Journal of Convex Analysis
Volume 33 (2026), No. 3&4, 925–952

High Order Symmetrised Fractional Variation for Signal and Image Analysis

Alessandro Lanza
Department of Mathematics, University of Bologna, Italy
alessandro.lanza2@unibo.it

Antonio Leaci
Dip. di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce, Italy
antonio.leaci@unisalento.it

Serena Morigi
Department of Mathematics, University of Bologna, Italy
serena.morigi@unibo.it

Franco Tomarelli
Dipartimento di Matematica, Politecnico di Milano, Italy
franco.tomarelli@polimi.it

Dedicated to the memory of Hedy Attouch.

Received: January 28, 2025
Accepted: June 15, 2025

We introduce and study a variational model for signal and image denoising based on Riemann-Liouville fractional derivatives of every positive order higher than zero. Both the one-dimensional and two-dimensional cases are studied. The model exploits an L^1 fitting data term together with both right and left Riemann-Liouville fractional derivatives as regularizing terms, with the aim of achieving an orientation independent analysis.
To provide evidence of effectiveness for the proposed model we introduce a discretisation based on a second-order consistent Grünwald Letnikov scheme and show some numerical simulations aiming to denoise images corrupted by impulsive noise, which can be well modeled by the Laplace distribution.

Keywords: Symmetrized fractional variation, Riemann-Liouville fractional derivatives, Grünwald-Letnikov formulas, discretization of fractional derivatives, calculus of variations, image denoising, Abel equation, bounded variation functions.

2020 Mathematics Subject Classification: 26A33, 26A45, 65K10.

1. Introduction

Starting from the classical notion of Riemann-Liouville fractional derivatives and related Grünwald-Letnikov derivative ([50]), and referring to their distributional counterpart introduced in [38], here we introduce and study the symmetrised fractional Sobolev spaces $W^{s,1}_*$ and symmetrised fractional bounded variation spaces BV^s_* for every real order $s>0$. In particular: for $0<s<1$ we recover spaces BV^s introduced in [38] and [39]; for $1<s<2$ and $\tau = s-1$ we find intermediate spaces

ISSN 0944-6532 / $ 2.50

$BH_*^\tau = BV_*^s$ between the space BV of functions of bounded variation and the space BH of functions with bounded hessian (see [12, 25, 44, 51]).

Suitable seminorms in these spaces BV_*^s (see (4.2),(4.3) and Theorem 3.9) provide tuned noise filtering estimators. Referring to these estimators and a fidelity term in L^1 norm, we introduce our model of Symmetrised Fractional Variation for signal analysis in dimensions 1 and 2, denoted by SFV-L^1 or shortly SFV (see sections 4 and 5). This model was previously tackled in 1 dimension and with the restriction $0< s<1$ in [39], then it was numerically implemented in [33].

The L^1 fidelity term is of great interest in practical denoising applications. In fact, it is well-known to be suitable in general for noise corruptions of impulsive type and, in particular, it is optimal for the important class of additive white noises having Laplace distribution. While the choice of the fidelity norm should be informed by the statistical properties of the noise, the choice of the regularizer is determined by a priori knowledge of the expected regularity of the computed solution. In denoising model with fractional derivative-based regularizer, the quality of the reconstruction depends on the choice of fractional order parameter s. In [33] a preliminary study on an automatic fractional order selection method has been proposed. It relies on the Whiteness Principle: the residual corresponding to the solution provided by the model is maximally white, say uncorrelated: [6, 34, 35, 36]. Preliminary studies have shown as the choice of a parameter $s>1$ turns out to be numerically very effective when aiming to denoise corrupted images of smooth oscillating features. In this respect all numerical experiments provide evidence for effectiveness of the automatic choice of the derivative fractional order $s > 1$ for denoising images corrupted by additive Laplace impulsive noise.

The numerical implementation of the presented Symmetrised Fractional Variation model (SFV) for signal analysis in the two-dimensional case is the object of last sections, where the relationship between a second-order consistent Grünwald-Letnikov scheme and the Riemann-Liouville fractional derivatives is exploited.

Concerning the mathematical background in image denoising and segmentation, the nonconvex variational models by Mumford-Shah and Blake-Zisserman functionals ([18, 26]) were extensively used.

Another classical model for image analysis is the Rudin-Osher-Fatemi model [48] also known as ROF or TV (total variation), which in turn relies on a convex minimization, together with its many variants [22, 46, 54]: in TV model, the regularizing term is a first order term, namely the total variation. To avoid artefacts as staircasing which appears in TV approach, the total variation term has been tuned by adding second order derivatives (Total Generalized Variation, or shortly TGV, in [9]), or by extending to fractional TV-norm, as in [40]. Analogous models have been considered with different second order terms for denoising ([29]) and for segmentation and inpainting ([13, 14, 15, 16, 17, 18, 19, 20]). All these models use integer-order derivatives in the regularization term.

To take advantage of nonlocal properties of fractional derivatives, some fractional order TV-like models have been proposed for signal denoising [3, 21, 28, 45, 46]. However, these approaches deal mainly with the finite dimensional framework (after discretization) and few among them ([38, 39]) propose an infinite dimensional approach together with a suitable mathematical analysis. We mention the work by Zhang and Chen [54] which provides an anisotropic total fractional-order variation model for image restoration in a continuous setting, and Golbaghi et al. in [32] that introduced a variable-order total fractional variation model in a suitable normed space $BV^{s(x)}$.

For related results concerning fractional derivatives we mention also [5, 11, 23, 24, 27, 30, 43].

In [39] we introduced and analysed an isotropic fractional variation model with fidelity term in L^2; in [33] we considered a fidelity term in L^1 and fractional orders $s \in (0,1)$ providing 1D numerical simulations; in the present paper we consider any real $s > 1$ and show some 2D numerical experiment.

2. Notations

Without loss of generality, we consider the 1D and 2D signal domains respectively as the interval $(0,1) \subset \mathbb{R}$ and the square $(0,1)\times(0,1) \subset \mathbb{R}^2$. Γ and H denote the Euler Gamma function and the Heaviside function ([42]), respectively.

For every $s > 0$, $[s]$ denotes the integer part of s, say the greatest integer among the ones less than or equal to s; $\lceil s \rceil$ denotes the ceiling of s, say the least integer greater than or equal to s: $\lceil s \rceil = [s] + 1$; $\mathrm{Frac}(s)$ denotes the fractional part of s, $\mathrm{Frac}(s) = s - [s]$. Moreover $x^+ = \max\{x, 0\}$ for every $x \in \mathbb{R}$.

The notation d/dx stands for the classical *pointwise derivative*, D_x or shortly D denotes the *distributional derivative*, alternative short notations: $v' = D_x v$ and $v^{(k)} = D_x^k v$.

$W^{1,1}(0,1) = \{u \in L^1(0,1) \mid Du \in L^1(0,1)\}$ denotes the *Gagliardo-Sobolev space* ([1, 4, 10]). Note that in the literature (see e.g. [50]) the space $W^{1,1}(0,1)$ is also referred to as the space of absolutely continuous functions $AC(0,1)$.

We denote by $\mathcal{M}$ the space of measures μ on $\mathbb{R}$ such that $\mu\big((0,1)\big)$ is finite.

$\mathcal{M}_+$ denotes the real-valued measures μ on $\mathbb{R}$ such that $\mu\big((-\infty,1)\big)$ is finite.

$\mathcal{M}_-$ denotes the real-valued measures μ on $\mathbb{R}$ such that $\mu\big((0,+\infty)\big)$ is finite.

A Borel measure is a measure defined on the Borel σ algebra.

A Borel measure which is finite on compact sets is called Radon measure.

$\mathcal{D}'(A)$ denotes the space of distributions on the open set $A \subset \mathbb{R}^n$.

3. Riemann-Liouville calculus for measures in the 1d case: space BV^s_*

We recall the definition of the Riemann-Liouville fractional calculus ([50]): fractional integrals for L^1-functions which can be represented by convolutions and related fractional derivatives in the distributional framework, as like as their bilateral versions (Riesz potentials and their distributional derivatives). For additional details concerning a bilateral approach we refer to [5, 37, 38, 39].

Definition 3.1. Let $u \in L^1(0,1)$. For every $s > 0$ the left-side, right-side and symmetrised *Riemann-Liouville fractional integrals* are defined respectively as

$$I^s_+[u](x) := \frac{1}{\Gamma(s)} \int_0^x \frac{u(t)}{(x-t)^{1-s}} dt \qquad 0 \le x \le 1, \tag{3.1}$$

$$I^s_-[u](x) := \frac{1}{\Gamma(s)} \int_x^1 \frac{u(t)}{(t-x)^{1-s}} dt \qquad 0 \le x \le 1. \tag{3.2}$$

$$I^s[u](x) := \frac{1}{2\,\Gamma(s)} \int_0^1 \frac{u(t)}{|t-x|^{1-s}} dt \qquad 0 \le x \le 1. \tag{3.3}$$

We emphasize that referring to the trivial extension of u $(u = 0$ a.e. on $\mathbb{R}\backslash(0,1))$, all integrals in (3.1)–(3.3) are convolutions on $\mathbb{R}$, though we often consider only their values restricted to the open set $(0,1)$ (see [37],[38]): denoting the Heaviside function by H we get

$$I^s_+[u] = u * \frac{H(x)}{\Gamma(s)|x|^{1-s}}\,, \quad I^s_-[u] = u * \frac{H(-x)}{\Gamma(s)|x|^{1-s}}\,, \quad I^s[u] = u * \frac{1}{2\Gamma(s)|x|^{1-s}}$$

namely

$$I^s_+[u](x) = \frac{1}{\Gamma(s)} \int_0^1 \frac{u(t)\,H(x-t)}{|x-t|^{1-s}}\, dt \quad x \in \mathbb{R},$$

$$I^s_-[u](x) = \frac{1}{\Gamma(s)} \int_0^1 \frac{u(t)\,H(t-x)}{|x-t|^{1-s}}\, dt \quad x \in \mathbb{R}.$$

All the convolutions above are well defined functions in $L^1_{loc}(\mathbb{R})$ (with supports respectively in $[0,+\infty)$, $(-\infty,1]$ and $\mathbb{R}$) since $u \in L^1(0,1)$ and $\operatorname{spt} u \subset [0,1]$, while $|x|^{s-1} \in L^1_{loc}(\mathbb{R})$.

