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Lipschitzness of the Width and Diameter Functions of Convex Bodies in $\mathbb{R}^n$

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Lipschitz constants for the width and diameter functions of a convex body in $\mathbb{R}^n$ are found in terms of its diameter and thickness (maximum and minimum of both functions). Also, a dual approach to thickness is proposed.

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1. Basic definitions and facts

Let K be a convex body in $\mathbb{R}^n$ ($n \geq 2$), i.e. a convex compactum with nonempty interior. As is well-known, $\mathbb{R}^n \setminus K$ is a union of (affine) half-spaces, and any hyperplane which intersects K without intersecting its interior is called a *supporting hyperplane*. It is interesting to measure the width of a convex body by looking at how much two parallel supporting hyperplanes must be distant from each other. This depends continuously on the direction of those hyperplanes. The present paper aims at improving the results about the variation of the width, and of analogous quantities. First we give more precise definitions.

Definition 1.1. For any vector $u \in S^{n-1}$ (the unit sphere in $\mathbb{R}^n$),

- the *width* of K in direction u, denoted $w_K(u)$, is the distance between the two supporting hyperplanes of K which are orthogonal to u;
- the *diameter* of K in direction u, denoted $d_K(u)$, is the maximal length of a chord of K parallel to u.

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The width may equivalently be defined using the support function

$$h_K(u) := \max\{\langle x, u\rangle : x \in K\},$$

by setting $w_K(u) := h_K(u) + h_K(-u)$, see for instance [1, Section 0.6]. A chord $[a; b]$ of K is of maximal length in a given direction if and only if the supporting hyperplanes at a and b are parallel and distinct, i.e. the chord is an *affine diameter* [4, 3.1].

Note that w_K and d_K are continuous functions on S^{n-1}, $w_K \geq d_K$, and that $\max w_K = \max d_K = \delta_K := \operatorname{diam} K$. The number $\omega_K = \min w_K$ is called the *thickness* of K. It turns out that $\omega_K = \min d_K$.

We point out that $|AB| = d_K(\overrightarrow{AB}/|AB|)$ $(A, B \in \partial K)$ exactly when K admits parallel supporting hyperplanes at A and B. Such a chord $[AB]$ is called *diametral.* It is called *double normal* if the supporting hyperplanes can be chosen orthogonal to AB. If $|AB| = d_K(\overrightarrow{AB}/|AB|) = \omega_K$, then $[AB]$ is a double normal chord.

The facts listed above can be found e.g. in [4], and partly in [2, Lemma 3.1.2, p. 64] (see also [3] about w_K and ω_K).

It is known that w_K is a Lipschitz function w.r.t. the Euclidean norm $||\cdot||$ with constant δ_K (see [3, Proposition]). This constant does not take into account the fact that for some convex bodies, the width can vary much less; for instance, there are convex bodies of constant width, the obvious example being the Euclidean ball. A comprehensive survey about convex bodies of constant width is [2].

In Section 2, we will prove that w_K and d_K are Lipschitz functions with constants

$$M_K = \sqrt{\delta_K^2 - \omega_K^2} \quad \text{resp.} \quad N_K = \frac{\delta_K}{\omega_K}\sqrt{\delta_K^2 - \omega_K^2},$$

(which vanish for bodies of constant width) w.r.t. $\rho(u, v) = \arccos |\langle u, v\rangle|$, the spherical distance on the projective space $S^{n-1}/\pm I$, which is a pseudo-distance on S^{n-1}. Note that $\rho(u, v) \leq \frac{\pi}{2\sqrt{2}}||u \pm v||$. Those results are refined in Section 4.

In Section 3, we study yet another notion of width; while we previously were looking at chords parallel to each other, so passing through the same point at infinity in the projective space, now we consider chords all passing through a given point in affine space. This leads to some interesting continuity phenomena.

2. Lipschitzness of w_K and d_K

For $u, v \in S^{n-1}, u \neq \pm v$, denote

$$\Delta w_K(u, v) := \frac{|w_K(u) - w_K(v)|}{\rho(u, v)}, \quad \Delta d_K(u, v) := \frac{|d_K(u) - d_K(v)|}{\rho(u, v)}.$$

Theorem 2.1. *For any convex body K in $\mathbb{R}^n$ and $u, v \in S^{n-1}$ we have:*

(1) $\Delta w_K(u, v) \leq M_K$;

(2) $\Delta d_K(u, v) \leq N_K$.

Proof. Since K is the intersection of convex polytopes $K_1 \supset K_2 \supset \ldots$ and since $w_{K_i} \downarrow w_K$, $d_{K_i} \downarrow d_K$, $\delta_{K_i} \downarrow \delta_K$, and $\omega_{K_i} \downarrow \omega_K$, we may assume that K is a convex polytope.

