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Duality of Non-Exposed Faces

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Given any polar pair of convex bodies we study its conjugate face maps and we characterize conjugate faces of non-exposed faces in terms of normal cones. The analysis is carried out using the positive hull operator which defines lattice isomorphisms linking three Galois connections. One of them assigns conjugate faces between the convex bodies. The second and third Galois connection is defined between the touching cones and the faces of each convex body separately. While the former is well-known, we introduce the latter in this article for any convex set in any finite dimension. We demonstrate our results about conjugate faces with planar convex bodies and planar self-dual convex bodies, for which we also include constructions.

Keywords: Convex body, polar, conjugate face, non-exposed point, singular face, dual, self-dual

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1. Non-exposed faces and dual convex bodies

Duality of faces of a dual pair of closed convex cones was studied in [2] with regard to the lattice of the inclusion ordering. This duality corresponds to the conjugate face map between faces of a polar pair of convex bodies. In this article we study the restriction of the conjugate face map to non-exposed faces. E.g. it will become clear that a face which is conjugate to a non-exposed face is singular (its normal cone has at least dimension two). We prove that such faces are fully characterized by a so-called *incomplete* normal cone.

Incomplete normal cones of planar convex bodies have a simple description by so-called *mixed* and *free corners*. Examples are given in Figure 2.1 and 2.2. The conjugate face map restricts to a surjective map from the non-exposed points of a planar convex body onto the mixed and free corners of the polar convex body

$$\{\textit{non-exposed points}\} \longrightarrow\!\!\!\!\!\rightarrow \{\text{mixed corners and free corners}\}\,. \qquad (1)$$

The idea underlying this article is to use (1), and its generalization in any dimension, to study non-exposed faces of a projection of the state space of the matrix algebra $\mathrm{Mat}(N,\mathbb{C})$. The polar convex body of a projection is an affine section of that state space, see §2.4 in [17]. Its singular points (with incomplete normal cone) may be studied by analyzing an associated determinantal variety, using techniques of

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algebraic geometry. Our interest in non-exposed points of projected state spaces lies in quantum information theory, they seem to cause discontinuities in certain information measures [8].

Planar projections of state spaces are studied in operator theory under the name of numerical range, see e.g. [5] and the references therein. The question when numerical range has non-exposed points was solved in [13] for $N = 3$. Recently, numerical range was studied in [6] from the point of view of convex algebraic geometry whose aim is to use techniques from algebraic geometry for studying convex semialgebraic sets. Important examples of such sets are spectrahedra which generalize the state space of $\mathrm{Mat}(N, \mathbb{C})$ and which are popular in optimization. Current questions in the field are concerned with convex duality and non-exposed faces, see e.g. [11, 14]. Our interest in self-dual convex bodies is influenced by the present discussion of self-duality in the axiomatic foundations of quantum theory [7, 10, 18].

This article is organized as follows. Constructions for dual convex bodies and a general construction for planar self-dual convex bodies are explained in §2. A Galois connection between touching cones and faces of a convex set is defined in §3. In §4 we study conjugate faces of any polar pair of convex bodies. We demonstrate our results in §5 with planar convex bodies. In particular we give a general construction for planar self-dual convex bodies without non-exposed points.

2. Constructions of dual convex bodies

We introduce constructions for dual and self-dual convex bodies (mainly in dimension two). They are used to generate examples to demonstrate non-exposed points and their relation to the singular points studied in §5.

In the n-dimensional Euclidean vector space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ we denote the norm of $u \in \mathbb{R}^n$ by $|u| := \sqrt{\langle u, u \rangle}$. In $\mathbb{R}^n$ we shall use the standard scalar product. The *polar* of a subset $C \subset \mathbb{R}^n$ is $C^\circ := \{u \in \mathbb{R}^n \mid \langle u, v \rangle \leq 1 \ \forall v \in C\}$ and the *dual* of C is $C^* := \{u \in \mathbb{R}^n \mid 1 + \langle u, v \rangle \geq 0 \ \forall v \in C\} = -C^\circ$. The subset $C \subset \mathbb{R}^n$ is *self-dual* if $C^* = C$. We denote the *interior* of C by $\mathrm{int}(C)$ and its boundary by $\partial(C) := C \setminus \mathrm{int}(C)$.

The first construction is Corollary 16.5.2 in [12]:

Construction 2.1. *For any family $\{C_i\}_{i \in I}$ of convex sets in $\mathbb{R}^n$ (I is an index set) we have*

$$(\textit{convex hull of } \{C_i \mid i \in I\})^* = \bigcap \{C_i^* \mid i \in I\}\,. \tag{2}$$

Example 2.2. The convex set $C \subset \mathbb{R}^2$ depicted in Figure 2.1 c) is the convex hull of the unit disk $D := \{u \in \mathbb{R}^2 \mid |u| \leq 1\}$ and of the point $\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$. We have $D^* = D$ (e.g. using (2)) and $\{\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)\}^* = \{(x, y) \in \mathbb{R}^2 \mid y \geq -\frac{1}{2}\}$. The dual $C^* = D^* \cap \{\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)\}^*$ is depicted in Figure 2.1 d).

In the sequel let $K \subset \mathbb{R}^n$ denote a *convex body*, i.e. a convex and compact subset, and let $0 \in \mathrm{int}(K)$. By Theorem 1.6.1 in [15] the polar K° is a convex body with $0 \in \mathrm{int}(K^\circ)$ and $(K^\circ)^\circ = K$. Obviously the dual K^* is a convex body with

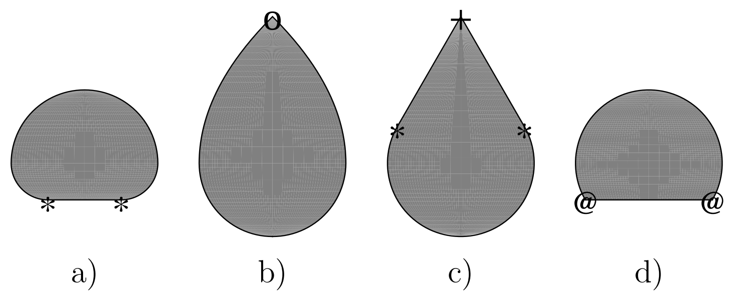


Figure 2.1: The convex sets a) and b) likewise c) and d) are duals of each other. Markings indicate non-exposed points (*), polyhedral corners (+), mixed corners (@) and free corners (o). The drawings have equal scaling and their origin is aligned vertically.

$0 \in \operatorname{int}(K^*)$ and $(K^*)^* = K$. A second construction for the dual convex body arises from the *support function* of a convex $C \subset \mathbb{R}^n$ in the direction $u \in \mathbb{R}^n$,

$$h_C(u) := \sup\{\langle x, u\rangle \mid x \in C\}\,.$$

The *radial function* of the convex body K is

$$\rho_K(u) := \sup\{\lambda \geq 0 \mid \lambda u \in K\}\,.$$

Theorem 1.7.6 in [15] shows for all $u \in \mathbb{R}^n$ that $\rho_{K^\circ}(u) = 1/h_K(u)$ holds, hence

$$\rho_{K^*}(u) = 1/h_K(-u)\,. \tag{3}$$

This equation includes $\rho_{K^*}(0) = \infty$ and $h_K(0) = 0$ with the convention of $1/0 = \infty$.

Construction 2.3. *The boundary of the dual convex body K^* is parametrized from the unit sphere by the support function of K,*

$$S^{n-1} := \{u \in \mathbb{R}^n \mid |u| = 1\} \;\to\; \partial K^*\,, \qquad u \;\mapsto\; \rho_{K^*}(u)u = u/h_K(-u)\,.$$

Proof. The map $u \mapsto \rho_{K^*}(u)u$ defined on the unit sphere S^{n-1} extends to a positively homogeneous function $\mathbb{R}^n \to \mathbb{R}^n$ by setting $0 \mapsto 0$ and $u \mapsto \rho_{K^*}(\frac{u}{|u|})u$ for $u \neq 0$. The Theorem of Sz. Nagy (see e.g. §VIII.1 in [3]) shows that this function, called *radial projection*, is a homeomorphism between the unit ball and K^*. In particular, $S^{n-1} \to \partial K^*$, $u \mapsto \rho_{K^*}(u)u$ is a parametrization of the boundary of K^*. The radial function of K^* is expressed by the support function of K in (3). □

Example 2.4. The convex body in Figure 2.1 a) appears at $a = b = \frac{1}{2}$ in a family of convex bodies $K \subset \mathbb{R}^2$ defined for $a, b > 0$. The dual convex body K^* is depicted in Figure 2.1 b) for $a = b = \frac{1}{2}$. We denote $u(\alpha) := \left(\begin{smallmatrix}\cos(\alpha)\\ \sin(\alpha)\end{smallmatrix}\right)$ for $\alpha \in \mathbb{R}$. The boundary of K consists of the segment between $\left(\begin{smallmatrix}-a\\ -b\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}a\\ -b\end{smallmatrix}\right)$, one half arc and two quarter arcs

$$c:\; [0, 2\pi) \;\to\; \mathbb{R}^2\,, \quad \alpha \;\mapsto\; \begin{cases} (a+b)u(\alpha) & \text{for } 0 \leq \alpha < \pi\,,\\ \left(\begin{smallmatrix}-a\\ 0\end{smallmatrix}\right) + bu(\alpha) & \text{for } \pi \leq \alpha < \frac{3}{2}\pi\,,\\ \left(\begin{smallmatrix}a\\ 0\end{smallmatrix}\right) + bu(\alpha) & \text{for } \frac{3}{2}\pi \leq \alpha < 2\pi\,.\end{cases}$$

By Construction 2.3 we have for $\alpha \in [0, 2\pi)$

$$\rho_{K^*}(u(\alpha)) = \begin{cases} (a\cos(\alpha) + b)^{-1} & \text{for } 0 \leq \alpha < \frac{\pi}{2}\,, \\ (-a\cos(\alpha) + b)^{-1} & \text{for } \frac{\pi}{2} \leq \alpha < \pi\,, \\ (a + b)^{-1} & \text{for } \pi \leq \alpha \leq 2\pi\,. \end{cases}$$

We define faces and conjugate faces and we prove technical assertions for §5.