Remark 3.2. If $u \in L^1(0,1)$ and $s > 0$ then, due to Fubini-Tonelli Theorem and hence associativity of the convolution, both I^s_+ and I^s_- fulfil the semigroup property:

$$I^s_+\left[I^\sigma_+[u]\right] = I^{s+\sigma}_+[u]\,, \qquad I^s_-\left[I^\sigma_-[u]\right] = I^{s+\sigma}_-[u]\,. \tag{3.4}$$

Hence, for every $s>0$ and $u \in L^1(0,1)$, both $I^{1+s}_+[u]$ and $I^{1+s}_-[u]$ belong to $AC(0,1)$ and

$$D_x\, I^{1+s}_+[u] = \frac{d}{dx}\, I^{1+s}_+[u] = I^s_+[u]\,, \qquad \text{a.e. on } (0,1) \tag{3.5}$$

$$D_x\, I^{1+s}_-[u] = -\frac{d}{dx}\, I^{1+s}_-[u] = -I^s_-[u]\,, \qquad \text{a.e. on } (0,1)\,. \tag{3.6}$$

The convolution representation of fractional integrals for integrable functions leads us to introduce in a natural way the fractional integral for a Borel measure.

Definition 3.3. For every $s > 0$ and μ Borel measure with support in $[0,+\infty)$ we introduce the left-side Riemann-Liouville fractional integrals of μ, by setting

$$I^s_+[\mu](x) \;=\; \mu * \frac{H(x)}{\Gamma(s)|x|^{1-s}}\;. \tag{3.7}$$

For every $s > 0$ and ν Borel measure with support in $(-\infty, 1]$ we introduce the right-side Riemann-Liouville fractional integrals of ν, by setting

$$I_-^s[\nu](x) \ := \nu * \frac{H(-x)}{\Gamma(s)|x|^{1-s}} \tag{3.8}$$

In the degenerate case $s=0$ we set

$$I_+^0[\mu] = \mu \qquad I_-^0[\nu] = -\,\nu\,. \tag{3.9}$$

Note that the semigroup property is fulfilled by these extended definition (3.7),(3.8) of fractional integral operators too:

$$I_+^s\left[I_+^\sigma[\mu]\right] = I_+^{s+\sigma}[\mu]\,, \qquad I_-^s\left[I_-^\sigma[\nu]\right] = I_-^{s+\sigma}[\nu]\,, \tag{3.10}$$

indeed this identities can be shown by reproducing the argument in the proof of Proposition 1 in [38].

Lemma 3.4. *Assume that μ is a Borel measure on $\mathbb{R}$ and $s>0$.*
If $\operatorname{spt}\mu\subset[0,+\infty)$ *(recall that* $\operatorname{spt}\mu$ *denotes the support of the measure μ) then* (3.7) *defines a locally integrable function $I_+^s[\mu]$ with support contained in* $[0,+\infty)$ *and*

$$\int_0^1 \left| \, I_+^s[\mu](x) \, \right| \, dx \le \left(\frac{1}{\Gamma(s)} \int_0^2 x^{s-1}\, dx\right) \, |\mu|([0,1])\,. \tag{3.11}$$

If $\operatorname{spt}\mu \subset (-\infty, 1]$ *then* (3.8) *defines a locally integrable function $I_-^s[\mu]$ with support contained in* $[0,+\infty)$ *and*

$$\int_0^1 \left| \, I_-^s[\mu](x) \, \right| \, dx \le \left(\frac{1}{\Gamma(s)} \int_0^2 x^{s-1}\, dx\right) \, |\mu|([0,1])\,. \tag{3.12}$$

Proof. By Proposition 3.9.9. in [7] and by [49], the convolution of a Borel measure μ on $\mathbb{R}$ with a Lebesgue integrable function f is a Lebesgue integrable function fulfilling $\|\mu * f\|_{L^1(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}\, |\mu|(\mathbb{R})$, such that

$$\int_{\mathbb{R}} \varphi(x)\, d(\mu * f)(x) \ = \ \iint_{\mathbb{R}^2} \varphi(x+y)\, f(y)\, d\mu(x)\, dy \qquad \forall \varphi \in C_0^\infty\,.$$

Moreover, the convolution of two distributions with both supports bounded from the left (or both supports bounded from the right) is well defined. Thus (3.11), (3.12) are proved. □

Next, we define the Riemann-Liouville fractional derivative.

Definition 3.5. (Distributional Riemann-Liouville fractional derivative.) Assume $u \in L^1(0,1)$ and $s>0$. Then, referring to [37] and [38], the left, right and bilateral Riemann-Liouville derivatives of u at $x \in (0,1)$ are defined by

$$\begin{aligned} D_+^s[u](x) &:= (D_x)^{\lceil s\rceil}\left[I_+^{\lceil s\rceil - s}[u]\right](x) = (D_x)^{\lceil s\rceil}\left[I_+^{1-\mathrm{Frac}(s)}[u]\right](x) = \\ &= \ \frac{1}{\Gamma(1-\mathrm{Frac}(s))}\,(D_x)^{\lceil s\rceil}\!\int_{-\infty}^x \frac{u(t)}{(x-t)^{\mathrm{Frac}(s)}}dt \end{aligned} \tag{3.13}$$

$$D^s_-[u](x) := -\left(D_x\right)^{\lceil s\rceil}\left[I_-^{\lceil s\rceil-s}[u]\right](x) = -\left(D_x\right)^{\lceil s\rceil}\left[I_-^{1-\mathrm{Frac}(s)}[u]\right](x) =$$

$$= \frac{-1}{\Gamma\big(1-\mathrm{Frac}(s)\big)}\left(D_x\right)^{\lceil s\rceil}\int_x^{+\infty}\frac{u(t)}{(t-x)^{\mathrm{Frac}(s)}}dt \tag{3.14}$$

$$D^s[u](x) := D_x I^{1-s}[u](x) = \frac{1}{2}\left(D^s_+[u](x) - D^s_-[u](x)\right), \tag{3.15}$$

where we exploited $\lceil s\rceil - s = 1+[s]-s = 1-\mathrm{Frac}(s)$ and, without relabeling, the symbol u denotes the trivial extension on $\mathbb{R}\backslash(0,1)$.

Moreover all the derivatives appearing in (3.13), (3.14) and (3.15) are understood in the distributional sense in $\mathcal{D}'(\mathbb{R})$, though we will work in $\mathcal{D}'(0,1)$ by handling their restrictions to the open set $(0,1)$. We refer to [38] for the related theory.

We recall that, if both u and Du belong to $L^2(0,1)$, then both $D^s_+[u]$ and $D^s[u]$ converge to Du in $L^2(0,1)$ while $D^s_-[u]$ converge to Du in $L^2(0,1)$, as $s \to 1_-$ (Lemma 2.3 in [37], where D^s was denoted by D^s_e).

Here we introduce the Sobolev and Bounded Variation spaces associated to the Riemann-Liouville fractional derivative of any order $s>0$, analogously to the ones already introduced in [38] for $0<s<1$ by extending the definition to derivatives of order higher than 1 too.

Definition 3.6. Riemann-Liouville fractional Sobolev space: $W^{s,1}_*$.

For $s>0$ we set $$W^{s,1}_*(0,1) := W^{s,1}_+(0,1)\cap W^{s,1}_-(0,1). \tag{3.16}$$

where, referring to Definition 3.1,

$$W^{s,1}_+(0,1) := \{u\in L^1(0,1):\ I_+^{\lceil s\rceil-s}[u]\in W^{\lceil s\rceil,1}(0,1)\}, \tag{3.17}$$

$$W^{s,1}_-(0,1) := \{u\in L^1(0,1):\ I_-^{\lceil s\rceil-s}[u]\in W^{\lceil s\rceil,1}(0,1)\}. \tag{3.18}$$

Note that, due to (3.9), when $s>0$ is an integer, we find, for every $0<k\in\mathbb{N}$,

$$W^{k,1}_+(0,1) = W^{k,1}_-(0,1) = W^{k,1}_*(0,1) = W^{k,1}(0,1) = \{u\in L^1(0,1)\mid D^k u\in L^1(0,1)\}.$$

Both $W^{s,1}_+(0,1)$ and $W^{s,1}_-(0,1)$ introduced in Definition 3.6 are Banach spaces, when endowed with their natural norms:

$$\|u\|_{W^{s,1}_+(0,1)} = \|u\|_{W^{[s],1}(0,1)} + \|\, D_x^{\lceil s\rceil}\, I_+^{\lceil s\rceil-s}[u]\,\|_{L^1(0,1)}, \tag{3.19}$$

$$\|u\|_{W^{s,1}_-(0,1)} = \|u\|_{W^{[s],1}(0,1)} + \|\, D_x^{\lceil s\rceil}\, I_-^{\lceil s\rceil-s}[u]\,\|_{L^1(0,1)}. \tag{3.20}$$

For every $s>1$, the bounded subsets of fractional Sobolev spaces $W^{s,1}_+$, $W^{s,1}_-$ and $W^{s,1}_*$ are relatively compact in the strong L^1 topology.