Proof of Part (1). Since w_K is already known to be continuous, it will be enough to prove the inequality outside of a set of dimension $n-2$ on the unit sphere. Let E be an edge of K (non-trivial line segment contained in ∂K such that its relative interior is not contained in the relative interior of any face of larger dimension), and H_E the hyperplane of vectors orthogonal to E. For further reference, call those *exceptional hyperplanes.* Let $u \in \mathcal{W}_K := S^{n-1} \setminus \bigcup_{E \text{ edge of } K} H_E$. Then u is not orthogonal to any face of ∂K of any dimension d, $1 \le d \le n-1$. For such a vector u, each of the two supporting hyperplanes H_1 and H_2 orthogonal to u intersects K at a single point, denoted respectively by A and B.

Lemma 2.2. *For any $u \in \mathcal{W}_K$ and A, B chosen as above,*

$$w'_K(u) := \limsup_{u',u''\to u} \Delta w_K(u', u'') = \sqrt{|AB|^2 - w_K^2(u)} \le M_K.$$

Proof. For any v close enough to u, the corresponding supporting hyperplanes intersect K at the same points A and B, so that $w_K(v) = |\langle AB, v\rangle|$ and

$$\frac{|w_K(u') - w_K(u'')|}{\|u' - u''\|} = \left| \left\langle AB, \frac{u' - u''}{\|u' - u''\|} \right\rangle \right|.$$

Note that $\langle u' - u'', (u' + u'')/2\rangle = 0$ and that $(u' + u'')/2$ tends to u. Passing to a subsequence, we may assume that $\frac{u'-u''}{\|u'-u''\|} \to \tilde{u}$, with $\tilde{u} \in S^{n-1}$, $\langle \tilde{u}, u\rangle = 0$. Let $\tilde{A}$ be the unique point in the supporting hyperplane going through B such that $A\tilde{A}$ is parallel to u; then $\langle AB, \tilde{u}\rangle = \langle \tilde{A}B, \tilde{u}\rangle$. So, observing that $\rho(u', u'') \sim \|u' - u''\|$ as $\|u' - u''\| \to 0$,

$$\begin{aligned}\lim_{u',u''\to u} \Delta w_K(u', u'') &= \lim_{u',u''\to u} \left| \left\langle AB, \frac{u' - u''}{\|u' - u''\|} \right\rangle \right| \\ &= \langle \tilde{A}B, \tilde{u}\rangle \le |\tilde{A}B| = \sqrt{|AB|^2 - |A\tilde{A}|^2} = \sqrt{|AB|^2 - w_K^2(u)}.\end{aligned}$$

We can obtain the case of equality above if we choose $u' = u$ and u'' so that $\tilde{u}$ is parallel to $(\tilde{A}B)$, which shows that the limes superior is equal to the righthand side. □

Any two vectors u, v in the same connected component of $\mathcal{W}_K$ can be joined within the component by a geodesic (arc of a great circle). If the inequality in Theorem 2.1(1) was violated, by dichotomy, we could find arbitrarily small arcs $\overset{\frown}{u'u''}$ within the geodesic segment where the inequality would be violated, but since we have $\rho(u', u'') \sim \|u' - u''\|$ as both quantities tend to 0, this would violate Lemma 2.2.

If u or v lies on an exceptional hyperplane, and the other in an adjacent component, we obtain the same result by continuity. If u and v belong to different components, the shorter of the two arcs on the great circle containing them will intersect the exceptional hyperplanes in a finite number of points, and we conclude again by continuity. If u and v are on distinct exceptional hyperplanes, the same method works. Finally, if u and v lie on the same exceptional hyperplane, we may perturb one of them slightly, apply the previous estimate, and conclude again by continuity.

Proof of Part (2). As in Part (1), we exclude a finite set of exceptional vector hyperplanes from the unit sphere. This time those are the hyperplanes H_D generated by a set D of vertices of K with $\#D = n$. When $u \in \mathcal{R}_K := S^{n-1} \setminus \bigcup_{D \subset V(K), \#D=n} H_D$, we can choose a diametral chord for u passing through a vertex A of K; its other extremity B will have to be in the interior of an $(n-1)$-dimensional face of K. Denote by $\tilde{d}_K(u)$ the distance between the parallel supporting hyperplanes of K, H_1 and H_2, that contain A and B respectively.

Lemma 2.3. *For any* $u \in \mathcal{R}_K$*,*

$$d'_K(u) := \limsup_{u',u'' \to u} \Delta d_K(u', u'') = \frac{d_K(u)}{\tilde{d}_K(u)} \sqrt{d_K^2(u) - \tilde{d}_K^2(u)} \le N_K.$$

Proof. There is a neighborhood of u, $\mathcal{N} \subset \mathcal{R}_K$, such that for any $t \in \mathcal{N}$, a diametral chord for t also passes through A and the face containing B. Assume $u', u'' \in \mathcal{N}$.

