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Scaled Relative Graphs of Normal Matrices

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The Scaled Relative Graph (SRG) is a geometric tool that maps the action of a multi-valued nonlinear operator onto the 2D plane, used to analyze the convergence of a wide range of iterative methods. As the SRG includes the spectrum for linear operators, we can view the SRG as a generalization of the spectrum to multi-valued nonlinear operators. In this work, we further study the SRG of linear operators and characterize the SRG of block-diagonal and normal matrices.

Keywords: Scaled relative graph, non-Euclidean geometry, hyperbolic geometry, normal matrix.

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1. Introduction

The Scaled Relative Graph (SRG) is a geometric tool that maps the action of a multi-valued nonlinear operator onto the extended complex plane, analogous to how the spectrum maps the action of a linear operator to the complex plane. The SRG can be used to analyze convergence of a wide range of iterative methods expressed as fixed-point iterations.

Scaled relative graph. For a matrix $A \in \mathbb{R}^{n\times n}$, define $z_A\colon \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ with

$$z_A(x) = \frac{\|Ax\|}{\|x\|} \exp[i\angle(Ax, x)],$$

where

$$\angle(a, b) = \begin{cases} \arccos\left(\frac{a^T b}{\|a\|\|b\|}\right) & \text{if } a \neq 0,\, b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

denotes the angle in $[0, \pi]$ between a and b. The SRG of a matrix $A \in \mathbb{R}^{n\times n}$ is

$$\mathcal{G}(A) = \left\{ z_A(x), \overline{z_A}(x) : x \in \mathbb{R}^n,\, x \neq 0 \right\}.$$

This definition of the SRG, specific to (single-valued) linear operators, coincides with the more general definition for nonlinear multi-valued operators provided in [13]. Ryu, Hannah, and Yin showed the SRG generalizes spectrum in the following sense.

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Fact 1.1. (Theorem 3.1 of [13]) *If* $A \in \mathbb{R}^{n\times n}$ *and* $n=1$ *or* $n\geq 3$, *then* $\Lambda(A) \subseteq \mathcal{G}(A)$.

2D geometric illustrations have been used by Eckstein and Bertsekas [4, 5], Giselsson [6, 7], Banjac and Goulart [1], and Giselsson and Moursi [8] to qualitatively understand the convergence of optimization algorithms. The SRG was first presented as a rigorous formulation of such illustrations in Hannah and Yin's technical report [9] and was further expanded upon in the follow-up work by Ryu, Hannah, and Yin [13].

Contributions. Prior work [10, 13] focused on the SRG of *nonlinear* multi-valued operators. For linear operators, Ryu, Hannah, and Yin [13] established $\mathcal{G}(A)$ includes $\Lambda(A)$, as stated in Fact 1.1, but did not characterize when and how $\mathcal{G}(A)$ enlarges $\Lambda(A)$. In this work, we further study the SRG of linear operators. In particular, we fully characterize the SRG of block-diagonal and normal matrices as a certain polygon in hyperbolic (non-Euclidean) geometry, under the Poincaré half-plane model.

Preliminaries. Let $A \in \mathbb{R}^{n\times n}$. Write $\Lambda(A)$ for the spectrum, the set of eigenvalues, of A. A is *normal* if $A^TA = AA^T$. Given matrices $A_1, \ldots, A_m$, write $\mathrm{Diag}(A_1, \ldots, A_m)$ for the block-diagonal matrix with m blocks. For $z \in \mathbb{C}$, write $\overline{z}$ for its complex conjugate. For a set $S \subseteq \mathbb{C}$, write $S^+ = \{z \in S \,|\, \mathrm{Im}\, z \geq 0\}$. In particular, write $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z \geq 0\}$ and $\mathcal{G}^+(A) = \{z_A(x) : x \in \mathbb{R}^n,\, x \neq 0\}$. Note $z_A(x) \in \mathbb{C}^+$ for all nonzero $x \in \mathbb{R}^n$. For $z_1, z_2 \in \mathbb{C}$, define

$$[z_1, z_2] = \{\theta z_1 + (1-\theta) z_2 : \theta \in [0,1]\},$$

i.e., $[z_1, z_2]$ is the line segment connecting z_1 and z_2.

