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A Note on Faces of Convex Sets

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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The faces of a convex set owe their relevance to an interplay between convexity and topology that is systematically studied in the work of Rockafellar. Infinite-dimensional convex sets are excluded from this theory as their relative interiors may be empty. Shirokov and the present author answered this issue by proving that every point in a convex set lies in the relative algebraic interior of the face it generates. This theorem is proved here in a simpler way, connecting ideas scattered throughout the literature. This article summarizes and develops methods for faces and their relative algebraic interiors and applies them to spaces of probability measures.

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1. Introduction

A *face* of a convex set K in a real vector space V is a convex subset of K including every pair of points in K that are the endpoints of some open segment intersected by this convex subset. One-point faces are in a one-to-one correspondence with extreme points and are useful in functional analysis [1, 5, 8, 15, 21]. Larger faces play a minor role beyond finite dimensions. A notable exception is Alfsen and Shultz' work on operator algebras [2].

Faces of finite-dimensional convex sets owe their success to an interplay of convexity and topology that appears in Grünbaum's work [12, Section 2.4] and is systematically studied by Rockafellar [17]. A central notion is the *relative interior* $\mathrm{ri}(K)$ of K, the interior of K in the induced topology on the affine hull $\mathrm{aff}(K)$ of K. Lacking monotonicity, the operator ri is not the interior operator of a topology. Imagine a side K of a triangle L in the Euclidean plane V; although $K \subset L$ holds, $\mathrm{ri}(K)$ and $\mathrm{ri}(L)$ are disjoint and nonempty. Rockafellar and Wets [18, p. 75] stress that the closure of $\mathrm{ri}(K)$ includes K in a Euclidean space V. Notably, $K \neq \emptyset$ implies $\mathrm{ri}(K) \neq \emptyset$.

The last assertion is false in a Hausdorff topological vector space, as the interior of K can be empty and the affine hull of K can still be equal to V.

(1) A first example is the convex set $K = \{x \in V : \ell(x) > 0\}$ defined by a discontinuous linear functional $\ell : V \to \mathbb{R}$, because $\ker(\ell)$ is dense in V.

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(2) The closed convex set $K = \{f \in V : f \geq 0 \text{ a.e.}\}$ in the Banach space $V = L^p([0,1])$ of p-integrable real functions, $1 \leq p < \infty$, has no interior points. Every function $f \in K$ is the limit of $f \cdot 1_{(1/n,1]} - 1_{[0,1/n]}$ as $n \to \infty$, where 1_A denotes the indicator function on a measurable set A.

Avoiding empty interiors, Borwein and Goebel [7, Thm. 2.8] study modified interiors of convex sets in Banach spaces. This article, instead, focuses on those faces of a convex set that happen to have nonempty relative interiors.

Among all vector topologies on V, the greatest interior (with respect to inclusion) of a convex set is achieved by the *finest locally convex topology* $\mathfrak{T}_\omega(V)$. We deduce this from the continuity of the map $\mathbb{R} \to V$, $\lambda \mapsto x + \lambda v$, for every $x, v \in V$, bearing in mind that a point $x \in K$ lies in the interior of K for $\mathfrak{T}_\omega(V)$, if and only if for every line g in V containing x, the intersection $g \cap K$ includes an open segment containing x [8, II.26]. The set of all such points x is called the *algebraic interior* [5, 9] or "core" [9, 15] of K.

Example (1) above ceases to exist in the topology $\mathfrak{T}_\omega(V)$, which renders every linear functional continuous [8, II.26]. The empty interior of Example (2) persists in the topology $\mathfrak{T}_\omega(V)$, as the interior of every closed convex subset of a Banach space is the algebraic interior of that set [9, p. 46]. Another example with empty algebraic interior is the set of univariate polynomials with real coefficients and positive leading coefficients, which Barvinok examines in several revealing exercises [5, III.1.6]. By contrast, if $\dim(V) < \infty$ then $\mathfrak{T}_\omega(V)$ is the Euclidean topology and, again, $K \neq \emptyset$ implies $\mathrm{ri}(K) \neq \emptyset$.

The relative interior $\mathrm{ri}(K)$ for the topology $\mathfrak{T}_\omega(V)$ can be described in terms of the Euclidean topology on a line. Shirokov and the present author [20] define the *relative algebraic interior* $\mathrm{rai}(K)$ of K as the set of points $x \in K$ such that for every line g in $\mathrm{aff}(K)$ containing x, the intersection $g \cap K$ includes an open segment containing x. This definition differs from that of the algebraic interior just in the affine space confining the lines. Holmes [15] calls $\mathrm{rai}(K)$ the "intrinsic core". We prove $\mathrm{ri}(K) = \mathrm{rai}(K)$ for $\mathfrak{T}_\omega(V)$ in Section 3. Topological vector spaces and relative interiors are ignored subsequently; instead, relative algebraic interiors are used consistently.

In their studies of constrained density operators [20], Shirokov and this author employ the *face of K generated by* a point $x \in K$. This is the smallest face of K containing x, which we denote by $F_K(x)$. A basic property is that $x \in \mathrm{rai}(F_K(x))$ holds for all $x \in K$.

This key result is deduced in Section 4 from two elementary assertions, whereas our prior proof unnecessarily employs the Kuratowski-Zorn lemma. The first elementary assertion (Proposition 4.1) is Alfsen's formula [1, p. 121]

$$F_K(x) = \{y \in K \mid \exists \epsilon > 0\colon x + \epsilon(x - y) \in K\}\ . \tag{1}$$

The second one (Proposition 4.4) is Borwein and Goebel's observation [7] that a point $x \in K$ lies in $\mathrm{rai}(K)$ if and only if for all $y \in K$ there is $\epsilon > 0$ such that $x + \epsilon(x - y) \in K$. A point x satisfying the latter proposition is called an *internal point* [11] or a "relatively absorbing point" [7] of K.

Consequences of $x \in \mathrm{rai}(F_K(x))$, $x \in K$, are organized roughly as follows. A review in Section 5 and new findings in Section 6 extend selected results from Section 6 and 18 of Rockafellar's monograph [17]. Sections 7 and 8 refer to Sections 2–4 of Dubins' paper [11] on infinite-dimensional convexity.

The methods of this paper are suitable to study the space of probability measures on a measurable space. The face generated by a probability measure is described in Section 9. The face generated by a Borel probability measure μ on $\mathbb{R}^d$ is related to Csiszár and Matúš' notion [10] of the *convex core* of μ in Section 10. Section 11 studies measures on the set of natural numbers.

2. Main definitions

Throughout this paper, V denotes a real vector space and K a convex subset of V, unless stated otherwise. If $x \neq y$ are distinct points of V, then

$$(x, y) = \{(1-\lambda)x + \lambda y \colon \lambda \in (0,1)\}$$

resp.

$$[x, y] = \{(1-\lambda)x + \lambda y \colon \lambda \in [0,1]\}$$

is called the *open segment* resp. *closed segment* with endpoints x, y. Each of the symbols $(x, x) = [x, x]$ denotes the singleton $\{x\}$ containing $x \in V$.

An *extreme set* [3] of K is a subset E of K including the closed segment $[x, y]$ for all points $x \neq y$ in K for which the open segment (x, y) intersects E. A point $x \in K$ is an *extreme point* of K if $\{x\}$ is an extreme set of K. A *face* of K is a convex extreme set of K. Clearly, any union or intersection of extreme sets of K is an extreme set of K. Since any intersection of convex sets is convex, any intersection of faces of K is a face of K. In particular, the intersection of all faces containing a point $x \in K$ is a face of K, which is called the *face of K generated by x*, and which is denoted by $F_K(x)$.

The *algebraic interior* of K is the set of all points x in K such that for every line g in V containing x, the intersection $g \cap K$ includes an open segment containing x [5]. The convex set K is *algebraically open* if it is equal to its algebraic interior. The *relative algebraic interior* $\mathrm{rai}(K)$ of K is the set of all points x in K such that for every line g in $\mathrm{aff}(K)$ containing x, the intersection $g \cap K$ includes an open segment containing x [20]. The convex set K is *relative algebraically open* if $K = \mathrm{rai}(K)$. A point $x \in K$ is an *internal point* [11] of K if for all $y \neq x$ in K there is $\epsilon > 0$ such that $x + \epsilon(x - y) \in K$.

3. The finest locally convex topology

One of the innovations of this paper is that Theorem 4.5 is a theorem of Zermelo-Fraenkel set theory. However, the use of the relative interior in a topological vector space creates a new dependence on the axiom of choice by Remark 3.2. Corollary 3.3 allows us to avoid this problem simply by dismissing topological vector spaces altogether, and relying on relative algebraic interiors of convex sets instead. This comes at the modest price that some theorems of topological vector spaces require a proof in this paper. For example, the theorem that the interior of a convex set is convex [8, II.14] would make Lemma 5.2 superfluous. The assertion that the interior of a set is open would essentially supersede Theorem 6.6 (up to questions regarding affine hulls).

