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Convexity of Generators of L^p-like Paranorms

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Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure, and $S_+ = S_+(\Omega, \Sigma, \mu)$ denote the set of all μ-integrable simple functions $\mathbf{x} : \Omega \to \mathbb{R}_+$ having support $\Omega(\mathbf{x})$ of positive measure. Then, for an arbitrary bijection $\varphi : (0, \infty) \to (0, \infty)$, the functional $\mathbf{P}_\varphi : S_+ \to \mathbb{R}_+$ given by $\mathbf{P}_\varphi(\mathbf{x}) := \varphi^{-1}\left(\int_{\Omega(\mathbf{x})} \varphi \circ x d\mu\right)$ is well defined. The results presented support the conjecture that subadditivity of $\mathbf{P}_\varphi$ implies the convexity of φ. The case of superadditivity of $\mathbf{P}_\varphi$ is also discussed.

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1. Introduction

To describe the investigation, denote by (Ω, Σ, μ) an arbitrary measure space, by $S(\Omega, \Sigma, \mu)$ the set of all μ-integrable simple functions $\mathbf{x} : \Omega \to \mathbb{R}$, and put

$$S_+(\Omega, \Sigma, \mu) := \{\mathbf{x} \in S(\Omega, \Sigma, \mu) : \mathbf{x} \geq 0\}.$$

For an arbitrary bijection $\varphi : (0, \infty) \to (0, \infty)$, the functional

$$\mathbf{P}_\varphi : S(\Omega, \Sigma, \mu) \to [0, \infty), \quad \mathbf{P}_\varphi(\mathbf{x}) := \begin{cases} \varphi^{-1}\left(\int_{\Omega(x)} \varphi \circ |\mathbf{x}| \, d\mu\right) & \text{if } \mu(\Omega(\mathbf{x})) > 0 \\ 0 & \text{if } \mu(\Omega(\mathbf{x})) = 0 \end{cases},$$

where $\Omega(\mathbf{x}) := \{\omega \in \Omega : \mathbf{x}(\omega) \neq 0\}$ is the support of $\mathbf{x} \in S(\Omega, \Sigma, \mu)$, is well-defined.

If a bijection φ generating $\mathbf{P}_\varphi$ is a power function such that $\varphi(t) = \varphi(1) t^p$ for some $p \geq 1$, then $\mathbf{P}_\varphi$, being the L_p-norm, satisfies the Minkowski triangle inequality, in particular it is subadditive in $S_+(\Omega, \Sigma, \mu)$:

$$\mathbf{P}_\varphi(\mathbf{x} + \mathbf{y}) \leq \mathbf{P}_\varphi(\mathbf{x}) + \mathbf{P}_\varphi(\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in S_+(\Omega, \Sigma, \mu), \tag{1}$$

and φ is convex.

Under rather weak regularity condition on φ (which can be omitted if the range of measure is enough rich), inequality (1) implies that φ is increasing (see Remark 1 and accompanying conjecture). Therefore, for the convenience, in this paper we can assume that φ is an increasing homeomorphism of $(0, \infty)$.

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Recall the following converse of the Minkowski inequality: *if* (Ω, Σ, μ) *is a measure space such that for some* $A, B \in \Sigma$ *we have*

$$0 < \mu(A) < 1 < \mu(B) < \infty,$$

then (1) *holds if and only if* φ *is a power function and* $\varphi(t) = \varphi(1)\, t^p$ *for some* $p \geq 1$*; so, if and only if* P_φ *is the* L^p*-norm* [4]. If the measure space is such that, for every $A \in \Sigma$,

$$\mu(A) \neq 0 \Longrightarrow \mu(A) \geq 1$$

(for instance, when (Ω, Σ, μ) a counting measure space), then inequality (1) is satisfied, if φ is convex (with respect to arithmetic mean) and geometrically convex, i.e. convex with respect to geometric mean, which means that

$$\varphi\left(\sqrt{st}\right) \leq \sqrt{\varphi(s)\,\varphi(t)}, \qquad s, t > 0,$$

(see Mulholland [11], and [6]). In the remaining possible case, if (Ω, Σ, μ) is such that, for every $A \in \Sigma$,

$$\mu(A) \neq \infty \Longrightarrow \mu(A) \leq 1,$$

inequality (1) holds, if the two-variable function

$$(0, \infty)^2 \ni (s, t) \longmapsto \varphi\left(\varphi^{-1}(s) + \varphi^{-1}(t)\right)$$

is concave, which implies the convexity of φ (see [4]).