2. Arc-edge polygon and arc-convexity

Consider points $z_1, z_2 \in \mathbb{C}^+$. If $\mathrm{Re}\, z_1 \neq \mathrm{Re}\, z_2$, let $\mathbb{C}(z_1, z_2)$ be the circle in $\mathbb{C}$ through z_1 and z_2 with the center on the real axis. We can construct $\mathbb{C}(z_1, z_2)$ by finding the center as the intersection of the perpendicular bisector of $[z_1, z_2]$ and the real axis. If $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$ but $z_1 \neq z_2$, let $\mathbb{C}(z_1, z_2)$ be the line extending $[z_1, z_2]$. If $z_1 = z_2$, then $\mathbb{C}(z_1, z_2)$ is undefined. If $\mathrm{Re}\, z_1 \neq \mathrm{Re}\, z_2$, let $\mathrm{Arc_{min}}(z_1, z_2) \subseteq \mathbb{C}^+$ be the arc of $\mathbb{C}(z_1, z_2)$ between z_1 and z_2 in the upper-half plane. (If $\mathrm{Im}\, z_1 > 0$ or $\mathrm{Im}\, z_2 > 0$, then $\mathrm{Arc_{min}}(z_1, z_2) \subseteq \mathbb{C}^+$ is the minor arc of $\mathbb{C}(z_1, z_2)$ between z_1 and z_2. If $\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2 = 0$, then $\mathrm{Arc_{min}}(z_1, z_2)$ is a semicircle in $\mathbb{C}^+$.) If $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$ but $z_1 \neq z_2$, let $\mathrm{Arc_{min}}(z_1, z_2) = [z_1, z_2]$. If $z_1 = z_2$, then $\mathrm{Arc_{min}}(z_1, z_2) = \{z_1\}$. For $z_1, z_2 \in \mathbb{C}^+$ such that $\mathrm{Re}\, z_1 \neq \mathrm{Re}\, z_2$, let $\mathrm{Disk}(z_1, z_2)$ and $\mathrm{Disk}^\circ(z_1, z_2)$ respectively be the closed and open disks enclosed by $\mathbb{C}(z_1, z_2)$. Figure 1 illustrates these definitions.

For $m \geq 1$ and $z_1, \ldots, z_m \in \mathbb{C}^+$, we call $\mathrm{Poly}(z_1, z_2, \ldots, z_m)$ an *arc-edge polygon* and define it as follows. For $m = 1$, let $\mathrm{Poly}(z_1) = \{z_1\}$. For $m \geq 2$, let

$$S = \bigcup_{1\leq i,j\leq m} \mathrm{Arc_{min}}(z_i, z_j)$$

and

$$\mathrm{Poly}(z_1, \ldots, z_m) = S \cup \{\text{region enclosed by } S\}.$$

Figure 2 illustrates this definition. Note $\mathrm{Poly}(z_1, z_2) = \mathrm{Arc_{min}}(z_1, z_2)$. The "region enclosed by S" is the union of all regions enclosed by non-self-intersecting continuous loops (Jordan curves) within S. Since S is a connected set, we can alternatively define $\mathrm{Poly}(z_1, \dots, z_m)$ as the smallest simply connected set containing S.

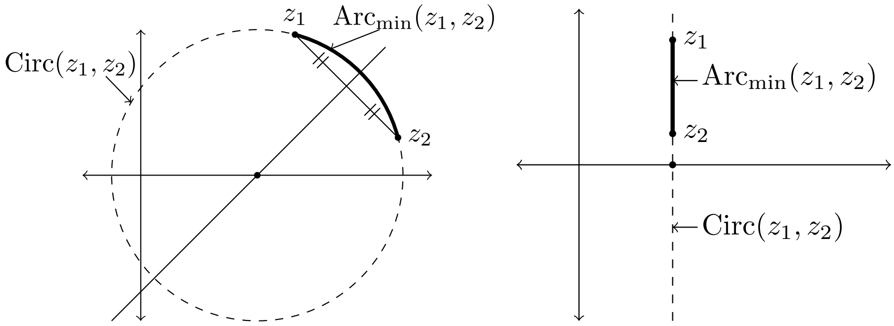


Figure 1: Illustration of $\mathbb{C}(z_1, z_2)$ and $\mathrm{Arc}_{\min}(z_1, z_2)$.