A *fundamental system of neighborhoods* of a point x in a topological space is any set $\mathfrak{S}$ of neighborhoods of x such that for each neighborhood U of x there is a neighborhood $W \in \mathfrak{S}$ such that $W \subset U$. A topological real vector space is *locally convex* if there exists a fundamental system of neighborhoods of 0 that are convex sets [8, II.23].

It is known that a convex subset $C \subset V$ is open for the finest locally convex topology $\mathfrak{T}_\omega(V)$ on V if and only if C is algebraically open; a subset $U \subset V$ is open for $\mathfrak{T}_\omega(V)$ if and only if U is a union of convex open subsets of V [8, II.26]. Every nonempty affine subspace $A \subset V$ is isomorphic to the vector space of translations $A - A = \{x - y \mid x, y \in A\}$ by means of the affine isomorphism $\alpha : x \mapsto x - x_0$ defined by some point $x_0 \in A$. A topology is defined on A for which a subset $U \subset A$ is open if and only if $\alpha(U)$ is open for $\mathfrak{T}_\omega(A - A)$. This topology does not depend on x_0 and is denoted by $\mathfrak{T}_\omega(A)$. The topology induced on A by $\mathfrak{T}_\omega(V)$ is denoted by $\mathfrak{T}_\omega(V)|A$. By definition, a subset $U \subset A$ is open for $\mathfrak{T}_\omega(V)|A$ if and only if there exists an open set W for $\mathfrak{T}_\omega(V)$ such that $U = A \cap W$. Let $S, S' \subset V$ be linear subspaces. Then S' is a *complementary subspace of* S *in* V if $S \cap S' = \{0\}$ and if for all $x \in V$ there is $s \in S$ and $s' \in S'$ such that $x = s + s'$.

Proposition 3.1. *If A is an affine subspace of V, then $\mathfrak{T}_\omega(A) = \mathfrak{T}_\omega(V)|A$.*

Proof. Using the isomorphism $A \cong A - A$, we assume that 0 lies in A. Clearly, the induced topology $\mathfrak{T}_\omega(V)|A$ is locally convex, which shows that $\mathfrak{T}_\omega(A)$ is finer than $\mathfrak{T}_\omega(V)|A$. Conversely, let U be a convex open subset of A for the topology $\mathfrak{T}_\omega(A)$. Let B be a complementary subspace of A in V. It is easy to see that the convex set $U + B = \{u + b : u \in U, b \in B\}$ is open for $\mathfrak{T}_\omega(V)$ and that $A \cap (U + B) = U$ holds. This shows that $\mathfrak{T}_\omega(V)|A$ is finer than $\mathfrak{T}_\omega(A)$ and completes the proof. □

Remark 3.2. The complementary subspace used in Proposition 3.1 exist by the axiom of choice. Conversely, the existence of a complementary subspace for all subspaces of all vector spaces over the reals (or over any other field) implies a weak version of the axiom of choice [6, Lemma 2] that is equivalent to the full axiom of choice in Zermelo-Fraenkel set theory [16, Thm. 9.1]. □

Corollary 3.3. *We have* $\mathrm{ri}(K) = \mathrm{rai}(K)$ *for the topology* $\mathfrak{T}_\omega(V)$.

Proof. This follows immediately from Proposition 3.1. □

4. Every point lies in the relative algebraic interior of the face it generates

This section provides proofs for Alfsen's formula (1) describing the face generated by a point, and Borwein and Goebel's observation [7] that a point is an internal point of K if and only if it lies in $\mathrm{rai}(K)$. An immediate corollary is Shirokov and the present author's theorem [20] that every point lies in the relative algebraic interior of the face it generates. Let $x \in K$ and define

$$\begin{aligned} S_K(x) &= \{y \in K \mid \exists \epsilon > 0 \colon x + \epsilon(x - y) \in K\} \,, \\ C_K(x) &= \{y \in V \mid \exists \epsilon > 0 \colon x - \epsilon(x - y) \in K\} \,, \\ A_K(x) &= \{y \in V \mid \exists \epsilon > 0 \colon x \pm \epsilon(x - y) \in K\} \,. \end{aligned}$$

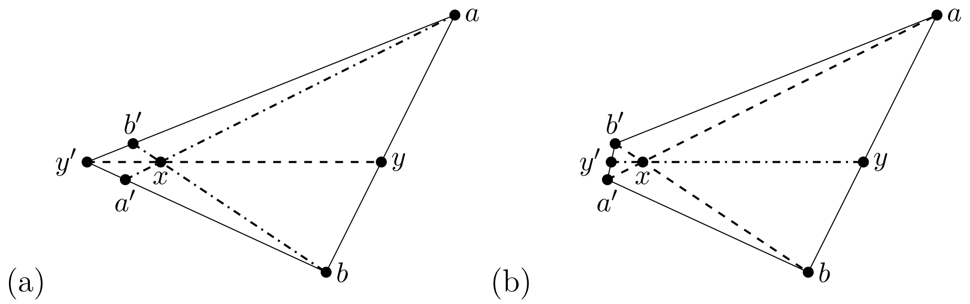

Figure 4.1: Configurations (in the plane) for Proposition 4.1.
The set $S_K(x)$ is (a) an extreme set and (b) convex.

Proposition 4.1. (Alfsen) *For all $x \in K$ we have $F_K(x) = S_K(x)$.*

Proof. Let y be a point in $S_K(x)$ and $y \neq x$. Then there is $\epsilon > 0$ such that x lies in the open segment with endpoints y and $x + \epsilon(x - y)$. As $F_K(x)$ is an extreme set containing x, it follows $y \in F_K(x)$, which proves $S_K(x) \subset F_K(x)$.

We finish the proof by showing that $S_K(x)$ is a face of K. Thus, we consider distinct points $a \neq b$ in K and a point y in the open segment (a, b), see Figure 4.1. Let $\eta \in (0, 1)$ such that $y = (1 - \eta)a + \eta b$. The coefficients for the following constructions are obtained from Menelaus' theorem [4, 13].

We show that $S_K(x)$ is an extreme set, see Figure 4.1(a). Assuming $y \in S_K(x)$, there is $\epsilon > 0$ such that $y' := x + \epsilon(x - y)$ lies in K. Let $\epsilon_a = \epsilon(1 - \eta)/(1 + \epsilon\eta)$ and $\xi_a = \epsilon\eta/(1 + \epsilon\eta)$. Then

$$a' := x + \epsilon_a(x - a) = (1 - \xi_a)y' + \xi_a b$$

lies in K, and hence $a \in S_K(x)$. Similarly, $b \in S_K(x)$.

We show that $S_K(x)$ is convex, see Figure 4.1(b). We assume $a, b \in S_K(x)$. Let $\epsilon_a, \epsilon_b > 0$ such that $a' := x + \epsilon_a(x - a)$ and $b' := x + \epsilon_b(x - b)$ lie in K; and let

$$\epsilon = \frac{\epsilon_a \epsilon_b}{(1 - \eta)\epsilon_b + \eta\epsilon_a} \quad \text{and} \quad \xi = \frac{\eta\epsilon_a}{(1 - \eta)\epsilon_b + \eta\epsilon_a}\,.$$

Then

$$y' := x + \epsilon(x - y) = (1 - \xi)a' + \xi b'$$

lies in K, and hence $y \in S_K(x)$. □

The union in Corollary 4.2 extends over all closed segments in K, whose respective open segments contain x, and the singleton $\{x\} = (x, x) = [x, x]$.

Corollary 4.2. *Every point $x \in K$ is an internal point of $F_K(x)$ and we have $F_K(x) = \bigcup_{y,z \in K, x \in (y,z)} [y, z]$.*

Proof. The first assertion follows from the definition of $S_K(x)$ and from Proposition 4.1, which proves $F_K(x) = S_K(x)$. Regarding the second assertion, the inclusion "$\supset$" holds because $F_K(x)$ is an extreme set of K containing x. The inclusion "$\subset$" is implied by a proof of $S_K(x) \subset \bigcup_{y,z \in K, x \in (y,z)} [y, z]$. Let $y \in S_K(x)$. Then there is $\epsilon > 0$ such that $z := x + \epsilon(x - y) \in K$. Hence, $x = \frac{1}{\epsilon+1}(\epsilon y + z) \in (y, z)$ and $y \in [y, z]$ complete the proof. □

Note that $C_K(x) = \mathrm{cone}(K - x) + x$ holds for all $x \in K$, where $\mathrm{cone}(C)$ denotes the set $\bigcup_{\lambda \geq 0}\{\lambda x : x \in C\}$ for every convex set $C \subset V$ containing the origin. Borwein and Goebel [7, p. 2544] suggest that a preliminary step to Proposition 4.4 should be a proof that x is an internal point of K if and only if $\mathrm{aff}(K) = C_K(x)$ holds. Instead, we use the following Lemma 4.3.