Definition 2.5.

1. A *face* of a convex subset $C \subset \mathbb{R}^n$ is a convex subset $F \subset C$ such that $x, y, z \in C$, $y \in F$ and $y \in]x, z[:= \{(1-\lambda)x + \lambda z \mid 0 < \lambda < 1\}$ implies $x, z \in F$.
2. If $u \in \mathbb{R}^n$ is non-zero then we define $H_C(u) := \{x \in \mathbb{R}^n \mid \langle x, u\rangle = h_C(u)\}$. If $C \cap H_C(u) \neq \emptyset$ then $H_C(u)$ is an affine hyperplane called *supporting hyperplane* and $C \cap H_C(u)$ is an *exposed face* of C. By definition $\emptyset$ and C are exposed faces of C. A face which is not an exposed face is called a *non-exposed face.*
3. If $\{x\}$ is a face of C for $x \in C$ then x is an *extremal point.* In the following we will identify extremal points with their faces. If the extremal point $x \in C$ is an exposed face then x is an *exposed point*, otherwise x is a *non-exposed* point.
4. The *conjugate face* $\mathcal{C}(F)$ of a subset $F \subset K$ is a subset of the polar convex body:
$$\mathcal{C}(F) = \mathcal{C}_K(F) := \{v \in K^\circ \mid \langle v, u\rangle = 1\ \forall u \in F\}\,. \tag{4}$$

Remark 2.6. Exposed faces of a convex subset $C \subset \mathbb{R}^n$ are faces of C, see e.g. §18 in [12]. It is a common practice to use the conjugate face mapping $\mathcal{C}$ without reference to the convex body K and write e.g. $\mathcal{C}^2(F)$, see §2.2 in [15].

We denote $H^\pm := \{(x, y)^T \in \mathbb{R}^2 \mid \pm y \geq 0\}$.

Lemma 2.7. *Let $L \subset \mathbb{R}^2$ be a convex body.*

1. *A point $x \in L \setminus H^\mp$ is an extremal point of L if and only if x is an extremal point of $L \cap H^\pm$.*
2. *Let $\rho_L(\pm \binom{1}{0}) = h_L(\pm \binom{1}{0})$, i.e. L has maximal x-extension on the x-axis.*
   a) *The support functions satisfy $h_L|_{H^\pm} = h_{L\cap H^\pm}|_{H^\pm}$.*
   b) *For every $u \in \mathbb{R}^2 \setminus H^\mp$ the supporting hyperplanes satisfy $H_L(u) = H_{L\cap H^\pm}(u)$.*
   c) *If $p \in L \setminus H^\mp$ is an exposed point of L then there is $u \in \mathbb{R}^2 \setminus H^\mp$ such that $\{p\} = L \cap H_L(u)$.*
   d) *If $0 \in \operatorname{int}(L)$ and $F \subset L$ such that $F \setminus H^\mp \neq \emptyset$, then $\mathcal{C}_L(F) \subset H^\pm$.*
3. *For $i = 1, 2$ let $L_i \subset \mathbb{R}^2$ be a convex body with $0 \in \operatorname{int}(L_i)$ and let $c_\pm > 0$ such that $\rho_{L_i}(\pm \binom{1}{0}) = h_{L_i}(\pm \binom{1}{0}) = c_\pm$. Then $L := (L_1 \cap H^+) \cup (L_2 \cap H^-)$ is a convex body with $0 \in \operatorname{int}(L)$ and $L^* = (L_2^* \cap H^+) \cup (L_1^* \cap H^-)$.*

Proof. The proof of *1.* and *2.* is written for $(\pm, \mp) = (+, -)$, $(\pm, \mp) = (-, +)$ is analogous. To show part *1.* let $x \in L \setminus H^-$. If x is an extremal point of L then it is trivially an extremal point of $L \cap H^+$. Conversely let x be an extremal point of $L \cap H^+$ and let $y, z \in L$ with $x \in]y, z[$. The case $y, z \in L \setminus H^+$ is impossible since $x \in H^+$. If $y, z \in L \cap H^+$ then $y = z = x$ follows as desired. Finally, if $y \in L \cap H^+$ and $z \in L \setminus H^+$, then $]y, z[$ intersects the x-axis in a point $p \neq x$. Then as before $y = p = x$ and this implies $z = x$.

To prove part *2a*) we show for $u = (u_x, u_y) \in H^+$ that $\langle \cdot, u\rangle$ is maximized on L at a point in $L \cap H^+$. Let $p = (p_x, p_y) \in L \cap H^-$. Assuming $\pm u_x \geq 0$ we show $\langle p, u\rangle \leq \langle \pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \rho_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)), u\rangle$. Since L satisfies $h_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = \rho_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ we have

$$\pm p_x = \langle p, \pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\rangle \leq h_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = \rho_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$$

and

$$\begin{aligned}\langle p, u\rangle = p_x u_x + p_y u_y \leq p_x u_x = (\pm p_x)(\pm u_x) &\leq \rho_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))(\pm u_x) \\ = \langle \pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \rho_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)), u\rangle\,. &\end{aligned} \tag{5}$$

The assertion *2b*) holds because $u \notin H^-$ and $p \notin H^+$ imply $p_y u_y < 0$ and then a strict inequality follows in (5).

We show *2c*). Since p is an exposed point of L, there exists a non-zero vector u with $\{p\} = L \cap H_L(u)$. By contradiction we show $u \notin H^-$. By *2a*) there is point $q \in L \cap H^-$ that lies on the hyperplane $H_L(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$. Since $p \notin H^-$ the vector u is not aligned with the x-axis. If we assume $u \notin H^+$ then *2b*) shows $p \notin H_L(u)$.

For *2d*) we show that $F \setminus H^- \neq \emptyset$ implies $\mathcal{C}(F) \subset H^+$ by proving $p \notin \mathcal{C}(F)$ for every $p = (p_x, p_y)^T$ in $L^\circ \setminus H^+$. We have $p_y < 0$ and there exists $u = (u_x, u_y)^T \in F$ such that $u_y > 0$. Since $0 \in \operatorname{int}(L)$ the polar L° is a convex body and by (3) it satisfies $\rho_{L^\circ}(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = h_{L^\circ}(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$. Assuming $\pm u_x \geq 0$ the strict inequality $\langle p, u\rangle < \langle \pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \rho_{L^\circ}(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)), u\rangle$ follows from (5) with L replaced by L°. Since $\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \rho_{L^\circ}(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \in L^\circ$ and $u \in L$ we have $\langle \pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \rho_{L^\circ}(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)), u\rangle \leq 1$ hence $\langle p, u\rangle < 1$ shows $p \notin \mathcal{C}(F)$.

We show part *3*. Clearly L is compact and $0 \in \operatorname{int}(L)$. To show convexity let $x, y \in L$ and $[x, y] := \{(1-\lambda)x + \lambda y \mid \lambda \in [0, 1]\}$. If $x, y \in H^\pm$ then $[x, y] \subset L$ by convexity of L_1 and L_2 Otherwise $[x, y]$ intersects the x-axis in a point p and the pairs $\{x, p\}$ and $\{p, y\}$ satisfy the previous assumption. Using (3) and *2b*) we have

$$\rho_{L^*}(u) = h_L(-u)^{-1} = h_{L_2}(-u)^{-1} = \rho_{L_2^*}(u)$$

for all $u \in H^+$. Similarly for $u \in H^-$ we have $\rho_{L^*}(u) = \rho_{L_1^*}(u)$. □

The following construction of planar self-dual convex bodies joins half of a convex body with half of its dual convex body. By part *b*) the construction is general.

Construction 2.8.

a) *Let $K \subset \mathbb{R}^2$ satisfy $\rho_K(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = h_K(\pm \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = e^{\pm\lambda}$ for some $\lambda \in \mathbb{R}$. Then $(K \cap H^+) \cup (K^* \cap H^-)$ is a self-dual convex body.*

b) *For every planar self-dual convex body K exists a rotation $\psi \in SO(2)$ such that $\psi(K)$ satisfies the assumptions in a).*

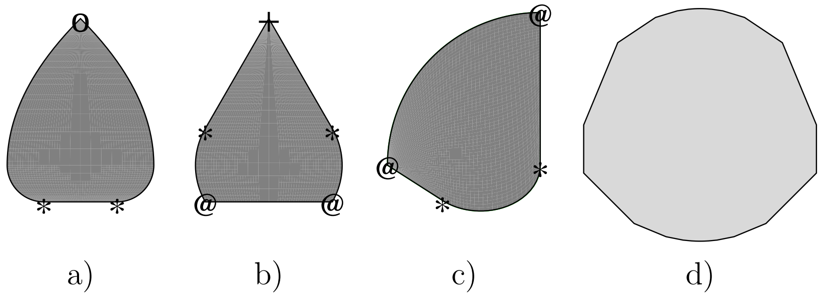


Figure 2.2: The depicted convex sets are self-dual. Markings are explained in Figure 2.1. Drawings a)–c) have equal scaling and their origin is aligned vertically.