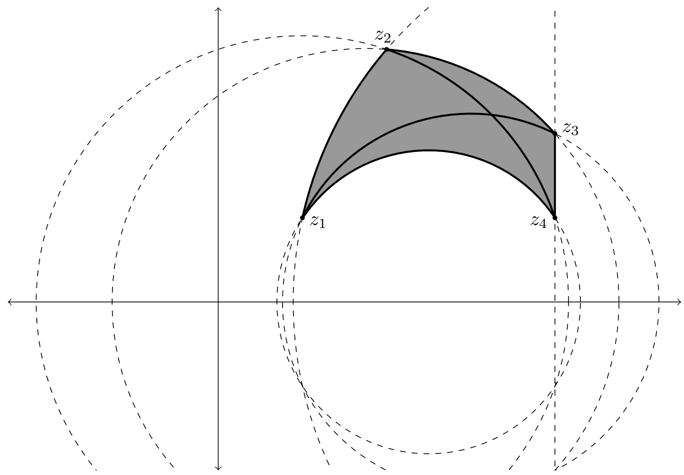


Figure 2: The shaded region illustrates the arc-edge polygon $\mathrm{Poly}(z_1, z_2, z_3, z_4)$ for $z_1 = 1 + i$, $z_2 = 2 + 3i$, $z_3 = 3 + 3i$, and $z_4 = 4 + i$. The solid arcs illustrate $\mathrm{Arc}_{\min}(z_i, z_j)$ dashed circles illustrate $\mathbb{C}(z_i, z_j)$ for $i, j = 1, \dots, m$.

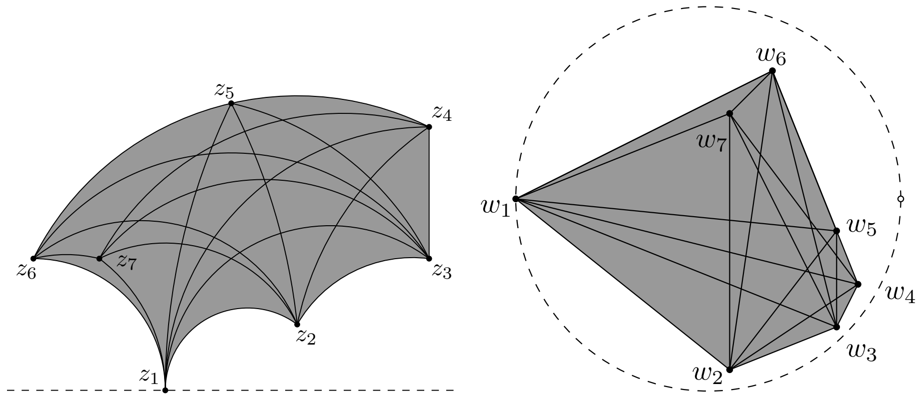


Figure 3: Illustration of $f \circ g$ and Lemma 2.1. The one-to-one map $f \circ g$ of (2) maps $\mathrm{Poly}(z_1, \dots, z_7)$ (a hyperbolic polygon) into a Euclidean polygon. We denote the mapped points as $w_i = f(g(z_i))$ for $i = 1, \dots, 7$. The equivalent Euclidean geometry tells us that $\mathrm{Poly}(z_1, \dots, z_7)$ is "convex" and can be enclosed by the curve through

$z_1 \to z_2 \to z_3 \to z_4 \to z_6 \to z_1$. Note that z_5 and z_7 are not necessary in the description of the boundary.

This construction of $\mathrm{Arc}_{\min}$ gives rise to the classical *Poincaré half-plane model* of hyperbolic (non-Euclidean) geometry, where a $\mathrm{Arc}_{\min}(z_1, z_2)$ and $\mathbb{C}(z_1, z_2) \cap \mathbb{C}^+$ are, respectively, the "line segment" between z_1 and z_2 and the "line" through z_1 and z_2 in the hyperbolic space [3, 12]. The *Beltrami–Klein model* maps the Poincaré half-plane model onto the unit disk and $\mathrm{Arc}_{\min}$ to straight line segments [11, 2]. Specifically, the one-to-one map

$$f \circ g \colon \mathbb{C}^+ \to \{z \in \mathbb{C} : |z| \le 1,\, z \neq 1\}$$

defined by
$$f(z) = \frac{2z}{1+|z|^2}, \qquad g(z) = \frac{z-i}{z+i}$$

maps the Poincaré half-plane model to the Beltrami–Klein model while mapping hyperbolic line segments $\mathrm{Arc}_{\min}$ into Euclidean straight line segments. The Beltrami–Klein model demonstrates that any qualitative statement about convexity in the Euclidean plane is equivalent to an analogous statement in the Poincaré half-plane model. See Figure 3.