Lemma 4.3. *Let x be an internal point of K. Then $\mathrm{aff}(K) \subset A_K(x)$ holds.*

Proof. Let $y \neq x$ be a point in $\mathrm{aff}(K)$. It suffices to find $\epsilon > 0$ such that the two points $x \pm \epsilon(x - y)$ are both contained in K.

Since $x, y \in \mathrm{aff}(K)$ and since $y \neq x$, there exist $y_i \in K$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, not all numbers α_i being zero, such that

$$y = x + \textstyle\sum_i \alpha_i y_i \quad \text{and} \quad \textstyle\sum_i \alpha_i = 0\,.$$

By the assumption that x is an internal point of K, there is $\epsilon_i > 0$ such that

$$y_i' := x + \epsilon_i(x - y_i)$$

is contained in K, $i = 1, \ldots, n$. Let $\|\alpha\|_1 = \sum_{i=1}^n |\alpha_i|$ and let ϵ be the minimum of $\min_i \epsilon_i/\|\alpha\|_1$ and $1/\|\alpha\|_1$. Then ϵ is strictly positive. We have

$$\begin{aligned} x \pm \epsilon(x - y) &= x \mp \epsilon \sum_i \alpha_i y_i = x \pm \epsilon \sum_i \alpha_i(x - y_i) \\ &= \sum_i \frac{|\alpha_i|}{\|\alpha\|_1} \underbrace{(x \pm \mathrm{sgn}(\alpha_i)\epsilon\|\alpha\|_1(x - y_i))}_{z_i :=}\,. \end{aligned}$$

If $\pm\,\mathrm{sgn}(\alpha_i) = -1$, then $z_i \in [y_i, x] \subset K$ holds because of $\epsilon\|\alpha\|_1 \leq 1$. If $\mathrm{sgn}(\alpha_i) = 0$, then $z_i = x \in K$. If $\pm\,\mathrm{sgn}(\alpha_i) = +1$, then $z_i \in [x, y_i'] \subset K$ because $\epsilon\|\alpha\|_1 \leq \epsilon_i$. This shows that the points $x \pm \epsilon(x - y)$ are convex combinations of points in K, and therefore are themselves points in K. □

Proposition 4.4 is mentioned on p. 2544 of [7].

Proposition 4.4. (Borwein and Goebel) *For all $x \in K$, the following assertions are equivalent.*

(1) *The point x is an internal point of K.*
(2) *We have $x \in \mathrm{rai}(K)$.*
(3) *We have $A_K(x) = C_K(x)$.*
(4) *We have $C_K(x) = \mathrm{aff}(K)$.*

Proof. If x is an internal point of K, then Lemma 4.3 shows $\mathrm{aff}(K) \subset A_K(x)$. This implies (2) by the definition of the relative algebraic interior. It also implies (3) and (4), because the inclusions $A_K(x) \subset C_K(x) \subset \mathrm{aff}(K)$ are trivial. As (2) $\Rightarrow$ (1) is clear, it suffices to prove (n) $\Rightarrow$ (1) for $n = 3, 4$.

Assume (3) is true and let $y \neq x$ be a point of K. Since $K \subset C_K(x)$, the point y lies in $A_K(x)$. This provides $\epsilon > 0$ such that $x + \epsilon(x - y)$ lies in K. Hence, x is an internal point of K.

Assume (4) is true and let $y \neq x$ be a point of K. Then $z := 2x - y$ lies in $\operatorname{aff}(K)$ and hence in $C_K(x)$. This provides $\epsilon > 0$ such that

$$x + \epsilon(x - y) = x - \epsilon(x - z) \in K$$

and shows that x is an internal point of K. □

This section's main result is a novel proof for Theorem 2.3 in [20], which states the following:

Theorem 4.5. (Weis and Shirokov) *For all $x \in K$ we have $x \in \operatorname{rai}(F_K(x))$.*

Proof. The point x is an internal point of $F_K(x)$ by Corollary 4.2. It lies in the relative algebraic interior of $F_K(x)$ by Proposition 4.4. □

Corollary 4.6. *For all $x \in K$ we have $\operatorname{aff}(F_K(x)) = A_K(x)$.*

Proof. Theorem 4.5 and Proposition 4.4 prove $\operatorname{aff}(F_K(x)) = A_{F_K(x)}(x)$. The inclusion $A_{F_K(x)}(x) \subset A_K(x)$ is clear. Conversely, if $y \in A_K(x)$, then there is $\epsilon > 0$ such that $x \pm \epsilon(x - y) \in K$. Since $F_K(x)$ is an extreme set of K containing x, this implies $x \pm \epsilon(x - y) \in F_K(x)$, and hence $y \in A_{F_K(x)}(x)$. □

5. Review on the face generated by a point

Here we review some of our prior work from [20, Section 2]. Lemma 5.1(1) matches [17, Thm. 18.1]. See Lemma 2.1 in [20] for a proof.

Lemma 5.1. *Let $C \subset K$ be a convex subset of K, let $E \subset K$ be an extreme set of K, let $F \subset K$ be a face of K, and let $x \in K$ be a point in K. Then*

(1) $\operatorname{rai}(C) \cap E \neq \emptyset \implies C \subset E$,
(2) $x \in F \iff F_K(x) \subset F$,
(3) $x \in \operatorname{rai}(F) \implies F = F_K(x)$.

Lemma 5.2 is proved in Lemma 2.2 in [20].

Lemma 5.2. *The complement $K \setminus \operatorname{rai}(K)$ of the relative algebraic interior $\operatorname{rai}(K)$ is an extreme set of K and $\operatorname{rai}(K)$ is a convex set.*

Whereas Lemma 5.1 and Lemma 5.2 are rather easy to prove, the remainder of this section relies on Theorem 4.5.

Corollary 5.3. *Let $S \subset K$. The following assertions are equivalent.*

(1) *S is an extreme set of K.*
(2) *S includes the face $F_K(x)$ of K generated by any point x in S.*
(3) *S is the union of the faces $F_K(x)$ of K generated by the points x in S.*

Proof. See Corollary 2.5 in [20]; a proof is provided for easy reference. (1) ⇒ (2) follows from Lemma 5.1(1) as $x \in \operatorname{rai}(F_K(x))$ holds for all $x \in S$ by Theorem 4.5. (2) ⇒ (3) follows from $x \in F_K(x)$ for all $x \in S$. (3) ⇒ (1) follows directly from the definition of an extreme set. □

Corollary 5.4, and Theorem 6.8 below, match [17, Thm. 18.2]. Let

$$\mathfrak{U}_1 = \{\mathrm{rai}(F_K(x))\colon x \in K\}\,, \text{ and}$$
$$\mathfrak{U}_2 = \{\mathrm{rai}(F)\colon F \text{ is a face of } K\} \setminus \{\emptyset\}\,.$$

We recall that a partition of K is a family of nonempty subsets of K whose elements are mutually disjoint and whose union is K.

Corollary 5.4. *We have $\mathfrak{U}_1 = \mathfrak{U}_2$, the family $\mathfrak{U}_2$ is a partition of K, and*

$$\{F \textit{ is a face of } K\colon \mathrm{rai}(F) \neq \emptyset\} \longrightarrow \mathfrak{U}_2, \quad F \longmapsto \mathrm{rai}(F), \ \textit{ is a bijection.}$$

Proof. See Corollary 2.6 in [20]; a proof is provided for easy reference. The union of the family $\mathfrak{U}_1$ covers K as $x \in \mathrm{rai}(F_K(x))$ by Theorem 4.5. Since $\mathfrak{U}_1 \subset \mathfrak{U}_2$ is clear, proving that the elements of $\mathfrak{U}_2$ are mutually disjoint implies that $\mathfrak{U}_1 = \mathfrak{U}_2$ and that $\mathfrak{U}_2$ is a partition of K. Let F, G be faces of K and let $x \in \mathrm{rai}(F) \cap \mathrm{rai}(G)$. Then Lemma 5.1(3) shows $F = F_K(x) = G$. This also shows that the map in question is injective. Its surjectivity is clear. □

Remark 5.5. (Partitions of extreme sets) (1) Every extreme set E of K is the union of the family $\{\mathrm{rai}(F_K(x)) : x \in E\}$ by Corollary 5.3(2) and Theorem 4.5. This family is a partition of E by Corollary 5.4.

(2) There exist subfamilies of the partition $\mathfrak{U}_1$ in Corollary 5.4, whose union is not an extreme set of K. An example is the subfamily having as its only element the open unit interval $(0, 1)$ when $K = [0, 1]$.