Let H be the *strip* of $\mathbb{R}^n$ determined by H_1 and H_2 (that is, the intersection between two distinct half spaces determined by H_1 or H_2 that contains K). Consider the points $B' \in H_2$ (resp. $B'' \in H_2$) such that AB' (resp. AB'') is parallel to u' (resp. u''), then $d_K(u') = |AB'|$, $d_K(u'') = |AB''|$. Assume as we may that $|AB''| \ge |AB'|$. In $\triangle AB'B''$, let h be the distance from A to the line $(B'B'')$ and α, β', β'' be the angles at the vertices A, B', B'' respectively. Again we use that $\rho(u', u'') \sim \|u' - u''\|$. Then by the law of sines and $\alpha + \beta' + \beta'' = \pi$,

$$\begin{aligned}
\frac{|AB''| - |AB'|}{\|u' - u''\|} &= \frac{|AB'|(\sin\beta' - \sin\beta'')}{(\sin\beta'')(2\sin\alpha/2)} = |AB'| \frac{\sin\frac{\beta'-\beta''}{2}\cos\frac{\beta'+\beta''}{2}}{\sin\beta''\sin\alpha/2} \\
&= |AB'| \frac{\cos(\beta'' + \frac{\alpha}{2})}{\sin\beta''} \le |AB'|\cot\beta'' = |AB'| \sqrt{\frac{|AB''|^2}{h^2} - 1} \le |AB'| \sqrt{\frac{|AB''|^2}{\tilde{d}_K(u)^2} - 1},
\end{aligned}$$

since $h \ge \tilde{d}_K(u)$. Passing to the limit, we obtain that N_K is an upper bound for the limes superior we are considering.

Observe that as $u', u'' \to u$, then $\alpha \to 0$, $|AB'|, |AB''| \to d_K(A)$, but when the unit vectors parallel to $(B'B'')$ admit a limit v, $h \to \operatorname{dist}(A, L)$ where L is the line through B parallel to v. If we choose $A' \in H_B$ such that $|AA'| = \tilde{d}_K(u)$, and u', u'' so that B', B'' tend to B along the line (BA'), then $h \to \tilde{d}_K(u)$, and

$$\lim_{u',u'' \to u} \Delta d_K(u', u'') = \frac{d_K(u)}{\tilde{d}_K(u)} \sqrt{d_K^2(u) - \tilde{d}_K^2(u)}.$$

□

We finish the proof of Part (2) in the same way as that of Part (1), substituting Lemma 2.3 for Lemma 2.2. □

Remark 2.4. Notice that to obtain Theorem 2.1, it is enough to use the upper bound for the derivative in Lemmas 2.3 and 2.2.

Open Question 2.5. Can one replace $\rho(u, v)$ by $||u - v||$ in Theorem 2.1?

The constants we have obtained are the best possible in the following sense.

Theorem 2.6. *Let $\delta \geq \omega > 0$ and let*

$$\mathcal{K}_{\delta,\omega} := \{K \subset \mathbb{R}^n \text{ convex body such that } \delta_K = \delta, \omega_K = \omega\}.$$

Then we have

$$\max_{K\in\mathcal{K}_{\delta,\omega}} \sup_{u\neq\pm v\in S^{n-1}} \Delta w_K(u,v) = \sqrt{\delta^2-\omega^2}, \quad \max_{K\in\mathcal{K}_{\delta,\omega}} \sup_{u\neq\pm v\in S^{n-1}} \Delta d_K(u,v) = \frac{\delta}{\omega}\sqrt{\delta^2-\omega^2}.$$

Proof. That the maximum in each equality is bounded above by the righthand side follows immediately from Theorem 2.1.

To prove the converse inequality, consider the following example. By scaling, it is enough to assume $\delta = 2$. For $\omega \in (0,2]$, let $K := \overline{B}(0,1) \cap \{-\omega/2 \leq x_1 \leq \omega/2\}$. Clearly $\omega_K = 2$. Since $\pi_{\mathbb{R}^2\times\{0\}}(K) \subset K \cap (\mathbb{R}^2 \times \{0\})$, and since the tangent space to $\partial B(0,1)$ at any point $u_\theta := (\cos\theta, \sin\theta, 0, \dots, 0)$ contains $\{(0,0)\} \times \mathbb{R}^{n-2}$, it is enough to work on the case $n = 2$.

Let $\omega = 2\cos\alpha$, $\alpha \in [0,\pi/2)$. Then it is easy to see that for $\pi/2 \geq \theta \geq \alpha$, $w_K(u_\theta) = 2$, and for $0 \leq \theta \leq \alpha$, $w_K(u_\theta) = 2\cos(\alpha-\theta)$. Therefore $w_K = 2\cos\alpha = \omega$, and $w_K'(u_0) = 2\sin\alpha = \sqrt{4-\omega^2}$, which proves the first part of the theorem.

