Journal of Convex Analysis
Volume 33 (2026), No. 1&2, 335–360

Relaxation for a Degenerate Functional with Linear Growth in the Onedimensional Case

Valeria Chiadò Piat
Dip. di Scienze Matematiche G. Lagrange, Politecnico di Torino, Italy
valeria.chiadopiat@polito.it

Virginia De Cicco
Dip. di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Italy
virginia.decicco@uniroma1.it

Anderson Melchor Hernandez
Dipartimento di Matematica, Università di Bologna, Italy
anderson.melchor@unibo.it

Received: December 4, 2024
Accepted: March 25, 2025

We study the relaxation of a degenerate functional with linear growth, depending on a weight w that does not exhibit doubling or Muckenhoupt-type conditions. In order to obtain an explicit representation of the relaxed functional and its domain, our main tools for are Sobolev inequalities with double weight.

Keywords: Lower semicontinuity, relaxation, degenerate variational integrals, weight, Poincaré inequality.

2020 Mathematics Subject Classification: 26A15, 49J45.

1. Introduction

In this work, we focus on the study of an integral functional in one dimension with linear growth, allowing for a degenerate weight w. We aim to provide an explicit relaxation formula for the functional

$$F(u) := \begin{cases} \displaystyle\int_\Omega |u'| w \mathrm{d}x & \text{if } u \in \mathrm{A}C(\overline{\Omega}), \\ +\infty & \text{if } u \in X \setminus \mathrm{A}C(\overline{\Omega}), \end{cases} \tag{1}$$

where Ω is an open bounded set in $\mathbb{R}$, u' denotes the derivative of u, w is a non-negative, locally integrable function, $\mathrm{A}C(\overline{\Omega})$ is the space of absolutely continuous functions on $\overline{\Omega}$, and X is a topological space comprising measurable functions which will be introduced later on. We will find an explicit expression of the lower semicontinuous envelope of F, that is denoted by $\overline{F}$ with respect to a suitable convergence.

Several studies have focused on the investigation of functionals with p-growth for $1<p<+\infty$ within different functional frameworks; see, for example, [10, 16, 17, 21]. Nevertheless, there are few works dedicated to the analysis of functionals with linear

ISSN 0944-6532 / $ 2.50

growth like (1) above (see, for instance, [7] and references therein). In the recent work [11], we have analyzed the p-version of the functional F, defined as

$$F_p(u) := \begin{cases} \displaystyle\int_\Omega |u'|^p w \,\mathrm{d}x & \text{if } u \in \mathrm{AC}(\overline{\Omega}), \\ +\infty & \text{if } u \in X \setminus \mathrm{AC}(\overline{\Omega}), \end{cases} \tag{2}$$

where w does not exhibit doubling or Muckenhoupt-type conditions, [22]. In that case, we have conducted the analysis in weighted Sobolev spaces; we refer to [2, 3, 4, 19] for general approaches to the definition of these spaces. Let us briefly explain our strategy in the case $1 < p < +\infty$, and what is different in the present case $p = 1$. We first proved Poincaré inequalities involving w and an auxiliary weight $\hat{w}_p$ that corrects the weight in the zones where w is strongly degenerate (i.e. $w^{-\frac{1}{p-1}}$ is not summable). Specifically, we showed that the p-norm of the gradient term of a generic function u weighted by w, is greater up to a suitable constant than the p-norm of u weighted by $(\hat{w}_p)^{p-1}$. Subsequently, assuming that w is finitely degenerate (see [11, Definition 2.1]), and in view of such a Poincaré inequality with two different weights, we proceeded to choose $X = L^p((\hat{w}_p)^{p-1})$, and showed that AC-functions are dense, in a suitable Sobolev space $W \subseteq X$. As a consequence, we were able to determine the finiteness domain of the relaxed functional $\overline{F}_p$ by performing the relaxation in the strong topology of X.

In the present work, we follow some of the previous ideas, but we cannot apply verbatim such methodology. The first reason is that for a functional with linear growth like (1), it is necessary to work with BV like spaces, rather than Sobolev spaces, and the second reason is that the functional in this case can be interpreted as a pairing.

A class of weighted bounded variation functions denoted $\mathrm{BV}(\Omega; w)$ in any dimension ($\Omega \subset \mathbb{R}^n$) is introduced in [6] (see Section B where we recall the definitions and the results of [6]).

By requiring that $w > 0$ and w belongs to the Muckenhoupt class A_1, (see Definition B.1 below) it is possible to define a weighted $\mathrm{BV}(\Omega, w)$-space. A priori such weight w is only a.e. defined, but it is not restrictive to assume that condition A_1 holds for any point in Ω (this is possible since it can be proved that there exists a further weight lower semicontinuous $\tilde{w}$ that defines the same weighted BV-space, and satisfies A_1 at any point, see Lemma B.3 below). Moreover a density theorem holds true in $\mathrm{BV}(\Omega, w)$ (see Theorem B.6 below) and by assuming the local growth condition (44) a Poincaré inequality holds (see Theorem B.7 below).

In the present paper, although confining the study to the onedimensional case, we follow another approach. We will deal with a weight $w \geq 0$ (and so it admits large degeneration), that does not belong to the Muckenhoupt class A_1 (and so it is only a.e. defined) and does not satisfy any doubling condition. We will consider a new category of spaces that we denote as $\mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ inspired to some BV like space recently introduced in [13], although this approach forces us to assume some regularity of the weight, i.e. w is a BV_{loc} within the largest open set where $\frac{1}{w}$ is bounded.

More precisely, we say that $u \in \mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ if it is a Borel function that belongs to $L^1_{\mathrm{loc}}(\Omega, w) \cap L^1_{\mathrm{loc}}(\Omega, |Dw|)$, such that the Anzellotti pairing (w, Du), defined below is a Radon measure (see [5] for its original definition). Moreover, under suitable assumptions this class is a Banach space.

Under the assumption $w \in BV_{\mathrm{loc}}(\Omega)$, the distributional definition of the pairing is the following

$$\langle (w, Du), \varphi \rangle := -\int_\Omega u^{\frac{1}{2}} \varphi \,\mathrm{d}Dw - \int_\Omega u\varphi' w \,\mathrm{d}x, \quad \text{for } \varphi \in C^\infty_c(\Omega). \tag{3}$$

Here φ' denotes the derivative of φ, Du denotes the distributional derivative of u, and $u^{\frac{1}{2}}$ the precise representative of u (see Section A for a more detail explanation). The space $\mathrm{BV}^w(\Omega)$ was introduced in [13] because it is the natural functional space where the distributional derivative defined in (3) is a Radon measure.

In the present work, we find that $\mathrm{BV}^w(\Omega)$ is the natural ambient space in which an explicit formula for $\overline{F}$ can be expressed. Therefore, to ensure a suitable behavior of (3), we restrict our analysis to the following setup. We assume that w is a nonnegative function such that w is locally integrable in Ω. Our objective is to demonstrate that, under these conditions, the relaxed functional can be expressed by means of a pairing, as studied in [13] and [12]. This pursuit is built upon innovative concepts introduced in those works, where $\mathrm{BV}^w(\Omega)$ spaces, consisting of functions that satisfy divergence-measure properties, are larger than the conventional $BV(\Omega)$-spaces in [1], or the weighted $BV(\Omega, w)$-spaces in [6]. By following [16, 11], our chosen space X comprises $W^{1,1}$-functions with a degenerate weight w. The pairing of such functions u with w consists in a Radon measure within the largest open set where $\frac{1}{w}$ is bounded. This requires the introduction of an additional weight, denoted as $\hat{w}$. This corrective function addresses the singularities inherent in the respective weight w. Moreover, in this scenario, we also prove a weighted Poincaré inequality involving w and $\hat{w}$.

Subsequently, in Section 3, we assume that w is finitely degenerate (see Definition 2.1 below) and the stronger condition that the weight w belongs to $W^{1,1}_{\mathrm{loc}}$ within the largest open set where $\frac{1}{w}$ is bounded. We then relax F with respect to a weak convergence involving $\hat{w}$ and $|Dw|$, which we will refer to as $(\hat{w}, Dw)$-convergence. This is similar to the $(w, \frac{1}{2})$-convergence introduced in [13] (see Definition A.5 before). The main difference lies in the choice of the $L^1(\hat{w})$-weak convergence rather than $L^1(w)$-weak convergence.

As we will see in Section 2, our analysis is based on a suitable decomposition of the open set Ω into disjoint subsets where the reciprocal $1/w$ of the weight w is locally bounded. The boundaries of these sets are a countable family of points. This fact is typical of dimension 1. The subsequent construction of the auxiliary weight and the remaining results are strongly tied to this property. In higher dimensions, this procedure becomes more intricate, as various situations can arise, for instances, cases where points must be replaced by surfaces, and thus the construction of $\hat{w}$ becomes more involved. Moreover, the strategy relying on the density of AC-functions must be carefully replaced.

This work is structured as follows. In Section 2, we define $\hat{w}$ and prove the validity of weighted Poincaré inequalities, see Theorem 2.10 below. Thanks to this result, we are allowed in Section 3, once introduced our $(\hat{w}, Dw)$-convergence, to prove a compactness theorem with respect to this convergence and to prove our relaxation theorem, see Definition 3.2, and Theorem 3.6, respectively. Lastly, in Section A, we revisit some fundamental concepts from geometric measure theory, applicable to all dimensions $n \geq 1$, and we recall the notion of pairing as studied in [13]. In Section B, we recall some similar results about weighted Poincaré inequalities when w belongs to the Muckenhoupt class A_1, obtained in [6].

2. Poincaré inequalities with double weight

Let $\Omega = (a, b)$ be a bounded open interval. In what follows, we make the following structural assumptions:

(H1) $w \geq 0$;

(H2) $w \in L^1_{\text{loc}}(\Omega)$.

Here we denote by $I_{\Omega,w}$ the biggest open bounded set contained in Ω such that $\frac{1}{w}$ is $L^\infty_{\text{loc}}(I_{\Omega,w})$-function. Then $I_{\Omega,w}$ can be written in a unique way as the union of pairwise disjoint open intervals $(a_i, b_i) \subset \Omega$, that is,

$$I_{\Omega,w} = \bigcup_{i=1}^{N_w} (a_i, b_i),$$

with $1 \leq N_w \leq +\infty$. Furthermore, since $\frac{1}{w} \in L^\infty_{\text{loc}}(I_{\Omega,w})$, for every $i = 1, \ldots, N_w$ and $K \Subset (a_i, b_i)$ there exists a nonnegative constant $c_{i,K}$ such that

$$\frac{1}{w(x)} \leq c_{i,K} \text{ for a.e. } x \in K. \tag{4}$$

Definition 2.1. (i) If $I_{\Omega,w} = \emptyset$, we put $N_w := 0$.

(ii) If $1 \leq N_w < \infty$ we say that w is *finitely degenerate* in Ω.

(iii) If $N_w = +\infty$ we say that w is *not finitely degenerate* in Ω.

Examples 2.2. Let us consider the following examples.

(I) Let $w(x) = (1 - x^2)^2$ defined in the interval $(-2, 2)$: then, $I_{\Omega,w} = (-2, -1) \cup (-1, 1) \cup (1, 2)$, and w is finitely degenerate with $N_w = 3$.