(3) If the present concept of an extreme set is replaced with that underlying Dubins' work [11], then the union of any subfamily of $\mathfrak{U}_1$ will be an extreme set, cf. Theorem 8.1. □

Corollary 5.6 and Corollary 5.7 are similar to [20, Corollary 2.7]. Proofs are provided for easy reference.

Corollary 5.6. *Let F be a face of K and let $x \in K$. The following statements are equivalent.*

(1) *We have $x \in \mathrm{rai}(F)$.*
(2) *We have $F = F_K(x)$.*
(3) *We have $\mathrm{rai}(F) = \mathrm{rai}(F_K(x))$.*

Proof. (1) ⇒ (2) is Lemma 5.1(3). (2) ⇒ (3) is clear. (3) ⇒ (1) is implied by Theorem 4.5. □

Corollary 5.7. *Let $x, y \in K$. The following statements are equivalent.*

(1) *We have $x \in \mathrm{rai}(F(y))$.*
(2) *We have $F_K(y) = F_K(x)$.*
(3) *We have $\mathrm{rai}(F_K(y)) = \mathrm{rai}(F_K(x))$.*
(4) *We have $x \in F_K(y)$ and $y \in F_K(x)$.*

Proof. (1) ⇔ (2) ⇔ (3) is the special case of Corollary 5.6 when F is replaced with $F_K(y)$. (2) ⇒ (4) is clear, and (4) ⇒ (2) follows from Lemma 5.1(2). □

Proposition 5.8 complements [11, Thm. 4.3] but is not equivalent to it, as different concepts of a "face" are in use (see Section 8 below).

Proposition 5.8. *Let $K, L \subset V$ be convex sets and let $x \in K \cap L$. Then*

$$F_{K\cap L}(x) = F_K(x) \cap F_L(x)\,, \tag{1}$$

$$\mathrm{rai}\,\big(F_{K\cap L}(x)\big) = \mathrm{rai}\,\big(F_K(x)\big) \cap \mathrm{rai}\,\big(F_L(x)\big)\,, \tag{2}$$

$$\mathrm{aff}\,\big(F_{K\cap L}(x)\big) = \mathrm{aff}\,\big(F_K(x)\big) \cap \mathrm{aff}\,\big(F_L(x)\big)\,. \tag{3}$$

Proof. See [20, Proposition 2.13]. □

Corollary 5.9. *Let $K, L \subset V$ be convex sets and let F be a nonempty face of $K \cap L$ with* $\mathrm{rai}(F) \neq \emptyset$*. Then F is the intersection of a face of K and a face of L. A sufficient condition for* $\mathrm{rai}(F) \neq \emptyset$ *is that F have finite dimension.*

Proof. Let $x \in \mathrm{rai}(F)$. Lemma 5.1(3) shows $F = F_{K\cap L}(x)$ and Proposition 5.8(1) proves the first claim. If $\dim(F) < \infty$, then $\mathfrak{T}_\omega(\mathrm{aff}(F))$ is the Euclidean topology on $\mathrm{aff}(F)$ [8, II.26]. Hence, the relative interior of F is nonempty by Theorem 6.2 in [17]. Corollary 3.3 shows that the relative algebraic interior is nonempty, too. □

Is the assumption of $\mathrm{rai}(F) \neq \emptyset$ in Corollary 5.9 necessary?

Open Problem 5.10. *Are there convex sets K and L and a face F of $K \cap L$ that can not be written as the intersection of a face of K and a face of L?*

The analogue of Proposition 5.8 for infinitely many convex sets is wrong.

Example 5.11. We consider the open segment $(-\epsilon, 1+\epsilon)$ for every $\epsilon > 0$. The intersection $\bigcap_{\epsilon>0}(-\epsilon, 1+\epsilon)$ is the closed unit interval $[0,1]$. The only face of $(-\epsilon, 1+\epsilon)$ containing the extreme point 0 of $[0,1]$ is $(-\epsilon, 1+\epsilon)$ itself. Therefore, $\{0\}$ is not an intersection of faces of $\{(-\epsilon, 1+\epsilon)\}_{\epsilon>0}$, as such an intersection includes $[0,1]$. More examples are obtained by replacing some or all of the open segments $(-\epsilon, 1+\epsilon)$ with closed segments $[-\epsilon, 1+\epsilon]$.

6. Novel results on faces and relative algebraic interiors

This section is inspired by properties of relative interiors of convex sets in finite dimensions [17]. Various insights into relative algebraic interiors are facilitated by Proposition 6.1 and Corollary 6.2. The unions in these statements extend over all open segments in K containing x, and the singleton $\{x\} = (x, x)$. Proposition 6.1 matches part of [17, Thm. 6.1].

Proposition 6.1. *For all $x \in K$ we have* $\mathrm{rai}(F_K(x)) = \bigcup_{y,z\in K, x\in(y,z)}(y,z)$*.*

Proof. The point x lies in the left-hand side of the equation by Theorem 4.5 and in the right-hand side by definition. Let $a \in K$ and $a \neq x$. Then

$$\begin{aligned} a \in \mathrm{rai}(F_K(x)) &\overset{\text{Cor. 5.7(4)}}{\iff} a \in F_K(x) \text{ and } x \in F_K(a) \\ &\overset{\text{Cor. 4.2}}{\iff} \exists y, z \in K : x \in (a, y) \text{ and } a \in (x, z) \\ &\overset{a\neq x}{\iff} \exists y, z \in K : x, a \in (y, z) \end{aligned}$$

proves the claim. □

Corollary 6.2. *Let* $x \in K$. *Then* $\mathrm{rai}(K) = \bigcup_{y,z\in K, x\in(y,z)}(y,z)$ *holds if and only if* $x \in \mathrm{rai}(K)$.

Proof. Corollary 5.6 shows that $x \in \mathrm{rai}(K)$ is equivalent to $\mathrm{rai}(K) = \mathrm{rai}(F_K(x))$. Now, the claim follows from Proposition 6.1. □

Corollary 6.3 matches part of [17, Thm. 6.2].

Corollary 6.3. *If* $\mathrm{rai}(K) \neq \emptyset$, *then* $\mathrm{aff}(\mathrm{rai}(K)) = \mathrm{aff}(K)$ *holds.*

Proof. Let $x \in \mathrm{rai}(K)$ and $y \in \mathrm{aff}(K)$. By the definition of the relative algebraic interior, there is some $\epsilon > 0$ such that $x \pm \epsilon(x-y) \in K$. Corollary 6.2 shows that $y_\pm = x \pm \frac{\epsilon}{2}(x-y) \in \mathrm{rai}(K)$, which implies

$$y = \left(\frac{1}{2} - \frac{1}{\epsilon}\right) y_+ + \left(\frac{1}{2} + \frac{1}{\epsilon}\right) y_- \in \mathrm{aff}(\mathrm{rai}(K))\,.$$

The opposite inclusion is obvious. □

Corollary 6.4 matches part of [17, Thm. 6.5].

Corollary 6.4. *Let* $K, L \subset V$ *be convex sets and* $\mathrm{rai}(K) \cap \mathrm{rai}(L) \neq \emptyset$. *Then*

$$\mathrm{rai}(K \cap L) = \mathrm{rai}(K) \cap \mathrm{rai}(L) \quad \textit{and} \quad \mathrm{aff}(K \cap L) = \mathrm{aff}(K) \cap \mathrm{aff}(L)\,.$$

The intersection of two relative algebraically open convex sets is relative algebraically open.

Proof. Let $x \in \mathrm{rai}(K) \cap \mathrm{rai}(L)$. Then $K = F_K(x)$ and $L = F_L(x)$ by Lemma 5.1(3) and the first claim follows from Proposition 5.8. If K and L are relative algebraically open, then

$$K \cap L = \mathrm{rai}(K) \cap \mathrm{rai}(L) = \mathrm{rai}(K \cap L)$$

shows that the intersection $K \cap L$ is relative algebraically open. □

Theorem 6.5 matches part of [17, Thm. 6.6].

Theorem 6.5. *Let* $\mathrm{rai}(K) \neq \emptyset$ *and let* $\alpha : V \to W$ *be an affine map to a real vector space* W. *Then* $\mathrm{rai}(\alpha(K)) = \alpha(\mathrm{rai}(K))$.