Unfortunately bounded subsets of fractional Sobolev spaces $W^{s,1}_+$, $W^{s,1}_-$ and $W^{s,1}_*$ lack compactness properties in L^1 for their highest order derivatives $D^s_+[u]$ and $D^s_-[u]$.

This leads us to introduce the fractional bounded variation counterparts of these spaces in a symmetric framework.

Definition 3.7. (Riemann-Liouville fractional bounded variation space BV^s_*)

For every $s>0$, we set
$$BV^s_* \ = BV^s_+ \cap BV^s_- \tag{3.21}$$
where, referring to Definition 3.1,
$$BV^s_+ := \{u \in L^1(0,1):\ I_+^{\lceil s\rceil - s}[u] \in W^{\lceil s\rceil,1}(0,1),\ D_x^{\lceil s\rceil} I_+^{\lceil s\rceil - s}[u] \in \mathcal{M}_+\}\,, \tag{3.22}$$
$$BV^s_- := \{u \in L^1(0,1):\ I_-^{\lceil s\rceil - s}[u] \in W^{\lceil s\rceil,1}(0,1),\ D_x^{\lceil s\rceil} I_-^{\lceil s\rceil - s}[u] \in \mathcal{M}_-\}\,. \tag{3.23}$$

Since $D^s_+[u]$ and $D^s_-[u]$ denote the left and right distributional RL fractional derivatives of u, we have $D^s_+[u]=0$ on the whole half-line $(-\infty,0)$ and $D^s_-[u]=0$ on the whole half-line $(1,+\infty)$.

Remark 3.8. We emphasize that for $0<s<1$ the spaces defined by (3.22) and (3.23) coincide respectively with the spaces BV^s_+ and BV^s_- introduced by Definition 2.4 of [33], say we could replace $\mathcal{M}_+$ with $\mathcal{M}$ in (3.22) and $\mathcal{M}_-$ with $\mathcal{M}$ in (3.23): in fact, the replacement of $\mathcal{M}$ with $\mathcal{M}_+$ relies only in the addition of the term

$I^{1-s}_+[u](0_+)\,\delta(x)$; moreover, if u belongs to BV^s_+ according to Definition 2.4 of [33], then $u\in L^1(0,1)$, $I^{1-s}_+[u]\in L^1(0,1)$, $D_x I^{1-s}_+[u]\in\mathcal{M}$, hence $|I^{1-s}_+[u](0_+)|$ is estimated by $\|I^{1-s}_+[u]\|_{L^1(0,1)}$ together with $\|D_x I^{1-s}_+[u]\|_{\mathcal{M}(0,1)}$; thus u belongs to BV^s_+ according to present definition (3.22).

Respectively, replacing $\mathcal{M}_-$ with $\mathcal{M}$ consists only in the addition of the term $-\,I^{1-s}_-[u](1_-)\,\delta(x-1)$. In consequence one can repeat the argument used to replace $\mathcal{M}_+$ with $\mathcal{M}$. □

In the same perspective of previous remark, feasible equivalent definitions of spaces BV^s_+ and BV^s_- with $s>1$ are postponed to a subsequent theorem: see Remark 3.13.

We introduce the notations $\|\cdot\|_{T+}$ and $\|\cdot\|_{T-}$ for natural norms in $\mathcal{M}_+$ and $\mathcal{M}_-$:
$$\|D^s_+[v]\|_{T_+} \ :=\ \|D^s_+[v]\|_{\mathcal{M}(0,1)} + |I^{1-s}_+[v](0_+)| \ =\ \|D^s_+[v]\|_{\mathcal{M}(-\infty,1)}\,, \tag{3.24}$$
$$\|D^s_-[v]\|_{T_-} \ :=\ \|D^s_-[v]\|_{\mathcal{M}(0,1)} + |I^{1-s}_-[v](1_-)| \ =\ \|D^s_-[v]\|_{\mathcal{M}(0,+\infty)}\,, \tag{3.25}$$
the symbol $\|\cdot\|_{\mathcal{M}(E)}$ represents total variation on the set E.

Theorem 3.9. *Assume $s>0$. Then the space BV^s_* can be equivalently defined as*
$$BV^s_* \ =\ \left\{ u \in L^1(0,1):\ \|\,D^s_+[u]\,\|_{T_+} < +\infty\,,\ \|\,D^s_-[u]\,\|_{T_-} < +\infty \right\} \tag{3.26}$$
and $BV^s_*(0,1)$ is a Banach space when endowed with the essential norm
$$\|u\|_{BV^s_*} := \|u\|_{L^1(0,1)} + \|\,D^s_+[u]\,\|_{T_+} + \|\,D^s_-[u]\,\|_{T_-}\,, \tag{3.27}$$
which turns out to be equivalent to the natural norm
$$|\!|\!|\,u\,|\!|\!|_{BV^s_*} \ :=\ \|u\|_{BV^s_*} + \|\,I_+^{1-\mathrm{Frac}(s)}[u]\,\|_{W^{[s],1}(0,1)} + \|\,I_-^{1-\mathrm{Frac}(s)}[u]\,\|_{W^{[s],1}(0,1)}\,. \tag{3.28}$$
If $0<s<1$ then for every $q\in[1,1/(1-s))$, there is $C=C(s,q)$, such that
$$\|u\|_{L^q(0,1)} \ \le\ C(s,q)\,\|u\|_{BV^s(0,1)}\,. \tag{3.29}$$
If $s=k=1,2,\dots$, then $BV^k_*=\{v\in L^1: v^{(k-1)}\in BV\}$. In particular $BV^1_*=BV(0,1)$.

If $s>1$*, then* $u\in BV_*^s$ *entails* $(D_x)^h u\in L^1(0,1)$ *for all positive integer* $h\leq [s]$ *and*

$$\| u \|_{W^{[s],1}(0,1)} \leq C\, \|u\|_{BV_*^s} . \tag{3.30}$$

For fixed $s>0$*, every* $u \in BV_*^s(0,1)$ *fulfills both representations on* $\mathbb{R}$

$$u = I_+^s\left[D_+^s[u]\right] = D_+^s\left[I_+^s[u]\right], \tag{3.31}$$

$$u = I_-^s\left[D_-^s[u]\right] = D_-^s\left[I_-^s[u]\right]. \tag{3.32}$$

For $s>0$*, the highest order derivative fulfill*

$$D_+^s[u] = D_+^{\mathrm{Frac}(s)}\left[D_x^{[s]}[u]\right] = D_x^{[s]}\left[D_+^{\mathrm{Frac}(s)}[u]\right] \in \mathcal{M}_+ \quad \forall u \in BV_+^s. \tag{3.33}$$

$$D_-^s[u] = D_-^{\mathrm{Frac}(s)}\left[D_x^{[s]}[u]\right] = D_x^{[s]}\left[D_-^{\mathrm{Frac}(s)}[u]\right] \in \mathcal{M}_- \quad \forall u \in BV_-^s. \tag{3.34}$$

Proof. If $0<s<1$, then completeness, estimate (3.29) together with the representations

$$u(x) = I_+^s\left[\widetilde{D_+^s[u]}\right](x) + \frac{I_+^{1-s}[u](0_+)}{\Gamma(s)}\,(x^+)^{s-1} \qquad \text{a.e } x \in (0,1),$$

$$u(x) = I_-^s\left[\widetilde{D_-^s[u]}\right](x) + \frac{I_-^{1-s}[u](1_-)}{\Gamma(s)}\,((1-x)^+)^{s-1} \qquad \text{a.e } x \in (0,1)$$

were proved by (51)–(53) of Theorem 1 in [38]; here $\widetilde{D_+^s[u]}$ and $\widetilde{D_-^s[u]}$ respectively denote the restrictions on $(0,+\infty)$ and $(-\infty,1)$:

$$D_+^s[u] = \widetilde{D_+^s[u]} + I_+^{1-s}[u](0_+)\delta(x),\ D_-^s[u] = \widetilde{D_-^s[u]} - I_-^{1-s}[u](1_-)\delta(x-1).$$

Thus, left equalities in (3.31)–(3.32) hold true for $0<s<1$: they are a distributional equivalent formulations, since $I_+^{1-s}[u](0_+)\delta(x)$ and $-I_-^{1-s}[u](1_-)\delta(x-1)$ respectively are the atomic parts of $D_+^s[u]$ and $D_-^s[u]$.