On the other hand, for $\pi/2 \geq \theta \geq \alpha$, $d_K(u_\theta) = 2$, and for $0 \leq \theta \leq \alpha$, $d_K(u_\theta) = \frac{\omega}{\cos\theta}$. So (with analogous notation) $d_K'(u_\alpha) = \omega\frac{\sin\alpha}{\cos^2\alpha} = \frac{2}{\omega}\sqrt{4-\omega^2}$. □

3. Dual approach to ω_K

One may define a function e_K in a dual way to d_K, replacing vectors by points. More precisely, for any point O in $\mathbb{R}^n$ set $e_K(O)$ to be the diameter of K w.r.t. O, i.e. the maximal length of a chord of K cut by a line through O. Equivalently, it can be defined from the *radial function* [1, Section 0.7] $\rho_K(u) := \max\{c : cu \in K\}$ by setting $e_K := \max_{u\in S^{n-1}}(\rho(u) + \rho(-u))$.

It is clear that $\delta_K = \max\limits_{O\in\mathbb{R}^n} e_K(O) = \limsup\limits_{O\to\infty} e_K(O)$.

Proposition 3.1. *The function e_K is upper semicontinuous on $\mathbb{R}^n$ and its restrictions to K° and $\mathbb{R}^n \setminus K$ are continuous.*

Proof. Let $l(\Delta)$ denote the length of a segment Δ. Suppose to get a contradiction that upper semicontinuity fails at a point O, and let L be a line realizing the maximum in e_K. Then there are $\varepsilon > 0$, a sequence of points O_k and unit vectors u_k such that if we write $L_k := O_k + \mathbb{R}u_k$, then $l(L_k \cap K) \geq l(L \cap K) + \varepsilon$. Passing to a subsequence, we can assume that $L_k \cap K$ converges in the Hausdorff sense to a line segment S contained in $L' \cap K$ for some line L' with $O \in L'$. Then $l(L' \cap K) \geq l(L \cap K) + \varepsilon$, which contradicts the maximality of L.

If $O \notin K$, choose a line L_O such that the maximum chord length is attained. Since K has non empty interior and is convex, we can find a line segment arbitrarily close to $L_O \cap K$, such that a δ-neighborhood of it, U, is contained in K. Taking O' close enough to O, which has a positive distance from K, we can obtain a line L' so that $l(L' \cap U) \geq l(L_O \cap K) - \varepsilon$.

If $O \in K^\circ$, a neighborhood of $L_O \cap K^\circ$ is contained in K°, and for O' close enough to O and L' passing through O' and parallel to L_O, $l(L' \cap K) \geq l(L_O \cap K) - \varepsilon$. □

Recall that a point $O \in K$ is said to be *extreme* if for any line segment $S \subset K$ such that $O \in S$, then O is an endpoint of S.

Proposition 3.2. *The function e_K is continuous at $O \in \partial K$ if and only if there exist a chord S of maximal length so that O is an endpoint of S; equivalently, if there exists $O' \in \partial K$ such that $e_K(O) = |OO'|$.*

In the case of continuity, O' must be an extreme point of K.

It follows in particular that e_K is continuous at any extreme point of K.

Proof. First, suppose that there exists a chord S of maximal length so that O is an endpoint of $[OO'] := S$. Then, given $\varepsilon > 0$, there exists a ball $B(A_1, \varepsilon_1) \subset K \cap B(O', \varepsilon/2)$, with $\varepsilon_1 > 0$. So the convex hull C of $\{O\} \cup B(A_1, \varepsilon_1)$ is contained in K. For $\varepsilon_2 > 0$ small enough and $|OP| < \varepsilon_2$ we obtain

$$l((A_1P) \cap C) \geq |OA_1| - \varepsilon/2 \geq |OO'| - \varepsilon.$$

If $O' \in S' \subset \partial K$ was not extreme, looking at a point $A \in S'$ on one or the other side of O', we could obtain $|OA| > |OO'| = e_K(O)$, thus violating the maximality assumption.

Conversely, assume that for any chord $S := [O'O'']$ containing O such that $e_K(O) = l(S) = |OO'| + |OO''|$ one has $O \neq O'$, $O \neq O''$. Without loss of generality assume that $|OO'| \leq |OO''|$. We set $r := \frac{1}{2} \min |OO'| \leq \frac{1}{4} e_K(O)$, where the minimum is taken over all chords S as above. Then by compactness there exists a $\delta > 0$ such that

$$\sup\{|OA| : A \in K, |AO'| \geq r, |AO''| \geq r, \forall O', O''\} \leq e_K(O) - \delta.$$