Proof. Let $x \in \mathrm{rai}(K)$. First, we show $\alpha(x) \in \mathrm{rai}(\alpha(K))$. Let $y \neq \alpha(x)$ be a point in $\alpha(K)$ and choose any $y' \in \alpha|_K^{-1}(y)$. Since x is an internal point of K, there exists $\epsilon > 0$ such that $x + \epsilon(x - y')$ lies in K. Applying α shows that $\alpha(x) + \epsilon(\alpha(x) - y)$ lies in $\alpha(K)$, hence $\alpha(x)$ is an internal point of $\alpha(K)$. The implication (1) ⇒ (2) of Proposition 4.4 implies $\alpha(x) \in \mathrm{rai}(\alpha(K))$. Second, Corollary 6.2 proves

$$\begin{aligned}\alpha(\mathrm{rai}(K)) &= \alpha\left(\bigcup\nolimits_{y,z\in K, x\in(y,z)}(y,z)\right) = \bigcup\nolimits_{y,z\in K, x\in(y,z)}(\alpha(y),\alpha(z)) \\ &= \bigcup\nolimits_{y',z'\in\alpha(K), \alpha(x)\in(y',z')}(y',z')\,.\end{aligned}$$

As $\alpha(x) \in \mathrm{rai}(\alpha(K))$, the last expression of this equation equals $\mathrm{rai}(\alpha(K))$, again by Corollary 6.2. □

Theorem 6.6 matches Problem 2 in [5, III.1.6].

Theorem 6.6. *The set* $\mathrm{rai}(K)$ *is a relative algebraically open convex set.*

Proof. The convexity of $\mathrm{rai}(K)$ is provided by Lemma 5.2. That $\mathrm{rai}(K)$ is relative algebraically open can be proved by showing for all $x \in \mathrm{rai}(K)$ that $x \in \mathrm{rai}(\mathrm{rai}(K))$, or equivalently that x is an internal point of $\mathrm{rai}(K)$, as per the implication (1) $\Rightarrow$ (2) of Proposition 4.4. Let $y \neq x$ lie in $\mathrm{rai}(K)$. Corollary 6.2 provides $a, b \in K$ such that x and y are in the open segment (a, b). Relabel a and b, if necessary, such that $x = (1-\lambda)a + \lambda b$ and $y = (1-\mu)a + \mu b$ for scalars $0 < \lambda < \mu < 1$. Then $\frac{1}{2}(a+x)$ lies in $\mathrm{rai}(K)$, again by Corollary 6.2, and

$$x + \frac{\lambda}{2(\mu-\lambda)}(x-y) = \frac{1}{2}(a+x)$$

shows that x is an internal point of $\mathrm{rai}(K)$. □

Corollary 6.7 matches part of [11, Thm. 2.1], see also Corollary 8.2 below.

Corollary 6.7. *For every $x \in K$, the set $\mathrm{rai}(F_K(x))$ is the greatest relative algebraically open convex subset (with respect to inclusion) of K that contains x.*

Proof. Theorem 6.6 shows that $\mathrm{rai}(F_K(x))$ is a relative algebraically open convex set, which contains x by Theorem 4.5. Let $C \subset K$ be relative algebraically open and convex, and let $x \in C$. For each $y \in C$, Corollary 6.2 provides $a, b \in C$ such that $x, y \in (a, b)$. Hence, Proposition 6.1 shows $y \in \mathrm{rai}(F_K(x))$. □

Theorem 6.8 generalizes [17, Thm. 18.2].

Theorem 6.8. *Let $K \neq \emptyset$. The family of maximal relative algebraically open convex subsets (with respect to inclusion) of K is*

$$\mathfrak{U} = \{\mathrm{rai}(F_K(x))\colon x \in K\} = \{\mathrm{rai}(F)\colon F \text{ is a face of } K\} \setminus \{\emptyset\}\,.$$

The family $\mathfrak{U}$ is a partition of K. Each nonempty relative algebraically open convex subset of K is included in a unique element of $\mathfrak{U}$.

Proof. Let C be a nonempty relative algebraically open convex subset of K and let x be a point in C. Then $C \subset \mathrm{rai}(F_K(x))$ holds by Corollary 6.7. The set C can not be included in any other element of $\mathfrak{U}$ because $\mathfrak{U}$ is a partition by Corollary 5.4, which also proves the equality between the two descriptions of $\mathfrak{U}$.

Let C be a relative algebraically open convex subset of K. We show that the elements of $\mathfrak{U}$ are maximal. Assume that C includes $\mathrm{rai}(F_K(x))$ for some $x \in K$. Then C contains x by Theorem 4.5 and $C \subset \mathrm{rai}(F_K(x))$ follows from Corollary 6.7. We show that K has no other maximal relative algebraically open convex subsets. Assume that C is maximal. As $\mathfrak{U} \neq \emptyset$, the set C contains some point $x \in K$. This implies $C = \mathrm{rai}(F_K(x))$ by Corollary 6.7. □

7. Novel results on generators of faces

This section is motivated by theorems in [11], but differs from them due to the nonequivalent concepts of a "face" (see Section 8).

Lemma 7.1. *Let K_i be a convex subset of V, let $x_i \in \mathrm{rai}(K_i)$, and let $\lambda_i > 0$, $i = 1, \ldots, n$, such that $\lambda_1 + \ldots + \lambda_n = 1$. Then the point $\lambda_1 x_1 + \ldots + \lambda_n x_n$ lies in the relative algebraic interior of the convex hull of $K_1 \cup \cdots \cup K_n$.*

Proof. Let $n = 2$, let $\lambda := \lambda_2 \in (0,1)$, $x := (1-\lambda)x_1 + \lambda x_2$, and let C denote the convex hull of $K_1 \cup K_2$. Below, we construct for every $y \in C$ a number $\eta > 0$ such that $x + \eta(x - y)$ lies in C. This shows that x is an internal point of C, and the implication (1) $\Rightarrow$ (2) of Proposition 4.4 proves $x \in \mathrm{rai}(C)$.

As $y \in C$, there is $\mu \in [0,1]$ and there are $y_1 \in K_1$ and $y_2 \in K_2$ such that $y = (1-\mu)y_1 + \mu y_2$. Since x_i is an internal point of K_i, there exists $\epsilon > 0$ such that $z_i := x_i + \epsilon(x_i - y_i)$ lies in K_i, $i = 1, 2$. We distinguish two cases. If $\mu \leq \lambda$, then $\nu := 1 - \mu + \epsilon(\lambda - \mu)$ and $\eta := \epsilon(1-\lambda)/\nu$ are positive and

$$x + \eta(x - y) = \tfrac{1}{\nu}\left[\epsilon(\lambda - \mu)y_2 + (1-\lambda)(1-\mu)z_1 + \lambda(1-\mu)z_2\right] \in K\,.$$

If $\mu \geq \lambda$, then $\nu := \mu + \epsilon(\mu - \lambda)$ and $\eta := \epsilon\lambda/\nu$ are positive and

$$x + \eta(x - y) = \tfrac{1}{\nu}\left[\epsilon(\mu - \lambda)y_1 + (1-\lambda)\mu z_1 + \lambda\mu z_2\right] \in K\,.$$

Induction extends the claim from $n = 2$ to all $n \in \mathbb{N}$. □

Theorem 7.2 matches [11, (3.1)].

Theorem 7.2. *Let $C \subset K$ be convex. Then $\bigcup_{x\in C} F_K(x)$ is a face of K.*

Proof. The set $E := \bigcup_{x\in C} F_K(x)$ is a union of extreme sets, and hence an extreme set itself. We show that E is convex. Let $a_i \in E$ and let $c_i \in C$ such that $a_i \in F_K(c_i)$, $i = 1, 2$. By Corollary 4.2, there is a point $b_i \in K$ such that c_i lies in the relative algebraic interior of $[a_i, b_i]$, $i = 1, 2$. Lemma 7.1 shows that the point $c := (c_1 + c_2)/2$ lies in the relative algebraic interior of the convex hull D of $\{a_1, a_2, b_1, b_2\}$. Since C is convex, it contains c. Hence c lies in E, and Lemma 5.1(1) proves $D \subset E$. It follows that $[a_1, a_2] \subset D \subset E$. □

We define the *face generated by a subset $S \subset K$* as the smallest face of K containing S. We denote this face by $F_K(S)$. Corollary 7.3 matches [11, (3.3)].

Corollary 7.3. *Let $S \subset K$. Then $F_K(S) = \bigcup_{x\in C} F_K(x)$, where C is the convex hull of S.*

Proof. The union $F := \bigcup_{x\in C} F_K(x)$ is a face of K by Theorem 7.2. Let G be any face containing S. As G is convex it includes C and it also includes the face $F_K(x)$ for every $x \in C$ by Lemma 5.1(2). This proves $F \subset G$. □

Corollary 7.4. *Let $x_i \in K$, $i = 1, \ldots, n$. Let $S = \bigcup_{i=1}^n F_K(x_i)$. Let $\lambda_i > 0$, $i = 1, \ldots, n$, such that $\lambda_1 + \ldots + \lambda_n = 1$, and put $x = \lambda_1 x_1 + \ldots + \lambda_n x_n$. Then*

$$F_K(S) = F_K(\{x_1, \ldots, x_n\}) = F_K(x)\,.$$

Proof. As the inclusions $F_K(S) \supset F_K(\{x_1, \ldots, x_n\}) \supset F_K(x)$ follow from Corollary 7.3, it suffices to prove $F_K(S) \subset F_K(x)$. Lemma 7.1 shows that x lies in the relative algebraic interior of the convex hull C of $\{x_1, \ldots, x_n\}$. Hence C is included in $F_K(x)$ by Lemma 5.1(1). Lemma 5.1(2) then shows that $F_K(x_i)$ is included in $F_K(x)$ for all $i = 1, \ldots, n$, which proves $F_K(S) \subset F_K(x)$. □

Corollary 7.5 matches [11, (4.7)].