The right equalities in (3.31)–(3.32), namely $D_+^s\left[I_+^s[u]\right]=u$ and $D_-^s\left[I_-^s[u]\right]=u$, are straightforward consequencies of definitions when for $0<s<1$:

$$D_\pm^s[I_\pm^s[u]] = DI_\pm^{1-s}[I_\pm^s[u]] = DI^1[u] = u.$$

Thanks to (3.9), the case when s is a natural number is classical and well known.

Thus we are left to examine the case of non integer values of s for $s>1$.

To this aim we choose $u \in BV_*^s$ according to definition (3.26) and we have to show that such u fulfills definition (3.21) too; the converse is straightforward.

For $k=[s]>1$, by semigroup property of fractional integrals we have

$$I_+^s[u] = I_+^{\mathrm{Frac}(s)}\left[I_+^{[s]}[u]\right] = I_+^{\mathrm{Frac}(s)}\left[I_+^k[u]\right],$$

$$I_+^{[s]+1-s}[u] = I_+^{\lceil s\rceil - s}[u] = I_+^{1-\mathrm{Frac}(s)}[u].$$

By the definition of fractional derivative, if $1\leq k=[s]$ we obtain

$$D_+^s[u] = D^{k+1}\left[I_+^{1-\mathrm{Frac}(s)}[u]\right] = D^k D\left[I_+^{1-\mathrm{Frac}(s)}[u]\right] = D^k D_+^{\mathrm{Frac}(s)}[u]$$

say

$$D_+^s[u] = D^k D_+^{\mathrm{Frac}(s)}[u] \qquad \text{for } 1\leq k=[s]. \tag{3.35}$$

Moreover, for $0<\sigma<1$, we obtain

$$D\, I_+^{1-\sigma}[u] \;=\; D_+^{\sigma}[u] \;= D(u * \frac{H(x)}{\Gamma(\sigma)|x|^{\sigma}}) \;=\; Du * \frac{H(x)}{\Gamma(\sigma)|x|^{\sigma}} \;=\; I_+^{1-\sigma}[Du]\,,$$

hence D commutes with $I_+^{1-\sigma}$ when both u and Du are Borel measures in $\mathcal{M}_+$. Thus, by iteration, whenever both u and $D^k u$ are Borel measures in $\mathcal{M}_+$ with support in $[0,+\infty)$ we have

$$D^k\, D_+^{\mathrm{Frac}(s)}[u] \;=\; D_+^{\mathrm{Frac}(s)}[D^k u]\,. \tag{3.36}$$

Identities (3.35), (3.36) together entail, for $u \in L^1_{loc}$ with $\operatorname{spt} u \subset [0,+\infty)$ and $D^k u$ Borel measure,

$$D_+^{s}[u] \;=\; D_+^{\mathrm{Frac}(s)}[D^k u] \qquad \text{for } 1 \le k = [s]\,. \tag{3.37}$$

By (3.37) and the definition of BV^s we obtain

$$D_+^{\mathrm{Frac}(s)}\left[D_x^k[u]\right] = D_+^{s}[u] = \mu \in \mathcal{M}_+ \qquad \forall u \in BV_+^s\,, \tag{3.38}$$

where μ is a Radon measure with support on $[0,+\infty)$, hence it is a Borel measure on $\mathbb{R}$. In particular (3.33) is proved.

The relationships (3.36)–(3.37) for $u \in BV_+^s$ together with the semigroup property of fractional integrals and $u = I_+^{\sigma}\left[D_+^{\sigma}[u]\right]$ (namely the left equality in (3.31) already proved only for $0<\sigma<1$) entail, for every $s>0$,

$$I_+^{s}\left[D_+^{s}[u]\right] = I_+^{[s]}\left[I^{\mathrm{Frac}(s)}\left[D^{\mathrm{Frac}(s)}\left[D_+^{[s]}[u]\right]\right]\right] = I_+^{[s]}\left[D_+^{[s]}[u]\right] = u\,,$$

that is the left equality in (3.31) for every $s>0$.

The right equality in (3.31), namely $D_+^s\left[I_+^s[u]\right] = u$ for every $s>0$, follows by the fact that it is already proved for $0<s<1$ taking into account (3.35):

$$D_+^s[I_+^s[u]] = D^{[s]}[D_+^{\mathrm{Frac}(s)}[I_+^{\mathrm{Frac}(s)}[I_+^{[s]}[u]]]] = D^{[s]} I_+^{[s]}[u] = u.$$

By applying fractional integral $I_+^{\mathrm{Frac}(s)}$ (as defined by (3.7)) to both sides of (3.38), taking into account that convolution of a measure with an L^1 function belongs to L^1 (Lemma 3.4) and replacing u with $D^k u$ and s with $\mathrm{Frac}(s)$ in (3.31), we get:

$$D_x^k u(x) = I_+^{\mathrm{Frac}(s)}\left[D_+^{\mathrm{Frac}(s)}\left[D_x^k u\right]\right] = I_+^{\mathrm{Frac}(s)}\left[D^s u\right] = I_+^{\mathrm{Frac}(s)}[\mu] \in L^1(0,1)\,. \tag{3.39}$$

Still exploiting Lemma 3.4, we get $\operatorname{spt} D_x^k u \subset [0,+\infty)$, $D_x^k[u] \in L^1(0,1)$, hence (3.38) entails $D_x^k[u] \in BV_+^{\mathrm{Frac}(s)}$.

By the same procedure one can show (3.32), (3.34) and $D_x^k[u] \in BV_-^{\mathrm{Frac}(s)}$. Thus the equivalence of definitions (3.21) and (3.26) is proved.

We are left to show estimate (3.30) and equivalence of the norms (3.27) and (3.28): actually we show a finer result, namely the equivalence of these norms:

$$\|u\|_{BV_+^s} := \|u\|_{L^1(0,1)} + \|\, D_+^s[u]\,\|_{T_+} \quad \text{and} \quad \|u\|_{BV_-^s} := \|u\|_{L^1(0,1)} + \|\, D_-^s[u]\,\|_{T_-}$$

respectively with

$$|\!|\!|\, u \,|\!|\!|_{BV_+^s} := \|u\|_{BV_+^s} + \|\, I_+^{\lceil s\rceil - s}[u]\,\|_{W^{[s],1}} \quad \text{and} \quad |\!|\!|\, u \,|\!|\!|_{BV_-^s} := \|u\|_{BV_-^s} + \|\, I_-^{\lceil s\rceil - s}[u]\,\|_{W^{[s],1}}\,.$$

By (3.39), convolution formulation of the fractional integral $I_+^{\mathrm{Frac}(s)}[\mu]$ and (3.11)

$$\begin{aligned}\|D_x^k u\|_{L^1(0,1)} &\leq \frac{\|x^{\mathrm{Frac}(s)-1}\|_{L^1(0,2)}}{\Gamma\big(\mathrm{Frac}(s)\big)}\ \|\mu\|_{T_+} = \frac{2^{\mathrm{Frac}(s)}}{\Gamma\big(1+\mathrm{Frac}(s)\big)}\,\|D_+^s[u]\|_{T_+} \leq \\ &\leq \frac{2^{\mathrm{Frac}(s)}}{\Gamma\big(1+\mathrm{Frac}(s)\big)}\,\|\mu\|_{T_+}\,.\end{aligned}$$

Standard interpolation inequalities in the Sobolev space $W^{k,1}(0,1)$ provide the estimate in L^1 for $D^h u$, $h{=}0,1,...k$, hence (3.30) is proved.

Eventually, about equivalence of norms (3.27) and (3.28) we observe that, due to (3.30), for $u \in BV_*^s$ and $0 \leq h \leq [s]$

$$D^h I_+^{1-\mathrm{Frac}(s)}[u] = D^h \,\frac{H(x)\,x^{-s}}{\Gamma(1-\mathrm{Frac}(s))}\, * \,u = \frac{H(x)\,x^{-s}}{\Gamma(1-\mathrm{Frac}(s))}\, * \,D^h u \in L^1(0,1)$$

and exploiting (3.11)

$$\|\,D^h I_+^{1-\mathrm{Frac}(s)}[u]\|_{L^1(0,1)} \ \leq\ \frac{2^{\mathrm{Frac}(s)}}{\Gamma\big(1+\mathrm{Frac}(s)\big)}\,\|D^h u\|_{T_+} \ \leq\ \frac{2^{\mathrm{Frac}(s)}}{\Gamma\big(1+\mathrm{Frac}(s)\big)}\,\|\mu\|_{T_+}.$$

The same analysis can be performed to achieve estimates on $D^h I_-^{1-\mathrm{Frac}(s)}[u]$, though obtaining the same estimate with $\|\mu\|_{T_+}$ replaced by $\|D_-^s u\|_{T_-}$.