(II) Let $w(x) = 1 + \sin \frac{1}{x}$ defined in the open interval $(0, 1)$: since $w(x_i) = 0$ if $x_i = \frac{1}{\pi(\frac{3}{2}+2i)}$, $i \in \mathbb{N}$, we have that $I_{\Omega,w} = \bigcup_{i\in\mathbb{N}}(x_{i+1}, x_i)$ and w is not finitely degenerate, i.e. $N_w = +\infty$.

(III) Let $w(x) = |x|^\alpha$ with $\alpha > 1$ defined in the interval $(-1, 1)$.

2.1. An auxiliary weight

Let $\hat{w} : \Omega \to [0, +\infty[$ be defined as

$$\hat{w}(x) := \lim_{x\to a_i^+} \left(\left\|w^{-1}\right\|_{L^\infty\left(\left(x, \frac{a_i+b_i}{2}\right)\right)}\right)^{-1} \qquad \text{if } x = a_i$$

$$\hat{w}(x) := \left(\left\|w^{-1}\right\|_{L^\infty\left(\left(x, \frac{a_i+b_i}{2}\right)\right)}\right)^{-1} \qquad \text{if } a_i < x \leq \frac{3a_i + b_i}{4}$$

$$\hat{w}(x) := \left(\|w^{-1}\|_{L^\infty((\frac{3a_i+b_i}{4},\frac{a_i+3b_i}{4}))}\right)^{-1} \qquad \text{if } \frac{3a_i+b_i}{4} \le x \le \frac{a_i+3b_i}{4}$$

$$\hat{w}(x) := \left(\|w^{-1}\|_{L^\infty((\frac{a_i+b_i}{2},x))}\right)^{-1} \qquad \text{if } \frac{a_i+3b_i}{4} \le x < b_i$$

$$\hat{w}(x) := \lim_{x\to b_i^-} \left(\|w^{-1}\|_{L^\infty((\frac{a_i+b_i}{2},x))}\right)^{-1} \qquad \text{if } x = b_i$$

$$\hat{w}(x) := 0 \qquad \text{if } x \in \Omega \setminus \overline{I}_{\Omega,w}\,.$$

Remark 2.3. At first glance, the definition of $\hat{w}$ may seem subtle. Nevertheless, it is an important function with nice regularity properties, as presented in the next proposition, and it allows us to prove the validity of a Poincaré inequality with weights w and $\hat{w}$, respectively. It is also worth noting that a similar definition of the function $\hat{w}$ was already considered in [11], in the case where the functional F defined in (1) is replaced by (2). Instead, the present work addresses the case $p = 1$ separately, because the tools used in [11] were developed in a Sobolev context, whereas here we need tools beyond BV(Ω)-spaces recently developed in [12, 13, 14, 15].

In the following figures, we illustrate the behavior of the function $\hat{w}$ for a specific choice of w, while in Proposition 2.4, we prove some of its mathematical properties.

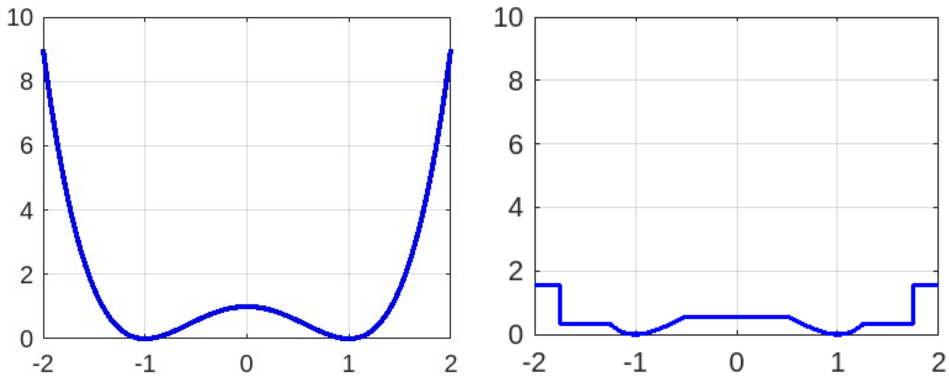

Figure 1: In the figure on the left hand side, we have the profile of $w(x) = (1-x^2)^2$ for $x \in (-2, 2)$, while in the right hand side, we have its associated weight $\hat{w}$. In this case, we note that $N_w = 3$.

Let us collect some properties of the function $\hat{w}$ in the following Proposition.

Proposition 2.4. (Properties of $\hat{w}$) *Suppose that* (H1)–(H2) *hold true.*

(i) *For each* $i = 1, \dots, N_w$, $\hat{w}$ *is constant in* $[\frac{3a_i+b_i}{4}, \frac{a_i+3b_i}{4}]$, *increasing in* $[a_i, \frac{3a_i+b_i}{4}]$ *and so a* BV*-function in* $\left[a_i, \frac{3a_i+b_i}{4}\right)$, *decreasing in* $[\frac{a_i+3b_i}{4}, b_i]$ *and so a* BV*-function in* $\left(\frac{a_i+3b_i}{4}, b_i\right]$. *Moreover, it holds that*

$$0 < \hat{w}(x) \le \sup_{y\in(a_i,b_i)} \hat{w}(y) =: L_i < \infty \quad \forall\, x \in (a_i, b_i)\,, \tag{5}$$

$$M_{i,K} := \inf_{x\in K} \hat{w}(x) > 0 \textit{ for each } x \in K \Subset (a_i, b_i), \tag{6}$$

and $\hat{w}(a_i) = 0$ *(respectively* $\hat{w}(b_i) = 0$*) if and only if* $\frac{1}{w} \notin L^\infty((a_i, \frac{a_i+b_i}{2}))$ *(respectively* $\frac{1}{w} \notin L^\infty((\frac{a_i+b_i}{2}, b_i))$*).*

(ii) *If* $\frac{1}{w} \in L^\infty(\Omega)$*, then there exists a constant* $c > 0$ *such that*

$$0 < \frac{1}{c} \leq \hat{w}(x) \leq c \quad a.e.\ x \in \Omega\,.$$

(iii) *If* w *is finitely degenerate in* Ω*, i.e.* $1 \leq N_w < \infty$*, then there exists a constant* $c > 0$ *such that*

$$0 \leq \hat{w}(x) \leq c \quad a.e.\ x \in \Omega \text{ and } \hat{w} \in \mathrm{BV}(\Omega).$$

(iv) *If* w *is not finitely degenerate in* Ω*, i.e.* $N_w = \infty$*, then* $\hat{w} \in L^\infty_{\mathrm{loc}}(I_{\Omega,w})$*, and for each* $1 \leq i < +\infty$*, we get* $\hat{w} \in \mathrm{BV}((a_i, b_i))$*.*

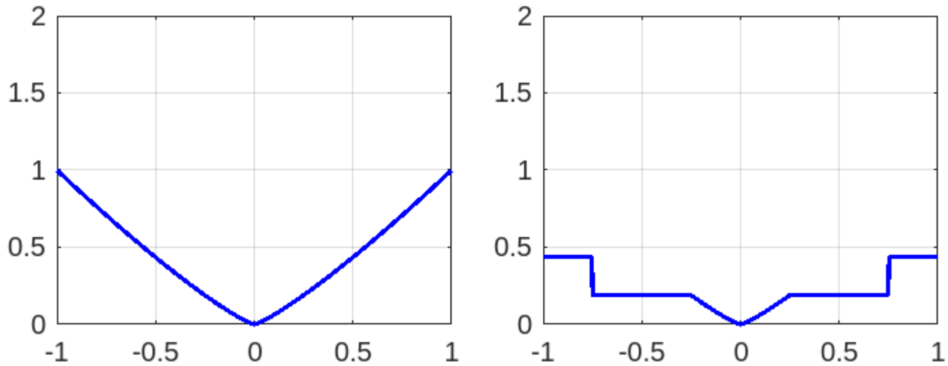

Figure 2: In the first figure on the left hand side, we have the profile of $w(x) = |x|^{1.2}$ for $x \in (-1, 1)$, $\alpha = 1.2$, while in the right hand side, we have its associated weight $\hat{w}$. In this case, we note that $N_w = 2$.

Remark 2.5. By definition $\hat{w} \leq w$ and fixed $i = 1, \ldots, N_w$ if the function w is increasing in $(a_i, \frac{3a_i+b_i}{4})$, then $\hat{w}(x) = w(x)$ a.e. in $(a_i, \frac{3a_i+b_i}{4})$. This is the case in Examples 2.2. In the case (I), the function w is increasing in $(-1, -\frac{1}{2})$ and in $(1, \frac{5}{4})$, while in the case (II) the function w is increasing in $(x_{i+1}, \frac{3x_{i+1}+x_i}{4})$.

On the contrary, if w admits an oscillating behaviour in a right neighborhood of some a_i, it can be happen that $\hat{w} \neq w$ in this neighborhood (see Example in Remark 2.7 below).

On the other hand, let us notice that, unlike the case $1 < p < +\infty$, our weight $\hat{w}$ involves the inverse of the L^∞-norm of w^{-1}. However, we can say that $\hat{w}$ is a $\mathrm{BV}_{\mathrm{loc}}$ function rather than absolutely continuous as it happens in the case $1 < p < +\infty$, [11, see Proposition 2.5 (ii)]. It is important to recognize that, in some sense, the conditions assumed in Proposition 2.4 are the analogue counterpart of those assumed in [11, Proposition 2.5 (ii)]. Specifically, while in such a Proposition, we required hypotheses to give a meaning to the integral of $w^{-\frac{1}{p-1}}$ for $1 < p < +\infty$, Proposition 2.4 involves the L^∞-norm of w^{-1}. □

In what follows, given $w \in \mathrm{BV}_{\mathrm{loc}}(I_{\Omega,w})$ we set

$$\mathrm{Dom}_w := \left\{ u : \Omega \to \mathbb{R} : u \in W^{1,1}_{\mathrm{loc}}(I_{\Omega,w}), u \in \mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w}) \right\}, \tag{7}$$

where the class $\mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w})$ has been defined in the Introduction. We note that this definition of Dom_w differs from the one in [11, formula 3]. Indeed, in [11, formula (3)], the definition of Dom_w does not require any regularity properties on the weight w.

In fact, we have that in the case $1 < p < +\infty$, Dom_w is defined as

$$\mathrm{Dom}_{w,p} := \left\{ u : \Omega \to \mathbb{R} : u \in W^{1,1}_{\mathrm{loc}}(I_{\Omega,w}), \int_{I_{\Omega,w}} |u'|^p \, w \, \mathrm{d}x < +\infty \right\}. \tag{8}$$

The importance of the functional spaces (7) and (8) is related to the relaxation result in Section 3 below and in [11, Section 3], respectively. For this reason we need to study the Poincaré inequality in Dom_w.