Corollary 7.5. *Let $K, L \subset V$ be convex sets and let K be relative algebraically open. Then every extreme set resp. face of $K \cap L$ is the intersection of K and an extreme set resp. face of L.*

Proof. Let E be an extreme set of $K \cap L$. Corollary 5.3(3) and Proposition 5.8(1) show

$$E = \bigcup_{x\in E} F_{K\cap L}(x) = \bigcup_{x\in E} (F_K(x) \cap F_L(x)) \,.$$

As K is relative algebraically open, Lemma 5.1(3) implies $F_K(x) = K$ for all $x \in K$, hence

$$E = \bigcup_{x\in E} (K \cap F_L(x)) = K \cap \bigcup_{x\in E} F_L(x) \,.$$

The set $\bigcup_{x\in E} F_L(x)$ is a union of extreme sets of L and hence an extreme set of L itself. If E is a face of $K \cap L$, then E is convex and Theorem 7.2 completes the proof. □

8. Dubins' terminology

A *d-extreme set* of K is a subset E of K including the open segment (x, y) for all points $x \neq y$ in K for which (x, y) intersects E. A point $x \in K$ is a *d-extreme point* of K if $\{x\}$ is a d-extreme set of K. A *d-face* of K is a convex d-extreme set of K. Clearly, any union or intersection of d-extreme sets of K is a d-extreme set of K. Hence, the intersection of all d-faces containing a point $x \in K$ is a d-face of K, which we call the *d-face of K generated by x*. This is the smallest d-face of K containing x.

Theorem 8.1. *A subset of K is a d-extreme set of K if and only if it is equal to the union $\bigcup_{x\in S} \mathrm{rai}(F_K(x))$ for some subset $S \subset K$. The d-face generated by $x \in K$ is $\mathrm{rai}(F_K(x))$. A point in K is a d-extreme point of K if and only if it is an extreme point of K.*

Proof. Let E be a d-extreme set of K and let $x \in E$. As $x \in \mathrm{rai}(F_K(x)) \subset E$ holds by Proposition 6.1, we have $E = \bigcup_{x\in E} \mathrm{rai}(F_K(x))$. Conversely, let $S \subset K$ be any subset and assume that a point y lies in the open segment (a, b) with endpoints $a \neq b$ in K and in the set $\mathrm{rai}(F_K(x))$ for some $x \in S$. The open segment (a, b) is included in $\mathrm{rai}(F_K(y))$ by Corollary 6.7 and hence in $\mathrm{rai}(F_K(x))$, as $\mathrm{rai}(F_K(y)) = \mathrm{rai}(F_K(x))$ holds by Corollary 5.7.

The set $\mathrm{rai}(F_K(x))$ contains x by Theorem 4.5 and is convex by Lemma 5.2. Hence, it is the smallest d-face containing x by the first part of this theorem.

That "d-extreme point" and "extreme point" are equivalent terms is implied by the fact that a singleton cannot contain a segment no matter whether it is an open segment or a closed segment. □

Corollary 8.2 provides an alternative proof of [11, Thm. 2.1].

Corollary 8.2. *For all $x \in K$, the d-face of K generated by x is equal to $\mathrm{rai}(F_K(x))$ and equal to the greatest relative algebraically open convex subset of K containing x.*

Proof. Theorem 8.1 shows that $\mathrm{rai}(F_K(x))$ is the d-face of K generated by x. That $\mathrm{rai}(F_K(x))$ is the greatest relative algebraically open convex subset of K containing x is proved in Corollary 6.7. □

By definition [11], an *elementary d-face* of K is a nonempty relative algebraically open d-face of K.

Corollary 8.3. *Let $K \neq \emptyset$. A subset of K is an elementary d-face of K if and only if it equals* $\mathrm{rai}(F_K(x))$ *for some $x \in K$.*

Proof. Let F be a d-face of K. By Theorem 8.1, F is a union of a subfamily of $\mathfrak{U} := \{\mathrm{rai}(F_K(x)) \colon x \in K\}$. If F is an elementary d-face of K, then F cannot be a union of more than one element of $\mathfrak{U}$, because the elements of $\mathfrak{U}$ are the maximal relative algebraically open convex subsets of K by Theorem 6.8. □

Corollary 8.4. *Let $K \neq \emptyset$. The following statements are equivalent.*

(1) *K is relative algebraically open.*

(2) *K has exactly two d-faces (which are $\emptyset$ and K).*

(3) *K has exactly two faces (which are $\emptyset$ and K).*

Proof. Assuming (1), $\mathrm{rai}(K) = K$ is nonempty. Hence, the partition of K

$$\mathfrak{U} = \{\mathrm{rai}(F_K(x)) \colon x \in K\}$$

contains $\mathrm{rai}(K)$ as one of its elements by Corollary 5.4. Hence, $\emptyset$ and K are the only d-faces of K by Theorem 8.1.

The statement (2) implies that K has at most two faces, as every face of K is a d-face of K. Since $K \neq \emptyset$, the convex set K has exactly two faces.

Assuming (3), the convex set K has only one nonempty face. Then $\{\mathrm{rai}(K)\}$ is a partition of K by Corollary 5.4, which implies $K = \mathrm{rai}(K)$. □

9. Examples 1: Spaces of probability measures

Let $\mathcal{P} = \mathcal{P}(\Omega, \mathcal{A})$ denote the convex set of probability measures on a measurable space $(\Omega, \mathcal{A})$. A probability measure $\lambda \in \mathcal{P}$ is *absolutely continuous* with respect to $\mu \in \mathcal{P}$, symbolically $\lambda \ll \mu$, if every μ-null set is a λ-null set. The measures are *equivalent*, $\lambda \equiv \mu$, if $\lambda \ll \mu$ and $\mu \ll \lambda$. If $\lambda \ll \mu$, then we denote by $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} : \Omega \to [0, \infty)$ the *Radon-Nikodym derivative* of λ with respect to μ, which is a measurable function satisfying $\lambda(A) = \int_A \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\,\mathrm{d}\mu$ for all $A \in \mathcal{A}$, see for example Halmos [14, Sec. 31].

Lemma 9.1. *For every $\mu \in \mathcal{P}$, the set $\{\lambda \in \mathcal{P} : \lambda \ll \mu\}$ is a face of $\mathcal{P}$.*

Proof. Let $D(\mu) := \{\lambda \in \mathcal{P} : \lambda \ll \mu\}$. The set $D(\mu)$ is an extreme set of $\mathcal{P}$. Let $\lambda_1, \lambda_2 \in \mathcal{P}$, $s \in (0,1)$, and $\lambda := (1-s)\lambda_1 + s\lambda_2$ be contained in $D(\mu)$. If A is a μ-null set, then A is a λ-null set and hence a λ_i-null set for $i = 1, 2$.

The set $D(\mu)$ is convex, as $(1-s)\lambda_1 + s\lambda_2$ has the Radon-Nikodym derivative $(1-s)\frac{\mathrm{d}\lambda_1}{\mathrm{d}\mu} + s\frac{\mathrm{d}\lambda_2}{\mathrm{d}\mu}$ with respect to μ for all $\lambda_1, \lambda_2 \in D(\mu)$ and $s \in [0,1]$. □

If $\mu \in \mathcal{P}$, then we say a proposition $\pi(\omega)$, $\omega \in \Omega$, is true *μ-almost surely*, which we abbreviate as *μ-a.s.*, if $\mu(\{\omega \in \Omega \colon \pi(\omega) \text{ is false}\}) = 0$.

Theorem 9.2. *Let $\lambda, \mu \in \mathcal{P}$. The following assertions are equivalent.*

(1) *The measure λ lies in the face $F_{\mathcal{P}}(\mu)$ of $\mathcal{P}$ generated by μ.*

(2) *There is $c \in [1, \infty)$ such that $\lambda(A) \leq c\,\mu(A)$ holds for all $A \in \mathcal{A}$.*

(3) *We have $\lambda \ll \mu$ and there is $c \in [1, \infty)$ such that $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \leq c$ holds μ-a.s..*

Proof. Proposition 4.1 shows that a probability measure $\lambda \in \mathcal{P}$ lies in $F_{\mathcal{P}}(\mu)$ if and only if there is $\epsilon > 0$ such that $\mu + \epsilon(\mu - \lambda) \in \mathcal{P}$. The latter condition is equivalent to the nonnegativity of the set function $\mu + \epsilon(\mu - \lambda)$, and hence to part (2) of the theorem. It remains to prove the equivalence (2) $\Leftrightarrow$ (3).