Proof. Assertion a) follows from (3) and Lemma 2.7.3 applied to the convex bodies $L_1 := K$ and $L_2 := K^*$. To show b) let u be an element of K with maximal norm $|u|$ in K and put $v := \frac{u}{|u|}$. Then $\rho_K(v) = |u|$ and $h_K(v) = \max_{w \in K} \langle w, v \rangle \leq \max_{w \in K} |w||v| = |u|$ by the Cauchy-Schwarz inequality. On the other hand, $h_K(v) \geq \langle u, v \rangle = |u|$ shows $\rho_K(v) = h_K(v)$. Since K is self-dual we get from (3) and with $(K^*)^* = K$

$$\rho_K(-v) = \rho_{K^*}(-v) = h_K(v)^{-1} = \rho_K(v)^{-1} = h_{K^*}(-v) = h_K(-v),$$

that is $\rho_K(\pm v) = h_K(\pm v) = e^{\pm\lambda}$ for some $\lambda \in \mathbb{R}$. For all $\psi \in SO(2)$ and $v \in \mathbb{R}^2$ the equalities $\rho_{\psi(K)}(\psi(v)) = \rho_K(v)$ and $h_{\psi(K)}(\psi(v)) = h_K(v)$ hold. The choice of ψ such that $\psi(v) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ completes the proof. $\square$

The self-dual convex body a) resp. b) in Figure 2.2 is generated by Construction 2.8 from the convex body b) resp. c) and its dual convex body a) resp. d) in Figure 2.1. We consider a less symmetric example.

Example 2.9. Let $a > 0$ and $K \subset \mathbb{R}^2$ have the upper part $K \cap H^+$ defined as the convex hull of $\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)$ and of the quarter arc consisting of all points $\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) + \frac{a^2+1}{a} u(\alpha)$ for $\alpha \in [\frac{\pi}{2}, \pi]$. Without specifying the lower part of K, Construction 2.3, Lemma 2.7.2 a) and Construction 2.8 provide a self-dual convex body $X := (K \cap H^+) \cup (K^* \cap H^-)$ with radial function

$$\begin{aligned} \rho_X(u(\alpha)) &= \rho_{K^*}(u(\alpha)) = h_K(-u(\alpha))^{-1} \\ &= \begin{cases} -a(a^2(\cos(\alpha) + \sin(\alpha)) + \sin(\alpha))^{-1} & \text{for } \pi \leq \alpha < \frac{3}{2}\pi, \\ a(a^2(1 - \cos(\alpha)) + 1)^{-1} & \text{for } \frac{3}{2}\pi \leq \alpha \leq 2\pi. \end{cases} \end{aligned}$$

For $a = \frac{4}{3}$ the self-dual convex body X is shown in Figure 2.2 c).

3. A Galois connection

We define a Galois connection between touching cones and faces of an arbitrary convex subset $C \subset \mathbb{R}^n$ which has not cardinality one. We will study two lattices of faces and two lattices of cones associated to C. We refer to [4] for general lattice theory and to [1, 9, 16] for the lattice theory of convex sets.

The *normal cone* at a point $x \in C$ is the set $N(x)$ of all vectors $u \in \mathbb{R}^n$ such that $\langle u, y - x\rangle \leq 0$ holds for all $y \in C$, i.e. u does not make an acute angle with $y - x$ for any $y \in C$. The whole space $\mathbb{R}^n$ is a normal cone by definition. A *touching cone* is any non-empty face of any normal cone of C. (Touching cones were first introduced in [15] by a different but equivalent definition).

The set of faces, exposed faces, touching cones resp. normal cones of C is denoted by $\mathcal{F}_C$, $\mathcal{E}_C$, $\mathcal{T}_C$ resp. $\mathcal{N}_C$. We have

$$\mathcal{E}_C \subset \mathcal{F}_C \quad \text{and} \quad \mathcal{N}_C \subset \mathcal{T}_C\,.$$

Each of these sets is a poset ordered by inclusion and a complete lattice of finite length where the infimum is the intersection, see e.g. §1.1 and §1.2 in [16]. We write these lattices in the form

$$\begin{array}{cc} (\mathcal{F}_C, \leq_{\mathcal{F}}, \vee_{\mathcal{F}}, \wedge_{\mathcal{F}})\,, & (\mathcal{E}_C, \leq_{\mathcal{E}}, \vee_{\mathcal{E}}, \wedge_{\mathcal{E}})\,, \\ (\mathcal{T}_C, \leq_{\mathcal{T}}, \vee_{\mathcal{T}}, \wedge_{\mathcal{T}})\,, & (\mathcal{N}_C, \leq_{\mathcal{N}}, \vee_{\mathcal{N}}, \wedge_{\mathcal{N}})\,. \end{array}$$

The infimum (supremum) of a subset $S \subset \mathcal{F}_C$ is denoted by $\bigwedge_{\mathcal{F}} S$ ($\bigvee_{\mathcal{F}} S$), the analogue notation is used for other lattices.

We consider a mapping $\theta : L \to M$ between two lattices $(L, \leq_L, \vee_L, \wedge_L)$ and $(M, \leq_M, \vee_M, \wedge_M)$. The mapping θ is

$$\begin{array}{rl} \textit{isotone} & \text{if } x \leq_L y \implies \theta(x) \leq_M \theta(y)\,, \quad (x, y \in L) \\ \textit{antitone} & \text{if } x \leq_L y \implies \theta(x) \geq_M \theta(y)\,, \\ \text{a } \textit{join-morphism} & \text{if } \theta(x \vee_L y) = \theta(x) \vee_M \theta(y)\,, \\ \text{a } \textit{meet-morphism} & \text{if } \theta(x \wedge_L y) = \theta(x) \wedge_M \theta(y)\,, \\ \text{a } \textit{dual join-morphism} & \text{if } \theta(x \vee_L y) = \theta(x) \wedge_M \theta(y) \\ \text{and a } \textit{dual meet-morphism} & \text{if } \theta(x \wedge_L y) = \theta(x) \vee_M \theta(y)\,. \end{array}$$

Join- and meet-morphisms are isotone, see [4] Chap. II.3. Hence dual join- and meet-morphisms are antitone. A *(dual) lattice-morphism* is a (dual) meet-morphism which is also a (dual) join-morphism.

The *relative interior* of C, denoted by $\text{ri}(C)$ is the interior of C in the topology of the affine hull $\text{aff}(C)$ of C. If $C \neq \emptyset$ then the translation vector space of the affine hull of C is denoted by $\text{lin}(C) := \text{aff}(C) - \text{aff}(C)$.

Definition 3.1.

1. To every touching cone we associate an exposed face

$$\Phi : \mathcal{T}_C \longrightarrow \mathcal{E}_C\,.$$

We put $\Phi(\text{lin}(C)^\perp) := C$, $\Phi(\mathbb{R}^n) := \emptyset$ and for touching cones $T \in \mathcal{T}_C \setminus \{\text{lin}(C)^\perp, \mathbb{R}^n\}$ Lemma 7.2 a) in [16] shows that the exposed face

$$\Phi(T) := C \cap H_C(u)$$

is well-defined for an arbitrary non-zero vector u in the relative interior of T.

2. To every face we associate a normal cone

$$\Psi : \mathcal{F}_C \longrightarrow \mathcal{N}_C .$$

We put $\Psi(\emptyset) := \mathbb{R}^n$. For faces $F \in \mathcal{F}_C \setminus \{\emptyset\}$ a normal cone is well-defined by the arguments provided in Definition 4.3 in [16]: We put

$$\Psi(F) := N(x)$$

for an arbitrary point x in the relative interior of F.

Remark 3.2.

1. The map $\Phi : \mathcal{T}_C \to \mathcal{E}_C$ is antitone, this follows from an intersection representation: If $T \in \mathcal{T}_C$ is a touching cone and $T \neq \mathrm{lin}(C)^\perp, \mathbb{R}^n$ then by Lemma 7.2 a) in [16] we have
$$\Phi(T) = \bigcap_{u \in T \setminus \{0\}} (C \cap H_C(u)) .$$
2. That $\Psi : \mathcal{F}_C \to \mathcal{N}_C$ is antitone is discussed in the paragraph following Definition 4.3 in [16]. For the sake of completeness we notice for faces $F \in \mathcal{F}_C$, $F \neq \emptyset$:
$$\Psi(F) = \bigcap_{x \in F} N(x) .$$
This follows from the inclusion $N(y) \subset N(x)$ valid for all y in the relative interior ri$F$ and $x \in F$, see (15)(ii) in [16], while $\Psi(F) = N(y)$ holds by Definition 3.1.2.
3. It is proved in Proposition 4.7 in [16] that the restrictions $\Phi|_{\mathcal{N}_C}$ and $\Psi|_{\mathcal{E}_C}$ are dual lattice isomorphisms, inverse to each other. The diagram
$$\begin{tikzcd} \mathcal{N}_C \arrow[r, hook, two heads, shift left, "\Phi"] & \mathcal{E}_C \arrow[l, hook, two heads, shift left, "\Psi"] \end{tikzcd} \qquad (6)$$
commutes.

To study Φ and Ψ we use the concepts of closure operation and of Galois connection, see e.g. §V.1 and §V.8 in [4].

Definition 3.3.