Remark 2.6. The space $\mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w})$ considered in the definition of Dom_w in (7) has been introduced recently in [13] in the general multidimensional setting. We recall the definition and the main properties of $\mathrm{BV}^w_{\mathrm{loc}}$ in Section A, with the details in the onedimensional case. We notice that

$$\mathrm{Dom}_w \subset L^1_{\mathrm{loc}}(I_{\Omega,w}, w) \cap L^1_{\mathrm{loc}}(I_{\Omega,w}, |\mathrm{D}w|),$$

and by the definition of pairing in (39) below

$$\langle (w, Du), \varphi \rangle := -\int_{I_{\Omega,w}} u\varphi \, \mathrm{d}\mathrm{D}w - \int_{I_{\Omega,w}} u\varphi' w \, \mathrm{d}x, \ \text{ for } \varphi \in C^\infty_c(I_{\Omega,w}), \ u \in \mathrm{Dom}_w \ .$$

Here we used that, since $u \in W^{1,1}_{\mathrm{loc}}(I_{\Omega,w})$ we have $u = u^*$ (recall that u^* is the precise representative of u, and since in the onedimensional case $W^{1,1}(I) = AC(\overline{I})$, we have that $u = u^* = u^{\frac{1}{2}}$ where $u^{\frac{1}{2}}$ is the trace of u as defined in (38)), the measure (w, Du) has the following expression

$$(w, Du)(I) = \int_I u'(x) w(x) \mathrm{d}x, \quad \text{for any } I \Subset I_{\Omega,w},$$

and, by definition of $\mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w})$, its total variation is finite

$$|(w, Du)|(I) < +\infty.$$

Let us note that we have used the symbol u' to denote the derivative of u. In what follows, we will maintain this notation and will subsequently use Du to denote the distributional derivative of u. Let us now give some comments about the ambient space $\mathrm{BV}^w_{\mathrm{loc}}$, and further weighted Sobolev spaces used in the literature.

- Note that when w is lower semicontinuous, and belongs to the Muckenhoupt class A_1 in Ω, it is possible to define the weighted space $\mathrm{BV}_{\mathrm{loc}}(\Omega, w)$ that consists of functions $u \in L^1(I; w)$ such that $\int_I w \mathrm{d}|Du| < +\infty$, for each $I \Subset \Omega$, see Section B below.
- Notice that $\mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ is defined by means of the Anzellotti pairing, whose definition requires the BV regularity of w. Hence, $\mathrm{BV}_{\mathrm{loc}}(\Omega, w)$ and $\mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ share similar properties, however, they are different spaces, as we will explain, not only by construction. Let us recall that by [6, Remark 5], one has that $\mathrm{BV}(\Omega, w) \subseteq \mathrm{BV}(\Omega)$ (and also $\mathrm{BV}_{\mathrm{loc}}(\Omega, w) \subseteq \mathrm{BV}_{\mathrm{loc}}(\Omega)$). A major difficulty in the definition of $\mathrm{BV}(\Omega; w)$ is that we need the Muckenhoupt class A_1 to hold at any point, rather than almost everywhere.

- Since in our context we do not assume that w belongs to the Muckenhoupt class A_1, a priori, we have that $\mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ and $\mathrm{BV}_{\mathrm{loc}}(\Omega, w)$ are not comparable. However, we may wonder whether $\mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ and $\mathrm{BV}_{\mathrm{loc}}(\Omega, w)$ are related (or if $\mathrm{BV}^w(\Omega)$ and $\mathrm{BV}(\Omega, w)$ are related). For the sake of a lean explanation, let us suppose that w is lower semicontinuous, and belongs to the Muckenhoupt class A_1 in Ω, and that $w \in L^\infty(\Omega)$. Then by Remark A.3 below, we have that $\mathrm{BV}_{\mathrm{loc}}(\Omega) \subset \mathrm{BV}^w_{\mathrm{loc}}(\Omega)$, and thus
$$\mathrm{BV}_{\mathrm{loc}}(\Omega, w) \subset \mathrm{BV}_{\mathrm{loc}}(\Omega) \subset \mathrm{BV}^w_{\mathrm{loc}}(\Omega). \tag{9}$$

Remark 2.7. Next, we show that $L^1(\Omega, \hat{w})$ is generally not contained in $L^1(\Omega, w)$. That is, we give an example of w such that there exists
$$u \in L^1(\Omega; \hat{w}) \;\text{ with }\; |(w, Du)|(\Omega) = \textstyle\int_\Omega |u'w| \mathrm{d}x < +\infty,$$
but $u \notin L^1(\Omega, w)$ and so $u \notin \mathrm{BV}^w(\Omega)$. Let us set $\Omega := (0, 2)$, and for each $h \in \mathbb{N}$, $h \neq 0$ define
$$I^1_h := \left(\frac{1}{h+1}, \frac{1}{2}\left(\frac{1}{h+1} + \frac{1}{h}\right)\right]; \;\; I^2_h := \left(\frac{1}{2}\left(\frac{1}{h+1} + \frac{1}{h}\right), \frac{1}{h}\right],$$
$$I^1 := \cup_{h=1}^\infty I^1_h; \;\; I^2 := \cup_{h=1}^\infty I^2_h; \;\; I_h := I^1_h \cup I^2_h.$$
Fix $1 < \beta < +\infty$, $0 < \gamma < 1$. We set w as
$$w(x) := \sum_{h=1}^{+\infty} h^{-2} x^\gamma \chi_{I^1_h}(x) + \sum_{h=1}^{+\infty} h^{-2} x^\beta \chi_{I^2_h}(x)$$
for every $x \in (0, 1)$ and $w(x) = w(2 - x)$ for every $x \in (1, 2)$. Note that $\|w\|_\infty \leq 1$, $I_{\Omega,w} = (0, 2)$ and $w \in BV((0, 2))$. Since we defined the function w by simmetry in the interval $(0, 2)$, it is enough to consider his behaviour only in the interval $(0, 1)$. Notice that
$$\frac{1}{\hat{w}(x)} = \begin{cases} h^2 \left(\frac{1}{2}\left(\frac{1}{h+1} + \frac{1}{h}\right)\right)^{-\beta} & \text{if } x \in \left(0, \frac{1}{2}\right) \cap I^1_h \\ h^2 x^{-\beta} & \text{if } x \in \left(0, \frac{1}{2}\right) \cap I^2_h, \\ 2^\beta h^2 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$
and so
$$\hat{w}(x) = \begin{cases} h^{-2} \left(\frac{1}{2}\left(\frac{1}{h+1} + \frac{1}{h}\right)\right)^{\beta} & \text{if } x \in \left(0, \frac{1}{2}\right) \cap I^1_h, \\ h^{-2} x^{\beta} & \text{if } x \in \left(0, \frac{1}{2}\right) \cap I^2_h, \\ 2^{-\beta} h^{-2} & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$
and $\hat{w}(0) = 0$. On the other hand, we set $u(x) = \frac{1}{x^3} \in W^{1,1}_{\mathrm{loc}}((0, 1))$. By definition of w and u, we get that $\int_0^1 u(x)w(x)\mathrm{d}x = +\infty$. Indeed, note that
$$\int_0^1 u(x)w(x)\mathrm{d}x = \sum_{h=1}^\infty h^{-2} \int_{I^1_h} x^{\gamma-3}\mathrm{d}x + \sum_{h=1}^\infty h^{-2} \int_{I^2_h} x^{\beta-3}\mathrm{d}x$$
$$\sim \sum_{h=1}^\infty h^{-\gamma} + \sum_{h=1}^\infty h^{-\beta}$$
which diverges because $\gamma < 1$.

On the other hand, we have that

$$\int_0^1 u(x)\hat{w}(x)\mathrm{d}x = \int_0^{\frac{1}{4}} u(x)\hat{w}(x)\mathrm{d}x + \int_{\frac{1}{4}}^1 u(x)\hat{w}(x)\mathrm{d}x,$$

and by definition of $\hat{w}$, u, and since the number of intervals of the form I_h^1, I_h^2 contained in $\left(\frac{1}{2},1\right)$ is finite, then the term $\int_{\frac{1}{2}}^1 u(x)\hat{w}(x)\mathrm{d}x$ is finite. Let us note that

$$\int_0^{\frac{1}{2}} u(x)\hat{w}(x)\mathrm{d}x = \sum_{h=1}^{\infty} h^{-\beta}$$

which are convergent because $\beta > 1$. Lastly, let us take a compact set $K \subset (0, 1/2)$ such that its interior is a non-empty set. Note that

$$|(w, Du)|(K) = \int_K |u'w|\mathrm{d}x \sim \sum_{h=1}^{\infty} h^{-\gamma-1} + \sum_{h=1}^{\infty} h^{-\beta-1}$$

which is finite because $\beta > 1$, $\gamma > 0$, and thus we are done.

2.2. A weighted Poincaré inequality

The following inequality is a first step into the proof of a Poincaré-type inequality in the domain $I_{\Omega,w}$.

Proposition 2.8. *Suppose that* (H1)–(H2) *hold true, and* $w \in \mathrm{BV}_{\mathrm{loc}}(I_{\Omega,w})$. *Fix* $1 \leq i \leq N_w$. *For all* $u \in \mathrm{Dom}_w$, *and any* η, x *such that* $a_i < \eta \leq x \leq \frac{a_i+b_i}{2}$ *we have*

$$|u(x) - u(\eta)|\,\hat{w}(\eta) \leq \int_\eta^x |u'(y)|w(y)\,\mathrm{d}y\,; \tag{10}$$

$$|u(\eta)|\hat{w}(\eta) \leq |u(x)|\hat{w}(\eta) + \int_{a_i}^x |u'(y)|w(y)\,\mathrm{d}y\,. \tag{11}$$

For every η, x *such that* $\frac{a_i+b_i}{2} \leq x \leq \eta < b_i$ *we have*

$$|u(x) - u(\eta)|\,\hat{w}(\eta) \leq \int_x^\eta |u'(y)|w(y)\,\mathrm{d}y\,; \tag{12}$$

$$|u(\eta)|\hat{w}(\eta) \leq |u(x)|\hat{w}(\eta) + \int_x^{b_i} |u'(y)|w(y)\,\mathrm{d}y\,. \tag{13}$$

Remark 2.9. By (13) we have $u\hat{w} \in L^\infty((\frac{a_i+b_i}{2}, b_i))$. Indeed, for every η such that $\frac{a_i+b_i}{2} \leq \eta < b_i$

$$|u(\eta)|\hat{w}(\eta) \leq \left|u\left(\frac{a_i+b_i}{2}\right)\right| L_i + \int_{\frac{a_i+b_i}{2}}^{b_i} |u'(y)|w(y)\,\mathrm{d}y < +\infty\,. \tag{14}$$

Proof. Fix $1 \leq i \leq N_w$. Let us consider the open set $(\eta, x) \subset (a_i, \frac{a_i+b_i}{2})$.