In the present paper, assuming that (Ω, Σ, μ) is a measure space with at least two disjoint sets of finite and positive measure (referred in the sequel nontrivial), we prove that *if* φ *satisfies inequality* (1) *that is sharp for all not proportional functions, and*

$$\limsup_{r \to \infty} \frac{\varphi(r)}{r} = \infty,$$

then φ *is convex* (Theorem 1). Moreover, without any additional conditions on φ, *if in the measure space there are two disjoint sets* $A, B \in \Sigma$ *such that*

$$\min(\mu(A), \mu(B)) \leq 1 \leq \mu(A) + \mu(B),$$

then φ *is convex* (Theorem 2). In the proof a method of Gauss-type invariant means is applied.

Both results strongly support our conjecture: if (Ω, Σ, μ) is a nontrivial measure space, then subadditivity of $\mathbf{P}_\varphi$ implies the convexity of φ.

The respective implication that superadditivity of $\mathbf{P}_\varphi$ implies the concavity of φ is easier for research and it is discussed in Section 3.

2. Results and proofs

Theorem 2.1. *Let* (Ω, Σ, μ) *be a measure space with at least two disjoint sets of finite and positive measure. Assume that an increasing bijection* $\varphi : (0, \infty) \to (0, \infty)$ *satisfies inequality* (1)*. If inequality* (1) *is sharp for all not proportional functions* $x, y \in S_+(\Omega, \Sigma, \mu)$*, and*

$$\limsup_{r \to \infty} \frac{\varphi(r)}{r} = \infty,$$

then φ *is convex.*

Proof. By the assumption there are two disjoint Σ-measurable sets $A, B \in \Sigma$ of finite and positive measure. Setting

$$\mathbf{x} = x_1\chi_A + x_2\chi_B, \qquad \mathbf{y} = y_1\chi_A + y_2\chi_B$$

in (1), where χ_A stands for the characteristic function of A, we get, for all positive real $x_1, x_2, y_1, y_2 > 0$,

$$\varphi^{-1}\left(a\varphi\left(x_1+y_1\right)+b\varphi\left(x_2+y_2\right)\right) \le \varphi^{-1}\left(a\varphi\left(x_1\right)+b\varphi\left(x_2\right)\right)+\varphi^{-1}\left(a\varphi\left(y_1\right)+b\varphi\left(y_2\right)\right).$$

where $a = \mu(A)$, $b = \mu(B)$. Since $\varphi(0+) = \lim_{t\to 0}\varphi(t) = 0$, putting $\varphi(0) := 0$, we hence get

$$\varphi^{-1}\left(a\varphi\left(x_1+y_1\right)+b\varphi(x_2+y_2)\right) \le \varphi^{-1}(a\varphi\left(x_1\right)+b\varphi\left(x_2\right))+\varphi^{-1}\left(a\varphi\left(y_1\right)+b\varphi\left(y_2\right)\right) \quad (2)$$

for all $x_1, x_2, y_1, y_2 \ge 0$. Setting here

$$x_2 = y_1 = 0, \quad x_1 = \varphi^{-1}\left(\frac{s}{a}\right), \quad y_2 = \varphi^{-1}\left(\frac{t}{b}\right),$$

we get

$$\varphi^{-1}(s+t) \le \varphi^{-1}(s) + \varphi^{-1}(t), \qquad s, t \ge 0.$$

Since φ is strictly increasing, it follows that φ is superadditive i.e. that

$$\varphi(s+t) \ge \varphi(s) + \varphi(t), \qquad s, t \ge 0.$$

Note that if $s, t > 0$ then the functions $\mathbf{x} = \varphi^{-1}\left(\frac{s}{a}\right)\chi_A$, $\mathbf{y} = \varphi^{-1}\left(\frac{t}{b}\right)\chi_B$ are not proportional, and in consequence, so are the vectors $(x_1, x_2) = \left(\varphi^{-1}\left(\frac{s}{a}\right), 0\right)$ and $(y_1, y_2) = \left(0, \varphi^{-1}\left(\frac{t}{b}\right)\right)$, and the above inequality is sharp, that is φ is strictly superadditive:

$$\varphi(s+t) > \varphi(s) + \varphi(t), \qquad s, t > 0.$$

Setting $y_2 = 0$ in (2) we have, for all $x_1, y_1, x_2 \ge 0$,

$$\varphi^{-1}\left(a\varphi\left(x_1+y_1\right)+b\varphi\left(x_2\right)\right) \le \varphi^{-1}\left(a\varphi\left(x_1\right)+b\varphi\left(x_2\right)\right)+\varphi^{-1}\left(a\varphi\left(y_1\right)\right). \quad (3)$$

Take arbitrary $s, t > 0$, $s < t$. By the strict monotonicity of φ,

$$\varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s) > 0,$$

so

$$y := \varphi^{-1}\left(\frac{1}{a}\varphi\left(\varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s)\right)\right) > 0,$$

is a well defined positive number. Note that

$$\varphi(y) < \frac{t-s}{2a}. \quad (4)$$

Indeed, by the definition of y and the strict superadditivity of φ, we have

$$\begin{aligned}\varphi(y) &= \frac{1}{a}\varphi\left(\varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s)\right) \\ &< \frac{1}{a}\left[\varphi\left(\varphi^{-1}\left(\frac{s+t}{2}\right)\right) - \varphi\left(\varphi^{-1}(s)\right)\right] = \frac{1}{a}\left(\frac{s+t}{2} - s\right) = \frac{t-s}{2a}.\end{aligned}$$

Note that there exists $x > 0$ such that

$$\varphi(x+y) - \varphi(x) > \frac{t-s}{2a}. \tag{5}$$

Indeed, in the opposite case we would have

$$\varphi(x+y) \le \varphi(x) + \frac{t-s}{2a},$$

and hence, for every $x > 0$,

$$\varphi(x+2y) = \varphi((x+y)+y) \le \varphi(x+y) + \frac{t-s}{2a} \le \varphi(x) + 2\frac{t-s}{2a},$$

and, by the induction,

$$\varphi(x+ny) \le \varphi(x) + n\frac{t-s}{2a}, \qquad x > 0,\ n \in \mathbb{N},$$

whence
$$\frac{\varphi(x+ny)}{x+ny} \le \frac{\varphi(x)}{x+ny} + \frac{n}{x+ny}\frac{t-s}{2a}, \qquad x > 0,\ n \in \mathbb{N}.$$

By the superadditivity of φ (see the respective result for subadditive functions [2], p. 248, Theorem 7.8.3), we have

$$\kappa := \lim_{r\to\infty} \frac{\varphi(r)}{r} = \sup\left\{\frac{\varphi(x)}{x} : x > 0\right\}.$$

Hence, letting $n \to \infty$ in the previous inequality, we get

$$\frac{\varphi(x+ny)}{x+ny} \le \frac{\varphi(x)}{x+ny} + \frac{n}{x+ny}\frac{t-s}{2a}, \qquad x > 0,\ n \in \mathbb{N},$$

that is $\kappa \le \frac{t-s}{2ay}$, which contradicts the assumption that $\limsup_{x\to\infty} \frac{\varphi(x)}{x} = \infty$. Consequently, inequality (5) holds true as desired.

Since $\varphi(0)=0$, the continuity of φ and inequalities (4) and (5) imply that there is x_1,

$$0 < x_1 < y \tag{6}$$

such that

$$\varphi(x_1+y) - \varphi(x_1) = \frac{t-s}{2a}.$$

Note that

$$x_2 := \varphi^{-1}\left(\frac{1}{b}\left(\frac{s+t}{2} - a\varphi(x_1)\right)\right)$$

is well defined, as, by the definition of x_1, (6), and the monotonicity of φ, we have

$$\frac{s+t}{2} - a\varphi(x_1) \ge \frac{s+t}{2} - a\varphi(y) = \frac{s+t}{2} - \varphi\left(\varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s)\right) > 0.$$