Let H be a supporting hyperplane for K at O, and let H^+ be the connected component of $\mathbb{R}^n \setminus H$ that does not meet K. Let $P \in H^+$, $A \in K$. For $|OP| \leq \varepsilon$, if A satisfies $|AO'| \geq r, |AO''| \geq r, \forall O', O''$, endpoints of chords of maximal length containing O, then $l((AP) \cap K) \leq |AP| \leq |OA| + \varepsilon \leq e_K(O) - \delta + \varepsilon \leq e_K(O) - \delta/2$ for ε small enough. If on the other hand there exists O' or O'' such that $|AO'|, |AO''| \leq r$, then (taking O' and O'' to be both endpoints of the same chord)

$$l((AP) \cap K) \leq |AP| \leq |OA| + \varepsilon \leq |OO''| + \frac{1}{2}|OO'| + \varepsilon \leq e_K(O) - r + \varepsilon \leq e_K(O) - \frac{r}{2},$$

for ε small enough. So $\liminf_{P \to O, P \in H^+} e_K(P) < e_K(O)$. □

Example 3.3. Let $K = \triangle ABC$. Then the set of points of continuity of e_K on ∂K reduces to $\{A, B, C\}$ if and only if $\triangle ABC$ is equilateral. □

Set now $\varepsilon_K = \inf e_K$. Observe that this infimum may fail to be attained. For example, if K is a parallelogram with heights $h_1 \leq h_2$, then $h_1 = \omega_K = \varepsilon_K < e_K(O)$ for any point O.

On the other hand, we have the following

Example 3.4. Let $K = \triangle ABC$, with heights $h_a \leq h_b, h_c$. Then $\varepsilon_K = \omega_K = h_a$. Moreover, $e_K(O) = \varepsilon_K$ for any point O on the line $(AA_1) \perp (BC)$ $(A_1 \in [BC])$ such that $\angle ABO \geq 90°$ and $\angle ACO \geq 90°$. □

Proof. Note that for any point O' there is a vertex D of K such that the line $(O'D)$ meets its opposite side at a point D'. Then $e_K(O') \geq |DD'| \geq h_d \geq h_a = \omega_K$.

It remains to prove that $e_K(O) = h_a$ for O as above. It is enough to show that if $E \in]AB[$, $E_1 \in]BA_1[$, and $O \in (EE_1)$, then $|EE_1| < |AA_1|$. The last follows by $|OE_1| > |OA_1|$ and $|OE| < |OA|$ (since $\angle AEO > \angle ABO \geq 90°$). □

The observations above suggest the following

Proposition 3.5. *For any convex body K one has $\omega_K = \varepsilon_K = \tilde{\varepsilon}_K := \liminf_{O\to\infty} e_K(O)$.*

Proof. Note that if $[AB]$ is a diametral chord, then $\lim_{AB\ni O\to\infty} e_K(O) = |AB|$ and hence $\varepsilon_K \leq \tilde{\varepsilon}_K \leq \omega_K$. On the other hand, for any point O there is a diametral chord $[AB]$ such that $O \in AB$ (see e.g. [4, 6.6]) and thus $\varepsilon_K \geq \min d_K = \omega_K$. □

4. Refinement of the Lipschitz constants

4.1. Variations of w_K

Recall that in order to obtain Lemma 2.2, we needed to assume $u \in \mathcal{W}_K$. For K a convex polytope, we will establish an analogous result for all $u \in S^{n-1}$. For any convex body K, we also derive a more precise upper bound for $w'_K(u)$ which depends on u.

As before, for a convex body K, denote by $H_1(u)$ and $H_2(u)$ the two supporting hyperplanes of K orthogonal to u, and set

$$s_K(u) := \max\{|A'B'| : A' \in K\cap H_1(u), B' \in K\cap H_2(u)\},\ p_K(u) := \sqrt{s_K^2(u) - w_K^2(u)}.$$

Lemma 4.1. *Let K_j be a decreasing sequence of convex bodies with limit K and let $u_j \in S^{n-1}$ be a sequence with $\lim_j u_j = u$. Then $\limsup_j s_{K_j}(u_j) \leq s_K(u)$.*

As a consequence, the analogous inequality holds for p_K, and both s_K and p_K are upper semicontinuous on S^{n-1}.

Proof. For any $x \in S^{n-1}$, denote by $H_1^j(x)$ and $H_2^j(x)$ the two supporting hyperplanes of K_j orthogonal to x.

We choose $A_j \in K \cap H_1^j(u_j)$, $B_j \in K \cap H_2^j(u_j)$ such that $s_{K_j}(u_j) = |A_jB_j|$.

Since s_{K_j} is a bounded sequence, passing to a subsequence K'_j we may assume that $s_{K'_j}$ converges to $\limsup_j s_{K_j}$, and again passing to a subsequence, that

$$A_j \to A^* \in H_1(u) \cap K,\ \ B_j \to B^* \in H_2(u) \cap K.$$

Then $s_K(u) \geq |A^*B^*| = \lim_j |A_jB_j| = \limsup_j s_{K_j}$.

To obtain the upper semicontinuity, it is enough to specialize to the case where $K_j = K$ for all j. □

For K any convex body, define $\hat{M}_K := \max_{u\in S^{n-1}} p_K(u)$. Since p_K is upper semicontinuous, the maximum in the definition is indeed attained.

For K a convex polytope, define $\tilde{M}_K := \sup_{u\in\mathcal{W}_K} p_K(u)$.