We have that

$$|u(x)-u(\eta)| \leq \left|\int_\eta^x u'(y)\mathrm{d}y\right| = \left|\int_\eta^x u'(y)w(y)\frac{1}{w(y)}\mathrm{d}y\right| \leq \int_\eta^x |u'(y)|\, w(y)\frac{1}{w(y)}\mathrm{d}y.$$

Taking the sup to $\frac{1}{w(y)}$ we obtain

$$|u(x)-u(\eta)| \leq \int_\eta^x |u'(y)|\, w(y)\mathrm{d}y \sup_{y\in\left(\eta,\frac{a_i+b_i}{2}\right)} \frac{1}{w(y)}.$$

From the above inequality, we may deduce (10). Further, since

$$|u(\eta)| \leq |u(x)| + |u(\eta)-u(x)|,$$

by (10), (11) follows. Similarly, (12) and (13) can be obtained. □

Theorem 2.10. (Poincaré type inequality on Dom_w) *Suppose that* (H1)–(H2) *hold true, and* $w \in \mathrm{BV}_{\mathrm{loc}}(I_{\Omega,w})$. *Then for every* $u \in \mathrm{Dom}_w$

$$\sum_{i=1}^{N_w} ⨍_{a_i}^{b_i} \left|u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right| \hat{w}(\eta)\, d\eta \leq \int_{I_{\Omega,w}} |u'(y)| w(y)\, \mathrm{d}y.$$

Remark 2.11. Let recall that since $u \in \mathrm{Dom}_w$, we need that $w \in \mathrm{BV}_{\mathrm{loc}}(I_{\Omega,w})$. However, the regularity of w does not play any role in the proof of Theorem 2.10, but it is necessary to define the ambient space $\mathrm{BV}^w_{\mathrm{loc}}$. Further, let us point out that the hypotheses of Theorem 2.10, and [11, Theorem 2.10] are different. Indeed, while the case $1 < p < +\infty$ requires a local sommability of $w^{-\frac{1}{p-1}}$, the case $p = 1$ requires the local boundedness of $\frac{1}{w}$. The results of both Theorems are formally analogous, but the auxiliary weights are different, and have different properties. Let us also note that we do not assume any local growth condition, as in Theorem B.7 below, where a weighted Poincaré inequality with a single weight is proved.

Proof. Fix $1 \leq i \leq N_w$. In (10) we take $x = \frac{a_i+b_i}{2}$, then

$$\left|u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right| \hat{w}(\eta) \leq \int_{a_i}^{\frac{a_i+b_i}{2}} |u'(y)| w(y)\, \mathrm{d}y.$$

By integrating with respect to η we obtain

$$\int_{a_i}^{\frac{a_i+b_i}{2}} \left|u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right| \hat{w}(\eta)\, \mathrm{d}\eta \leq \frac{b_i-a_i}{2} \int_{a_i}^{\frac{a_i+b_i}{2}} |u'(y)| w(y)\, \mathrm{d}y.$$

Similarly we have

$$\int_{\frac{a_i+b_i}{2}}^{b_i} \left|u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right| \hat{w}(\eta)\, \mathrm{d}\eta \leq \frac{b_i-a_i}{2} \int_{\frac{a_i+b_i}{2}}^{b_i} |u'(y)| w(y)\, \mathrm{d}y.$$

Therefore
$$\int_{a_i}^{b_i} \left|u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right| \hat{w}(\eta)\, \mathrm{d}\eta \leq (b_i-a_i) \int_{a_i}^{b_i} |u'(y)| w(y)\, \mathrm{d}y.$$

Hence $$⨍_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right| \hat{w}(\eta)\,\mathrm{d}\eta \le \int_{a_i}^{b_i} |u'(y)| w(y)\,\mathrm{d}y.$$

The conclusion follows since $u \in \mathrm{Dom}_w$ and so

$$\sum_{i=1}^{N_w} \int_{a_i}^{b_i} |u'(y)| w(y)\,\mathrm{d}y = \int_{I_{\Omega,w}} |u'(y)| w(y)\,\mathrm{d}y < +\infty. \qquad \square$$

We also have the following convergence result (see Proposition 9.3 in [13] for an analogous result).

Proposition 2.12. *Suppose that* (H1)–(H2) *hold true, let* $w \in \mathrm{BV}_{\mathrm{loc}}(I_{\Omega,w})$ *and let* $(u_k) \subset \mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w})$ *be a sequence of functions such that*

$$\sup_{k\in\mathbb{N}} |(w, Du_k)|\,(I_{\Omega,w}) < +\infty, \quad \sup_{k\in\mathbb{N}} u_k\left(\frac{a_i+b_i}{2}\right) < +\infty \tag{15}$$

for any $i = 1, \dots, N_w$. *Then for any interval* $K \Subset (a_i, b_i)$, *with* $i = 1, \dots, N_w$, *there exists* $u \in L^1(K, \hat{w}) \cap W^{1,1}(K)$ *and a subsequence* (u_{k_j}) *such that* $u_{k_j} \to u$ *in* $L^1(K, \hat{w})$.

Moreover, if the sequence $(u_k)_{k\in\mathbb{N}}$ *is uniformly bounded also in* $L^\infty(I_{\Omega,w})$, *i.e.*

$$\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(I_{\Omega,w})} + |(w, Du_k)|\,(I_{\Omega,w}) < +\infty, \tag{16}$$

then $u \in \mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w})$.

Proof. A first consequence of Theorem 2.10 and (15) is that

$$\sup_{k\in\mathbb{N}} \|u_k\|_{L^1(K,\hat{w})} < +\infty.$$

By (6), we can find a positive constant $M_{i,K} > 0$ such that $\hat{w}(x) > M_{i,K}$ for a.e. $x \in K$. Then

$$M_{i,K} \sup_{k\in\mathbb{N}} \int_K |u_k|\mathrm{d}x \le \sup_{k\in\mathbb{N}} \int_K |u_k|\hat{w}\mathrm{d}x < +\infty.$$

Moreover, since $\hat{w} \le w$

$$\begin{aligned} M_{i,K} \sup_{k\in\mathbb{N}} \int_K |u_k'|\mathrm{d}x &\le \sup_{k\in\mathbb{N}} \int_K |u_k'|\hat{w}\mathrm{d}x \le \sup_{k\in\mathbb{N}} \int_{I_{\Omega,w}} |u_k'|\hat{w}\mathrm{d}x \le \\ &\le \sup_{k\in\mathbb{N}} \int_{I_{\Omega,w}} |u_k'| w\mathrm{d}x \le \sup_{k\in\mathbb{N}} |(w, Du_k)|\,(I_{\Omega,w}) < +\infty. \end{aligned}$$

Then, $(u_k)_k$ is bounded in $W^{1,1}(K)$. By [8, Theorem 8.8, Remark 10] we can extract a subsequence still denoted $(u_k)_k$, and find $u \in W^{1,1}(K)$ such that

$$\|u_k - u\|_{L^1(K)} \to 0, \text{ as } k \to +\infty.$$

Furthermore, by (5) we can find a constant $L_i > 0$ such that

$$\int_K |u_k - u|\hat{w}\mathrm{d}x \le L_i \int_K |u_k - u|\mathrm{d}x \to 0 \text{ as } k \to +\infty,$$

and thus we have proved that

$$\|u_k - u\|_{L^1(K,\hat{w})} \to 0, \text{ as } k \to +\infty.$$

Finally, if the sequence $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(I_{\Omega,w})$, as in the proof of Proposition 9.3 in [13] we can conclude that $\|u\|_{L^\infty(K)} \leq C$, and this implies that $u \in L^1(K, |Dw|)$. Hence $u \in BV^w_{\mathrm{loc}}(I_{\Omega,w})$. □

Corollary 2.13. *Under the assumptions of Proposition 2.12, if $N_w < +\infty$, then there exists $u \in L^1(I_{\Omega,w}, \hat{w})$ and a subsequence (u_{k_j}) such that $u_{k_j} \to u$ in $L^1(I_{\Omega,w}, \hat{w})$. If* (16) *holds, then $u \in \mathrm{BV}^w(I_{\Omega,w})$.*

Proof. It suffices to use $M = \min\{M_1, \ldots, M_{N_w}\}$ and $L = \min\{L_1, \ldots, L_{N_w}\}$, instead of M_i and L_i, respectively. □

3. Relaxation for finitely degenerate weights

In this section, in addition to hypothesis (H1), (H2) introduced in the previous section, we also make the following assumption on the weight w.

(H3) $w \in W^{1,1}_{\mathrm{loc}}(I_{\Omega,w})$;

(H4) $1 \leq N_w < +\infty$.

Remark 3.1. We note that, as proven in [13, Proposition 5.1 (3)], under (H3), the space $\mathrm{BV}^w(I_{\Omega,w})$ is a Banach space, with the norm

$$||u||_{\mathrm{BV}^w(I_{\Omega,w})} := ||u||_{L^1(I_{\Omega,w},w)} + ||u||_{L^1(I_{\Omega,w},|Dw|)} + |D(uw)|(I_{\Omega,w}) \tag{17}$$

which is not generally the case.

3.1. The choice of the ambient space X and the convergence

Notice that, by the Poincaré inequality in Theorem 2.10, we have $\mathrm{Dom}_w \subset L^1(\Omega, \hat{w})$. In what follows, we set $X = L^1(\Omega, \hat{w})$ and we define the $(\hat{w}, Dw)$-convergence, as follows:

Definition 3.2. We say that a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ $(\hat{w}, Dw)$-converges to $u \in \mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ if

(i) $u_n \rightharpoonup u$ in $L^1_{\mathrm{loc}}(I_{\Omega,w}, \hat{w})$,

(ii) $u_n \rightharpoonup u$ in $L^1_{\mathrm{loc}}(I_{\Omega,w}, |Dw|)$.

Remark 3.3. This new convergence is a modification of the one introduced in [13] and guarantees the lower semicontinuity of the pairing functional (see Step 2 in the proof of Theorem 3.6).

Proposition 3.4. *Suppose that assumptions* (H1)–(H3) *hold true. Then* Dom_w *defined as in* (7) *is a Banach space endowed with the norm*

$$\|u\|_{\mathrm{Dom}_w} := \|u\|_{L^1(I_{\Omega,w},\hat{w})} + |(w, Du)|\,(I_{\Omega,w}). \tag{18}$$

Furthermore, the convergence in (18) *implies the* $(\hat{w}, Dw)$*-convergence.*

Proof. Notice that Dom_w is a linear subspace of $\mathrm{BV}^w_{\mathrm{loc}}(I_{\Omega,w})$, and by [13, Corollary 5.2] we can endow it with the norm

$$\|u\|_{\mathrm{BV}^w(I_{\Omega,w})} := |(w,Du)|\,(I_{\Omega,w}),\ u \in \mathrm{Dom}_w. \qquad \square \tag{19}$$

We also have the following compactness result which extends Proposition 9.3 in [13]. In what follows, we denote by $\mathcal{L}^1$ the unidimensional Lebesgue measure.

Proposition 3.5. *Suppose that* (H1)–(H3) *hold true, and*

$$\mathcal{L}^1(\Omega \setminus \mathrm{supp}(\hat{w})) = 0. \tag{20}$$

Let $(u_k) \subset \mathrm{Dom}_w$ *be a sequence of functions such that*

$$\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty} + \|u_k\|_{\mathrm{Dom}_w} < +\infty. \tag{21}$$

Then there exist $u \in \mathrm{Dom}_w$ *and a subsequence* (u_{k_j}) *such that, possibly up to a further subsequence,* $u_{k_j} \to u$ *in* $L^1_{\mathrm{loc}}(I_{\Omega,w}, |Dw|)$*; so that the sequence* $(u_{k_j})_{j\in\mathbb{N}}$ *locally* $(\hat{w}, Dw)$*-converges to* u *in* $I_{\Omega,w}$.