Putting $y_1 := y$, and taking into account the above definitions of x_1, x_2, we have

$$\begin{aligned} & a\varphi(x_1+y_1)+b\varphi(x_2) \\ = \; & a[\varphi(x_1+y_1)-\varphi(x_1)]+[a\varphi(x_1)+b\varphi(x_2)] \\ = \; & a\left[\frac{t-s}{2a}\right]+\left[a\varphi(x_1)+b\varphi\left(\varphi^{-1}\left(\frac{1}{b}\left(\frac{s+t}{2}-a\varphi(x_1)\right)\right)\right)\right] \\ = \; & \frac{t-s}{2}+\frac{s+t}{2}=t, \end{aligned}$$

and
$$a\varphi(x_1)+b\varphi(x_2)=\frac{s+t}{2}.$$

Therefore, setting the numbers x_1, x_2, y_1 into inequality (3), we obtain

$$\varphi^{-1}(t)\le\varphi^{-1}\left(\frac{s+t}{2}\right)+\varphi^{-1}\left(\frac{s+t}{2}\right)-\varphi^{-1}(s),$$

that is
$$\varphi^{-1}\left(\frac{s+t}{2}\right)\ge\frac{\varphi^{-1}(s)+\varphi^{-1}(t)}{2},$$

which shows that φ^{-1} is Jensen concave. Since φ is increasing, it follows that it is continuous and convex. This completes the proof. □

Theorem 2.2. *Let* (Ω,Σ,μ) *an arbitrary measure space with two disjoint sets* $A, B\in\Sigma$ *such that*

$$0<\min(\mu(A),\mu(B))\le 1\le\mu(A)+\mu(B).$$

If $\varphi:(0,\infty)\to(0,\infty)$ *is an increasing bijection satisfying inequality (1), then* φ *is convex.*

Proof. Assume first that

$$\min(\mu(A),\mu(B))<1<\mu(A)+\mu(B).$$

By the converse Minkowski inequality [4] (see also [5]), there is $p\ge 1$ such that $\varphi(t)=\varphi(1)t^p$ for all $t>0$. Consequently, φ is convex.

To simplify the notations in two remaining cases

$$\min(\mu(A),\mu(B))=1 \quad\text{and}\quad \mu(A)+\mu(B)=1,$$

we put
$$a:=\mu(A) \text{ and } b:=\mu(B)$$

and note that, similarly as in the previous proof, inequality (1) implies (2).

Consider the case $\min(a,b)=1$.

Without any loss of generality, we can assume $a=1$. Setting $a=1$ and $y_2=0$ in (2), we have

$$\varphi^{-1}(\varphi(x_1+y_1)+b\varphi(x_2))\le\varphi^{-1}(\varphi(x_1)+b\varphi(x_2))+y_1, \qquad x_1,x_2,y_1>0. \quad (7)$$

Take arbitrary fixed $s,t>0$, $s<t$. The numbers

$$x_1=\varphi^{-1}(s), \qquad x_2=\varphi^{-1}\left(\frac{t-s}{2b}\right)$$

are positive and, by the assumed monotonicity of φ, so is the number

$$y_1 = \varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s).$$

Setting them into inequality (7), we obtain

$$\begin{aligned} &\varphi^{-1}\left(\varphi\left(\varphi^{-1}(s) + \left(\varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s)\right)\right) + b\varphi\left(\varphi^{-1}\left(\frac{t-s}{2b}\right)\right)\right) \\ \leq\ &\varphi^{-1}\left(\varphi\left(\varphi^{-1}(s)\right) + b\varphi\left(\varphi^{-1}\left(\frac{t-s}{2b}\right)\right)\right) + \varphi^{-1}\left(\frac{s+t}{2}\right) - \varphi^{-1}(s) \end{aligned}$$

which simplifies to

$$\varphi^{-1}(t) + \varphi^{-1}(s) \leq 2\varphi^{-1}\left(\frac{s+t}{2}\right)$$

and shows that φ^{-1} is Jensen concave in $(0, \infty)$. Since φ is increasing, it follows that φ is Jensen convex. The continuity of φ implies its convexity (see, for instance, Kuczma [3]).

Now assume that $a + b = 1$.