If K is a convex polytope and $u \in \mathcal{W}_K$, $K \cap H_1(u)$ and $K \cap H_2(u)$ are singletons, and Lemma 2.2 says that $w'(u) = p_K(u)$.

Proposition 4.2. *For K a convex polytope, $\tilde{M}_K = \hat{M}_K$ and for any $u \in S^{n-1}$, $w'_K(u) = p_K(u) \leq \hat{M}_K$.*

Proof. To prove that $\tilde{M}_K = \hat{M}_K$, it will be enough to show :

$$\forall u \in S^{n-1} \setminus \mathcal{W}_K, \limsup_{u' \to u, u' \neq u} p_K(u') = p_K(u).$$

That the lefthand side is less or equal to $p_K(u)$ follows from Lemma 4.1.

To obtain the reverse inequality, let A_m, B_m be two points where the maximum in the definition of $s_K(u)$ is achieved. Let $\mathcal{P}$ be the affine plane generated by A_m, u, and $\overrightarrow{A_m B_m}$, $\pi_{\mathcal{P}}$ the orthogonal projection to $\mathcal{P}$, and $\mathcal{L}$ the $(n-2)$-dimensional vector space orthogonal to $\mathcal{P}$.

Then both $H_1(u)$ and $H_2(u)$ are parallel to $\mathcal{L} \oplus v$, where $v \in \mathrm{Span}(u, \overrightarrow{A_m B_m})$, $\|v\| = 1$, $v \perp u$. Furthermore we can choose $\langle \overrightarrow{A_m B_m}, v \rangle \geq 0$. Notice that

$$\pi_{\mathcal{P}}(H_1(u)) = H_1(u) \cap \mathcal{P} = A_m + \mathbb{R}v := L_A;$$
$$\pi_{\mathcal{P}}(H_2(u)) = H_2(u) \cap \mathcal{P} = B_m + \mathbb{R}v := L_B,$$

and because we have supporting hyperplanes, $\pi_{\mathcal{P}}(K)$ is contained in the closed strip between L_A and L_B. The line segment $\pi_{\mathcal{P}}(H_1(u) \cap K) \subset L_A$, and by our maximality hypothesis, for any $t < 0$, $A_m + tv \notin \pi_{\mathcal{P}}(H_1(u) \cap K)$. Similarly, $\pi_{\mathcal{P}}(H_2(u) \cap K) \subset L_B$, and for any $t > 0$, $B_m + tv \notin \pi_{\mathcal{P}}(H_2(u) \cap K)$.

Since $\pi_{\mathcal{P}}(K)$ is a closed polygonal convex body, if we orient the plane so that $\angle(\overrightarrow{A_m B_m}, v) > 0$, then there exists $\varepsilon_0 > 0$ with the property that for any unit vector $v' \in \mathrm{Span}\ (u, \overrightarrow{A_m B_m})$ with $0 < \angle(v, v') < \varepsilon_0$, then $A_m + \mathbb{R}v'$ is a supporting line for $\pi_{\mathcal{P}}(K)$, with $A_m + \mathbb{R}v' \cap \pi_{\mathcal{P}}(K) = \{A_m\}$, and $B_m + \mathbb{R}v'$ satisfies the analogous properties. Therefore $H'_1 := \pi_{\mathcal{P}}^{-1}(A_m + \mathbb{R}v')$ and $H'_2 := \pi_{\mathcal{P}}^{-1}(B_m + \mathbb{R}v')$ are parallel supporting hyperplanes for K, orthogonal to a vector u' which can be made arbitrarily close to u, such that $H'_1 \cap K \subset A_m + \mathcal{L}$ and $H'_2 \cap K \subset B_m + \mathcal{L}$.

Again by our maximality hypothesis, we have that

$$K \cap (A_m + \mathcal{L}) = \{A_m\}, \quad K \cap (B_m + \mathcal{L}) = \{B_m\},$$

so finally the supporting hyperplanes H'_1 and H'_2 intersect K only at A_m and B_m respectively, so $u' \in \mathcal{W}_K$ and $s_K(u') = s_K(u)$; it is easy to see that $s_K(u') \to s_K(u)$ as $u' \to u$, and we are done with the first part of the proposition.

Now consider again $u_0 \in S^{n-1} \setminus \mathcal{W}_K$. Taking $u \in \mathcal{W}_K$ close to u_0 such that $p_K(u)$ is close to $p_K(u_0)$, and u', u'' which tend to u and come close to the limes superior in Lemma 2.2, we see that $w'(u_0) \geq p_K(u_0)$ (perform a diagonal process to have u' and u'' tending to u_0).