Proof. By Proposition 2.12 for any interval $K \Subset (a_i, b_i)$, with $i = 1, \dots, N_w$, there exists $u \in L^1(K, \hat{w}) \cap W^{1,1}(K)$ and a subsequence (u_{k_j}) such that $u_{k_j} \to u$ in $L^1(K, \hat{w})$ and so $u_{k_j}(x) \to u(x)$ for $|\hat{w}|\mathcal{L}^1$-a.e. $x \in K$, and therefore $u_{k_j}(x) \to u(x)$ for $\mathcal{L}^1$-a.e. $x \in \mathrm{supp}(\hat{w}) \cap K$. Then by (20) $u_{k_j}(x) \to u(x)$ for $\mathcal{L}^1$-a.e. $x \in K$. Hence, since $|Dw| \ll \mathcal{L}^1$, we get

$$u_{k_j}(x) \to u(x) \ \text{ for } \ |Dw|\text{-a.e. } x \in K.$$

Since by (21), there exists $C > 0$ such that

$$|u_{k_j} - u| \le C \in L^1(K, |Dw|),$$

by Lebesgue's Dominated Convergence Theorem we conclude that $u_{k_j} \to u$ in $L^1(\Omega, |Dw|)$. This implies that $(u_{k_j})_{j\in\mathbb{N}}$ $(\hat{w}, Dw)$-converges to u in K. On the other hand, by Fatou's Lemma we obtain

$$\int_K |u|\,\mathrm{d}|Dw| = \int_K \liminf_{j\to+\infty} |u_{k_j}|\,\mathrm{d}|Dw| \le \liminf_{j\to+\infty} \int_K |u_{k_j}|\,\mathrm{d}|Dw| < +\infty.$$

Therefore, we have $u \in L^1_{\mathrm{loc}}(I_{\Omega,w}, |Dw|)$, and so $u \in BV^w_{\mathrm{loc}}(I_{\Omega,w})$. $\square$

3.2. Main result

We then consider

$$\overline{F}(u) := \inf\{\liminf_{k\to+\infty} F(u_k) : u_k \to u \ \text{ w.r.t. } \ (\hat{w}, Dw)-\text{convergence} \ \}$$

where
$$F(u) := \begin{cases} \displaystyle\int_\Omega |u'| w \,\mathrm{d}x & \text{if } u \in \mathrm{AC}(\overline{\Omega}), \\ +\infty & \text{if } u \in L^1(\Omega, \hat{w}) \setminus \mathrm{AC}(\overline{\Omega}), \end{cases}$$

and let
$$\mathscr{A}_{\overline{F}} := \{u \in L^1(\Omega, \hat{w}) : \overline{F}(u) < +\infty\}.$$

Note that for every $u \in \mathrm{AC}(\overline{\Omega})$ we have

$$\int_\Omega |u'| w \, \mathrm{d}x = \int_\Omega |(w, Du)|.$$

Theorem 3.6. *Suppose that* (H1)–(H4) *hold true. Then*

$$\mathscr{A}_{\overline{F}} = \mathrm{Dom}_w$$

where Dom_w *is defined by* (7) *and the following representation holds for the relaxed functional*

$$\overline{F}(u) = \begin{cases} |(w, Du)|(I_{\Omega,w}) & \text{if } u \in \mathrm{Dom}_w, \\ +\infty & \text{if } u \in L^1(\Omega, \hat{w}) \setminus \mathrm{Dom}_w. \end{cases} \tag{22}$$

Proof. Let us denote by $H(u)$ the right-hand side of the above formula (22), i.e.

$$H(u) := \begin{cases} |(w, Du)|(I_{\Omega,w}) & \text{if } u \in \mathrm{Dom}_w, \\ +\infty & \text{if } u \in L^1(\Omega, \hat{w}) \setminus \mathrm{Dom}_w. \end{cases}$$

In the following we will prove that $\overline{F} = H$ by showing the two inequalities.

Step 1. We first prove that $\overline{F} \leq H$. To this end, it is enough to show that

$$\overline{F}(u) \leq |(w, Du)|(I_{\Omega,w}) \text{ for all } u \in \mathrm{Dom}_w. \tag{23}$$

Suppose that $AC(\overline{\Omega})$ is dense in Dom_w with respect to (18). Then there exists a sequence (u_k) in $AC(\overline{\Omega})$ such that

$$\lim_{k \to +\infty} u_k = u \text{ in } AC(\overline{\Omega}) \text{ with respect to (18).}$$

Then, $\quad \overline{F}(u) \leq \lim_{k \to +\infty} \overline{F}(u_k) = \lim_{k \to +\infty} |(w, Du_k)|(I_{\Omega,w}) = |(w, Du)|(I_{\Omega,w}),$

which is (23). To complete the proof, we now need to show that $AC(\overline{\Omega})$ is actually dense in Dom_w with respect to (18), i.e., that for each $u \in \mathrm{Dom}_w$ there is $u_h \in AC(\overline{\Omega})$ such that

$$\begin{aligned} &\lim_{h \to \infty} u_h = u \text{ in } L^1(I_{\Omega,w}, \hat{w}) \text{ and} \\ &|(w, Du_h)|(I_{\Omega,w}) \to |(w, Du)|(I_{\Omega,w}) \text{ as } h \to +\infty. \end{aligned} \tag{24}$$

Since $u' \in L^1(I_{\Omega,w}, w)$, we can apply [9, Theorem 3.45] to find a sequence of functions $(v_h)_h \subset C^0_c(I_{\Omega,w}) \subset L^1(\Omega, w)$ such that

$$\|v_h - u'\|_{L^1(I_{\Omega,w},w)} = \sum_{i=1}^{N_w} \int_{a_i}^{b_i} |v_h - u'| \, w \, \mathrm{d}x \to 0 \text{ as } h \to +\infty\,. \tag{25}$$

Let us define, for given $h \in \mathbb{N}$, $\tilde{u}_h^{(i)} : (a_i, b_i) \to \mathbb{R}$, $i = 1, 2, \ldots, h$ as

$$\tilde{u}_h^{(i)}(x) := u\left(\frac{a_i + b_i}{2}\right) - \int_x^{\frac{a_i+b_i}{2}} v_h(y) \, dy\,, \qquad x \in (a_i, b_i). \tag{26}$$

We divide the proof into three cases, according to the structure of the set $I_{\Omega,w}$.

Case 1. Assume that $N_w = 1$. In this case $I_{\Omega,w} = (a_1, b_1)$. Let $(\tilde{u}_h^{(1)})_h$ the sequence defined in (26) for $i = 1$ and, for each h, let $u_h = \bar{u}_h : (a,b) \to \mathbb{R}$ defined as

$$\bar{u}_h(x) := \begin{cases} \tilde{u}_h^{(1)}(a_1) & \text{if } x \in [a, a_1], \\ \tilde{u}_h^{(1)}(x) & \text{if } x \in (a_1, b_1), \\ \tilde{u}_h^{(1)}(b_1) & \text{if } x \in [b_1, b]\,. \end{cases}$$

Then it is easy to see that $(\bar{u}_h)_h \subset AC(\overline{\Omega})$. Let us prove that

$$\int_a^b |\bar{u}_h - u|\, \hat{w}\, \mathrm{d}x \to 0 \text{ as } h \to \infty\,. \tag{27}$$

In fact, since $\hat{w} \equiv 0$ in $\Omega \setminus I_{\Omega,w}$,

$$\int_a^b |\bar{u}_h - u|\, \hat{w}\, \mathrm{d}x = \int_{a_1}^{b_1} |\bar{u}_h - u|\, \hat{w}\, \mathrm{d}x.$$

Applying the Poincaré type inequality (2.10) with $\tilde{u}_h - u$ instead of u and since $\tilde{u}_h\left(\frac{a_1+b_1}{2}\right) = u\left(\frac{a_1+b_1}{2}\right)$, we obtain

$$\begin{aligned}\int_{a_1}^{b_1} |\bar{u}_h - u|\, \hat{w}\, \mathrm{d}x &\le \int_{I_{\Omega,w}} |\bar{u}_h' - u'|\, w\, \mathrm{d}x = |(D(\bar{u}_h - u), w)|(I_{\Omega,w}) \\ &= \int_{I_{\Omega,w}} |v_h - u'|\, w\, \mathrm{d}x\,.\end{aligned}$$

Then $|(D(\bar{u}_h - u), w)|(I_{\Omega,w}) \to 0$, as $h \to +\infty$. Hence

$$\Big| |(D\bar{u}_h, w)|(I_{\Omega,w}) - |(Du, w)|(I_{\Omega,w}) \Big| \le |(D(\bar{u}_h - u), w)|(I_{\Omega,w}) \to 0, \text{ as } h \to +\infty. \tag{28}$$

Moreover, by (25) and (28), (27) follows.

Case 2. Assume now that $N_w = 2$. In this case $I_{\Omega,w} = (a_1, b_1) \cup (a_2, b_2)$, and assume that $b_1 \le a_2$.

Firstly, we consider the subcase $b_1 < a_2$. Let $(\tilde{u}_h^{(i)})_h$ the sequence defined in (26) for $i = 1, 2$ and, for each h, let $u_h = \bar{u}_h : \Omega \to \mathbb{R}$ defined as

$$\bar{u}_h(x) := \begin{cases} \tilde{u}_h^{(1)}(a_1) & \text{if } x \in [a, a_1), \\ \tilde{u}_h^{(1)}(x) & \text{if } x \in [a_1, b_1), \\ \dfrac{\tilde{u}_h^{(2)}(a_2) - \tilde{u}_h^{(1)}(b_1)}{a_2 - b_1}(x - b_1) + \tilde{u}_h^{(1)}(b_1) & \text{if } x \in [b_1, a_2), \\ \tilde{u}_h^{(2)}(x) & \text{if } x \in [a_2, b_2), \\ \tilde{u}_h^{(2)}(b_2) & \text{if } x \in [b_2, b]\,. \end{cases}$$

Notice that $(u_h)_h \subset AC(\overline{\Omega})$ and (24) holds. Indeed, it can be done by repeating the arguments of the 1st case and by observing that $\hat{w} \equiv 0$ in $\Omega \setminus \overline{I}_{\Omega,w}$.

Now, we consider the second subcase $b_1 = a_2$. Let $h \in \mathbb{N}$ such that

$$\frac{1}{h} < \min\left\{\frac{b_i - a_i}{4} : i = 1, 2\right\}.$$

$$\bar{w}(x) := \begin{cases} \int_{\frac{a_i+b_i}{2}}^{x} \hat{w}(y)\,\mathrm{d}y & \text{if } a_i \le x \le \frac{3a_i+b_i}{4}, \\ \int_{\frac{3a_i+b_i}{4}}^{\frac{a_i+3b_i}{4}} \hat{w}(y)\,\mathrm{d}y & \text{if } \frac{3a_i+b_i}{4} \le x \le \frac{a_i+3b_i}{4}, \\ \int_{x}^{\frac{a_i+b_i}{2}} \hat{w}(y)\,\mathrm{d}y & \text{if } \frac{a_i+3b_i}{4} \le x \le b_i, \\ 0 & \text{if } x \in \Omega \setminus \overline{I}_{\Omega,w}\,. \end{cases}$$

Note that by (5) $\hat{w} \in L^\infty((a_i, b_i))$ and so $\bar{w} \in L^\infty((a_i, b_i))$.