Lemma 2.1. *Let $z_1, \dots, z_m \in \mathbb{C}^+$ and $m \ge 1$. Then $\mathrm{Poly}(z_1, \dots, z_m)$ is "convex" in the following non-Euclidean sense:*

$$w_1, w_2 \in \mathrm{Poly}(z_1, \dots, z_m) \quad \Rightarrow \quad \mathrm{Arc}_{\min}(w_1, w_2) \subseteq \mathrm{Poly}(z_1, \dots, z_m).$$

If $\mathrm{Poly}(z_1, \dots, z_m)$ has an interior, then there is $\{\zeta_1, \dots, \zeta_q\} \subseteq \{z_1, \dots, z_m\}$ such that

$$\mathrm{Arc}_{\min}(\zeta_1, \zeta_2) \cup \mathrm{Arc}_{\min}(\zeta_2, \zeta_3) \cup \cdots \cup \mathrm{Arc}_{\min}(\zeta_{q-1}, \zeta_q) \cup \mathrm{Arc}_{\min}(\zeta_q, \zeta_1)$$

is a Jordan curve, and the region the curve encloses is $\mathrm{Poly}(z_1, \dots, z_m)$.

Proof. Let $w_1, \dots, w_m$ be in the unit complex disk. Consider the construction

$$\tilde{S} = \bigcup_{1 \le i,j \le m} [w_i, w_j]$$

and
$$\widetilde{\mathrm{Poly}}(w_1, \dots, w_m) = \tilde{S} \cup \{\text{region enclosed by } \tilde{S}\}.$$

This is the (Euclidean) 2D polyhedron given as the convex hull of $w_1, \dots, w_m$. The Euclidean convex hull has the properties analogous to those in the Lemma statement, and we use the map $(f \circ g)^{-1}$, where $f \circ g$ is as given by (2) to map the properties to our setup. □

3. SRGs of block-diagonal matrices

We characterize the SRG of block-diagonal matrices as follows.

Theorem 3.1. *Let $A_1, \dots, A_m$ be square matrices, where $m \ge 1$. Then*

$$\mathcal{G}^+(\mathrm{Diag}(A_1, \dots, A_m)) = \bigcup_{\substack{z_i \in \mathcal{G}^+(A_i)\\ i=1,\dots,m}} \mathrm{Poly}(z_1, \dots, z_m).$$

Proof. When $m = 1$, there is nothing to show. Assume $m \geq 2$.

Step 1. Let $A_i \in \mathbb{R}^{n_i \times n_i}$ and $u_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, m$. We use the notation $n = n_1 + \cdots + n_m$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^n, \qquad \mathbf{u}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n \quad \text{for } i = 1, \ldots, m,$$

and $\mathbf{A} = \mathrm{Diag}(A_1, \ldots, A_m) \in \mathbb{R}^{n \times n}$. Then we have

$$\begin{aligned} \mathcal{G}^+ (\mathrm{Diag}(A_1, \ldots, A_m)) &= \bigcup_{\mathbf{u} \in \mathbb{R}^n \backslash \{0\}} z_{\mathbf{A}}(\mathbf{u}) \\ &= \bigcup_{\substack{u_i \in \mathbb{R}^{n_i}, \, u_i \neq 0 \\ i=1,\ldots,m}} z_{\mathbf{A}} (\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) \backslash \{0\}) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \bigcup_{\substack{u_i \in \mathbb{R}^{n_i}, \, u_i \neq 0 \\ i=1,\ldots,m}} \mathrm{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1), \ldots, z_{\mathbf{A}}(\mathbf{u}_m)\right) &= \bigcup_{\substack{u_i \in \mathbb{R}^{n_i}, \, u_i \neq 0 \\ i=1,\ldots,m}} \mathrm{Poly}\left(z_{A_1}(u_1), \ldots, z_{A_m}(u_m)\right) \\ &= \bigcup_{\substack{z_i \in \mathcal{G}^+(A_i) \\ i=1,\ldots,m}} \mathrm{Poly}\left(z_1, \ldots, z_m\right). \end{aligned} \tag{2}$$

To clarify, $\mathbf{u}_i$ depends on u_i for $i = 1, \ldots, m$. In the following, we show

$$z_{\mathbf{A}} (\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) \backslash \{0\}) = \mathrm{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1), \ldots, z_{\mathbf{A}}(\mathbf{u}_m)\right) \tag{3}$$

for all $\mathbf{u}_i$ given by $u_i \neq 0$ for $i = 1, \ldots, m$. Once (3) is proved, (1) and (2) are equivalent and the proof is complete.