If $\lambda \ll \mu$ and if there is $c \geq 1$ such that $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(\omega) \leq c$ holds μ-a.s., then part (2) follows (with the same constant c),

$$\lambda(A) = \int_A \tfrac{\mathrm{d}\lambda}{\mathrm{d}\mu}\,\mathrm{d}\mu \leq c\,\mu(A) \quad \text{for all } A \in \mathcal{A}.$$

Conversely, if $\lambda \ll \mu$ is false, then there is $A \in \mathcal{A}$ such that $\lambda(A) > \mu(A) = 0$, making part (2) impossible. If $\lambda \ll \mu$ is true but $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ is not bounded μ-a.s., then for every $c > 0$ there is $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(\omega) > c$ holds for all $\omega \in A$.

Then
$$\lambda(A) = \int_A \tfrac{\mathrm{d}\lambda}{\mathrm{d}\mu}\,\mathrm{d}\mu > c\,\mu(A)$$

proves that part (2) fails. □

Dubins in [11] asserts that a probability measure $\lambda \in \mathcal{P}$ is contained in the smallest d-face of $\mathcal{P}$ generated by $\mu \in \mathcal{P}$ if and only if $\lambda \ll \mu$ and there exists $c > 0$, such that for all $A \in \mathcal{A}$ we have $\mu(A) \leq c\,\lambda(A) \leq c^2\mu(A)$. Corollary 8.2 translates this assertion into Corollary 9.3.

Corollary 9.3. (Dubins) *Let $\lambda, \mu \in \mathcal{P}$. The following statements are equivalent.*

(1) *The measure λ lies in* $\mathrm{rai}(F_{\mathcal{P}}(\mu))$.

(2) *There is $c \geq 1$ such that $\mu(A)/c \leq \lambda(A) \leq c\,\mu(A)$ for all $A \in \mathcal{A}$.*

(3) *We have $\lambda \equiv \mu$ and there are $c_1, c_2 \in [1,\infty)$ such that $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \leq c_1$ holds μ-a.s. and $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \leq c_2$ holds λ-a.s..*

(4) *We have $\lambda \ll \mu$ and there is $c \in [1,\infty)$ such that $\frac{1}{c} \leq \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \leq c$ holds μ-a.s..*

Proof. Corollary 5.7 shows that $\lambda \in \mathrm{rai}(F_{\mathcal{P}}(\mu))$ is equivalent to $\lambda \in F_{\mathcal{P}}(\mu)$ and $\mu \in F_{\mathcal{P}}(\lambda)$, so the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from those of Theorem 9.2. Note that λ-a.s. is the same as μ-a.s. if $\lambda \equiv \mu$. Hence, (3) implies that $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \cdot \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = 1$ and hence $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = (\frac{\mathrm{d}\mu}{\mathrm{d}\lambda})^{-1} \geq 1/c_2$ holds μ-a.s., see for example [14, Thm. A, p. 133]. Conversely, if $\lambda \ll \mu$ and if there is $c \in [1,\infty)$ such that $\frac{1}{c} \leq \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ holds μ-a.s., then

$$\lambda(A) = \int_A \tfrac{\mathrm{d}\lambda}{\mathrm{d}\mu}\,\mathrm{d}\mu \geq \tfrac{1}{c}\,\mu(A) \quad \text{for all } A \in \mathcal{A}$$

implies that $\mu \equiv \lambda$ and that $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = (\frac{\mathrm{d}\lambda}{\mathrm{d}\mu})^{-1} \leq c$ holds λ-a.s.. □

10. Examples 2: Convex cores

In a second example, we consider the Borel σ-algebra $\mathcal{B}(d)$ of $\mathbb{R}^d$. The *convex core* $\mathrm{cc}(\mu)$ of $\mu \in \mathcal{P} = \mathcal{P}(\mathbb{R}^d, \mathcal{B}(d))$ is the intersection of all convex sets $C \in \mathcal{B}(d)$ of full measure $\mu(C) = \mu(\mathbb{R}^d)$. The convex core was introduced in [10] to extend exponential families in a natural way, such that information projections become properly defined. The *mean* of μ is the integral $m(\mu) = \int_{\mathbb{R}^d} x\,\mathrm{d}\mu(x) \in \mathbb{R}^d$, provided that each coordinate function is μ-integrable; otherwise, μ does not have a mean.

Theorem 10.1. (Csiszár and Matúš) *Let $\mu \in \mathcal{P}(\mathbb{R}^d, \mathcal{B}(d))$ have a mean. Then the convex core of μ equals $\mathrm{cc}(\mu) = m(\{\lambda \in \mathcal{P} \colon \lambda \ll \mu\})$. Moreover, to each $a \in \mathrm{cc}(\mu)$ there exists $\lambda \in \mathcal{P}$ with $\lambda \ll \mu$ and mean $m(\lambda) = a$ such that $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ is bounded μ-a.s..*

Theorem 10.1 is proved in Theorem 3 of [10]. We derive from it a description of the relative algebraic interior of the convex core.

Corollary 10.2. *Let $\mu \in \mathcal{P}(\mathbb{R}^d, \mathcal{B}(d))$ have a mean. Then $\mathrm{cc}(\mu) = m(F_{\mathcal{P}}(\mu))$. The relative algebraic interior of $\mathrm{cc}(\mu)$ is $\mathrm{rai}(\mathrm{cc}(\mu)) = m(\mathrm{rai}(F_{\mathcal{P}}(\mu)))$, which equals*

$$\mathrm{rai}(\mathrm{cc}(\mu)) = m\left(\left\{\lambda \in \mathcal{P} \mid \lambda \equiv \mu,\, \exists c \in (1,\infty) \colon \frac{1}{c} \leq \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \leq c \ \mu\text{-}a.s.\right\}\right) .$$

Proof. Theorem 9.2 and Theorem 10.1 show that $\mathrm{cc}(\mu) = m(F_{\mathcal{P}}(\mu))$. As μ lies in the relative algebraic interior of $F_{\mathcal{P}}(\mu)$ by Theorem 4.5, we obtain

$$\mathrm{rai}(\mathrm{cc}(\mu)) = m(\mathrm{rai}(F_{\mathcal{P}}(\mu)))$$

from Theorem 6.5. Corollary 9.3 completes the proof. □

The characterization of $\mathrm{rai}(\mathrm{cc}(\mu))$ in Corollary 10.2 is somewhat stronger than that in Lemma 5 of [10], which ignores the lower bound $0 < \frac{1}{c} \leq \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ μ-a.s.. Lemma 5 of [10] also shows $\mathrm{rai}(\mathrm{cc}(\mu)) = m(\{\lambda \in \mathcal{P} \colon \lambda \equiv \mu\})$, which cannot be deduced from Theorem 10.1 with the methods developed in this paper, without the assistance of other methods.

11. Examples 3: Discrete probability measures

In a third example, we consider the discrete σ-algebra $2^{\mathbb{N}}$ of all subsets of $\mathbb{N}$. Consider the Banach space $\ell^1 = \{x : \mathbb{N} \to \mathbb{C} \mid \|x\|_1 < \infty\}$ of absolutely summable sequences endowed with the ℓ^1-norm $\|x\|_1 = \sum_{n=1}^{\infty} |x(n)|$. We study $\mathcal{P} = \mathcal{P}(\mathbb{N}, 2^{\mathbb{N}})$ in terms of the set of probability mass functions

$$\Delta_{\mathbb{N}} = \{p : \mathbb{N} \to \mathbb{R} \mid \forall n \in \mathbb{N} \colon p(n) \geq 0 \text{ and } \|p\|_1 = 1\} .$$

The map $\mathcal{P} \to \Delta_{\mathbb{N}}$ that maps a probability measure $\mu \in \mathcal{P}$ to its Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ with respect to the counting measure ν, is an affine isomorphism. We endow $\mathcal{P}$ with the *distance in variation*

$$\|\mu - \lambda\| := 2 \sup\nolimits_{A \subset \mathbb{N}} |\mu(A) - \lambda(A)| , \quad \lambda, \mu \in \mathcal{P} .$$

Then $\mathcal{P} \to \Delta_{\mathbb{N}}$ is an isometry, as $\|\mu - \lambda\| = \|\frac{\mathrm{d}\mu}{\mathrm{d}\nu} - \frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\|_1$ holds [19, Section 3.9]. The *support* of $p \in \Delta_{\mathbb{N}}$ is $\mathrm{spt}(p) := \{n \in \mathbb{N} \colon p(n) > 0\}$. Lemma 11.1 is proved in [20, Lemma 2.12], and follows also from Theorem 9.2 and Corollary 9.3.