1. A *closure operation* on a set I is an operator $X \to \mathrm{cl}(X)$ on the subsets of I such that for all $X, Y \subset I$ we have
$$\begin{array}{ll} X \subset \mathrm{cl}(X) & \text{(Extensive)} \\ \mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X)) & \text{(Idempotent)} \\ \text{if } X \subset Y, \text{ then } \mathrm{cl}(X) \subset \mathrm{cl}(Y) & \text{(Isotone)} \end{array}$$
Subsets $X \subset I$ with $X = \mathrm{cl}(X)$ are called *closed sets* with respect to cl.
2. Let $(L, \leq_L)$ and $(M, \leq_M)$ be any posets and let $\theta : L \to M$, $\phi : M \to L$ be maps such that for all $l_1, l_2 \in L$ and $m_1, m_2 \in M$ we have
$$l_1 \leq_L l_2 \text{ implies } \theta(l_1) \geq_M \theta(l_2) ,$$
$$m_1 \leq_M m_2 \text{ implies } \phi(m_1) \geq_L \phi(m_2) ,$$
$$l_1 \leq_L \phi(\theta(l_1)) \quad \text{and} \quad m_1 \leq_M \theta(\phi(m_1)) .$$
Then θ and ϕ are said to define a *Galois connection* between L and M.

3. We define the *normal closure* as the operation on touching cones

$$\mathrm{cl}_{\mathcal{N}} : \mathcal{T}_C \longrightarrow \mathcal{N}_C\,, \quad T \longmapsto \bigcap_{\substack{N \in \mathcal{N}_C \\ T \subset N}} N \tag{7}$$

and the *exposed closure* as the operation on faces

$$\mathrm{cl}_{\mathcal{E}} : \mathcal{F}_C \longrightarrow \mathcal{E}_C\,, \quad F \longmapsto \bigcap_{\substack{G \in \mathcal{E}_C \\ F \subset G}} G\,. \tag{8}$$

Since $\mathcal{N}_C$ and $\mathcal{E}_C$ are complete lattices with the intersection as the infimum, the normal closure and the exposed closure are closure operations in the sense of Definition 3.3.1. The closed sets of $\mathrm{cl}_{\mathcal{N}}$ are the normal cones and the closed sets of $\mathrm{cl}_{\mathcal{E}}$ are the exposed faces.

These closures can equivalently be defined by the mappings Φ and Ψ between touching cones and faces.

Lemma 3.4.

1. *Every touching cone $T \in \mathcal{T}_C$ has normal closure $\mathrm{cl}_{\mathcal{N}}(T) = \Psi \circ \Phi(T)$. In particular $T \leq_{\mathcal{T}} \Psi \circ \Phi(T)$ holds.*
2. *Every face $F \in \mathcal{F}_C$ has exposed closure $\mathrm{cl}_{\mathcal{E}}(F) = \Phi \circ \Psi(F)$. In particular $F \leq_{\mathcal{F}} \Phi \circ \Psi(F)$ holds.*

Proof. We prove part *1.* Let $T \in \mathcal{T}_C$ be a touching cone and let $N \in \mathcal{N}_C$ be a normal cone. By Remark 3.2.1 and 2 the maps $\Phi : \mathcal{T}_C \to \mathcal{E}_C$ and $\Psi : \mathcal{F}_C \to \mathcal{N}_C$ are antitone hence the composition $\Psi \circ \Phi$ is isotone. Its restriction $\Psi \circ \Phi|_{\mathcal{N}_C}$ is the identity mapping by (6) hence

$$T \subset N \implies \Psi \circ \Phi(T) \subset \Psi \circ \Phi(N) = N\,.$$

This implication has two consequences. Firstly, the inclusion $\Psi \circ \Phi(T) \subset \mathrm{cl}_{\mathcal{N}}(T)$ into the normal closure (7) follows. Secondly we have

$$\mathrm{cl}_{\mathcal{N}}(T) \overset{\text{def.}}{=} \bigcap_{\substack{N \in \mathcal{N}_C \\ T \subset N}} N \subset \bigcap_{\substack{N \in \mathcal{N}_C \\ \Psi \circ \Phi(T) \subset N}} N = \Psi \circ \Phi(T)$$

where the last equality holds because $\Psi \circ \Phi(T) \in \mathcal{N}_C$. This shows $\Psi \circ \Phi(T) = \mathrm{cl}_{\mathcal{N}}(T)$. The inclusion $T \subset \mathrm{cl}_{\mathcal{N}}(T)$ is obvious. The proof of part 2 is analogous. □

Their link to the closure operations enables us to analyze Φ and Ψ.

Lemma 3.5.

1. *The assignment of exposed faces to touching cones $\Phi : \mathcal{T}_C \to \mathcal{E}_C$ is a dual join-morphism. For all touching cones $T, U \in \mathcal{T}_C$ we have $\Phi(T \wedge_{\mathcal{T}} U) \geq_{\mathcal{E}} \Phi(T) \vee_{\mathcal{E}} \Phi(U)$ and $\Phi(T) = \Phi(\mathrm{cl}_{\mathcal{N}}(T))$.*
2. *The assignment of normal cones to faces $\Psi : \mathcal{F}_C \to \mathcal{N}_C$ is a dual join-morphism. For all faces $F, G \in \mathcal{F}_C$ we have $\Psi(F \wedge_{\mathcal{F}} G) \geq_{\mathcal{N}} \Psi(F) \vee_{\mathcal{N}} \Psi(G)$ and $\Psi(F) = \Psi(\mathrm{cl}_{\mathcal{E}}(F))$.*

Proof. We prove part *1.* in five steps. As Φ is antitone by Remark 3.2, for all touching cones $T, U \in \mathcal{T}_C$ follows

$$\begin{array}{rcll} T \wedge_{\mathcal{T}} U \leq_{\mathcal{T}} T, U & \Longrightarrow & \Phi(T \wedge_{\mathcal{T}} U) \geq_{\mathcal{E}} \Phi(T), \Phi(U) & \\ & \Longrightarrow & \Phi(T \wedge_{\mathcal{T}} U) \geq_{\mathcal{E}} \Phi(T) \vee_{\mathcal{E}} \Phi(U) & \text{and secondly} \\ T \vee_{\mathcal{T}} U \geq_{\mathcal{T}} T, U & \Longrightarrow & \Phi(T \vee_{\mathcal{T}} U) \leq_{\mathcal{E}} \Phi(T), \Phi(U) & \\ & \Longrightarrow & \Phi(T \vee_{\mathcal{T}} U) \leq_{\mathcal{E}} \Phi(T) \wedge_{\mathcal{E}} \Phi(U) \,. & \end{array}$$

Thirdly, by (6) and Lemma 3.4.1 we have

$$\Phi(T) = \Phi \circ \Psi \circ \Phi(T) = \Phi(\mathrm{cl}_{\mathcal{N}}(T)) \,.$$

Fourthly, as $\mathcal{N}_C$ is a complete lattice with the restricted partial order from $\mathcal{T}_C$,

$$T \vee_{\mathcal{T}} U \leq_{\mathcal{T}} \mathrm{cl}_{\mathcal{N}}(T) \vee_{\mathcal{T}} \mathrm{cl}_{\mathcal{N}}(U) \leq_{\mathcal{T}} \mathrm{cl}_{\mathcal{N}}(T) \vee_{\mathcal{N}} \mathrm{cl}_{\mathcal{N}}(U) \,.$$

Finally, by step two, step three, since $\Phi|_{\mathcal{N}_C} : \mathcal{N}_C \to \mathcal{E}_C$ is a dual lattice isomorphism and by step four we have

$$\begin{aligned} \Phi(T \vee_{\mathcal{T}} U) \leq_{\mathcal{E}} & \; \Phi(T) \wedge_{\mathcal{E}} \Phi(U) = \Phi(\mathrm{cl}_{\mathcal{N}}(T)) \wedge_{\mathcal{E}} \Phi(\mathrm{cl}_{\mathcal{N}}(U)) \\ = & \; \Phi(\mathrm{cl}_{\mathcal{N}}(T) \vee_{\mathcal{N}} \mathrm{cl}_{\mathcal{N}}(U)) \leq_{\mathcal{E}} \Phi(T \vee_{\mathcal{T}} U) \,. \end{aligned}$$

This completes the proof of part *1.*, part *2.* is analogous. □

Example 3.6. The dual join morphisms in Lemma 3.5 are no dual lattice morphisms in general. A counterexample for Ψ is given by the two non-exposed faces of the convex set in Figure 2.1 a) or c) which is also a counterexample against a lattice morphism of the exposed closure in Lemma 3.7. The convex bodies in Figure 2.1 b) or d) are counterexamples for Φ and for the normal closure, because b) is dual to a) and d) is dual to c), see Proposition 4.1.

The closure operations inherit properties from Φ and Ψ.

Lemma 3.7.

1. *The normal closure* $\mathrm{cl}_{\mathcal{N}} : \mathcal{T}_C \to \mathcal{N}_C$ *is a join morphism such that for all touching cones* $T, U \in \mathcal{T}_C$ *we have* $\mathrm{cl}_{\mathcal{N}}(T \wedge_{\mathcal{T}} U) \leq_{\mathcal{N}} \mathrm{cl}_{\mathcal{N}}(T) \wedge_{\mathcal{N}} \mathrm{cl}_{\mathcal{N}}(T)$.
2. *The exposed closure* $\mathrm{cl}_{\mathcal{E}} : \mathcal{F}_C \to \mathcal{E}_C$ *is a join morphism such that for all faces* $F, G \in \mathcal{F}_C$ *we have* $\mathrm{cl}_{\mathcal{E}}(F \wedge_{\mathcal{F}} G) \leq_{\mathcal{E}} \mathrm{cl}_{\mathcal{E}}(F) \wedge_{\mathcal{E}} \mathrm{cl}_{\mathcal{E}}(G)$.