To obtain the reverse inequality, for any $\varepsilon > 0$, by the upper semicontinuity of p_K and the first part of the proposition, there is a neighborhood U of u_0 such that for any $u \in U$, $w'(u_1) \leq p_K(u_0) + \varepsilon$. If $u', u'' \in U$, we can connect them by an arc γ of a great circle and reasoning as in the proof of Lemma 2.2,

$$|w_K(u') - w_K(u'')| \leq \rho(u', u'') \sup_{u \in \gamma} w'(u) \leq (1 + \varepsilon)\|u' - u''\|(p_K(u_0) + \varepsilon),$$

reducing U if needed. This proves that $w'(u_0) \leq p_K(u_0)$. □

As a consequence of Theorem 2.1 and the proof above, we obtain the following.

Corollary 4.3. *For any convex polytope $K \subset \mathbb{R}^n$,*

$$\sup_{u \neq \pm v \in S^{n-1}} \Delta w_K(u,v) = \tilde{M}_K = \hat{M}_K \leq M_K.$$

Proposition 4.4. *For any convex body $K \subset \mathbb{R}^n$ one has*

$$\sup_{u \neq \pm v \in S^{n-1}} \Delta w_K(u,v) \leq \hat{M}_K \quad \text{and} \quad w'_K(u) \leq p_K(u).$$

Proof. We will say that $\mathcal{A} \subset S^{n-1}$ is convex whenever it is the intersection of a convex cone with S^{n-1}, i.e. $\mathbb{R}_+\mathcal{A}$ is convex, i.e. for any two points $a, b \in \mathcal{A}$, the geodesic segment (arc of a great circle) from a to b is contained in $\mathcal{A}$.

Now, take a sequence of polytopes K_j decreasing to K, and a closed convex subset $\mathcal{N} \subset S^{n-1}$ such that for any $u, v \in \mathcal{N}$, $\rho(u,v) \leq \pi/2$. We claim that

$$\sup_{u \neq v \in \mathcal{N}} \Delta w_K(u,v) \leq \max_{\mathcal{N}} p_K.$$

Indeed, by the proof of Theorem 2.1(1), carried out on geodesics remaining within $\mathcal{N}$,

$$|w_{K_j}(u) - w_{K_j}(v)| \leq \rho(u,v) \max_{\mathcal{N}} p_{K_j}.$$

We claim that $\limsup_j \max_{\mathcal{N}} p_{K_j} \leq \max_{\mathcal{N}} p_K$. Indeed, take a sequence $(u_j)_j \subset \mathcal{N}$ such that $\lim_j p_{K_j}(u_j) = \limsup_j \max_{\mathcal{N}} p_{K_j}$, and $u_j \to u \in \mathcal{N}$, then Lemma 4.1 implies that

$$\lim_j p_{K_j}(u_j) \leq p_K(u) \leq \max_{\mathcal{N}} p_K,$$

and the claim is proved.

To get the statement over the whole sphere, recall that $w_K(u) = w_K(-u)$, and notice that for any $u, v \in S^{n-1}$, $\rho(u,v) \leq \pi/2$ or $\rho(u,-v) \leq \pi/2$, and we can choose a convex set $\mathcal{N}$ accordingly.

To get the statement about $w'_K(u)$, take a sequence of convex closed neighborhoods of u in S^{n-1} converging to $\{u\}$, and apply the upper semicontinuity of p.

Note that we could also deduce the statement on Δw_K from the one on w'_K. □

Open question 4.5. Can the inequalities in Proposition 4.4 be replaced by equalities for all convex bodies?

4.2. Variations of d_K

In analogy to the beginning of Subsection 4.1, we extend the notations defined before Lemma 2.3. Given $u \in S^{n-1}$, for any diametral chord $[AB]$ we have $|AB| = d_K(u)$ and there exists some parallel supporting hyperplanes for K at A and B. We define

$$r_K(u) := \inf \left\{ \text{dist } (H_1, H_2) : \begin{array}{l} [AB] \text{ diametral chord and } H_1 \ni A, \\ H_2 \ni B \text{ supporting hyperplanes} \end{array} \right\},$$

and

$$q_K(u) := d_K(u) \sqrt{\frac{d_K^2(u)}{r_K^2(u)} - 1}.$$

For K a convex polytope and $u \in \mathcal{R}_K$, Lemma 2.3 says that $d'_K(u) = q_K(u)$.

Open question 4.6. For K a convex polytope, do we have

$$\sup_{u\in\mathcal{R}_K} q_K(u) = \max_{u\in S^{n-1}} q_K(u)?$$

Even without an answer to that question, we can recover a result analogous to Proposition 4.4.

Lemma 4.7. *Let K_j be a decreasing sequence of convex bodies with limit K and let $u_j \in S^{n-1}$ be a sequence with $\lim_j u_j = u$. Then $\liminf_j r_{K_j}(u_j) \geq r_K(u)$.*

As a consequence, the reverse inequality holds for q_K, r_K is lower semicontinuous and q_K upper semicontinuous on S^{n-1}.

