which is parallel to L and intersects $\operatorname{int} K$. Due to $L \cap K = [o, z]$, the section $K \cap N$ is bounded (if $K \cap N$ were unbounded and whence contained a halfline h, then $K \cap L$ would contain a translate of h). Choose a subspace $l' \in \mathcal{F} \setminus \{l\}$ so close to l that $S_{l'}(K)$ meets $K \cap N$ at some points x_1 and x_2 which are strictly separated by M. By condition *3*), there is a plane H' that intersects $K + l'$ and satisfies the equality $H' \cap \operatorname{bd}(K + l') = S_{l'}(K)$. Due to the construction above, H' should contain the set $X = [o, z] \cup \{x_1, x_2\}$, which is impossible since X is not planar. The obtained contradiction shows that F is a halfline. $\square$

Assertions 4.3, 4.4, and 4.6 imply the following corollary.

Assertion 4.7. $\operatorname{bd} K = \operatorname{cl} C$, *where C is the union of all exposed halflines and exposed cones of K.* $\square$

Assertion 4.8. *K is the closed convex hull of its exposed halflines, and any two such halflines lie in a common plane.*

Proof. The first part of Assertion 4.8 immediately follows from Klee [9, Assertion 3.6] and Assertion 4.7 above. Assume, for contradiction, the existence of exposed halflines h_1 and h_2 of K whose union $h_1 \cup h_2$ does not lie in a plane. Denote by z_1 and z_2 the endpoints of h_1 and h_2, respectively. Choose planes P_1 and P_2 such that $K \cap P_1 = h_1$ and $K \cap P_2 = h_2$.

We observe that P_1 and P_2 are not parallel, since otherwise h_1 should be a translate of h_2 (see Assertion 4.5), implying that $h_1 \cup h_2$ lies in a plane. Therefore $P_1 \cap P_2$ is a line, m. Next, we state that none of the halflines h_1 and h_2 is parallel to m. Indeed, assuming that h_1 is parallel to m, we would obtain that the halfline $h'_1 = (z_2 - z_1) + h_1$ lies in $K \cap P_2$, in contradiction with the assumption $K \cap P_2 = h_2$. Denote by g_i the line through z_i which is parallel to m, and by G_i the closed halfplane of P_i that contains h_i and is bounded by g_i. Let N_1 and N_2 be parallel planes through g_1 and g_2, respectively, such that both N_1 and N_2 meet $\operatorname{int} K$.

Denote by l the 1-dimensional subspace parallel to m. From the argument above it follows that $S_l(K) \cap G_i = h_i$, $i = 1, 2$, and $S_l(K) \setminus (h_1 \cup h_2)$ lies in the open slab between N_1 and N_2. We claim that $S_l(K)$ is sharp. Indeed, assume for a moment that $S_l(K)$ contains a line segment $[x, y]$ which is parallel to l. Clearly, $[x, y]$ lies between N_1 and N_2. Let F be an exposed face of K that contains $[x, y]$. Due to Assertions 4.4 and 4.5 above, F is a convex cone. We observe that $F \cap (\operatorname{rint} G_1 \cup \operatorname{rint} G_2) = \varnothing$. Indeed, if F contained a point $u \in \operatorname{rint} G_i$, then the triangle $G_i \cap \operatorname{Conv}\{x, u, y\}$ would lie in $K \cap P_i$, in contradiction with $K \cap P_i = h_i$. Hence F lies in the closed slab between N_1 and N_2. Since a 2-dimensional cone containing the segment $[x, y]$ and lying in $\operatorname{bd} K$ cannot be embedded between N_1 and N_2, the face F is not 2-dimensional. So, F is a halfline parallel to l. Then $\operatorname{rec} K$ contains a halfline h with apex o which is parallel to l. In this case, the halflines $h'_1 = z_1 + h$ and $h'_2 = z_2 + h$ satisfy the inclusions $h'_1 \subset K \cap P_1$ and $h'_2 \subset K \cap P_2$. Hence $h'_1 = h_1$ and $h'_2 = h_2$, implying that h_1 and h_2 are parallel. The last is in contradiction with the choice of h_1 and h_2. Therefore l is sharp for K.