Let $u_h = \bar{u}_h : \Omega \to \mathbb{R}$ defined as

$$\bar{u}_h(x) := \begin{cases} \tilde{u}_h^{(1)}(a_1) & \text{if } x \in [a, a_1), \\ \tilde{u}_h^{(1)}(x) & \text{if } x \in [a_1, \frac{a_1+b_1}{2}), \\ u(x) & \text{if } x \in [\frac{a_1+b_1}{2}, b_1 - \frac{1}{h}), \\ u(x)\dfrac{\bar{w}(x)}{|\bar{w}(b_1 - \frac{1}{h})|} & \text{if } x \in [b_1 - \frac{1}{h}, b_1) \\ u(x)\dfrac{\bar{w}(x)}{|\bar{w}(a_2 + \frac{1}{h})|} & \text{if } x \in [a_2, a_2 + \frac{1}{h}), \\ u(x) & \text{if } x \in [a_2 + \frac{1}{h}, \frac{a_2+b_2}{2}), \\ \tilde{u}_h^{(2)}(x) & \text{if } x \in [\frac{a_2+b_2}{2}, b_2), \\ \tilde{u}_h^{(2)}(b_2) & \text{if } x \in [b_2, b]\,. \end{cases}$$

Then $(u_h)_h \subset AC(\overline{\Omega})$ and (24) holds. Indeed, in order to prove (24), we now prove that

$$\int_{\frac{a_1+b_1}{2}}^{b_1} |\bar{u}_h - u|\,\hat{w}\,\mathrm{d}x \to 0 \text{ as } h \to \infty\,, \tag{29}$$

and

$$\int_{\frac{a_1+b_1}{2}}^{b_1} |\bar{u}_h'|\,w\,\mathrm{d}x \le C < +\infty\,, \tag{30}$$

since the proof of the analogous conditions on $(a_2, \frac{a_2+b_2}{2})$ are similar. Indeed, we have

$$\int_{\frac{a_1+b_1}{2}}^{b_1} |\bar{u}_h - u|\,\hat{w}\,\mathrm{d}x = \int_{b_1-\frac{1}{h}}^{b_1} u\left(1 - \frac{\bar{w}(x)}{|\bar{w}(b_1 - \frac{1}{h})|}\right)\hat{w}(x)\,\mathrm{d}x.$$

Notice that $\bar{w}$ is decreasing in $[\frac{a_1+3b_1}{4}, b_1]$, and by (5)

$$0 \le 1 - \frac{\bar{w}(x)}{|\bar{w}(b_1 - \frac{1}{h})|} = \frac{|\bar{w}(b_1 - \frac{1}{h})| - \bar{w}(x)}{|\bar{w}(b_1 - \frac{1}{h})|} \le \frac{2L_1}{|\bar{w}(b_1 - \frac{1}{h})|} =: \tilde{c}_h, \quad x \in \left(b_1 - \frac{1}{h}, b_1\right). \tag{31}$$

Note that $|\bar{w}(b_1 - \frac{1}{h})| \to |\bar{w}(b_1)| \neq 0$. Indeed,

$$\bar{w}(b_1) = \int_{b_1}^{\frac{a_1+b_1}{2}} \hat{w}(y)\,\mathrm{d}y = \int_{b_1}^{\frac{a_1+b_1}{2}} \left(\left\|w^{-1}\right\|_{L^\infty((\frac{a_1+b_1}{2}, y))}\right)^{-1} \mathrm{d}y < 0.$$

This implies that

$$\int_{\frac{a_1+b_1}{2}}^{b_1} |\bar{u}_h - u|\,\hat{w}\,\mathrm{d}x \le \tilde{c}_h \int_{b_1-\frac{1}{h}}^{b_1} u\,\hat{w}\,\mathrm{d}x \to 0 \text{ as } h \to +\infty\,.$$

This proves (29). On the other hand, in order to prove (30) we note that

$$\bar{u}_h'(x) := \begin{cases} u'(x) & \text{if } x \in [\frac{a_1+b_1}{2}, b_1 - \frac{1}{h}), \\ \dfrac{1}{|\bar{w}(b_1-\frac{1}{h})|}\,(u'(x)\bar{w}(x) + u(x)\bar{w}'(x)) & \text{if } x \in [b_1 - \frac{1}{h}, b_1). \end{cases}$$

Therefore

$$\int_{\frac{a_1+b_1}{2}}^{b_1} |\bar{u}_h'|\,w\,\mathrm{d}x = \int_{\frac{a_1+b_1}{2}}^{b_1-\frac{1}{h}} |u'|\,w\,\mathrm{d}x + \int_{b_1-\frac{1}{h}}^{b_1} \frac{1}{|\bar{w}(b_1-\frac{1}{h})|}\,|u'\bar{w} + u\bar{w}'|\;w\,\mathrm{d}x$$

$$\le \int_{\frac{a_1+b_1}{2}}^{b_1} |u'|\,w\,\mathrm{d}x + \int_{\frac{a_1+b_1}{2}}^{b_1} \frac{|\bar{w}|}{|\bar{w}(b_1-\frac{1}{h})|}|u'|w\,\mathrm{d}x + \int_{b_1-\frac{1}{h}}^{b_1} \frac{1}{|\bar{w}(b_1-\frac{1}{h})|}|u|\,|\bar{w}'|\,w\,\mathrm{d}x.$$

Notice that the second integral is finite by (31). Let us prove that the last integral tends to 0. Indeed,

$$\bar{w}' = -\hat{w} \quad \text{ a.e. in } \left(b_1 - \frac{1}{h}, b_1\right)$$

and, since $u\hat{w}$ is bounded in $(b_1 - 1/h, b_1)$ (see Remark 2.9), we obtain

$$\int_{b_1-\frac{1}{h}}^{b_1} \frac{1}{|\bar{w}(b_1-\frac{1}{h})|}|u||\bar{w}'|\,w\,\mathrm{d}x = \int_{b_1-\frac{1}{h}}^{b_1} \frac{1}{|\bar{w}(b_1-\frac{1}{h})|}|u||\hat{w}|w\,\mathrm{d}x$$

$$\le C\frac{1}{|\bar{w}(b_1-\frac{1}{h})|}\int_{b_1-\frac{1}{h}}^{b_1} w\,\mathrm{d}x \to 0 \text{ as } h \to +\infty.$$

Case 3. In the general case we have $I_{\Omega,w} = \bigcup_{i=1}^{N_w}(a_i, b_i)$ with $b_i \le a_{i+1}$, for every $i = 1, \dots, N_{w-1}$, it is sufficient to repeat the arguments of the 2nd case for every $i = 1, \dots, N_{w-1}$.

Step 2. We now prove that $H \le \overline{F}$. To this end, since

$$\overline{F} = \sup\{G : G \text{ lower semicontinuous and } G \le F\},$$

its is enough to show that H is lower semicontinuous and $H \le F$. The last inequality is trivially true, so, we now need to prove the lim inf inequality for H. Let $u_h \to u$ with respect to the $(\hat{w}, Dw)$-convergence in $I_{\Omega,w}$. Then we have that $u_h \rightharpoonup u$ weakly in $L^1(I_{\Omega,w}, \hat{w})$. By Mazur Lemma there exists a function $f : \mathbb{N} \to \mathbb{N}$ and a sequence $\{\alpha_{k,h} : h \le k \le f(h)\}$ such that $\alpha_{k,h} \ge 0$, and

$$\sum_{k=h}^{f(h)} \alpha_{k,h} = 1 \quad \text{such that the sequence} \quad v_h := \sum_{k=h}^{f(h)} \alpha_{k,h} u_k$$

strongly converges to u in $L^1(I_{\Omega,w}, \hat{w})$ and $L^1(I_{\Omega,w}, |Dw|)$.

Notice that (4) and the definition of $\hat{w}$ imply that $\mathcal{L}^1(I_{\Omega,w}\setminus\mathrm{supp}(\hat{w})) = 0$. Then $v_h(x) \to u(x)$ for $\mathcal{L}^1$-a.e. $x \in I_{\Omega,w}$. Since $w \in W^{1,1}_{\mathrm{loc}}(I_{\Omega,w}) \subset L^\infty_{\mathrm{loc}}(I_{\Omega,w})$ and (4) hold true, then for all compact $K \Subset I_i := (a_i, b_i)$ one gets

$$\frac{1}{c_{i,K}}\int_K |v_h - u|\mathrm{d}x \le \int_K |v_h - u| w\mathrm{d}x \le C\int_K |v_h - u|\mathrm{d}x,$$

for some positive constant C. Then $v_h \to u$ strongly in $L^1_{\mathrm{loc}}(I_{\Omega,w}, w)$, and thus weakly in $L^1_{\mathrm{loc}}(I_{\Omega,w}, w)$. Hence, since $v_h \to u$ strongly in $L^1_{\mathrm{loc}}(I_{\Omega,w}, |Dw|)$, and thus weakly in $L^1_{\mathrm{loc}}(I_{\Omega,w}, |Dw|)$ we conclude that v_h $(w, \frac{1}{2})$-converges to u in the sense of Definition A.5. Therefore, we may apply Theorem A.6 to conclude the desired lower semicontinuity inequality. Indeed, by (41) we get

$$\liminf_{h\to+\infty} H(v_h) \ge \lim_{h\to+\infty} |(w, Dv_h)|(I_{\Omega,w}) \ge |(w, Du)|(I_{\Omega,w}) = H(u). \tag{32}$$

Now let us prove that (32) holds true for u_h. Suppose by contradiction that (32) is not true for u_h. By the definition of lim inf we have that

$$C_1 := \sup\left\{\inf\left\{\int_{I_{\Omega,w}} w|u'_m|\mathrm{d}x : m \ge h\right\}; h \in \mathbb{N}\right\} < \int_{I_{\Omega,w}} w|u'|\mathrm{d}x, \tag{33}$$

$$C_2 := \sup\left\{\inf\left\{\int_{I_{\Omega,w}} w|v'_j|\mathrm{d}x : j \ge h'\right\}; h' \in \mathbb{N}\right\} \ge \int_{I_{\Omega,w}} w|u'|\mathrm{d}x. \tag{34}$$

In (34), we use the definition of sup, so that for all $\varepsilon > 0$, there exists $h' \in \mathbb{N}$ such that

$$\inf\left\{\int_{I_{\Omega,w}} w|v'_j|\mathrm{d}x : j \ge h'\right\} > C_2 - \varepsilon \ge \int_{I_{\Omega,w}} w|u'|\mathrm{d}x - \varepsilon.$$

Moreover, by (33), for all $h \in \mathbb{N}$, we get

$$\inf\left\{\int_{I_{\Omega,w}} w|u'_m|\mathrm{d}x : m \ge h\right\} < \int_{I_{\Omega,w}} w|u'|\mathrm{d}x.$$

It implies that $\int_{I_{\Omega,w}} w|u'|\mathrm{d}x$ is not the infimum, so that there exists $\delta > 0$, such that for each $m \ge h$

$$\int_{I_{\Omega,w}} w|u'_m|\mathrm{d}x + \delta < \int_{I_{\Omega,w}} w|u'|\mathrm{d}x, \tag{35}$$

and the same inequality holds true for all $m' \ge m \ge h$. Now let us choose such $h \ge h'$. Then

$$\begin{aligned}\int_{I_{\Omega,w}} w|u'|\mathrm{d}x - \varepsilon &\le \inf\left\{\int_{I_{\Omega,w}} w|v'_j|\mathrm{d}x : j \ge h'\right\} \le \inf\left\{\sum_{k=j}^{f(j)} \alpha_{k,j}\int_{I_{\Omega,w}} w|u'_k|\mathrm{d}x : j \ge h'\right\}\\ &\le \inf\left\{\sum_{k=j}^{f(j)} \alpha_{k,j}\left(\int_{I_{\Omega,w}} w|u'|\mathrm{d}x - \delta\right) : j \ge h'\right\} = \int_{I_{\Omega,w}} w|u'|\mathrm{d}x - \delta.\end{aligned}$$

Then $\delta \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get that $\delta \le 0$, and thus a contradiction because $\delta > 0$. □

A. A pairing beyond BV

In this section, we recall the notion of pairing (w, Du) for functions u that may not be of bounded variation, and we introduce the larger space $BV^w(\Omega)$, where this pairing make sense. In the definition, we will use a precise representative $u^{\frac{1}{2}}$ defined for functions $u \in L^1_{\mathrm{loc}}(\Omega)$.