Step 2. We show the following intermediate result: for all nonzero $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}\mathbf{v} \rangle = \langle \mathbf{A}\mathbf{u}, \mathbf{A}\mathbf{v} \rangle = 0, \tag{4}$$

we have

$$z_{\mathbf{A}}(\mathrm{span}(\mathbf{u}, \mathbf{v}) \backslash \{0\}) = \mathrm{Arc}_{\min}(z_{\mathbf{A}}(\mathbf{u}), z_{\mathbf{A}}(\mathbf{v})). \tag{5}$$

First, consider the case $\mathrm{Re}\, z_{\mathbf{A}}(\mathbf{u}) \neq \mathrm{Re}\, z_{\mathbf{A}}(\mathbf{v})$. Let the circle $\mathbb{C}(z_{\mathbf{A}}(\mathbf{u}), z_{\mathbf{A}}(\mathbf{v}))$ be centered at $(t, 0)$ with $t \in \mathbb{R}$ and radius $r \geq 0$. Then $z_{\mathbf{A}}(\mathbf{u})$ and $z_{\mathbf{A}}(\mathbf{v})$ satisfy

$$(\mathrm{Re}\, z_{\mathbf{A}}(\mathbf{u}) - t)^2 + (\mathrm{Im}\, z_{\mathbf{A}}(\mathbf{u}))^2 = r^2$$

$$(\mathrm{Re}\, z_{\mathbf{A}}(\mathbf{v}) - t)^2 + (\mathrm{Im}\, z_{\mathbf{A}}(\mathbf{v}))^2 = r^2.$$

This is equivalent to

$$\langle \mathbf{Au}, \mathbf{Au}\rangle - 2t\langle \mathbf{Au}, \mathbf{u}\rangle + (t^2 - r^2)\langle \mathbf{u}, \mathbf{u}\rangle = 0$$
$$\langle \mathbf{Av}, \mathbf{Av}\rangle - 2t\langle \mathbf{Av}, \mathbf{v}\rangle + (t^2 - r^2)\langle \mathbf{v}, \mathbf{v}\rangle = 0.$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\mathbf{w} = \alpha_1\mathbf{u}+\alpha_2\mathbf{v}$. Assume $\mathbf{w} \neq 0$. Using (4) and basic calculations, we have

$$\langle \mathbf{Aw}, \mathbf{Aw}\rangle - 2t\langle \mathbf{Aw}, \mathbf{w}\rangle + (t^2 - r^2)\langle \mathbf{w}, \mathbf{w}\rangle = 0,$$

and this is equivalent to $(\operatorname{Re} z_{\mathbf{A}}(\mathbf{w}) - t)^2 + (\operatorname{Im} z_{\mathbf{A}}(\mathbf{w}))^2 = r^2$.

Therefore $z_{\mathbf{A}}(\mathbf{w}) = z_{\mathbf{A}}(\alpha_1\mathbf{u} + \alpha_2\mathbf{v}) \in \mathbb{C}(z_{\mathbf{A}}(\mathbf{u}), z_{\mathbf{A}}(\mathbf{v}))$. Notice that

$$\operatorname{Re}\, z_{\mathbf{A}}(\mathbf{w}) = \frac{\langle \mathbf{Aw}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle} = \frac{\alpha_1^2\langle \mathbf{Au}, \mathbf{u}\rangle + \alpha_2^2\langle \mathbf{Av}, \mathbf{v}\rangle}{\alpha_1^2\langle \mathbf{u}, \mathbf{u}\rangle + \alpha_2^2\langle \mathbf{v}, \mathbf{v}\rangle}$$

fills the interval $[\operatorname{Re}\, z_A(\mathbf{u}), \operatorname{Re}\, z_A(\mathbf{v})]$ as α_1 and α_2 varies. So we have

$$\bigcup_{\substack{\alpha_1,\alpha_2\in\mathbb{R}\\ \alpha_1\mathbf{u}+\alpha_2\mathbf{v}\neq 0}} z_{\mathbf{A}}(\alpha_1\mathbf{u} + \alpha_2\mathbf{v}) = \mathrm{Arc}_{\min}(z_{\mathbf{A}}(\mathbf{u}), z_{\mathbf{A}}(\mathbf{v}))$$

and we conclude (5).

Next, consider the case $\operatorname{Re} z_{\mathbf{A}}(\mathbf{u}) = \operatorname{Re} z_{\mathbf{A}}(\mathbf{v})$. Note that

$$\operatorname{Re} z_{\mathbf{A}}(\mathbf{u}) = \frac{\langle \mathbf{Au}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}, \qquad \operatorname{Re} z_{\mathbf{A}}(\mathbf{v}) = \frac{\langle \mathbf{Av}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}.$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\mathbf{w} = \alpha_1\mathbf{u}+\alpha_2\mathbf{v}$. Assume $\mathbf{w} \neq 0$. Using (4) and basic calculations, we have

$$\operatorname{Re}\, z_{\mathbf{A}}(\mathbf{w}) = \frac{\langle \mathbf{Aw}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle} = \operatorname{Re}\, z_{\mathbf{A}}(\mathbf{u}) = \operatorname{Re}\, z_{\mathbf{A}}(\mathbf{v}).$$