By condition *3)*, there is a plane H satisfying the equality (10). Hence the set $h_1 \cup h_2 = S_l(K) \cap (P_1 \cup P_2)$ lies in H, contrary to our assumption. □

Our final step in the proof of Proposition 4.2 (see Assertion 4.10) uses the following elementary statement.

Assertion 4.9. *A family $\mathcal{R}$ of lines in $\mathbb{R}^3$ has the property that any two lines from $\mathcal{R}$ belong to a plane if and only if either of the following assertions holds: (i) all lines from $\mathcal{R}$ lie in the same plane, (ii) any two lines from $\mathcal{R}$ are parallel, (iii) all lines from $\mathcal{R}$ have a common point.* □

Assertion 4.10. *K is a convex cone.*

Proof. Denote by $\mathcal{H}$ the family of exposed halflines of K and by $\mathcal{R}$ the family of lines containing the halflines from $\mathcal{H}$. Let H and R be the unions of the halflines from $\mathcal{H}$ and the lines from $\mathcal{R}$, respectively. Due to Assertion 4.8, $\mathcal{R}$ satisfies one of conditions (i)–(iii) from Assertion 4.9 and

$$K = \mathrm{cl}\,(\mathrm{Conv}\, H). \tag{15}$$

We claim that $\mathcal{R}$ satisfies condition (iii). Indeed, assuming that (i) holds and denoting by P the plane containing all lines from $\mathcal{R}$, we would obtain from (15) the inclusion $K \subset P$, which is impossible. Assume for a moment that $\mathcal{R}$ satisfies condition (ii). Then (15) implies that K lies within the convex solid cylinder $D = \mathrm{cl}\,(\mathrm{Conv}\, R)$. Since K is line-free, any two halflines from $\mathcal{H}$ are translates of each other. Choose a point $z \in \mathrm{int}\, D \setminus K$ and denote by h the halfline with apex z which is a translate of any given halfline from $\mathcal{H}$. Then the point of intersection of h and $\mathrm{bd}\, K$ does not belong to a halfline which lies in $\mathrm{bd}\, K$, contradicting Assertion 4.7.

So, $\mathcal{R}$ satisfies condition (iii) from Assertion 4.9. Denote by p the common point of all lines from $\mathcal{R}$. We claim that no halfline from $\mathcal{H}$ contains p in its relative interior. Indeed, assume for a moment that p is a relatively interior point of a halfline $h_1 \in \mathcal{H}$. Choose a halfline $h_2 \in \mathcal{H} \setminus \{h_1\}$ and denote by P_2 a plane such that $K \cap P_2 = h_2$. From $p \in \mathrm{rint}\, h_1 \subset K$ we conclude that P_2 should contain h_1. Hence $h_1 \subset K \cap P_2 = h_2$. Then $h_1 = h_2$ because h_1 is an exposed halfline. The latter is in contradiction with the choice of h_2.

Next, we state that $p \in K$. Indeed, assuming the opposite, choose a halfline h with apex p which intersects $\mathrm{int}\, K$. If u is the point of intersection of h and $\mathrm{bd}\, K$, then u does not belong to any halfline from $\mathcal{H}$, in contradiction with Assertion 4.7. Hence p is a common endpoint of all halflines from $\mathcal{H}$, which implies that K is a cone with apex p. □

Let $n > 3$. We continue the proof of *3)* $\Rightarrow$ *4)* assuming that it holds for all $m \leq n-1$, where $n \geq 4$. Let K be a convex solid in $\mathbb{R}^n$ that satisfies condition *3)*.