A.1. Precise representatives

Firstly, we recall some basic definitions and results about the precise representatives of $u \in L^1_{\mathrm{loc}}(\Omega)$ (see [1, Sections 3.6 and 4.5]), where $\Omega \subset \mathbb{R}^n$ is an open set.

We say that a function $u \in L^1_{\mathrm{loc}}(\Omega)$ has an *approximate limit* $z \in \mathbb{R}$ at $x \in \Omega$ if

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n\left(B_r(x)\right)} \int_{B_r(x)} |u(y) - z| \, dy = 0\,;$$

and we say that x is a *Lebesgue point* of u. The set $S_u \subset \Omega$ of points where this property does not hold is called the *approximate discontinuity set* of u, and $\mathcal{L}^n(S_u) = 0$. For any $x \in \Omega \setminus S_u$ the approximate limit z is uniquely determined and is denoted by $z =: \tilde{u}(x)$. Let $u = \chi_E$, for a measurable set $E \subset \mathbb{R}^n$; in this case the approximate limit at a point $x \in \mathbb{R}^n$ is also called *density* of E at x, and it is defined by

$$D(E; x) := \lim_{r\to 0^+} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))}$$

whenever this limit exists.

For every Borel function $u : \Omega \to \mathbb{R}$, we denote the *sublevel and superlevel sets* of u as

$$\{u < t\} = \{x \in \Omega : u(x) < t\} \quad \text{and} \quad \{u > t\} = \{x \in \Omega : u(x) > t\},$$

and we give the definition of the *approximate liminf and limsup* at a point $x \in \Omega$ in the following way

$$u^-(x) := \sup\left\{t \in \overline{\mathbb{R}} \colon D(\{u < t\}; x) = 0\right\},$$

and

$$u^+(x) := \inf\left\{t \in \overline{\mathbb{R}} \colon D(\{u > t\}; x) = 0\right\}$$

(see [1, Definition 4.28]), where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

We note further that $u^+, u^- : \Omega \to [-\infty, +\infty]$ are Borel functions and that the set $S_u^* := \{x \in \Omega : u^-(x) < u^+(x)\}$ satisfies

$$\mathcal{L}^n(S_u^*) = 0,$$

so that $u^+(x) = u^-(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ (see [1, Definition 4.28]). If $u \in L^1_{\mathrm{loc}}(\Omega)$, we have

$$u^+(x) = u^-(x) = \tilde{u}(x) \quad \text{for all } x \in \Omega \setminus S_u,$$

and so $S_u^* \subset S_u$. Therefore, in $\Omega \setminus S_u^*$ we shall write $\tilde{u}(x) := u^+(x) = u^-(x)$, with an abuse of notation.

On the other hand, for every $u \in L^1_{\mathrm{loc}}(\Omega)$, we say that $x \in \Omega$ is an *approximate jump point* of u if there exist $a, b \in \mathbb{R}$, $a \neq b$, and a unit vector $\nu \in \mathbb{R}^n$ such that

$$\begin{aligned} &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B^i_r(x))} \int_{B^i_r(x)} |u(y) - a|\, dy = 0, \\ &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B^e_r(x))} \int_{B^e_r(x)} |u(y) - b|\, dy = 0, \end{aligned} \tag{36}$$

where

$$B^i_r(x) := \{y \in B_r(x) : (y - x)\cdot \nu > 0\}, \quad B^e_r(x) := \{y \in B_r(x) : (y - x)\cdot \nu < 0\}.$$

The triplet (a, b, ν), uniquely determined by (36) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^i(x), u^e(x), \nu_u(x))$. We observe that

$$u^-(x) = \min\{u^i(x), u^e(x)\} \text{ and } u^+(x) = \max\{u^i(x), u^e(x)\} \quad \text{for all } x \in J_u.$$

Finally, for $u \in L^1_{\mathrm{loc}}(\Omega)$ we define the *precise representative* of u in $x \in \Omega$ as

$$u^*(x) := \lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} u(y)\, dy,$$

whenever the limit exists. It is then clear that

$$u^*(x) = \begin{cases} \tilde{u}(x) & x \in \Omega \setminus S_u, \\ \dfrac{u^i(x) + u^e(x)}{2} & x \in J_u. \end{cases} \tag{37}$$

A priori, it is not clear whether u^* is well posed in $S_u \setminus J_u$, in general. However, for $u \in BV_{\mathrm{loc}}(\Omega)$, it is well known that we have $\mathscr{H}^{n-1}(S_u \setminus J_u) = 0$, so that $u^*(x)$ exists for $\mathscr{H}^{n-1}$-a.e $x \in \Omega$ and, up to a $\mathscr{H}^{n-1}$-negligible set, is given by (37).

Finally, for every Borel function we define the *representative* $u^{\frac{1}{2}} : \Omega \to \overline{\mathbb{R}}$ as

$$u^{\frac{1}{2}}(x) := \begin{cases} \frac{1}{2}(u^-(x) + u^+(x)) & \text{if } x \in \Omega \setminus Z_u \\ 0 & \text{if } x \in Z_u \end{cases} \tag{38}$$

where $Z_u := \{x \in \Omega : u^+(x) = +\infty \text{ and } u^-(x) = -\infty\}$ and

$$u^{\frac{1}{2}}(x) = \tilde{u}(x) \text{ for all } x \in \Omega \setminus S^*_u.$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$, we notice that $u^{\frac{1}{2}}(x) = \tilde{u}(x)$ for all $x \in \Omega \setminus S_u$ and $u^{\frac{1}{2}}(x) = u^*(x)$ for all $x \in \Omega \setminus (S_u \setminus J_u)$, but we might have $u^*(x) \neq u^{\frac{1}{2}}(x)$ for some $x \in S_u \setminus J_u$ (see Example in [13] Section 2.2).

A.2. Pairing in the n-dimensional case

In this subsection, we need to recall a general notion of pairing for divergence measure fields, as introduced in [13, Section 3].

We define $\mathcal{DM}^1_{\rm loc}(\Omega)$ as the space of all vector fields $w \in L^1_{\rm loc}(\Omega, \mathbb{R}^n)$ whose divergence div$w$ in the sense of distributions belongs to $\mathcal{M}_{\rm loc}(\Omega)$.

First of all, we need suitable ambient classes of summable functions, which naturally depend on the chosen Borel field w. Given $w \in \mathcal{DM}^1_{\rm loc}(\Omega)$, we set

$$X^w(\Omega) := \left\{ u \text{ Borel function} : u \in L^1(\Omega, w), u^{\frac{1}{2}} \in L^1(\Omega, |\mathrm{div} w|) \right\},$$

$$X^w_{\rm loc}(\Omega) := \left\{ u \text{ Borel function} : u \in L^1_{\rm loc}(\Omega, w), u^{\frac{1}{2}} \in L^1_{\rm loc}(\Omega, |\mathrm{div} w|) \right\}.$$

We now recall the definition of pairing for functions in $X^w_{\rm loc}(\Omega)$.

Definition A.1. Let $w \in \mathcal{DM}^1_{\rm loc}(\Omega)$, and $u \in X^w_{\rm loc}(\Omega)$. We define the pairing between w and u as the distribution

$$(w, Du) : C^\infty_c(\Omega) \to \mathbb{R}$$

acting as

$$\langle (w, Du), \varphi \rangle := -\int_\Omega u^{\frac{1}{2}} \varphi \,\mathrm{d}\mathrm{div} w - \int_\Omega u \nabla\varphi \cdot w \,\mathrm{d}x \qquad \text{for } \varphi \in C^\infty_c(\Omega). \tag{39}$$

A.3. Pairing in the one-dimensional case

Let us notice that for $n = 1$, $\Omega \subset \mathbb{R}$ and we have div$w = Dw$. Furthermore, we have that

$$\mathcal{DM}^1_{\rm loc}(\Omega) = \mathrm{BV}_{\rm loc}(\Omega) \subset L^\infty_{\rm loc}(\Omega)$$

and $$\langle (w, Du), \varphi \rangle = -\int_\Omega u^{\frac{1}{2}} \varphi \,\mathrm{d}Dw - \int_\Omega u \nabla\varphi \cdot w \,\mathrm{d}x \qquad \text{for } \varphi \in C^\infty_c(\Omega). \tag{40}$$

Then, recalling that if $u \in \mathrm{BV}_{\rm loc}(\Omega)$, then $u^*(x) = u^{\frac{1}{2}}(x)$ for every $x \in \Omega$ and by Proposition 3.5 (3) in [13] since $\mathrm{BV}_{\rm loc}(\Omega) \subset L^\infty_{\rm loc}(\Omega)$

$$\{u \in \mathrm{BV}_{\rm loc}(\Omega) : u^{\frac{1}{2}} \in L^1_{\rm loc}(\Omega, |Dw|)\} = \mathrm{BV}_{\rm loc}(\Omega).$$

We will also need the following classes of functions which are the analogue of BV-type functions when working with the pairing.

Definition A.2. Given $w \in \mathrm{BV}_{\rm loc}(\Omega)$, we define the classes

$$\mathrm{BV}^w(\Omega) := \{u \in X^w(\Omega) : (w, Du) \in \mathcal{M}(\Omega)\},$$

$$\mathrm{BV}^w_{\rm loc}(\Omega) := \{u \in X^w_{\rm loc}(\Omega) : (w, Du) \in \mathcal{M}_{\rm loc}(\Omega)\}.$$

Remark A.3. By Proposition 3.5 in [13] since $w \in L^\infty_{\rm loc}(\Omega)$, then

$$\mathrm{BV}_{\rm loc}(\Omega) \subseteq BV^w_{\rm loc}(\Omega).$$

Remark A.4. As noted in [13, Remark 3.4], the set $\mathrm{BV}^w_{\rm loc}(\Omega)$ is not a linear space. This is due to the fact that the pairing is, in fact, a nonlinear operation in the second component, representing a departure from the classical BV-setup. Nevertheless, if $w \in W^{1,1}_{\rm loc}(\Omega)$, then $\mathrm{BV}^w_{\rm loc}(\Omega)$ is a linear space (see Corollary 5.3 in [13]).