Notice that $\quad |z_{\mathbf{A}}(\mathbf{w})|^2 = \dfrac{\langle \mathbf{Aw}, \mathbf{Aw}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle} = \dfrac{\alpha_1^2\langle \mathbf{Au}, \mathbf{Au}\rangle + \alpha_2^2\langle \mathbf{Av}, \mathbf{Av}\rangle}{\alpha_1^2\langle \mathbf{u}, \mathbf{u}\rangle + \alpha_2^2\langle \mathbf{v}, \mathbf{v}\rangle}$

fills the interval $[|z_{\mathbf{A}}(\mathbf{u})|^2, |z_{\mathbf{A}}(\mathbf{v})|^2]$ as α_1 and α_2 varies. So $\operatorname{Im}\, z_{\mathbf{A}}(\mathbf{w})$ fills the interval $[\operatorname{Im}\, z_{\mathbf{A}}(\mathbf{u}), \operatorname{Im}\, z_{\mathbf{A}}(\mathbf{v})]$ as α_1 and α_2 varies, and we conclude

$$z_{\mathbf{A}}(\mathrm{span}(\mathbf{u}, \mathbf{v})\backslash\{0\}) = [z_{\mathbf{A}}(\mathbf{u}), z_{\mathbf{A}}(\mathbf{v})] = \mathrm{Arc}_{\min}(z_{\mathbf{A}}(\mathbf{u}), z_{\mathbf{A}}(\mathbf{v})).$$

Step 3. We show

$$z_{\mathbf{A}}\left(\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m)\backslash\{0\}\right) \subseteq \mathrm{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1), \ldots, z_{\mathbf{A}}(\mathbf{u}_m)\right) \tag{6}$$

by induction. Clearly

$$z_{\mathbf{A}}\left(\mathrm{span}(\mathbf{u}_1)\backslash\{0\}\right) = \mathrm{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1)\right).$$

Now assume (6) holds for $m-1$. By (5), we have

$$z_{\mathbf{A}}\left(\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m)\backslash\{0\}\right) = \bigcup_{\zeta\in z_{\mathbf{A}}(\mathrm{span}(\mathbf{u}_1,\ldots,\mathbf{u}_{m-1})\backslash\{0\})} \mathrm{Arc}_{\min}(\zeta, z_{\mathbf{A}}(\mathbf{u}_m)).$$

By the induction hypothesis, $\zeta \in z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_{m-1})\backslash\{0\}\right)$, implies

$$\zeta \in \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_{m-1})\right) \subseteq \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right).$$

By construction, $z_{\mathbf{A}}(\mathbf{u}_m) \in \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right)$.

"Convexity" of Lemma 2.1 implies

$$\bigcup_{\zeta \in z_{\mathbf{A}}(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_{m-1})\backslash\{0\})} \operatorname{Arc}_{\min}(\zeta, z_{\mathbf{A}}(\mathbf{u}_m)) \subseteq \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right),$$

and we conclude (6).

Step 4. We show

$$z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m)\backslash\{0\}\right) \supseteq \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right). \tag{7}$$

First, consider the case where $\operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right)$ has no interior. In 2D Euclidean geometry, a polygon has no interior when it is a single line segment or a point. The Beltrami–Klein model provides us with an equivalent statement in hyperbolic geometry: the "polygon" can be expressed as

$$\operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right) = \operatorname{Arc}_{\min}(z_{\mathbf{A}}(\boldsymbol{\mu}_1), z_{\mathbf{A}}(\boldsymbol{\mu}_2))$$

where $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \{\mathbf{u}_1,\dots,\mathbf{u}_m\}$. By the reasoning of Step 2, we conclude

$$\begin{aligned} z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m)\backslash\{0\}\right) &\supseteq z_{\mathbf{A}}\left(\operatorname{span}(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\backslash\{0\}\right) \\ &= \operatorname{Arc}_{\min}(z_{\mathbf{A}}(\boldsymbol{\mu}_1), z_{\mathbf{A}}(\boldsymbol{\mu}_2)) = \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right). \end{aligned}$$

Next, consider the case where $\operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right)$ has an interior. In this case, $\dim\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m) \geq 3$ by the arguments of Step 2. Assume for contradiction that there is a $z \in \operatorname{Poly}\left(z_{\mathbf{A}}(\mathbf{u}_1),\dots,z_{\mathbf{A}}(\mathbf{u}_m)\right)$ but $z \notin z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m)\backslash\{0\}\right)$.