Thus it is shown the equivalence of natural and essential norms and the fact that BV_*^s with the essential norm (3.27) defines a Banach space. □

Remark 3.10. By inspection of the previous proof, we can replace measures (wherever they appear) with L^1 functions, thus proving claims for the fractional Sobolev spaces defined in Definition 3.6, which are analogous to the claims in the theorem above about BV_*^s functions: precisely, for every $s > 0$ we find

$$W_+^{s,1}(0,1) \ = \ \{u \in L^1(0,1) \ : \ D_+^s[u] = D_x^{\lceil s\rceil}\, I_+^{\lceil s\rceil - s}[u] \in L^1(0,1)\ \}\,, \tag{3.40}$$

$$W_-^{s,1}(0,1) \ = \ \{u \in L^1(0,1) \ : \ D_-^s[u] = D_x^{\lceil s\rceil}\, I_-^{\lceil s\rceil - s}[u] \in L^1(0,1)\ \}\,. \tag{3.41}$$

Moreover, both Banach spaces $W_+^{s,1}(0,1)$ and $W_-^{s,1}(0,1)$ can be endowed with these essential norms (that are equivalent to the natural ones):

$$\|u\|_{W_+^{s,1}(0,1)} \ = \ \|u\|_{L^1(0,1)} + \|\,D_x^{\lceil s\rceil}\, I_+^{\lceil s\rceil - s}[u]\,\|_{L^1(0,1)}\,, \tag{3.42}$$

$$\|u\|_{W_-^{s,1}(0,1)} \ = \ \|u\|_{L^1(0,1)} + \|\,D_x^{\lceil s\rceil}\, I_-^{\lceil s\rceil - s}[u]\,\|_{L^1(0,1)}\,. \tag{3.43}$$

Remark 3.11. Actually by the previous proof we have also shown that the even unilateral definitions (3.22) and (3.23) can be equivalently replaced respectively by

$$BV_+^s := \big\{u \in L^1(0,1)\colon\ D_x^{\lceil s\rceil} I_+^{1-\mathrm{Frac}(s)}[u] \in \mathcal{M}_+\,\big\}\,, \tag{3.44}$$

$$BV_-^s := \{u \in L^1(0,1):\ D_x^{\lceil s\rceil} I_-^{1-\mathrm{Frac}(s)}[u] \in \mathcal{M}_-\,\}\,. \tag{3.45}$$

Remark 3.12. For $1 < s < 2$ we find the functions with fractional Hessian of bounded variation
$$BH_*^\tau \;=\; BV_*^s\,, \qquad \text{where } \tau = s-1 \tag{3.46}$$
say the intermediate spaces between the space BV of functions of bounded variation and the space $BH := BV_*^2$ of functions with bounded Hessian (see [12, 25, 44]).

Remark 3.13. Analogously to the discussion made in Remark 3.8, even for $s > 1$ we could replace $\mathcal{M}_+$ with $\mathcal{M}$ in (3.22) and $\mathcal{M}_-$ with $\mathcal{M}$ in (3.23), still obtaining equivalent definitions of the spaces BV_+^s and BV_-^s: this depends strongly on the condition $I_\pm^{\lceil s\rceil}[u] \in W^{[s],1}(0,1)$ (where $s > 1$ entails $[s] \geq 1$), which is implied by $u \in BV_*^s$ as defined in (3.26), as it is proved in Theorem 3.9.

Remark 3.14. Notwithstanding Remarks 3.8, 3.13, stating that $\|\cdot\|_{L^1} + \|D_+^s[\,\cdot\,]\|_{T_+}$ is a norm on BV_+^s and that $\|\cdot\|_{L^1} + \|D_-^s[\,\cdot\,]\|_{T_-}$ is a norm on BV_-^s, we can replace neither T_+ nor T_- with $\mathcal{M}$ therein.

For instance, take $s \in (1,2)$, hence $\lceil s\rceil = 2$, and set $u(x) = (x^+)^{\mathrm{Frac}(s)}/\,\Gamma(\mathrm{Frac}(s)+1)$. Then, by exploiting $I_+^{1-\sigma}[x^\sigma] = \Gamma(\sigma+1)\,x^+$ for $x \in (-\infty,1)$ and $\sigma \in (0,1)$, with the choice $\sigma = \mathrm{Frac}(s)$, we get
$$I_+^{\lceil s\rceil - s}[u] \;=\; I_+^{1-\mathrm{Frac}(s)}[u] \;=\; x^+ \qquad x \in (-\infty,1)\,,$$
$$D_+^s[u] \;=\; D_x^{\lceil s\rceil}\left[I_+^{\lceil s\rceil - s}[u]\right] \;=\; D_x^2\,x^+ \;=\; \delta \qquad \mathcal{D}'(-\infty,1)\,,$$
whereas
$$D_+^s[u] \;=\; 0 \qquad \mathcal{D}'(0,1)\,.$$
Analogous counterexample can be shown in BV_-^s: say
$$u(x) = \left((1-x)^+\right)^{\mathrm{Frac}(s)}/\,\Gamma\big(\mathrm{Frac}(s)+1\big) \text{ for } x \in (0,+\infty),\ u(x) = 0 \text{ for } x < 1.$$

Remark 3.15. We note that if s is a positive noninteger real number, then neither $(x^+)^{s-1}$ nor $((1-x)^+)^{s-1}$ belong to BV_*^s (see [33], Remark 3.2 for $0 < s < 1$, that easily extends to the the case $s > 1$).

Moreover the space spanned by $\left\{(x^+)^{s-1}\right\}$ is the kernel of the fractional derivative D_+^s under restriction to $\mathcal{D}'(0,1)$ (see [38], Lemma 8), though the fractional derivative $D_+^s\,[(x^+)^{s-1}]$ has a singular term at $x = 0$ as precisely shown in the next example, where we set $x^+ = \max\{x,0\}$ and $(x^+)^0 := H(x)$ for $x \in \mathbb{R}$.

Example 3.16. $(x^+)^{s-1} \in BV_+^s \backslash BV_-^s$; $((1-x)^+)^{s-1} \in BV_-^s \backslash BV_+^s$.

Indeed we consider the trivial extension of functions for negative arguments and use again the notations $(x^+)^0 := H(x)$ for $x \in \mathbb{R}$, with the aim of taking into account possible atomic parts of $D^{[s]}$ and D_+^s at $x = 0$.

Taking into account the semigroup property of fractional integrals and identity
$$I_+^{1-\alpha}[(x^+)^\beta] = \frac{\Gamma(1+\beta)}{\Gamma(2+\beta-\alpha)}(x^+)^{1+\beta-\alpha} \quad \text{for } \alpha \in (0,1) \text{ and } \beta > -1,$$
we choose $\alpha = \mathrm{Frac}(s)$ and $\beta = \mathrm{Frac}(s) - 1$ therein, thus
$$D^{[s]}(x^+)^{s-1} \;=\; \prod_{j=0}^{[s]-1}\big(j + \mathrm{Frac}(s)\big)(x^+)^{\mathrm{Frac}(s)-1},$$
$$I_+^{1-\mathrm{Frac}(s)}\left[D^{[s]}(x^+)^{s-1}\right] \;=\; \prod_{j=0}^{[s]-1}\big(j + \mathrm{Frac}(s)\big)\,\Gamma\big(\mathrm{Frac}(s)\big)\,H(x) \tag{3.47}$$

and, by (3.37), (3.38),

$$\begin{aligned}
D_+^s\left[(x^+)^{s-1}\right] = D_+^{\mathrm{Frac}(s)}\left[D^{[s]}\,(x^+)^{s-1}\right] =& \\
&= \Big(\prod_{j=0}^{[s]-1}\big(j+\mathrm{Frac}(s)\big)\Big)\, D_+^{\mathrm{Frac}(s)}\left[(x^+)^{\mathrm{Frac}(s)-1}\right] \\
&= \Big(\prod_{j=0}^{[s]-1}\big(j+\mathrm{Frac}(s)\big)\Big)\, D\, I_+^{1-\mathrm{Frac}(s)}\left[(x^+)^{\mathrm{Frac}(s)-1}\right] = \\
&= \Big(\prod_{j=0}^{[s]-1}\big(j+\mathrm{Frac}(s)\big)\Big)\,\Gamma\big(\mathrm{Frac}(s)\big) DH = \\
&= \Big(\prod_{j=0}^{[s]-1}\big(j+\mathrm{Frac}(s)\big)\Big)\,\Gamma\big(\mathrm{Frac}(s)\big)\,\delta,
\end{aligned}$$

that is $(x^+)^{s-1}$ belongs to BV_+^s.
By the previous remark $(x^+)^{s-1}$ does not belong to BV_-^s.
The properties of $((1-x)^+)^{s-1}$ are achieved in the same way.

Theorem 3.17. (Compactness in $BV_*^s(0,1)$) (see [38]) *Assume that* $s>0$ *and*

$$\|u_n\|_{BV_*^s(0,1)} \le C\,. \tag{3.48}$$

Then, there exist $u \in BV_*^s$ *and a subsequence such that, without relabeling,*

$$\begin{cases}
\text{(i)} \quad I_+^{[s]}[u_n] \rightharpoonup I_+^{[s]}[u] & \textit{weakly in } L^q(0,1)) \quad \forall q \in [1, 1/(1-\mathrm{Frac}(s)))\,, \\
\text{(ii)} \quad I_+^{\lceil s\rceil - s}[u_n] \to I_+^{\lceil s\rceil - s}[u] & \textit{strongly in } L^p(0,1)\,,\ \forall p < +\infty\,, \\
\text{(iii)} \quad I_-^{\lceil s\rceil - s}[u_n] \to I_-^{\lceil s\rceil - s}[u] & \textit{strongly in } L^p(0,1)\,,\ \forall p < +\infty\,.
\end{cases}$$

$$I_+^{\lceil s\rceil - s}[u_n] \to I_+^{\lceil s\rceil - s}[u]\,, \quad I_-^{\lceil s\rceil - s}[u_n] \to I_-^{\lceil s\rceil - s}[u] \quad \textit{weakly in } BV(0,1). \tag{3.49}$$

Proof. By Theorem 3.9 we have uniform boundedness of $D_*^{[s]}[u_n]$ in $BV_*^{\mathrm{Frac}(s)}$.
Thus the claims follow by Theorem 11 of [38], (3.13), (3.14) and the completeness of $BV_*^{\mathrm{Frac}(s)}$ □

Next we extend Proposition 3 in [38] about Abel integral equation in the distributional setting to the case of a left-side fractional integral I_+^s in the space BV_+^s with $s > 1$.