Lemma 11.1. *For every $p \in \Delta_{\mathbb{N}}$ we have*

$$F_{\Delta_{\mathbb{N}}}(p) = \left\{q \in \Delta_{\mathrm{spt}(p)} \colon \sup\nolimits_{n \in \mathrm{spt}(p)} q(n)/p(n) < \infty\right\} ,$$

$$\mathrm{rai}(F_{\Delta_{\mathbb{N}}}(p)) = \left\{q \in F_{\Delta_{\mathbb{N}}}(p) \colon \inf\nolimits_{n \in \mathrm{spt}(p)} q(n)/p(n) > 0\right\} .$$

Example 11.2. (Faces with empty relative algebraic interiors) For all subsets $I \subset \mathbb{N}$ we define

$$\Delta_I := \{p \in \Delta_{\mathbb{N}} \colon \mathrm{spt}(p) \subset I\} \quad \text{and} \quad \Delta_{I,\mathrm{fin}} := \{p \in \Delta_I \colon \nu(\mathrm{spt}(p)) < \infty\} .$$

The sets Δ_I and $\Delta_{I,\text{fin}}$ are faces of $\Delta_{\mathbb{N}}$, see [20, Example 2.10]. If I is an infinite set, then $\text{rai}(\Delta_I) = \text{rai}(\Delta_{I,\text{fin}}) = \emptyset$ holds by Lemma 5.1(3), as each of the faces Δ_I and $\Delta_{I,\text{fin}}$ is strictly larger than the face generated by any of its points p. This is clear if $J := \text{spt}(p)$ is finite. Otherwise, if J is infinite, let $p_H(n) := p(n)/(\sqrt{r_n} + \sqrt{r_{n+1}})$ for $n \in \mathbb{N}$, where $r_n := \sum_{m \geq n} p(m)$. Then $p_H \in \Delta_J \subset \Delta_I$ but $p_H \notin F_{\Delta_{\mathbb{N}}}(p)$ holds by Lemma 11.1. □

In [20, Section 2], we raise the question as to whether $\Delta_{\mathbb{N}}$ has other faces with empty relative algebraic interiors, aside from those described in Example 11.2. Example 11.3 shows the answer is yes.

Example 11.3. (Another face with empty relative algebraic interior) For every $s > 1$, let $p_s : \mathbb{N} \to \mathbb{R}$, $n \mapsto \zeta(s)^{-1} \cdot n^{-s}$, where $\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$ is the Euler-Riemann zeta function. For all $s, t > 1$, Lemma 11.1 shows that the probability mass function p_s is included in $F_{\Delta_{\mathbb{N}}}(p_t)$ if and only if $s \geq t$. Hence, Lemma 5.1(2) proves

$$F_{\Delta_{\mathbb{N}}}(p_s) \subset F_{\Delta_{\mathbb{N}}}(p_t) \iff s \geq t, \qquad s, t > 1.$$

By Lemma 11.4 below, for every $t \geq 1$, the union

$$F_t := \textstyle\bigcup_{s>t} F_{\Delta_{\mathbb{N}}}(p_s)$$

is a face of $\Delta_{\mathbb{N}}$ and $\text{rai}(F_t) = \emptyset$. Let $t > 1$. Then F_t is included in $F_{\Delta_{\mathbb{N}}}(p_t)$, which is properly included in $\Delta_{\mathbb{N}}$ by Ex. 11.2. We also have $F_t \neq \Delta_{I,\text{fin}}$ and $F_t \neq \Delta_I$ for all $I \subset \mathbb{N}$, because F_t contains the point p_{t+1} of support $\mathbb{N}$.

Lemma 11.4. *Let $\{x_\alpha\}_{\alpha \in A}$ be a set of points in a convex set K indexed by a totally ordered set A that has no greatest element, such that $\alpha \leq \beta$ if and only if $F_K(x_\alpha) \subset F_K(x_\beta)$ holds for all $\alpha, \beta \in A$. Then $F = \bigcup_{\alpha \in A} F_K(x_\alpha)$ is a face of K and $\text{rai}(F) = \emptyset$.*

Proof. The set F is convex. If $a, b \in F$, then $a \in F_K(x_\alpha)$ and $b \in F_K(x_\beta)$ for some $\alpha, \beta \in A$. Both points x_α and x_β lie in $F_K(x_{\max(\alpha,\beta)})$, hence the closed segment with endpoints a, b lies in $F_K(x_{\max(\alpha,\beta)}) \subset F$. The set F is an extreme set. If the open segment with endpoints $a \neq b$ in K intersects F, then it intersects $F_K(x_\alpha)$ for some $\alpha \in A$. It follows that $a, b \in F_K(x_\alpha) \subset F$.

Assume there is $a \in \text{rai}(F)$. Since $a \in F$, there is $\alpha \in A$ with $a \in F_K(x_\alpha)$. Lemma 5.1(1) implies $F \subset F_K(x_\alpha)$, which shows that α is the greatest element of A. This is excluded from the assumptions. □

Whereas $\Delta_{\mathbb{N}}$ is closed in the ℓ^1-norm [21], the face $F_{\Delta_{\mathbb{N}}}(p)$ is not closed for any $p \in \Delta_{\mathbb{N}}$ of infinite support. Indeed, Lemma 11.5 shows that the closure of $F_{\Delta_{\mathbb{N}}}(p)$ is $\Delta_{\text{spt}(p)}$ whereas $F_{\Delta_{\mathbb{N}}}(p)$ is strictly included in $\Delta_{\text{spt}(p)}$ by Ex. 11.2. Let the function $e_n \in \ell^1$ be defined by $e_n(m) = 1$ if $n = m$ and $e_n(m) = 0$ otherwise, $m, n \in \mathbb{N}$.

Lemma 11.5. *The closure of any face F of $\Delta_{\mathbb{N}}$ in the ℓ^1-norm is $\Delta_{I(F)}$, where $I(F) := \bigcup_{p \in F} \text{spt}(p)$. For every $p \in \Delta_{\mathbb{N}}$ we have $I(F_{\Delta_{\mathbb{N}}}(p)) = \text{spt}(p)$.*

Proof. First, the face $\Delta_{I,\text{fin}}$ of functions with finite support is dense in Δ_I for all $I \subset \mathbb{N}$. To see this, let $I = \mathbb{N}$ (without loss of generality) and let $p \in \Delta_{\mathbb{N}}$. Then $(p_k)_{k \in \mathbb{N}} \subset \Delta_{\mathbb{N},\text{fin}}$, defined by

$$p_k(n) = \begin{cases} p(n) & \text{if } n < k, \\ \sum_{m \geq k} p(m) & \text{if } n = k, \\ 0 & \text{else,} \end{cases} \qquad k, n \in \mathbb{N}\,,$$

converges to p, as $\|p - p_k\|_1 = 2 \sum_{m>k} p(m)$ for all $k \in \mathbb{N}$.

Second, if F is a face of $\Delta_{\mathbb{N}}$ then $\Delta_{I(F),\mathrm{fin}} \subset F \subset \Delta_{I(F)}$ holds. The right inclusion is obvious. To see the left inclusion, let $n \in I(F)$, and let $p \in F$ such that $n \in \mathrm{spt}(p)$. By Lemma 11.1, we have $e_n \in F_{\Delta_{\mathbb{N}}}(p)$. Then $e_n \in F$ follows from Lemma 5.1(2). This implies $\Delta_{I(F),\mathrm{fin}} \subset F$ as F is convex.

The preceding two arguments prove the first assertion. The second assertion is a special case of the first one and follows from Lemma 11.1. □

Lemma 11.5 shows that the norm closed faces of $\Delta_{\mathbb{N}}$ are in a one-to-one correspondence with the subsets of $\mathbb{N}$. This assertion is a special case of a more general property of von Neumann algebras [2]. The space ℓ^1 is the predual of the von Neumann algebra

$$\ell^\infty = \{x : \mathbb{N} \to \mathbb{C} \mid \sup_{n\in\mathbb{N}} |x(n)| < \infty\}\,.$$

The set $\Delta_{\mathbb{N}} \subset \ell^1$ is the *normal state space* of ℓ^∞, and the subsets I of $\mathbb{N}$ are in a one-to-one correspondence with the projections in ℓ^∞, that is to say, with functions $\mathbb{N} \to \{0, 1\}$. In a general von Neumann algebra, there is an order preserving isomorphism between the norm closed faces of the normal state space and the projections in the algebra [2, Thm. 3.35].

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