Let $\zeta_1,\dots,\zeta_q$ be vertices provided by Lemma 2.1. There exists corresponding $\{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_q\} \subseteq \{\mathbf{u}_1,\dots,\mathbf{u}_m\}$ such that $\zeta_i = z_{\mathbf{A}}(\boldsymbol{\mu}_i)$ for $i = 1,\dots,q$. Define the curve

$$\boldsymbol{\eta}(t) : [1, q+1] \to \operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m) \cap S^{m-1},$$

where $S^{m-1} \subset \mathbb{R}^m$ is the unit sphere, as

$$\boldsymbol{\eta}(t) = \frac{\cos((t-p)\frac{\pi}{2})}{\|\boldsymbol{\mu}_p\|}\boldsymbol{\mu}_p + \frac{\sin((t-p)\frac{\pi}{2})}{\|\boldsymbol{\mu}_{p+1}\|}\boldsymbol{\mu}_{p+1}, \qquad \text{for } p \leq t \leq p+1.$$

where we regard $\boldsymbol{\mu}_{q+1}$ as $\boldsymbol{\mu}_1$. Then $\{\gamma(t)\}_{\{t\in[1,q+1]\}} = \{z_{\mathbf{A}}(\boldsymbol{\eta}(t))\}_{t\in[1,q+1]}$ encloses z.

Since $\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m) \cap S^{m-1}$ is simply connected, we can continuously contract $\{\boldsymbol{\eta}(t)\}_{t\in[1,q+1]}$ to a point in $\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m) \cap S^{m-1}$ and the curve under the map $z_{\mathbf{A}}$ continuously contracts to a point in $z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m)\backslash\{0\}\right)$.

However, this is not possible as $z \notin z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m)\backslash\{0\}\right)$ and $\{\gamma(t)\}_{\{t\in[1,q+1]\}}$ has a nonzero winding number around z. We have a contradiction and we conclude $z \in z_{\mathbf{A}}\left(\operatorname{span}(\mathbf{u}_1,\dots,\mathbf{u}_m)\backslash\{0\}\right)$. □

4. SRGs of normal matrices

We now use Theorem 3.1 to fully characterize the SRG of normal matrices.

Proposition 4.1. *Let* $A = \begin{bmatrix} a_1 & b_1 \\ b_2 & a_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$*. Then* $\mathcal{G}(A)$ *consists of two circles centered at* $\left(\frac{a_1+a_2}{2}, \pm\frac{b_1-b_2}{2}\right)$ *with radius* $\sqrt{\left(\frac{a_1-a_2}{2}\right)^2 + \left(\frac{b_1+b_2}{2}\right)^2}$.

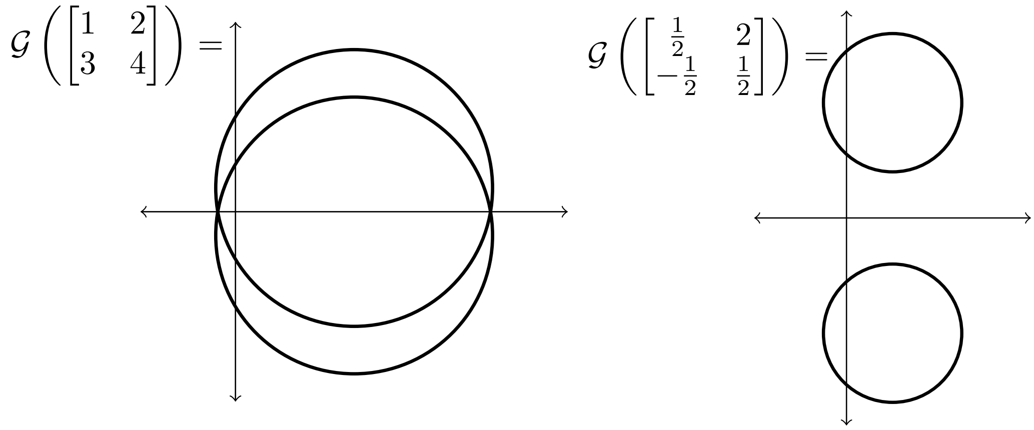


Figure 4: Illustration of Proposition 4.1.

Proof. Let $x_\theta = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} \in \mathbb{R}^2$, and $T\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right) = \begin{bmatrix}x_1\\ |x_2|\end{bmatrix}$.