Proof. We begin by proving that $\lim_j d_{K_j}(u_j) = d_K(u)$. Indeed, since $K \subset K_j$ for all j, $d_{K_j}(u_j) \geq d_K(u_j) \to d_K(u)$ since d_K is continuous. On the other hand, choosing A_j, B_j such that $d_{K_j}(u_j) = |A_jB_j|$, any convergent subsequences will tend respectively to A^*, B^* such that (A^*B^*) is parallel to u, so we deduce that $\limsup_j d_{K_j}(u_j) \leq |A^*B^*| \leq d_K(u)$ and we are done.

Considering a subsequence such that $d_{K_j}(u_j) = |A_jB_j| \to d_K(u)$, we have in consequence $A_j \in K \cap H_1^j$, $B_j \in K \cap H_2^j$ with dist $(H_1^j, H_2^j) = r_{K_j}(u_j)$. We can choose a further subsequence such that $r_{K_j}(u_j) \to \liminf_j r_{K_j}(u_j)$, and $H_1^j(u_j)$, $H_2^j(u_j)$ converge to hyperplanes H_1, H_2. Then H_1 and H_2 are supporting hyperplanes for K at A^* and B^* respectively, and $[A^*; B^*]$ is now a diametral chord for u, so that $\liminf_j r_{K_j}(u_j) = \text{dist } (H_1, H_2) \geq r_K(u)$. □

Proposition 4.8. *For any convex body $K \subset \mathbb{R}^n$ one has that*

$$\sup_{u\neq\pm v\in S^{n-1}} \Delta d_K(u,v) \leq \hat{N}_K \quad \textit{and} \quad d'_K(u) \leq q_K(u).$$

Proof. The proof follows the same lines as that of Proposition 4.4 (which makes no use of Proposition 4.2), replacing p_K by q_K and M_K by N_K. □

4.3. Examples

Example 4.9. Let $K = \triangle ABC$, with side lengths $a \leq b \leq c$, and heights h_a, h_b, h_c. Then $\delta_K = c$, $\omega_K = h_c$, $M_K = \sqrt{c^2 - h_c^2}$, $N_K = \frac{c}{h_c}\sqrt{c^2 - h_c^2}$.

Let $\hat{v}$ be a unit vector parallel to the side of length c, $\hat{u}$ be the unit vector orthogonal to the side of length b such that $\langle \hat{u}, \hat{v}\rangle \geq 0$, and γ the geodesic arc between them. Then

$$\lim_{\gamma\ni u',u''\to\hat{u}} \Delta w_K(u',u'') = \sup_{u\neq\pm v\in S^1} \Delta w_K(u,v) = \sqrt{c^2 - h_b^2} = \hat{M}_K$$

$$\lim_{\gamma\ni v',v''\to\hat{v}} \Delta d_K(v',v'') = \sup_{u\neq\pm v\in S^1} \Delta d_K(u,v) = \frac{c}{h_b}\sqrt{c^2 - h_b^2} = \hat{N}_K.$$

We see that $\hat{M}_K = M_K$ and $\hat{N}_K = N_K$ if and only if $b = c$.

Example 4.10. Let $K = [0, a_1] \times \cdots \times [0, a_n]$, $0 < a_1 \leq \cdots \leq a_n$. Then $\omega_K = a_1$ and $\delta_K = \sqrt{a_1^2 + \cdots + a_n^2}$. Let $u = e_1$, $v = (a_1 e_1 + \ldots a_n e_n)/\delta_K$, and γ be the geodesic joining u and v on S^{n-1}. One may check that

$$\lim_{\gamma \ni u', u'' \to u} \Delta w_K(u', u'') = M_K, \qquad \lim_{\gamma \ni v', v'' \to v} \Delta d_K(u', u'') = N_K.$$

Example 4.11. Let $K = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$. Then we get for $u = (\cos\alpha, \sin\alpha) \in S^1$:

$$w_K(u) = 2\sqrt{a^2 \cos^2\alpha + b^2 \sin^2\alpha}, \quad s_K(u) = 2\sqrt{\frac{a^4 \cos^2\alpha + b^4 \sin^2\alpha}{a^2 \cos^2\alpha + b^2 \sin^2\alpha}},$$

$$d_K(u) = \frac{2ab}{\sqrt{a^2 \sin^2\alpha + b^2 \cos^2\alpha}}, \quad r_K(u) = 2ab\sqrt{\frac{a^2 \sin^2\alpha + b^2 \cos^2\alpha}{a^4 \sin^2\alpha + b^4 \cos^2\alpha}},$$

$$p_K(u) = \frac{|(a^2 - b^2)\sin 2\alpha|}{\sqrt{a^2 \cos^2\alpha + b^2 \sin^2\alpha}}, \quad q_K(u) = \frac{ab|(a^2 - b^2)\sin 2\alpha|}{\sqrt{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)^3}},$$

$$\lim_{u', u'' \to u} \Delta w_K(u', u'') = p_K(u), \qquad \lim_{u', u'' \to u} \Delta d_K(u', u'') = q_K(u).$$

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