Proof. We prove part *1.* and choose touching cones $T, U \in \mathcal{T}_C$. By Lemma 3.4.1, Lemma 3.5.1, (6) and Lemma 3.4.1 we have

$$\begin{aligned} \mathrm{cl}_{\mathcal{N}}(T \vee_{\mathcal{T}} U) &= \Psi \circ \Phi(T \vee_{\mathcal{T}} U) = \Psi(\Phi(T) \wedge_{\mathcal{E}} \Phi(U)) \\ &= \Psi \circ \Phi(T) \vee_{\mathcal{N}} \Psi \circ \Phi(U) = \mathrm{cl}_{\mathcal{N}}(T) \vee_{\mathcal{N}} \mathrm{cl}_{\mathcal{N}}(U) \,. \end{aligned}$$

The same arguments as above prove

$$\begin{aligned} \mathrm{cl}_{\mathcal{N}}(T \wedge_{\mathcal{T}} U) &= \Psi \circ \Phi(T \wedge_{\mathcal{T}} U) \leq_{\mathcal{N}} \Psi(\Phi(T) \vee_{\mathcal{E}} \Phi(U)) \\ &= \Psi \circ \Phi(T) \vee_{\mathcal{N}} \Psi \circ \Phi(U) = \mathrm{cl}_{\mathcal{N}}(T) \vee_{\mathcal{N}} \mathrm{cl}_{\mathcal{N}}(U) \,, \end{aligned}$$

except the inequality $\leq_{\mathcal{N}}$ follows because Ψ is antitone by Remark 3.2.2 and because Lemma 3.5.1 shows $\Phi(T \wedge_{\mathcal{T}} U) \geq_{\mathcal{E}} \Phi(T) \vee_{\mathcal{E}} \Phi(U)$. The proof of part *2.* is analogous. □

We summarize a part of our results as follows.

Theorem 3.8. *Let $C \subset \mathbb{R}^n$ be any convex subset of cardinality not one. Then the map $\Phi : \mathcal{T}_C \to \mathcal{E}_C$ from touching cones to exposed faces and the map $\Psi : \mathcal{F}_C \to \mathcal{N}_C$ from faces to normal cones define a Galois connection between the touching cone lattice $\mathcal{T}_C$ and the face lattice $\mathcal{F}_C$.*

Proof. This follows from Lemma 3.4 and Lemma 3.5. □

We recover the dual lattice isomorphism (6) from an abstract theorem:

Remark 3.9. If $\theta : L \to M$, $\phi : M \to L$ is a Galois connection between complete lattices L and M, then the maps $\phi \circ \theta$ and $\theta \circ \phi$ are closure operations. Moreover, θ and ϕ restricts to a dual lattice isomorphism between the complete lattices of closed sets of $\phi \circ \theta$ and $\theta \circ \phi$. This is proved in §V.8 in [4].

4. Conjugate faces of a convex body

We study conjugate faces of a polar pair of convex bodies in a lattice theoretic perspective. This pair will be given by the convex body $K \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ and by its polar convex body $K^\circ \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(K^\circ)$.

In the following we consider the conjugate face map (4) in the restriction to the face lattice $\mathcal{F}_K$ of K,

$$\mathcal{C}_K : \mathcal{F}_K \to \mathcal{E}_{K^\circ} .$$

It is obvious by definition that $\mathcal{C}_K(\mathcal{F}_K)$ is included in the exposed face lattice $\mathcal{E}_{K^\circ}$. Similarly we consider the conjugate face map $\mathcal{C}_{K^\circ} : \mathcal{F}_{K^\circ} \to \mathcal{E}_K$.

It is well-known that the two conjugate face maps $\mathcal{C}_K$ and $\mathcal{C}_{K^\circ}$ define a Galois connection, see Definition 3.3, between the face lattices $\mathcal{F}_K$ and $\mathcal{F}_{K^\circ}$. The corresponding closure operations, see Remark 3.9, are the exposed closure operations (8)

$$\mathcal{C}_{K^\circ} \circ \mathcal{C}_K = \mathrm{cl}_\mathcal{E}|_{\mathcal{F}_K} \quad \text{and} \quad \mathcal{C}_K \circ \mathcal{C}_{K^\circ} = \mathrm{cl}_\mathcal{E}|_{\mathcal{F}_{K^\circ}} . \tag{9}$$

A proof of these statements is given in Theorem 2.1.4 in [15]. Equation (9) brings the Galois connection (9) in contact with the Galois connection in Theorem 3.8, once for K and once for K°. The latter consists of Φ assigning exposed faces to touching cones and Ψ assigning normal cones to faces. The arguments in this paragraph already integrate all solid and dashed arrows into the diagram in Proposition 4.1.

The dotted arrows in the diagram arise from the positive hull operator. The *positive hull* of $X \subset \mathbb{R}^n$ is $\mathrm{pos}(X) := \{\lambda x | \lambda \geq 0, x \in X\}$ unless $X = \emptyset$ where $\mathrm{pos}(\emptyset) := \{0\}$. Lemma 2.2.3 in [15] proves for faces $F \in \mathcal{F}_K$ (indeed for non-empty convex subsets of K)

$$\Psi(F) = \mathrm{pos} \circ \mathcal{C}_K(F) , \tag{10}$$

i.e. the normal cone $\Psi(F)$ is the positive hull of the conjugate face. It follows from (10), (9) and the dual lattice isomorphism (6) that we have a lattice isomorphism

$$\mathrm{pos}|_{\mathcal{E}_{K^\circ}} : \mathcal{E}_{K^\circ} \to \mathcal{N}_K . \tag{11}$$

Theorem 8.3 in [16] uses (11) and an elementary analysis of sections of normal cones to prove the lattice isomorphism

$$\mathrm{pos}|_{\mathcal{F}_{K^\circ}} : \mathcal{F}_{K^\circ} \to \mathcal{T}_K\,. \tag{12}$$

The inverse isomorphism is defined for $T \in \mathcal{T}_K$ with $T \neq \mathbb{R}^n$ by

$$T \mapsto \partial K^\circ \cap T \tag{13}$$

and by $\mathbb{R}^n \mapsto K^\circ$. Here ∂K° denotes the boundary of K°.

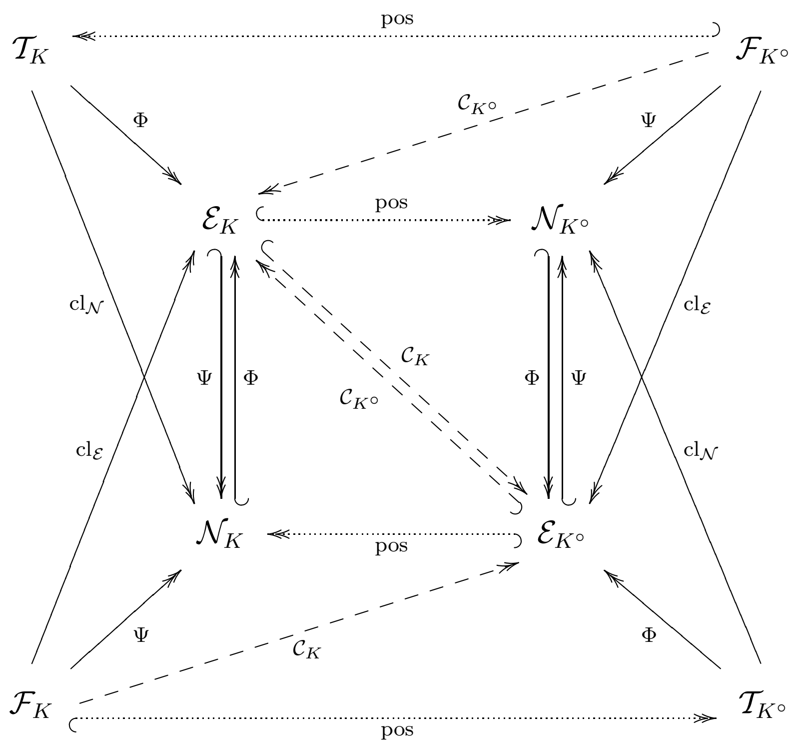
Proposition 4.1. *The following diagram commutes. The closure operations* $\mathrm{cl}_{\mathcal{N}}$ *and* $\mathrm{cl}_{\mathcal{E}}$ *are isotone join morphisms satisfying*

$$f(a \wedge b) \leq f(a) \wedge f(b) \quad \forall a, b\,,$$

their restrictions to normal cones resp. exposed faces is the identity map. The mappings Φ, Ψ *and the conjugate face maps* $\mathcal{C}_K$ *and* $\mathcal{C}_{K^\circ}$ *are antitone dual join morphism satisfying*

$$f(a \wedge b) \geq f(a) \vee f(b) \quad \forall a, b\,,$$

their restrictions to normal cones resp. exposed faces define dual lattice isomorphisms. The positive hull operator pos *defines lattice isomorphisms in the diagram.*



Proof. The commuting diagram was introduced in the above discussion except we have to show the equality of functions $\mathrm{cl}_{\mathcal{N}} \circ \mathrm{pos} = \mathrm{pos} \circ \mathrm{cl}_{\mathcal{E}}$ on the domain of the two face lattices $\mathcal{F}_K$ or $\mathcal{F}_{K^\circ}$. We will carry out the proof for $F \in \mathcal{F}_{K^\circ}$, the proof for faces of K is analogous. By (7), (12), (11), (11) and (8) we have

$$\begin{aligned}
\mathrm{cl}_{\mathcal{N}} \circ \mathrm{pos}(F) &= \textstyle\bigwedge_{\mathcal{N}}\{N \in \mathcal{N}_K \mid \mathrm{pos}(F) \leq_{\mathcal{T}} N\} \\
&= \mathrm{pos} \circ \mathrm{pos}^{-1} \left(\textstyle\bigwedge_{\mathcal{N}}\{N \in \mathcal{N}_K \mid F \leq_{\mathcal{F}} \mathrm{pos}^{-1}(N)\}\right) \\
&= \mathrm{pos} \left(\textstyle\bigwedge_{\mathcal{E}}\{\mathrm{pos}^{-1}(N) \in \mathcal{E}_{K^\circ} \mid F \leq_{\mathcal{F}} \mathrm{pos}^{-1}(N)\}\right) \\
&= \mathrm{pos} \left(\textstyle\bigwedge_{\mathcal{E}}\{G \in \mathcal{E}_{K^\circ} \mid F \leq_{\mathcal{F}} G\}\right) \\
&= \mathrm{pos} \circ \mathrm{cl}_{\mathcal{E}}(F)\,.
\end{aligned}$$

Lemma 3.7 shows that the closure operations have the claimed properties. This is shown for Φ and Ψ in Lemma 3.5. Since the conjugate face map $\mathcal{C}_K = \Phi \circ \mathrm{pos}$ is a composition of Φ with the positive hull lattice isomorphism (12) it has the claimed properties. The argument for $\mathcal{C}_{K^\circ}$ is analogous. □

Remark 4.2.