Obviously an analogous statement holds true for the case of a right-side fractional integral I_-^s in the space BV_-^s with $s > 1$.

Proposition 3.18. *Set* $s>0$, $f \in BV_+^s$ *and is a Laplace transformable distribution. Then, the Abel integral equation in the distributional framework*

$$I_+^s[u] \,=\, f \qquad \textit{in } \mathcal{D}'(\mathbb{R}), \tag{3.50}$$

admits a unique solution u *among Laplace transformable distributions, which in addition belongs to* $\mathcal{M}_+$*, say* u *is a real-valued measure on* $\mathbb{R}$ *with support contained in* $[0,+\infty)$ *and* $\|u\|_{T_+} < +\infty$*; this solution is given explicitly by*

$$u(x) \,=\, D_+^s[f](x) \quad \textit{in } \mathcal{D}'(\mathbb{R})\,. \tag{3.51}$$

More precisely, according to Definitions 3.3 *and* 3.5*, the left-hand side of* (3.50)

$$I_+^s[u] \;=\; u \,*\, \frac{1}{\Gamma(s)}\,\frac{H(x)}{|x|^{1-s}}$$

represents the distributional convolution whose evaluation is identically 0 *on* $(-\infty,0)$ *and possibly non-zero on* $[1,+\infty)$*, whereas the right-hand side of* (3.51)

$$D_+^s[f](x) = (D_x)^{\lceil s\rceil}\left[I_+^{1-\mathrm{Frac}(s)}[f]\right](x)$$

represents the left Riemann-Liouville fractional derivative of order s *defined on the whole real line where* D_x *denotes the distributional derivative on* $\mathbb{R}$*.*

Proof. Whenever necessary, we consider the trivial extension (namely, 0 valued) of u on $(-\infty,0)$, without relabeling the function name.

f is a Laplace-transformable distribution. We show that actually a Laplace transformable solution u exists. Indeed, if it exists, then its Laplace transform has to fulfill the transformed equation; thus by labelling $F(z)=\mathcal{L}\{f\}(z)$ and $U(z)=\mathcal{L}\{u\}(z)$ their Laplace transform evaluated at the complex variable z, we find

$$I_+^s[u](x) = \frac{1}{\Gamma(s)}\int_0^x \frac{u(t)\,dt}{(x-t)^{1-\alpha}} = \frac{1}{\Gamma(s)}\,\left(H(x)\,x^{s-1}\right)\,*\,u\,,$$

$$I_+^s[u](x) = f(x)\,,$$

$$\frac{1}{\Gamma(s)}\,\frac{\Gamma(s)}{z^s}\,U(z) \;=\; F(z)\,,$$

$$U(z) \;=\; z^s F(z) \;=\; z^{\lceil s\rceil}\,\frac{F(z)}{z^{1-\mathrm{Frac}(s)}}\,,$$

where z^α denotes the analytic branch of the multivalued function which is real valued on the intersection of $\mathbb{R}$ with the Laplace half-plane of convergence for f.

Reminding that $$\frac{F(z)}{z^{1-\mathrm{Frac}(s)}} = \mathcal{L}\left\{I_+^{1-\mathrm{Frac}(s)}[f]\right\}$$

$$\mathcal{L}\left\{D_x w(x)\right\} = z\,\mathcal{L}\left\{w\right\} \qquad \mathcal{L}\left\{D_x^{\lceil s\rceil} w(x)\right\} = z^{\lceil s\rceil}\,\mathcal{L}\left\{w\right\}$$

where w denotes the generic $\mathcal{L}$-transformable distribution and D_x denote the distributional derivative on $\mathbb{R}$, we evaluate $u=\mathcal{L}^{-1}\mathrm{U}$ and find

$$u \;=\; D_x^{\lceil s\rceil}\, I_+^{1-\mathrm{Frac}(s)}[f] \;=\; D_x^{\lceil s\rceil}\, I_+^{\lceil s\rceil-s}[f] \;=\; D_+^s[f]\,.$$

Such u (unique due to injectivity of $\mathcal{L}$ transform) is a Laplace transformable distribution, since under our assumptions $D_+^s[f]$ is a Laplace transformable measure. $\square$

Remark 3.19. Proposition 3.18 extends to the case of general $s>0$ the results in Proposition 2 and 3 of [38]: for comparison, we emphasize that there the fractional order was denoted by α and subject to the constraint $\alpha\in(0,1)$, moreover in [38] D_+^α denoted the distributional derivative of $I_+^{1-\alpha}$ in the open set $(0,1)$ while here

$$D_+^s[u](x) = (D_x)^{\lceil s\rceil}\left[I_+^{\lceil s\rceil-s}[u]\right](x) = (D_x)^{\lceil s\rceil}\left[I_+^{1-\mathrm{Frac}(s)}[u]\right](x)$$

where D_x denotes the distributional derivative on $\mathbb{R}$ (see Definition 3.5). Therefore, by (3.37), if $\mathcal{D}_x$ denotes the distributional derivative in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$, then formula (3.51) (which provides the solution to the Abel integral equation: find u for given $f \in BV_+^s$, $I_+^s[u] = f$ in $\mathcal{D}'(\mathbb{R})$) can be rewritten also as

$$u(x) \;=\; \mathcal{D}_+^s[f](x) + I_+^{\lceil s\rceil - s}[D_x^{[s]} f](0_+)\,\delta(x) \text{ in } \mathcal{D}'(\mathbb{R})\,.$$

We end this section by listing some embedding related to the space of bounded Hessian functions $BH(0,1)$ and the space of special functions of bounded variation $SBV(0,1)$ whose distributional derivative has not Cantor part ([1, 4]).

By (74),(78),(91),(93) of [38] we remind the space embedding

$$SBV(0,1) \subset \bigcap_{s\in(0,1)} W^{s,1}(0,1)\,,$$
$$BH(0,1) \;\subset\; W^{s,1}(0,1) \;\subset\; BV_*^s(0,1) \qquad \text{for } 1<s<2\,. \tag{3.52}$$

4. SFV : Symmetrised Fractional Variation with L^1 fidelity term (1D)

In [33] we introduced the one dimensional model of symmetrized fractional variation for signal denoising:

for fixed $\lambda>0$, $s>0$ and g in $L^1(0,1)$, we study the problem

$$\text{minimize} \quad \{\, \mathcal{G}(v) : \, v \text{ belonging to } BV_*^s(0,1)\} \tag{$\mathcal{P}_{\mathcal{G}}$}$$

where
$$\mathcal{G}(v) \;:=\; \|D_+^s[v]\|_{T_+} + \|D_-^s[v]\|_{T_-} + \lambda \int_0^1 |v(x) - g(x)|\, dx\,. \tag{4.1}$$

The parameter λ denotes the so-called regularization coefficient of the L^1 fitting term, s is the fractional order and g is the measured input. Furthermore $D_+^s[u]$ and $D_-^s[u]$ denote the left and right distributional RL fractional derivatives of u, thus $D_+^s[u] = 0$ on the whole half-line $(-\infty, 0)$ and $D_-^s[u] = 0$ on the whole half-line $(1, +\infty)$; precisely we set

$$\|D_+^s[v]\|_{T_+} \;:=\; \|D_+^s[v]\|_{\mathcal{M}(0,1)} + |I_+^{1-s}[v](0_+)| \;=\; \|D_+^s[v]\|_{\mathcal{M}(-\infty,1)}\,, \tag{4.2}$$

$$\|D_-^s[v]\|_{T_-} \;:=\; \|D_-^s[v]\|_{\mathcal{M}(0,1)} + |I_-^{1-s}[v](1_-)| \;=\; \|D_-^s[v]\|_{\mathcal{M}(0,+\infty)}\,, \tag{4.3}$$

where the symbol $\|\cdot\|_{\mathcal{M}(E)}$ represents total variation.

A preliminary analysis of functionals of the kind (4.1) was made in [33] under the assumption $0 < s < 1$. Here we can deal with the general case $s > 0$ and achieve similar conclusions, as stated by the next theorem.

Theorem 4.1. *If we assume that g belongs $L^1(0,1)$ and $s>0$, then the functional $\mathcal{G}$ has a nonempty domain and is convex but not strictly convex; moreover the problem $(\mathcal{P}_{\mathcal{G}})$ achieves a finite minimum.*

Proof. Convexity of $\mathcal{G}$ is straightforward, because it is the sum of convex terms. We prove the existence of minimizers by the direct methods of calculus of variations. The functional $\mathcal{G}$ is bounded from below since $\mathcal{G}(v) \geq 0$ for every v.