Take arbitrary $s, t > 0$. Setting $x_1 = y_2 = \varphi^{-1}(s)$, $x_2 = y_1 = \varphi^{-1}(t)$ in the inequality (2), and taking into account that $b = 1 - a$, we get

$$\varphi^{-1}(s) + \varphi^{-1}(t) \leq \varphi^{-1}(as + (1-a)t) + \varphi^{-1}((1-a)s + at)$$

(that is φ^{-1} is a-Wright convex). Putting

$$M(s,t) = as + (1-a)t, \qquad N(s,t) = (1-a)s + at, \qquad s, t > 0,$$

we can write this inequality as follows:

$$\varphi^{-1}(s) + \varphi^{-1}(t) \leq \varphi^{-1}(M(s,t)) + \varphi^{-1}(N(s,t)), \qquad s, t > 0. \tag{8}$$

Note that M and N are strict continuous means, and that the arithmetic mean $A(s,t) = \frac{s+t}{2}$ is Gauss-invariant with respect to the mean-type mapping (M, N), i.e. that $A \circ (M, N) = A$. It follows that (see [10], also [1]) the sequence $(M_n, N_n) := (M, N)^n$, $n \in \mathbb{N}$, of iterates of (M, N) converges (uniformly on compact subsets) to the mean-type mapping (A, A), that is

$$\lim_{n\to\infty} M_n(s,t) = \lim_{n\to\infty} N_n(s,t) = \frac{s+t}{2}, \qquad s, t > 0.$$

From (8), by induction, we have

$$\varphi^{-1}(s) + \varphi^{-1}(t) \leq \varphi^{-1}(M_n(s,t)) + \varphi^{-1}(N_n(s,t)), \qquad s, t > 0, \quad n \in \mathbb{N}.$$

Hence, by the continuity of φ^{-1}, letting $n \to \infty$, we obtain

$$\varphi^{-1}(s) + \varphi^{-1}(t) \leq 2\varphi^{-1}\left(\frac{s+t}{2}\right), \qquad s, t > 0,$$

which shows that φ^{-1} is Jensen concave. Consequently, similarly as in the preceding case, φ is convex. This completes the proof. □

Theorem 1 and Theorem 2 imply that an increasing bijection φ of $(0,\infty)$ satisfying inequality (1) can be not convex only if

$$\min(a,b) > 1 \qquad \text{or} \qquad a+b<1$$

and

$$\lim_{r\to\infty} \frac{\varphi(r)}{r} \text{ is finite.}$$

Example 2.3. Let (Ω,Σ,μ) be such that or every $A\in\Sigma$,

$$\mu(A)\neq\infty \Longrightarrow \mu(A)\leq 1.$$

If $\varphi(t)=\frac{t^2}{t+1}$ or $\varphi(t)=t\exp\left(-\frac{1}{t}\right)$, then the two variable function

$$(0,\infty)^2 \ni (s,t)\longmapsto \varphi\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right)$$

is concave, so $\mathbf{P}_\varphi$ is subadditive in $S_+(\Omega,\Sigma,\mu)$. In both cases φ we have

$$\lim_{r\to\infty}\frac{\varphi(r)}{r}=1,$$

and φ is a convex increasing bijection of $(0,\infty)$. □

Since we doubt the existence of an example of a nonconvex increasing bijection φ of $(0,\infty)$ satisfying (1) for a nontrivial measure space (Ω,Σ,μ), and such that $\lim_{r\to\infty}\frac{\varphi(r)}{r}$ is finite, the following is our:

Conjecture 2.4. *If* (Ω,Σ,μ) *is a nontrivial measure space and* φ *is a bijection of* $(0,\infty)$ *satisfying (1), then* φ *is convex.*

The following remarks support admitting in this conjecture all bijections.

Remark 2.5. The assumption in Theorem 1 and Theorem 2 that the bijection φ is increasing can be replaced by each of the following conditions:

(i) $\lim_{t\to 0}\varphi^{-1}(t)=0$;

(ii) ([9], Theorem 6) there is a set $C\subset(0,\infty)$ such that

$$\lim_{t\to 0}\varphi^{-1}|_C(t)=0$$

and the density of C at the point 0 with respect to the Lebesgue measure λ, defined by

$$\lambda_C(0):=\liminf_{h\to 0+}\frac{\lambda(C\cap(0,h])}{h},$$

is positive;