1. The convex bodies in Figure 2.1 a) or c) show that the conjugate face map is not a dual lattice morphism (see Example 3.6 for the other mappings.) Equality conditions of a dual join morphism in the inequality

$$f(a \wedge b) \geq f(a) \vee f(b) \quad \forall a, b$$

   were studied in [2] for face lattices of closed convex cones in relation to modularity of face lattices.
2. Although the exposed face lattices $\mathcal{E}_K$ and $\mathcal{E}_{K^\circ}$ are dually isomorphic by the conjugate face map, the face lattices $\mathcal{F}_K$ and $\mathcal{F}_{K^\circ}$ are not dually isomorphic in general. Examples are the dual pairs of convex bodies in Figure 2.1.

We notice two restricted isomorphisms of the conjugate face map. For their discussion we introduce further concepts. We call a non-empty face $F \in \mathcal{F}_K$ *singular* if its normal cone has dimension at least two, $\dim \Psi(F) \geq 2$. A non-empty face F is a *corner* of K if $\dim \Psi(F) = n$. A face F of K is a *facet* if $\mathrm{codim}(F) = 1$. Finally, we call a point $x \in K$ *smooth* if its normal cone has dimension one, $\dim N(x) = 1$.

Corollary 4.3. *The conjugate face $\mathcal{C}_K : \mathcal{F}_K \to \mathcal{E}_{K^\circ}$ restricts to a bijection*

$$\{\textit{smooth exposed points of } K\} \;\to\; \{\textit{smooth exposed points of } K^\circ\}\,.$$

Proof. The bijection is immediate from Proposition 4.1. □

For completeness we include the following well-known proposition.

Lemma 4.4. *All facets of K are exposed faces of K, all corners of K are exposed points of K.*

Proof. Indirectly, if a face F of K is not exposed, then $F \subsetneq \mathrm{cl}_{\mathcal{E}}(F)$. Now $\dim(F) < \dim \mathrm{cl}_{\mathcal{E}}(F) < n$ follows by [12] Corollary 18.1.3 and Lemma 4.6 in [16]. This shows that F is not a facet.

Let F be a corner of K. First, the face F is exposed: Its normal cone is $\Psi(F) = \Psi \circ \mathrm{cl}_{\mathcal{E}}(F))$ by Proposition 4.1. By contradiction, if $F \subsetneq \mathrm{cl}_{\mathcal{E}}(F)$, then $\mathrm{cl}_{\mathcal{E}}(F)$ contains a segment and its normal cone has dimension $\leq n-1$. This shows that F is an exposed face. Second, let x belong to the relative interior of F, then $F=\{x\}$: By Definition 3.1.2 of Ψ we have $N(x)=\Psi(F)$ and the proper inclusion $\{x\} \subsetneq F$ leads to a contradiction as before. □

Corollary 4.5. *The conjugate face map* $\mathcal{C}_K : \mathcal{F}_K \to \mathcal{E}_{K^\circ}$ *restricts to a bijection* $\{$*facets of* $K\} \to \{$*corners of* $K^\circ\}$. *The inverse map is the restriction of* $\mathcal{C}_{K^\circ} : \mathcal{F}_{K^\circ} \to \mathcal{E}_K$ *to the bijection* $\{$*corners of* $K^\circ\} \to \{$*facets of* $K\}$.

Proof. Since facets and corners are exposed faces by Lemma 4.4, we can use the decomposition of $\mathcal{C}_K|_{\mathcal{E}_K} : \mathcal{E}_K \to \mathcal{E}_{K^\circ}$ into the bijections $\mathcal{C}_K|_{\mathcal{E}_K} = \Phi \circ \mathrm{pos}|_{\mathcal{E}_K}$ in Proposition 4.1. Now it suffices to notice from (13) that $\mathrm{pos}|_{\mathcal{E}_K}$ restricts to a bijection between the facets of K and the normal cones ($\neq \mathbb{R}^n$) of K° of dimension n. □

We arrive at our main results. Let $(L, \leq_L)$ be a poset with greatest element 1. An element $x \in L$, $x \neq 1$ is a *coatom* of L if for all $y \in L$ the two conditions $x \leq_L y$ and $y \neq 1$ imply $y = x$. We consider for normal cones $N \in \mathcal{N}_K$ the principal ideal

$$\mathcal{T}_K(N) := \{T \in \mathcal{T}_K \mid T \leq_{\mathcal{T}} N\}.$$

It is clear that $\mathcal{T}_K(N)$ is a complete sublattice of the touching cone lattice $\mathcal{T}_K$ and that N is the greatest element in $\mathcal{T}_K(N)$. We call the normal cone N *complete* if all coatoms of the ideal $\mathcal{T}_K(N)$ are normal cones of K. Otherwise N is *incomplete.* We also consider for exposed faces $F \in \mathcal{F}_K$ the principal ideal

$$\mathcal{F}_K(F) := \{G \in \mathcal{F}_K \mid G \leq_{\mathcal{F}} F\},$$

which is the face lattice of F.

Theorem 4.6. *The conjugate face map* $\mathcal{C}_K : \mathcal{F}_K \to \mathcal{E}_{K^\circ}$ *restricts to a surjective map* $\mathcal{F}_K \backslash \mathcal{E}_K \to \{F \in \mathcal{E}_{K^\circ} \mid F$ *has an incomplete normal cone* $\}$. *It restricts further to a surjective map with range* $\mathcal{C}_K(\mathcal{F}_K \setminus \mathcal{E}_K)$ *and with domain equal to those non-exposed faces of* K *which are coatoms of* $\mathcal{F}_K(F)$ *for some exposed face* F *of* K. *The preimage of* $F \in \mathcal{E}_{K^\circ}$ *under* $\mathcal{C}_K$ *is* $\mathcal{C}_K^{-1}(F) = \mathrm{cl}_{\mathcal{E}}^{-1}(\mathcal{C}_{K^\circ}(F))$.

Proof. We use Proposition 4.1 extensively in the proof. About the preimage of an exposed face $F \in \mathcal{E}_{K^\circ}$ we notice for faces $G \in \mathcal{F}_K$ that

$$\mathcal{C}_K(G) = F \quad \Longleftrightarrow \quad \mathcal{C}_{K^\circ} \circ \mathcal{C}_K(G) = \mathcal{C}_{K^\circ}(F) \quad \Longleftrightarrow \quad \mathrm{cl}_{\mathcal{E}}(G) = \mathcal{C}_{K^\circ}(F).$$

We prove that the conjugate face of any non-exposed face has an incomplete normal cone. For a non-exposed face $F \in \mathcal{F}_K$ we consider the touching cone $T := \mathrm{pos}(F) \in \mathcal{T}_{K^\circ}$ and we consider the normal cone of its conjugate face

$$N := \Psi \circ \mathcal{C}_K(F) = \mathrm{cl}_{\mathcal{N}} \circ \mathrm{pos}(F) = \mathrm{cl}_{\mathcal{N}}(T).$$

Since $F \lneq_{\mathcal{F}} \mathrm{cl}_{\mathcal{E}}(F)$ the lattice isomorphism $\mathcal{F}_K \to \mathcal{T}_{K^\circ}$ of the positive hull operator pos implies $T \lneq_{\mathcal{T}} \mathrm{cl}_{\mathcal{N}}(T) = N$. By *Hausdorff's Maximal Principle* there exists a maximal chain C in the ideal $\mathcal{T}_{K^\circ}(N)$ including T and N, see Chap. VIII.7 in [4].

A proper inclusion $F_1 \subsetneq F_2$ of faces of K implies a dimension difference $\dim(F_1) < \dim(F_2)$ by Corollary 18.1.3 in [12]. Hence every chain in the face lattice $\mathcal{F}_K$ is finite and hence every chain in the touching cone lattice $\mathcal{T}_{K^\circ}$ is finite. So the penultimate element P in C exists and P is a coatom in $\mathcal{T}_{K^\circ}(N)$ because C is a maximal chain. By contradiction, if N is a complete normal cone, then $P \in \mathcal{N}_{K^\circ}$. Then $T \leq_{\mathcal{T}} P \lneq_{\mathcal{T}} N$ implies $\mathrm{cl}_{\mathcal{N}}(T) \leq_{\mathcal{N}} P$ and this contradicts $\mathrm{cl}_{\mathcal{N}}(T) = N$.