First, we eliminate the trivial cases when either every 1-dimensional subspace of $\mathbb{R}^n$ is recessional for K or every shadow-boundary $S_l(K)$ is empty. This occurs precisely when K is either a halfspace or a slab (see Lemma 3.1), with $\mathrm{bd}\, K$ being a degenerate convex quadric. So, we assume that K is neither a halfspace nor a slab. This gives $\dim(\mathrm{lin}\, K) \le n-2$. Since the union of all ordinary hyperspaces of K is dense in $\mathbb{R}^n \setminus (\mathrm{rec}\, K \cup (-\mathrm{rec}\, K))$, Lemma 3.2 implies that the union of all 1-dimensional sharp for K subspaces also is dense in $\mathbb{R}^n \setminus (\mathrm{rec}\, K \cup (-\mathrm{rec}\, K))$.

Next, consider the case $\mathrm{lin}\, K \ne \{o\}$. Choose a subspace $L \subset \mathbb{R}^n$ which is complementary to $\mathrm{lin}\, K$ and write $K = \mathrm{lin}\, K \oplus M$, where $M = K \cap L$ is a line-free closed convex set. Clearly, $\dim L = n - \dim(\mathrm{lin}\, K) \ge 2$. Let $l \subset L$ be a 1-dimensional subspace which is sharp for M. From $K = \mathrm{lin}\, K \oplus M$ it follows that l is also sharp for K. By condition *3*), there is a hyperplane H intersecting $K+l$ and satisfying the equality (10). From $S_l(K) = \mathrm{lin}\, K \oplus S_l(M)$ it follows that H contains a plane which is a translate of $\mathrm{lin}\, K$. Therefore, $L \not\subset H$ (since otherwise $\mathbb{R}^n = \mathrm{lin}\, K \oplus L \subset H$). Hence $G = H \cap L$ is a plane of dimension $\dim L - 1$. Furthermore, $G \cap \mathrm{rbd}\,(M+l) = S_l(M)$. Indeed, assuming the existence of a point

$$x \in [(G \cap \mathrm{rbd}\,(M+l)) \setminus S_l(M)] \cup [S_l(M) \setminus (G \cap \mathrm{rbd}\,(M+l))],$$

we would obtain

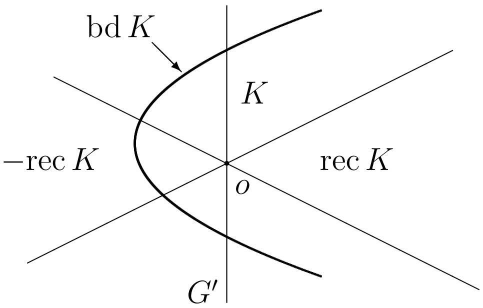
$$x \in [(H \cap \mathrm{bd}\,(K+l)) \setminus S_l(K)] \cup [S_l(K) \setminus (H \cap \mathrm{bd}\,(K+l))],$$

in contradiction with (10). Hence M satisfies condition *3*) within L. Since $\mathrm{lin}\, M = \{o\}$, the inductive assumption implies that M is either a 2-dimensional line-free closed convex set (if $m=2$), or a 3-dimensional line-free closed convex cone (if $m=3$), or $\mathrm{rbd}\, M$ is a line-free convex quadric (if $m \ge 3$). Therefore $K = \mathrm{lin}\, K \oplus M$ has one of the shapes *a*)–*c*).