(iii) [7] there exist $n\in\mathbb{N}$, $n>1$, and $A,B\in\Sigma$ such that

$$A\cap B=\emptyset,\quad \mu(A)=\frac{1}{n},\quad \mu(B)=n;$$

(iv) [7] there exist $m,n\in\mathbb{N}$, $n\neq m$, $n>1$, and $A,B,C\in\Sigma$ such that

$$A\cap B=\emptyset,\quad \mu(A)=\frac{m}{n},\quad \mu(B)=\frac{n}{m},\quad \mu(C)=n.$$ □

Remark 2.6. Let φ be an arbitrary monotonic bijection of $(0,\infty)$, and (Ω,Σ,μ) be a measure space with at least two disjoint sets $A, B \in \Sigma$ of a positive and finite measure $a = \mu(A)$ and $b = \mu(B)$. If $\mathbf{P}_\varphi$ satisfies (1), then φ is increasing. □

Indeed, setting the functions $\mathbf{x}, \mathbf{y} \in S_+(\Omega,\Sigma,\mu)$,

$$\mathbf{x} = x_1\chi_A + x_2\chi_B, \qquad \mathbf{y} = y_1\chi_A + y_2\chi_B$$

into (1), we get the inequality (2) for all $x_1, x_2, y_1, y_2 > 0$. If φ were decreasing, letting $y_2 \to 0$, by the continuity of φ, we would get

$$\varphi^{-1}\left(a\varphi(x_1+y_1) + b\varphi(x_2)\right) \le \varphi^{-1}\left(a\varphi(x_1) + b\varphi(x_2)\right), \qquad x_1, x_2, y_1 > 0,$$

which, letting $x_1 \to 0$, would give

$$\varphi^{-1}\left(a\varphi(y_1) + b\varphi(x_2)\right) \le 0, \qquad x_2, y_1 > 0,$$

which is a contradiction.

3. Superadditivity of $\mathbf{P}_\varphi$

Research of the class of all bijections φ satisfying the converse inequality (1) is much simpler, because we have the following (see [8]).

Remark 3.1. Let (Ω,Σ,μ) be a measure space with at least two disjoint sets of finite and positive measure, and φ be an arbitrary bijection of $(0,\infty)$. If

$$\mathbf{P}_\varphi(\mathbf{x}+\mathbf{y}) \ge \mathbf{P}_\varphi(\mathbf{x}) + \mathbf{P}_\varphi(\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in S_+(\Omega,\Sigma,\mu), \tag{9}$$

than φ is increasing. □

Proof. Let $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $a = \mu(A)$, $b = \mu(B)$ are finite and positive. For arbitrary $s, t > 0$, the functions

$$\mathbf{x} := \varphi^{-1}\left(\frac{s}{a}\right)\chi_A, \qquad \mathbf{y} := \varphi^{-1}\left(\frac{t}{b}\right)\chi_B$$

belong to $S_+(\Omega,\Sigma,\mu)$. Setting them into (9), by the definition of $\mathbf{P}_\varphi$, we get

$$\varphi^{-1}(s+t) \ge \varphi^{-1}(s) + \varphi^{-1}(t), \qquad s, t > 0,$$

so φ^{-1} is superadditive. Since φ^{-1} is positive, it follows that it is increasing. It follows that φ is increasing. □

Using this remark, and arguing similarly as in the proofs of Theorems 1 and Theorem 2, we obtain two results and formulate a conjecture.

Theorem 3.2. *Let (Ω,Σ,μ) be a measure space with at least two disjoint sets of finite and positive measure. Assume that a bijection $\varphi : (0,\infty) \to (0,\infty)$ satisfies inequality (9). If (9) is sharp for all not proportional functions $x, y \in S_+(\Omega,\Sigma,\mu)$*

and
$$\limsup_{r\to\infty} \frac{\varphi(r)}{r} = \infty,$$

then φ is concave.

Theorem 3.3. *Let* (Ω, Σ, μ) *an arbitrary measure space with two disjoint sets* $A, B \in \Sigma$ *such that*

$$0 < \min(\mu(A), \mu(B)) \leq 1 \leq \mu(A) + \mu(B).$$

If $\varphi : (0, \infty) \to (0, \infty)$ *is an arbitrary bijection satisfying inequality* (9)*, then* φ *is concave.*

Conjecture 3.4. *If* (Ω, Σ, μ) *is a nontrivial measure space and* φ *is a bijection of* $(0, \infty)$ *satisfying* (9)*, then* φ *is concave.*

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