We prove surjectivity for the second, smaller, restriction. It suffices to find for every exposed face $F \in \mathcal{E}_{K^\circ}$ with incomplete normal cone $N := \Psi(F)$ a coatom G of $\mathcal{F}_K(\mathcal{C}_{K^\circ}(F))$ which is a non-exposed face of K and to show $\mathcal{C}_K(G) = F$. There exists a coatom $T \in \mathcal{T}_{K^\circ}(\Psi(F))$ such that $T \notin \mathcal{N}_{K^\circ}$ and we put $G := \mathrm{pos}^{-1}(T)$. Since pos is a lattice isomorphism $\mathcal{F}_K \to \mathcal{T}_{K^\circ}$, the face G is a coatom of $\mathcal{F}_K(\mathcal{C}_{K^\circ}(F))$. Since pos restricts to a bijection $\mathrm{pos} : \mathcal{E}_K \to \mathcal{N}_{K^\circ}$ from the exposed faces to the normal cones, G is a non-exposed face of K. Finally

$$\mathcal{C}_K(G) = \Phi \circ \mathrm{pos}(G) = \Phi(T) = \Phi \circ \mathrm{cl}_{\mathcal{N}}(T) = \Phi(N) = F$$

follows because $\mathrm{cl}_{\mathcal{N}}(T) = N$ holds as T is a coatom of $\mathcal{T}_{K^\circ}(N)$. □

Remark 4.7.

1. A face with an incomplete normal cone is a singular face (with normal cone of dimension ≥ 2). Indeed, a one-dimensional normal cone is a closed ray r and its two non-empty faces $\{0\}$ and r are both normal cones.
2. If we apply the dual lattice isomorphism $\mathcal{C}_{K^\circ}$ to the second restriction in Theorem 4.6 then it says that the preimage $\mathrm{cl}_{\mathcal{E}}^{-1}(F)$ of an exposed face F of K under the exposed closure $\mathrm{cl}_{\mathcal{E}}$ contains a coatom of the face lattice of F whenever $\mathrm{cl}_{\mathcal{E}}^{-1}(F) \supsetneq \{F\}$.
   In higher dimensions $n \geq 4$, of course, $\mathrm{cl}_{\mathcal{E}}^{-1}(F)$ can contain non-exposed faces, which are not coatoms of the face lattice of F. An example is the direct sum of two copies of the convex body in Figure 2.1 a) or c).

5. Conjugate faces in dimension two

We study conjugate faces of a polar pair of planar convex bodies, in particular we study the conjugate faces of non-exposed points and we count special points of the two convex bodies. We characterize self-dual planar convex bodies without non-exposed faces and we provide a general construction for them.

Let $K \subset \mathbb{R}^2$ be a convex body with $0 \in \mathrm{int}(K)$ and polar convex body K°. For extremal points $x \in K$ there are two alternatives. They have a normal cone $N(x)$ of dimension $\dim N(x) = 1$ resp. $\dim N(x) = 2$,

$$x \text{ is smooth} \qquad \text{resp. } x \text{ is a corner.}$$

The normal cone of a corner $x \in K$ is a *salient*[1] convex cone i.e. a convex cone such that $N(x) \cap (-N(x)) = \{0\}$. It follows that $N(x)$ has two distinct one-dimensional rays r_1, r_2 as its faces. Three types of corners can be distinguished:

[1]If $N(x)$ contains a line, then K is included in a hyperplane in $\mathbb{R}^2$, see e.g. (15)(iv) in [16], and $\mathrm{int}(K) = \emptyset$ follows.

x is a *polyhedral* corner if	$r_1, r_2 \in \mathcal{N}_K$,
x is a *mixed* corner if	$r_1 \in \mathcal{N}_K$ or $r_2 \in \mathcal{N}_K$ but not both,
x is a *free* corner if	$r_1, r_2 \notin \mathcal{N}_K$.

All facets of K are one-dimensional, we call them *segments*. If an extremal point $x \in K$ lies on a segment $s \subset K$ we call x and s *incident*. Any non-empty face of K is either an extremal point, a segment or K itself. The extremal points and relative interiors of segments are a partition of the boundary ∂K, see Theorem 18.2 in [12]. The boundary ∂K is homeomorphic to the unit circle S^1 under a positively homogeneous map (see the Theorem of Sz. Nagy in Construction 2.3).

Remark 5.1 (Local classification of extremal points).

1. Segments and corners are exposed faces by Lemma 4.4 and the proof that every non-exposed point is incident with a unique segment is given in Remark 1.1 in [16]. This shows

$$\begin{aligned}\mathcal{F}_K \setminus \mathcal{E}_K &= \{\text{non-exposed points}\}\\ &= \{\text{smooth extremal points incident with a unique segment}\}\end{aligned}$$

   except the inclusion "$\supset$" in the second equality. This follows by contradiction from the dual lattice isomorphism $\Psi|_{\mathcal{E}_K} : \mathcal{E}_K \to \mathcal{N}_K$ between exposed faces and normal cones in (6): If x is an exposed point incident with a segment s, then $x \subsetneq s$ shows $N(x) \supsetneq \Psi(s)$. Then $\dim N(x) = 2$ so x is not smooth.
2. By the dual lattice isomorphism $\Psi|_{\mathcal{E}_K} : \mathcal{E}_K \to \mathcal{N}_K$, exposed points $x \in \mathcal{E}_K$ split into the three types of corners above and into smooth exposed points:

x is a polyhedral corner	$\Longleftrightarrow$	x is the intersection of two segments,
x is a mixed corner	$\Longleftrightarrow$	x is incident with a unique segment,
x is a free corner or a smooth exposed point	$\Longleftrightarrow$	x is not incident with a segment.

   (We have seen in part 1. that a smooth exposed point is not incident with any segment.) Examples are depicted in Figure 2.1 and 2.2.

To understand the conjugate face map we divide the non-exposed points in

$$\mathcal{F}_K^{\text{mixed}} := \{x \in \mathcal{F}_K \setminus \mathcal{E}_K \mid \mathcal{C}_K(x)\text{is a mixed corner of } K^\circ\}$$

and

$$\mathcal{F}_K^{\text{free}} := \{x \in \mathcal{F}_K \setminus \mathcal{E}_K \mid \mathcal{C}_K(x)\text{is a free corner of } K^\circ\}\,.$$

We show that $\{\mathcal{F}_K^{\text{mixed}}, \mathcal{F}_K^{\text{free}}\}$ is a partition of the non-exposed points $\mathcal{F}_K \setminus \mathcal{E}_K$.

Lemma 5.2. *The conjugate face map $\mathcal{C}_K : \mathcal{F}_K \to \mathcal{E}_{K^\circ}$ restricts to the surjection*

$$\mathcal{F}_K \setminus \mathcal{E}_K \;\longrightarrow\; \{\textit{mixed corners of } K^\circ\} \,\cup\, \{\textit{free corners of } K^\circ\}\,.$$

The restriction of $\mathcal{C}_K$ to $\mathcal{F}_K^{\text{mixed}}$ is 1-1 and the restriction to $\mathcal{F}_K^{\text{free}}$ is 2-1.

Proof. All one-dimensional normal cones of K° are complete. This shows

$$\begin{aligned}&\{F \in \mathcal{E}_{K^\circ} \mid F \text{ has an incomplete normal cone}\}\\ &= \{\text{mixed corners in } K^\circ\} \cup \{\text{free corners in } K^\circ\}\end{aligned}$$

so Theorem 4.6 proves the first claim. Proposition 4.1 and the positive hull isomorphism $\mathrm{pos} : \mathcal{F}_K \to \mathcal{T}_{K^\circ}$ show for mixed and free corners $x \in K^\circ$

$$\mathrm{pos} \circ \mathcal{C}_K^{-1}(x) = \mathrm{cl}_{\mathcal{N}}^{-1}(N) = \begin{cases} \{N, T\} & \text{if } x \text{ is a mixed corner,} \\ \{N, T_1, T_2\} & \text{if } x \text{ is a free corner,} \end{cases}$$

where $N := N(x)$ is the normal cone and T, T_1, T_2 are rays such that $T_1 \neq T_2$. The inverse (13) of pos gives

$$\mathcal{C}_K^{-1}(x) = \begin{cases} \{N \cap \partial K, T \cap \partial K\} & \text{if } x \text{ is a mixed corner,} \\ \{N \cap \partial K, T_1 \cap \partial K, T_2 \cap \partial K\} & \text{if } x \text{ is a free corner.} \end{cases}$$

As $\mathrm{pos} : \mathcal{E}_K \to \mathcal{N}_{K^\circ}$ is a bijection between exposed faces and normal cones, $N \cap \partial K$ is an exposed face and $T \cap \partial K$ and $T_1 \cap \partial K \neq T_2 \cap \partial K$ are non-exposed points. $\square$

We use a 10-tuple to label the cardinalities (possibly ∞) of special points and segments:

convex body	K	K°
non-exposed points	n	n°
polyhedral corners	p	p°
mixed corners	m	m°
free corners	f	f°
segments	s	s°

The following linear equations hold for all planar convex bodies K with $0 \in \mathrm{int}(K)$. From Lemma 5.2 follow the equations

$$n = m^\circ + 2f^\circ\,, \qquad n^\circ = m + 2f\,. \tag{14}$$

By Corollary 4.5 we have

$$s = p^\circ + m^\circ + f^\circ\,, \qquad s^\circ = p + m + f\,. \tag{15}$$

Counting endpoints of segments, we get from Remark 5.1

$$2s = n + 2p + m\,, \qquad 2s^\circ = n^\circ + 2p^\circ + m^\circ\,. \tag{16}$$

These equations span a five-dimensional space of linear functionals and on the other hand the examples in Figure 2.1 plus the example of a triangle provide five linearly independent data vectors.