Non-emptiness of domain follows by $0 \leq \inf \mathcal{G} \leq \mathcal{G}(0) = \lambda \|g\|_{L^1(0,1)} < +\infty$.

If $s>0$ is an integer number, then the existence of minimisers is a classical result.

Let v_n be a minimizing sequence. Then, by Theorem 3.17, $I_+^{[s]}[v_n]$ and $I_-^{[s]}[v_n]$ are bounded in $L^q(0,1)$ for $1\leq q<1/(1-s)$, hence they are bounded in $L^1(0,1)$.

Thus, taking into account that also v_n is bounded in $L^1(0,1)$, we obtain (up to subsequences and without relabeling): if $0<s<1$ then v_n converges weakly to some $\widetilde{v}$ in $L^q(0,1)$ for $1 \leq q < 1/(1-s)$; if $1 < s$ then, thanks to Theorem 3.9, v_n is a bounded seqence in $W^{[s],1}(0,1)$, hence v_n converges weakly to some $\widetilde{v}$ in $L^p(0,1)$ for $1\leq p<+\infty$.

The sequences of derivatives $D_+^s v_n$ and $D_-^s v_n$ are bounded respectively in the spaces $\mathcal{M}_+$ and $\mathcal{M}_-$. Therefore by (3.49) and Definition 3.5 there exists $\widetilde{v}$ in $BV^s(0,1)$, such that $D_+^s v_n \to D_+^s\widetilde{v}$, $D_-^s v_n \to D_-^s\widetilde{v}$ in the sense of weak* convergence of measures.

Eventually, the existence of a minimizer follows by the lower semicontinuity of total variation with respect to the weak* convergence, as stated by (3.49) since $D_+^s[v_n]$ and $D_-^s[v_n]$ are bounded sequences of measures and

$$\|\widetilde{v}-g\|_{L^1(0,1)} = \|\widetilde{v}-g\|_{\mathcal{M}(0,1)} \leq \liminf \|v_n-g\|_{\mathcal{M}(0,1)} = \liminf \|v_n-g\|_{L^1(0,1)} \,.$$

The fact that $\mathcal{G}$ is not strictly convex can be easily checked on constant functions: set $g\equiv 0$, then

$$\mathcal{G}(0) \,=\, 0, \qquad \mathcal{G}(c) \,=\, c\left(\lambda+\frac{2}{\Gamma(2-s)}\right), \qquad \forall\, c\in\mathbb{R}_+. \qquad \square$$

Moreover the minimizers fulfill the contrast equivariance as stated in the subsequent corollary.

Corollary 4.2. *For any $s>0$, $g\in L^1(0,1)$ and $\lambda>0$, the set $S(s,g,\lambda)$ of solutions to problem $\mathcal{P}_{\mathcal{G}}$ is nonempty, convex and closed in the strong $L^1(0,1)$ norm, and, for any $\beta>0$,*

$$S(s,\beta g,\lambda) = \beta\; S(s,g,\lambda) \qquad \text{[contrast equivariance]}\;. \tag{4.4}$$

Proof. The claims are straightforward consequences of the Theorem 4.1 and the definition (4.1) which makes the functional $\mathcal{G}$ convex, lower semicontinuous in the strong $L^1(0,1)$ topology and 1-homogeneous. In particular 1-homogeneity of $\mathcal{G}$ entails $\mathcal{G}_{\beta g}(\beta v) = \beta\,\mathcal{G}_g(v)$, where $\mathcal{G}_f$ denotes the functional $\mathcal{G}$ with g replaced by βg; hence the identity (4.4) follows. $\square$

Remark 4.3. When v belongs to $L^\infty(0,1)$ and $s\in(0,1)$, the evaluation of functional $\mathcal{G}$ defined in (4.1) simplifies as follows:

$$\mathcal{G}(v) \;=\; \|D_+^s[v]\|_{\mathcal{M}(0,1)} \;+\; \|D_-^s[v]\|_{\mathcal{M}(0,1)} + \lambda\int_0^1 |v(x)-g(x)|\,dx\,. \tag{4.5}$$

Indeed, in such case we have $I_+^{1-s}(0_+)=0$ and $I_+^{1-s}(1_-)=0$.

5. SFV : Symmetrised Fractional Variation with L^1 fidelity term (2D)

We set the image domain as $\Omega=(0,1)\times(0,1)$.

The symbols $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ represent the distributional partial derivatives in x and y.

5.1. Fractional integral and fractional derivative in 2D

According to [39] we introduce both partial fractional integrals and derivatives. Our framework is inspired by Beppo Levi and Lamberto Cesari approaches, respectively to Sobolev and Bounded Variation spaces (see the hystorical comments in [2]).

Definition 5.1. (Fractional partial integrals) For $\Omega = (0,1)\times(0,1)$, $u \in L^1(\Omega)$ and every $0<s<1$, we define the partial left-side and right-side Riemann-Liouville fractional integrals, by setting respectively

$$
\begin{aligned}
I^s_{x,+}[u](x,y) &:= \frac{1}{\Gamma(s)}\int_0^x \frac{u(t,y)}{(x-t)^{1-s}}dt \qquad 0\le x\le 1,\\
I^s_{x,-}[u](x,y) &:= \frac{1}{\Gamma(s)}\int_x^1 \frac{u(t,y)}{(t-x)^{1-s}}dt \qquad 0\le x\le 1.\\
I^s_{y,+}[u](x,y) &:= \frac{1}{\Gamma(s)}\int_0^y \frac{u(x,t)}{(y-t)^{1-s}}dt \qquad 0\le y\le 1,\\
I^s_{y,-}[u](x,y) &:= \frac{1}{\Gamma(s)}\int_y^1 \frac{u(x,t)}{(t-y)^{1-s}}dt \qquad 0\le y\le 1.
\end{aligned}
$$

Definition 5.2. (Distributional fractional partial derivative) For any u in $L^1(\Omega)$ and $0<s<1$, the left and right fractional partial derivatives of u at the point $(x,y)\in\Omega$ are set as follows

$$\left(\frac{\partial}{\partial x}\right)^s_+[u](x,y) := \frac{\partial}{\partial x}\left(I^{1-s}_{x,+}[u]\right)(x,y) \tag{5.1}$$

$$\left(\frac{\partial}{\partial x}\right)^s_-[u](x,y) := -\frac{\partial}{\partial x}\left(I^{1-s}_{x,-}[u]\right)(x,y) \tag{5.2}$$

$$\left(\frac{\partial}{\partial y}\right)^s_+[u](x,y) := \frac{\partial}{\partial y}\left(I^{1-s}_{y,+}[u]\right)(x,y) \tag{5.3}$$

$$\left(\frac{\partial}{\partial y}\right)^s_-[u](x,y) := -\frac{\partial}{\partial y}\left(I^{1-s}_{y,-}[u]\right)(x,y) \tag{5.4}$$

5.2. 2-dimensional SFV-L^1 model

We introduce the notation $|\mu|_{T(A)}$ for the total variation of μ on a Borel set $A\subset\mathbb{R}^2$: $\|\cdot\|_{T_{x,+}}=\|\cdot\|_{\mathcal{M}([0,1)_x)}$, $\|\cdot\|_{T_{x,-}}=\|\cdot\|_{\mathcal{M}((0,1]_x)}$, $\|\cdot\|_{T_{y,+}}=\|\cdot\|_{\mathcal{M}([0,1)_y)}$, $\|\cdot\|_{T_{y,-}}=\|\cdot\|_{\mathcal{M}((0,1]_y)}$, and $\Omega_{x,+}=[0,1)\times(0,1)$, $\Omega_{x,-}=(0,1]\times(0,1)$, $\Omega_{y,+}=(0,1)\times[0,1)$, $\Omega_{y,-}=(0,1)\times(0,1]$. The minimization problem $\mathcal{P}_{\mathbf{G}}$ reads as follows. Find u minimizing

$$
\begin{aligned}
\mathfrak{G}(u) &:= \lambda\|u-g\|_{L^1(\Omega)} + \\
&+\left|\left(\frac{\partial}{\partial x}\right)^s_+\right|_{T(\Omega_{x,+})} + \left|\left(\frac{\partial}{\partial y}\right)^s_+\right|_{T(\Omega_{y,+})} + \left|\left(\frac{\partial}{\partial x}\right)^s_-\right|_{T(\Omega_{x,-})} + \left|\left(\frac{\partial}{\partial y}\right)^s_-\right|_{T(\Omega_{y,-})}
\end{aligned} \tag{5.5}
$$

on $u\in L^1(\Omega)$ where $\left(\frac{\partial}{\partial x}\right)^s_+[u](\cdot,y)$, and $\left(\frac{\partial}{\partial x}\right)^s_-[u](\cdot,y)$ are measures of bounded variation for a.e. y and $\left(\frac{\partial}{\partial y}\right)^s_+[u](x,\cdot)$ and $\left(\frac{\partial}{\partial x}\right)^s_-[u](x,\cdot)$ are measures of bounded variation for a.e. x.