The stated result follows from

$$\mathcal{G}^+(A) = \{z_A(x_\theta) : \theta \in [0, 2\pi)\}$$

and the calculations

$$\begin{aligned} z_A(x_\theta) &= \begin{bmatrix} \frac{1}{2}\left(a_1 + a_2 + (a_1 - a_2)\cos(2\theta) + (b_1 + b_2)\sin(2\theta)\right) \\ \frac{1}{2}\left|-b_1 + b_2 + (b_1 + b_2)\cos(2\theta) - (a_1 - a_2)\sin(2\theta)\right| \end{bmatrix} \\ &= T\left(\begin{bmatrix} \frac{a_1+a_2}{2} \\ -\frac{b_1-b_2}{2} \end{bmatrix} + \underbrace{\begin{bmatrix} \cos(-2\theta) & -\sin(-2\theta) \\ \sin(-2\theta) & \cos(-2\theta) \end{bmatrix}}_{\text{rotation by } -2\theta} \begin{bmatrix} \frac{a_1-a_2}{2} \\ \frac{b_1+b_2}{2} \end{bmatrix} \right). \end{aligned}$$ □

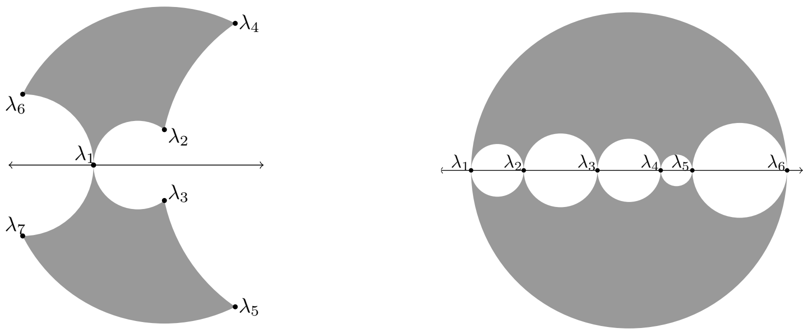
Proposition 4.2. *A matrix's SRG is invariant under orthogonal similarity transforms.*

Proof. Let $A \in \mathbb{R}^{n\times n}$. Let $Q \in \mathbb{R}^{n\times n}$ be orthogonal. For any nonzero $x \in \mathbb{R}^n$, we have

$$\begin{aligned} z_{QAQ^T}(x) &= \frac{\|QAQ^Tx\|}{\|x\|}\exp[i\angle(QAQ^Tx, x)] \\ &= \frac{\|AQ^Tx\|}{\|x\|}\exp[i\angle(AQ^Tx, Q^Tx)] = z_A(Q^Tx). \end{aligned}$$

Therefore,
$$\begin{aligned}\mathcal{G}(QAQ^T) &= \left\{z_{QAQ^T}(x), \overline{z_{QAQ^T}}(x) : x \in \mathbb{R}^n,\, x \neq 0\right\}\\ &= \{z_A(Qx), \overline{z_A}(Qx) : x \in \mathbb{R}^n,\, x \neq 0\}\\ &= \{z_A(x), \overline{z_A}(x) : x \in \mathbb{R}^n,\, x \neq 0\} = \mathcal{G}(A).\end{aligned}$$
□

Theorem 4.3. *If A is normal, then $\mathcal{G}^+(A) = \mathrm{Poly}(\Lambda(A) \cap \mathbb{C}^+)$.*



(a) SRG of an $n \times n$ normal matrix with one distinct real eigenvalue and three distinct complex conjugate eigenvalue pairs.

(b) SRG of an $n \times n$ symmetric matrix with distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_6$.

Figure 5: Illustration of Theorem 4.3. For normal matrices, multiplicity of eigenvalues do not affect the SRG.

Proof. A normal matrix is orthogonally similar to the real block-diagonal matrix
$$\begin{bmatrix} a_1 & b_1 & & & & & & \\ -b_1 & a_1 & & & & & & \\ & & \ddots & & & & & \\ & & & a_k & b_k & & & \\ & & & -b_k & a_k & & & \\ & & & & & \lambda_{k+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_m \end{bmatrix}.$$

Propositions 4.1 tells us
$$\mathcal{G}^+\left(\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}\right) = \{a_j + |b_j|i\} = \Lambda\left(\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}\right) \cap \mathbb{C}^+$$
for $j = 1, \ldots, k$. We conclude the stated result with Theorem 4.3 and Proposition 4.2. □

Corollary 4.4. *Let $A \in \mathbb{R}^{n \times n}$ be symmetric, and let $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ be the distinct (real) eigenvalues of A. If $m = 1$, then $\mathcal{G}^+(A) = \{\lambda_1\}$. If $m \geq 2$, then*
$$\mathcal{G}(A) = \mathrm{Disk}(\lambda_1, \lambda_m) \backslash \bigcup_{i=1}^{m-1} \mathrm{Disk}^\circ(\lambda_i, \lambda_{i+1}).$$

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