Since the pairing (w, Du) is affected by the pointwise value of $u^{\frac{1}{2}}$, then a suitable notion of convergence involving these representatives is introduced in [13].

Definition A.5. Let $w \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. We say that a sequence $(u_n)_{n\in\mathbb{N}} \subset X^w_{\mathrm{loc}}(\Omega)$ $(w, \frac{1}{2})$-converges to $u \in X^w_{\mathrm{loc}}(\Omega)$ if

(i) $u_n \rightharpoonup u$ in $L^1_{\mathrm{loc}}(\Omega, w)$,

(ii) $u_n^{\frac{1}{2}} \rightharpoonup u^{\frac{1}{2}}$ in $L^1_{\mathrm{loc}}(\Omega, |Dw|)$.

When $w \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$, then (ii) is equivalent to $u_n \rightharpoonup u$ in $L^1_{\mathrm{loc}}(\Omega, |Dw|)$.

The following lower semicontinuity of the pairing holds true.

Theorem A.6. [13, Theorem 4.3] *Let $w \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Then for every sequence $(u_n)_{n\in\mathbb{N}} \subset X^w_{\mathrm{loc}}(\Omega)$ and for every $u \in X^w_{\mathrm{loc}}(\Omega)$, and such that $(u_n)_n$ $(w, \frac{1}{2})$-converges to u, it holds*

$$\langle (w, Du), \varphi\rangle = \lim_{n\to+\infty} \langle (w, Du_n), \varphi\rangle \quad \textit{for all } \varphi \in C^1_c(\Omega)$$

in the sense of distributions. Further, if $u, u_n \in \mathrm{BV}^w_{\mathrm{loc}}(\Omega)$ for all $n \in \mathbb{N}$, then

$$|(w, Du)|(\Omega) \le \liminf_{n\to+\infty} |(w, Du_n)|(\Omega). \tag{41}$$

If

$$\sup_{n\in\mathbb{N}} |(w, Du_n)|(\Omega) < +\infty,$$

we get

$$|(w, Du_n)|(\Omega) \rightharpoonup |(w, Du)|(\Omega)$$

weakly in the sense of measures.

B. Weighted BV-spaces

In this part, for the sake of completeness, we recall the definition of weighted $\mathrm{BV}(\Omega; w)$-spaces introduced in [6], where the weight w belongs to the global Muckenhoupt's $A_1 := A_1(\Omega)$. Suppose that Ω is an open subset of $\mathbb{R}$, and let Ω_0 be a neighborhood of $\overline{\Omega}$.

Definition B.1. Let $w \in L^1_{\mathrm{loc}}(\Omega_0)$, $w > 0$. We say that $w \in A_1$ if there exists a constant $c > 0$ such that

$$w(x) \ge c \fint_{B(x,r)} w(y)\mathrm{d}y \text{ a.e. in any ball } B(x, r) \subset \Omega_0. \tag{42}$$

In[6], given $u \in L^1(\Omega; w)$, the weighted total variation of u with respect to w is defined as

$$TV(u; w) := \sup\left\{ \int_\Omega u\phi' \,\mathrm{d}x : \phi \in C^1_c(\Omega; \mathbb{R}), |\phi(x)| \le w(x) \text{ for all } x \in \Omega\right\}.$$

Denote by $\mathrm{BV}(\Omega; w)$ the set of all functions $u \in L^1(\Omega; w)$ for which $TV(u; w) < +\infty$, and we equip it with the norm

$$\|u\|_{\mathrm{BV}(\Omega,w)} := \|u\|_{L^1(\Omega;w)} + TV(u; w).$$

In particular, when $w \equiv 1$ we recover the usual space $\mathrm{BV}(\Omega)$. For a measurable set $B \subset \Omega$, we then define the perimeter in Ω as the weighted total variation of the characteristic function of B, that is, $\mathrm{Per}(B; w) := TV(\chi_B; w)$.

Remark B.2. Let us recall that in the definition of weighted Sobolev spaces, the weight is usually defined a.e. (almost everywhere) because functions in these spaces have derivatives that, as measures, are absolutely continuous with respect to the Lebesgue measure. Nevertheless, in the case of weighted BV-spaces, the situation is completely different. Indeed, derivatives can be concentrated on sets of null Lebesgue measure. A proper definition of a weighted BV-space requires a pointwise definition of w. In fact, requiring that $w \in A_1$ reflects this, as it captures a pointwise definition in each ball $B(x,r)$ for which the inequality (42) holds.

In [6], it is shown that it not necessary to assume that w is lower semicontinuous to define a weighted Sobolev space. However, in the case where $w \in A_1$, it is possible to show that we can find an auxiliary weight w^* that is lower semicontinuous and such that $\mathrm{BV}(\Omega; w) = \mathrm{BV}(\Omega; w^*)$.

Lemma B.3. [6, Lemma 3.1] *Suppose that $w \in A_1$. The following assertions hold true.*

(i) *Let us set $L_0(\Omega, \mathbb{R})$ the set of Lipschitz continuous functions with compact support. Define*

$$w^* := \sup_{\substack{\phi \in L_0(\Omega,\mathbb{R}) \\ |\phi| \le w}} |\phi|.$$

Then $\mathrm{BV}(\Omega; w) = \mathrm{BV}(\Omega; w^*)$.

(ii) *Let us consider the relaxed function w^{**} associated to w, that is,*

$$w^{**} := \sup \{ g : g : \Omega \to (0, +\infty) \textit{ is lower semicontinuous, and } g \le w \}.$$

Then $w^{**} = w^*$ *in* Ω, *and* $\mathrm{BV}(\Omega; w) = \mathrm{BV}(\Omega; w^*) = \mathrm{BV}(\Omega; w^{**})$.

(iii) $w^{**} \in A_1$.

Let us set
$$\widetilde{w}(x) := \sup_{r>0} \fint_{B(x,r)} w(y)\mathrm{d}y.$$

Since $w \in A_1$, note that $\widetilde{w} \in A_1$ with the same constant $c > 0$. Indeed, observe that

$$\fint_{B(x,r)} \widetilde{w}(y)\mathrm{d}y \le \frac{1}{c} \fint_{B(x,r)} w(y)\mathrm{d}y \le \frac{1}{c}\widetilde{w}(x).$$

Furthermore, since the integral is a continuous operation, then by taking the supremum of continuous functions we obtain a lower semicontinuous function, and $\widetilde{w} > 0$. Hence, in order to obtain suitable density results, it is customary to replace w with an appropriate lower semicontinuous function when defining weighted BV-spaces.

Definition B.4. Let $w \in A_1$, and define A_1^* as

$$A_1^* := \left\{ w \in A_1 : \begin{array}{l} w \text{ is lower semicontinuous, and} \\ \text{condition } A_1 \text{ is satisfied at any point} \end{array} \right\}.$$

The following holds true.

Proposition B.5. [6, Theorem 3.3] *Let $w \in A_1^*$, and $u \in \mathrm{BV}(\Omega; w)$. Then there exist a finite Radon measure $|Du|_w$ and a $|Du|_w$-measurable function $\sigma\colon \Omega \to \mathbb{R}$ such that $|\sigma(x)| = 1$ for $|Du|_w$-almost every $x \in \Omega$ and such that*

$$\int_\Omega u(x)\phi'(x)\,\mathrm{d}x = -\int_\Omega \frac{\phi(x)\sigma(x)}{w(x)}\,\mathrm{d}|Du|_w(x). \tag{43}$$

The measure $|Du|_w$ and the function σ are uniquely determined by (43) *and the weighted total variation $TV(u; w)$ is equal to $|Du|_w(\Omega)$.*

Note that, using (43), one can check that $|Du|_w = w|Du|$, so that

$$TV(u; w) = \int_\Omega w(x)\,\mathrm{d}|Du|(x).$$

Since the functional $TV(\cdot; w)$ is defined as the supremum of linear continuous functionals in $L^1(\Omega; w)$, it is lower semicontinuous with respect to the $L^1(\Omega; w)$ metric. The following density theorem for weighted BV functions holds true.

Theorem B.6. [6, Theorem 3.4] *Let Ω be an open subset of $\mathbb{R}$ with Lipschitz boundary. Suppose $w \in \mathrm{Lip}(\Omega)$, and $w \in A_1$. Then for every $u \in \mathrm{BV}(\Omega; w)$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R})$ such that $u_n \to u$ in $L^1(\Omega)$ and we have $\int_\Omega |u_n'|w\,\mathrm{d}x \to TV(u; w)$ as $n \to \infty$.*

A similar version of this density result can be found in [18, Proposition 2.4]. In what follows, we recall a Poincaré inequality proved in [6, Theorem 4.2].

Theorem B.7. [6, Theorem 4.2] *Let $u \in \mathrm{BV}(\Omega; w)$, with $w \in A_1^*$, and $q > 1$. Suppose that the local growth condition*

$$\left(\frac{\displaystyle\int_{B(x,r)} w(y)\mathrm{d}y}{\displaystyle\int_{B(x,s)} w(y)\mathrm{d}y}\right) \le c\left(\frac{r}{s}\right)^{\frac{q}{q-1}} \tag{44}$$

holds for any pair of balls $B(x, r) \subset B(x, s)$ in $\mathbb{R}$. Then there exist two positive constants C_1, C_2 such that the following inequalities hold true:

- $$\left(\fint_B |u - u_B|^q w(y)\mathrm{d}y\right)^{\frac{1}{q}} \le \frac{rC_1}{B} TV(u; w)(B)$$

  for all balls $B = B(x, r) \subset \mathbb{R}$, where $u_B := \fint_B u(y)\mathrm{d}y$, and

  $$TV(u; w)(B) := \int_B w(x)\,\mathrm{d}|Du|(x).$$

- *Suppose that*

$$\limsup_{R\to+\infty} R\left(\int_{B(x,R)} w(y)\mathrm{d}y\right)^{\frac{1}{q}-1} < +\infty.$$

Then

$$\|u\|_{L^q(\Omega;w)} \le C_2 TV(u;w)(\mathbb{R}).$$

Acknowledgments. The authors are members of the Istituto Nazionale di Alta Matematica (INdAM), GNAMPA Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, and are partially supported by the INdAM–GNAMPA 2023 Project *Problemi variazionali degeneri e singolari* and the INdAM–GNAMPA 2024 Project *Pairing e div-curl lemma: estensioni a campi debolmente derivabili e differenziazione non locale.*

Part of this work was undertaken while the first and third authors were visiting Sapienza University and SBAI Department in Rome. They would like to thank these institutions for the support and warm hospitality during the visits.

The research was in part carried out within the project: Geometric-Analytic Methods for PDEs and Applications (GAMPA), ref. 2022SLTHCE – cup E53D2300588 0006 - funded by the European Union - Next Generation EU within the PRIN 2022 program (D.D. 104 - 02/02/2022 Ministero dell'Università e della Ricerca). This manuscript reflects only the authors' views and opinions and the Ministry cannot be considered responsible for them.

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