If K is self-dual then five cardinalities (n, p, m, f, s) suffice to count the special points. If they are finite, then (14)–(16) reduce to

$$\boxed{s - p = n - f = \tfrac{1}{2}(n + m)} \tag{17}$$

while three linearly independent data vectors are available from Figure 2.2 a)–c).

In the following proposition the necessary condition of an odd number of segments is likely to be well-known. For completeness we include a proof.

Lemma 5.3. *If K is self-dual and has no non-exposed points ($n = 0$), then all corners of K are polyhedral ($m = f = 0$) and $s = p$. Either K is strictly convex ($s = 0$), a polytope with $s = 3, 5, 7, \ldots$ segments or $s = \infty$.*

Proof. If $n = 0$ then Lemma 5.2 implies $m = f = 0$ and (16) shows $s = p$. A two-dimensional convex body without boundary segments ($s = p = 0$) is strictly convex i.e. all boundary points are smooth exposed points.

We consider $0 < s < \infty$. Any endpoint of a segment is an exposed point (since $n = 0$) hence it is a polyhedral corner (since $m = 0$). As the number of segments s is finite, they are connected in a polygonal circuit. So K is a polytope, which must have at least three segments. We show that p is odd.

Like in Construction 2.8 *b*) we rotate the polytope K about the origin such that $x_- := -\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\rho_K(-\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ maximizes the Euclidean norm on K. Then x_- is an exposed point of K, hence a polyhedral corner of K and the normal cone $N(x_-)$ is a two-dimensional salient convex cone. The segments incident with x_- lie in the ball of radius $|x_-|$ about the origin, so $N(x_-)$ meets $\mathbb{R}^2 \setminus H^-$ and $\mathbb{R}^2 \setminus H^+$. By (10) we have

$$N(x_-) = \mathrm{pos} \circ \mathcal{C}_K(x_-)\,,$$

so the conjugate face $\mathcal{C}_K(x_-)$ is a segment meeting $\mathbb{R}^2 \setminus H^-$ and $\mathbb{R}^2 \setminus H^+$. If $r : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection $a \mapsto (-a)$, then for some $y \in \mathbb{R}^2 \setminus H^-$ and $z \in \mathbb{R}^2 \setminus H^+$

$$r \circ \mathcal{C}_K(x_-) = [y, z]$$

is a segment of $K^* = K$. In particular, the boundary point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\rho_K(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ of K is not a corner since it lies in the relative interior of $[y, z]$.

We consider the polygonal chain C in the boundary ∂K from x_- to y which lies in H^+. Its segments are in bijection to its vertices distinct from x_- (by assigning endpoints in the direction from x_- to y). As x_- is a corner, the segments of C are the segments $s \neq r \circ \mathcal{C}_K(x_-)$ of K meeting $K \setminus H^-$. As $y \in K \setminus H^-$ the vertices of C distinct from x_- are the corners of K lying in $K \setminus H^-$. This gives a bijection

$$\{\text{corners in } K \setminus H^-\} \cong \{\text{segments } s \neq r \circ \mathcal{C}_K(x_-) \text{ meeting } K \setminus H^-\} \tag{18}$$

between a subset of corners of K and a subset of segments of K. Similarly we have

$$\{\text{corners in } K \setminus H^+\} \cong \{\text{segments } s \neq r \circ \mathcal{C}_K(x_-) \text{ meeting } K \setminus H^+\}\,. \tag{19}$$

By Corollary 4.5 the map $r \circ \mathcal{C}_K$ restricts to a bijection between the segments and the corners of K, one pair of corresponding faces being

$$\{x_-, r \circ \mathcal{C}_K(x_-)\}\,. \tag{20}$$

The corners and segments (18), (19) and (20) of K are a complete list. Hence, according to Lemma 2.7.2 *d*) the map $r \circ \mathcal{C}_K$ is a bijection between (18) and (19). □

We provide a construction for planar self-dual convex bodies without non-exposed points. Part *b*) shows that the construction is general.

Construction 5.4.

(a) *Let K satisfy $\rho_K(\pm\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = h_K(\pm\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = e^{\pm\lambda}$ for some $\lambda \in \mathbb{R}$. Let K have no non-exposed points and let all corners of K be polyhedral. We assume that $x_- := -\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\rho_K(-\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ is a smooth extremal point of K if and only if $x_+ := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\rho_K(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ is a smooth extremal point of K. Then $L := (K\cap H^+)\cup(K^*\cap H^-)$ is a self-dual convex body without non-exposed points.*

(b) *For every planar self-dual convex body K without non-exposed points exists a rotation in $\psi \in SO(2)$ such that $\psi(K)$ satisfies the assumptions in a).*

Proof. To prove b) we consider a rotated convex body K according to Construction 2.8 b). As rotation is an isometry, K is self-dual and has no non-exposed faces. All corners of K are polyhedral by Lemma 5.3. Since K has maximal x-extension on the x-axis we have $x_+ \in r \circ \mathcal{C}_K(x_-)$. If x_- is a smooth exposed point then Corollary 4.3 shows that $x_+ = r \circ \mathcal{C}_K(x_-)$ is a smooth exposed point and *vice versa.*

We prove a). Construction 2.8 a) already shows that L is a self-dual convex body. We show that L has no non-exposed points. First we show that any extremal point x of L in $L \setminus H^-$ is an exposed point of L (the case $x \in L \setminus H^+$ is analogous). By Lemma 2.7.1 x is an extremal point of K, hence an exposed point of K. By Lemma 2.7.2 c) there exists $u \in \mathbb{R}^2 \setminus H^-$ such that $\{x\} = K \cap H_K(u)$. Then Lemma 2.7.2 b) shows that $\{x\} = L \cap H_L(u)$ is an exposed point of L.

We show that a non-exposed point x_- in L leads to a contradiction, the proof for x_+ is analogous. We will use that all corners of K^* are polyhedral and (since $(K^*)^* = K$) that K^* has no non-exposed points (this is proved in Lemma 5.2). By Remark 5.1 any extremal point of K or K^* is either a smooth exposed point or a polyhedral corner.

Since L has maximal x-extension on the x-axis we have $x_+ \in r \circ \mathcal{C}_L(x_-)$. If x_- is a non-exposed point then Lemma 5.2 shows that $r \circ \mathcal{C}_L(x_-)$ is a mixed or a free corner of L so

$$x_+ = r \circ \mathcal{C}_L(x_-)\,. \tag{21}$$

The contradiction that x_+ is incident with two segments in L completes the proof.

If x_- is a non-exposed point of L then x_- is incident with a segment $[x_-, y]$ of L say for $y \in L \setminus H^- = K \setminus H^-$ (the case $y \in L \setminus H^+$ is analogous by arguing with K^* in place of K). Since x_- is incident with a unique segment, the smallest exposed face (8) of L containing x_- is the segment

$$\mathrm{cl}_{\mathcal{E}}(x_-) = [x_-, y]\,. \tag{22}$$

We show that x_+ is incident with a segment of L in H^-. The extremal point y of L is an extremal point of K by Lemma 2.7.1. Hence y is a polyhedral corner of K and also of L. Corollary 4.5 shows that the face $s := r \circ \mathcal{C}_L(y)$ is a segment of L. By (21), by the equation $\mathcal{C}_L(x_-) = \mathcal{C}_L \circ \mathrm{cl}_{\mathcal{E}}(x_-)$ from Proposition 4.1, by (22) and since $r \circ \mathcal{C}_L$ is antitone, we obtain that x_+ is incident with the segment s,

$$x_+ = r \circ \mathcal{C}_L(x_-) = r \circ \mathcal{C}_L([x_-, y]) \subset s\,.$$

Lemma 2.7.2 *d*) shows $s \subset H^-$.

We find a segment of L in H^+ incident with x_+. If x_+ is a smooth exposed point of K then x_- is a smooth exposed point of K by Corollary 4.3. This is wrong as x_- is incident with the segment $[x_-, y]$ of L hence is included in a segment of K. Otherwise if x_+ is not a smooth exposed point of K it is a polyhedral corner of K or lies in the relative interior of a segment of K. In both cases x_+ is included in a segment of K meeting $K \setminus H^-$, hence is incident with a segment of L included in H^+. $\square$

We give an example of a planar self-dual convex body with $n = 0$ and $s = p = \infty$.

Example 5.5. Let $E := \{u(\alpha) \mid \alpha = (\frac{k}{2} \pm 2^{-m})\pi, k \in \{1,3\}, m \in \mathbb{N}\} \cup \{\alpha(\frac{\pi}{2}), \alpha(\frac{3}{2}\pi)\}$ for $u(\alpha) := \binom{\cos(\alpha)}{\sin(\alpha)}$. It follows from Carathéodory's theorem, see e.g. Theorem 17.2 in [12], that the convex hull K of E is compact. Since $E \subset S^1$, the convex body K has no non-exposed points. The two accumulation points $\alpha(\frac{\pi}{2})$ and $\alpha(\frac{3}{2}\pi)$ of E are approximated by points of E both counterclockwise and clockwise on S^1, hence they are smooth exposed points of K. This shows that all corners of K are polyhedral. The convex body $(K \cap H^+) \cup (K^* \cap H^-)$ is self-dual and has no non-exposed points by Construction 5.4 *a*), it is depicted in Figure 2.2 d).

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