Finally, consider the case $\mathrm{lin}\, K = \{o\}$ (so that K is line-free). Translating K on a suitable vector, we may suppose that $o \in \mathrm{int}\, K$. Choose an ordinary hypersubspace G that lies in $\mathbb{R}^n \setminus (\mathrm{rec}\, K \cup (-\mathrm{rec}\, K)) \cup \{o\}$ and a 1-dimensional subspace $l \subset G$ which is sharp for K (see Lemma 3.2). By condition *3*), there is a hyperplane H intersecting $K+l$ and satisfying the equality (10). Then H meets $\mathrm{bd}\,(K+l)$ because K is line-free, and Lemma 3.4 implies that H is not parallel to l. Hence $H \cap G$ is an $(n-2)$-dimensional plane in G and

$$\begin{aligned}(G \cap H) \cap \mathrm{rbd}\,(G \cap K + l) &= (G \cap H) \cap \mathrm{bd}\,(K+l)\\ &= G \cap S_l(K) = S_l(G \cap K).\end{aligned}$$

Therefore the $(n-1)$-dimensional compact convex set $G \cap K$ satisfies condition *3*) within the hypersubspace G. By the inductive assumption, $G \cap \mathrm{bd}\, K$ is an $(n-1)$-dimensional ellipsoid. Since any hypersubspace $G' \subset \mathbb{R}^n \setminus (\mathrm{rec}\, K \cup (-\mathrm{rec}\, K)) \cup \{o\}$ can be expressed as the limit of a sequence of ordinary for K hypersubspaces $G_1, G_2, \ldots$ from $\mathbb{R}^n \setminus (\mathrm{rec}\, K \cup (-\mathrm{rec}\, K)) \cup \{o\}$, and since $G_i \cap \mathrm{bd}\, K$ tends to $G' \cap \mathrm{bd}\, K$ when $i \to \infty$, we conclude that $G' \cap \mathrm{bd}\, K$ is an $(n-1)$-dimensional ellipsoid. By Theorem 2 from [17], the set $\mathrm{bd}\, K \setminus (-\mathrm{rec}\, K)$ is a piece of a convex quadric (see the picture below).



Continuously translating K such that o tends to $\mathrm{bd}\, K$ within $-\mathrm{rec}\, K$, we conclude that the whole hypersurface $\mathrm{bd}\, K$ is a convex quadric. □

5. Proof of Corollary 2.4

1) $\Rightarrow$ *2)*. Let a 1-dimensional non-recessional for K subspace l and a hyperplane H satisfy condition *1)* of the corollary. If l and H are parallel, then Lemma 3.3 implies that $\dim(\mathrm{lin}\, K) = n-2$ and K is the direct sum of $\mathrm{lin}\, K$ and a 2-dimensional unbounded line-free closed convex set M. Suppose that l and H are not parallel for any choice of a 1-dimensional non-recessional for K subspace l. Then, according to Lemma 3.4, (2) $\Leftrightarrow$ (10); whence K has one of the shapes *a)*–*c)* from Theorem 2.1. It remains to show that K has one of the shapes *a)*, *b′)*, and *c′)*. Since case *a)* trivially holds and the proof of case *c′)* is similar to that of *b′)*, we will consider case *b′)* only.

b′). Let $K = \mathrm{lin}\, K \oplus M$, where M is a 2-dimensional line-free closed convex set. Denote by L the plane containing M. Assume for a moment that M is bounded and not strictly convex. Choose a line segment $[x, z] \subset \mathrm{rbd}\, M$. Denote by l the 1-dimensional subspace parallel to $[x, z]$ and by $l_1, l_2 \subset L$ the lines which are parallel to l and properly support M. Then the set $S_l(M) = (M \cap l_1) \cup (M \cap l_2)$ does not lie on a line. Since l and H are not parallel, the set $H \cap L$ is a line, which cannot contain $S_l(M)$. Therefore $S_l(K) = \mathrm{lin}\, K \oplus S_l(M)$ does not contain $S_l(K) = \mathrm{lin}\, K \oplus S_l(M)$, in contradiction with condition 1). Hence M must be strictly convex.

2) $\Rightarrow$ *1)*. Let l be a 1-dimensional non-recessional for K subspace. Condition *2)* of the corollary implies that l is sharp for K. Therefore Theorem 2.1 gives the existence of a hyperplane H such that (10) holds. In particular, $S_l(K) \subset H$.

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