The functional $\mathfrak{G}$ equivalently reads as follows

$$\mathfrak{G}(u) := \lambda\int_\Omega \big|u(x,y)-g(x,y)\big|dx\,dy + \int_0^1 \Big\|\Big(\frac{\partial}{\partial x}\Big)_+^s[u]\Big\|_{T_{x,+}} dy + \int_0^1 \Big\|\Big(\frac{\partial}{\partial y}\Big)_+^s[u]\Big\|_{T_{y,+}} dx$$
$$+ \int_0^1 \Big\|\Big(\frac{\partial}{\partial x}\Big)_-^s[u]\Big\|_{T_{x,-}} dy + \int_0^1 \Big\|\Big(\frac{\partial}{\partial y}\Big)_-^s[u]\Big\|_{T_{y,-}} dx.$$

Lemma 5.3. *Assume that* $\mathfrak{G}(v_k) \le C < +\infty$.
Then there exists $v \in L^1(\Omega)$ *such that* $\big(\frac{\partial}{\partial x}\big)_\pm^s[v]$ *and* $\big(\frac{\partial}{\partial y}\big)_\pm^s[v]$ *are measures of bounded variation respectively on* $\Omega_{x,\pm}$ *and* $\Omega_{y,\pm}$, *and there is a subsequence* v_{h_k} *such that*

$$v_{h_k} \rightharpoonup v \qquad \text{weakly in } L^q(\Omega) \quad 1 \le q < 1/(1-s)\,, \tag{5.6}$$
$$\Big(\frac{\partial}{\partial x}\Big)_+^s[v_{h_k}] \rightharpoonup \Big(\frac{\partial}{\partial x}\Big)_+^s[v] \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,, \tag{5.7}$$
$$\Big(\frac{\partial}{\partial x}\Big)_-^s[v_{h_k}] \rightharpoonup \Big(\frac{\partial}{\partial x}\Big)_-^s[v] \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,, \tag{5.8}$$
$$\Big(\frac{\partial}{\partial y}\Big)_+^s[v_{h_k}] \rightharpoonup \Big(\frac{\partial}{\partial y}\Big)_+^s[v] \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,, \tag{5.9}$$
$$\Big(\frac{\partial}{\partial y}\Big)_-^s[v_{h_k}] \rightharpoonup \Big(\frac{\partial}{\partial y}\Big)_-^s[v] \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,. \tag{5.10}$$

Proof. Due to standard compactness properties, by subsequent extractions, we find $v_{h_k},\ v,\ \mu_{x,+},\ \mu_{x,-},\ \mu_{y,+},\ \mu_{y,-}$ s.t. (5.6) holds true together with

$$\Big(\frac{\partial}{\partial x}\Big)_+^s[v_{h_k}] \rightharpoonup \mu_{x,+} \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,,$$
$$\Big(\frac{\partial}{\partial x}\Big)_-^s[v_{h_k}] \rightharpoonup \mu_{x,-} \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,,$$
$$\Big(\frac{\partial}{\partial y}\Big)_+^s[v_{h_k}] \rightharpoonup \mu_{y,+} \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,,$$
$$\Big(\frac{\partial}{\partial y}\Big)_-^s[v_{h_k}] \rightharpoonup \mu_{y,-} \qquad \text{weak}^* \text{ in } \mathcal{M}(\Omega)\,.$$

The convergence $v_{h_k} \rightharpoonup v$ in $L^q(\Omega)$ $\forall q \in [1, 1/(1-s))$ entails $v_{h_k} \to v$ in $\mathcal{D}'(\Omega)$; therefore $I_+^{1-s}[v_{h_k}] \to I_+^{1-s}[v]$ in $\mathcal{D}'(\Omega)$ and

$$\Big(\frac{\partial}{\partial x}\Big)_+^s[v_{h_k}] = \frac{\partial}{\partial x} I_+^{1-s}[v_{h_k}] \to \frac{\partial}{\partial x} I_+^{1-s}[v] = \mu_{x,+} \qquad \text{in } \mathcal{D}'(\Omega),$$

then (5.7) follows.
Convergences (5.8),(5.9),(5.10) are deduced by the same argument. □

Theorem 5.4. *Assume* $g \in L^1(\Omega)$. *Then the functional* $\mathfrak{G}$ *defined in* (5.5) *is convex and achieves a finite minimum over its nonempty domain.*

Proof. Convexity (not strict) holds true since $\mathfrak{G}$ is the sum of five convex terms. It is a consequence of Lemma 5.3, since all terms are lower semicontinuous respectively in the convergences (5.6)-(5.10) warranted by sequential compactness of sublevels of functional $\mathfrak{G}$ (Lemma 5.3). □

6. From Riemann-Liouville to Grünwald-Letnikov fractional derivatives

In [33] we presented some numerical simulation for the 1D SFV model $(\mathcal{P}_{\mathcal{G}})$/(4.1).

Here we propose a computational method for minimizing the 2D SFV model (5.5) by exploiting a reliable approximation of Riemann-Liouville fractional derivatives based on Grünwald-Letnikov fractional derivatives. This approach is particularly advantageous for signal and image processing applications [21, 31, 33].

Definition 6.1. (Grünwald-Letnikov fractional derivative) Assume u is an analytic function in $[0,T]$ and s is a real number fulfilling $s>0$, then the left and right Grünwald-Letnikov derivative of order s of u at $0<x<T$ (respectively denoted as $\mathfrak{D}^s_+$ and $\mathfrak{D}^s_-$) are defined by:

$$\mathfrak{D}^s_+[u](x) = \lim_{\substack{h\to 0_+\\ hK=x}} \frac{1}{h^s}\,{}_{\mathrm{GL}}\Delta^s_+[u]\,, \qquad {}_{\mathrm{GL}}\Delta^s_+[u] := \sum_{k=0}^{K}(-1)^k\binom{s}{k}u(x-kh) \tag{6.1}$$

$$\mathfrak{D}^s_-[u](x) = \lim_{\substack{h\to 0_+\\ hK=x}} \frac{1}{h^s}\,{}_{\mathrm{GL}}\Delta^s_-[u]\,, \qquad {}_{\mathrm{GL}}\Delta^s_-[u] := \sum_{k=0}^{K}(-1)^k\binom{s}{k}u(x+kh)\,, \tag{6.2}$$

for a series of step sizes h with $xh\in\mathbb{N}$ and $x=hK$.

We will use in the following the approximation $\mathfrak{D}^s_\pm[u](x)\approx \frac{1}{h^s}\,{}_{\mathrm{GL}}\Delta^s_\pm[u](x)$, arising from the Grünwald-Letnikov Definition 6.1.

The standard left and right Grünwald difference operators ${}_{\mathrm{GL}}\Delta^s_+$, ${}_{\mathrm{GL}}\Delta^s_-$ approximate the left and right Riemann-Liouville fractional derivatives D^s_+ and D^s_- uniformly with first order accuracy; in formula, for $h\to 0$, we have

$$D^s_+[u](x) = \frac{1}{h^s}\,{}_{\mathrm{GL}}\Delta^s_+[u](x) + O(h), \tag{6.3}$$

$$D^s_-[u](x) = \frac{1}{h^s}\,{}_{\mathrm{GL}}\Delta^s_-[u](x) + O(h). \tag{6.4}$$

Here, following [52], we approximate the Riemann-Liouville fractional derivative via a weighted combination of the shifted Grünwald-Letnikov formulae, which achieve second order accuracy:

$$D^s_+[u](x) = \frac{1}{h^2}\,{}^{p,q}_{\mathrm{GL2}}\Delta^s_+[u](x) + O(h^2), \tag{6.5}$$

$${}^{p,q}_{\mathrm{GL2}}\Delta^s_+[u](x) := \frac{s-2q}{2(p-q)}\,{}^{p}_{\mathrm{GL}}\Delta^s_+[u](x) + \frac{2p-s}{2(p-q)}\,{}^{q}_{\mathrm{GL}}\Delta^s_+[u](x), \tag{6.6}$$

$$D^s_-[u](x) = \frac{1}{h^2}\,{}^{p,q}_{\mathrm{GL2}}\Delta^s_-[u](x) + O(h^2), \tag{6.7}$$

$${}^{p,q}_{\mathrm{GL2}}\Delta^s_-[u](x) := \frac{s-2q}{2(p-q)}\,{}^{p}_{\mathrm{GL}}\Delta^s_-[u](x) + \frac{2p-s}{2(p-q)}\,{}^{q}_{\mathrm{GL}}\Delta^s_-[u](x), \tag{6.8}$$

where p,q are integers and $p\neq q$, and

$${}^{p}_{\mathrm{GL}}\Delta^s_+[u](x) = \sum_{k=0}^{K}\omega^s_k\,u(x-(k-p)h) \tag{6.9}$$

$$ {}^{\;\;p}_{\mathrm{GL}}\Delta^s_- \left[u\right](x) = \sum_{k=0}^{K} \omega^s_k \, u(x+(k-p)h) \tag{6.10}$$

with coefficients ω^s_k defined as follows

$$\omega^s_k \;=\; (-1)^k \binom{s}{k} = \;(-1)^k \frac{\Gamma(s+1)}{\Gamma(k+1)\Gamma(s-k+1)} \;\;, \quad k \in \mathbb{N}. \tag{6.11}$$

The generalized binomial coefficients $\binom{s}{k}$, defined in (6.11) in terms of the Gamma function, can be computed by the following recurrence relationships

$$\omega^s_0 = 1; \quad \omega^s_k = \omega^s_{k-1} \cdot \left(1 - \frac{s+1}{k}\right) \;, \quad k = 1, 2, \ldots . \tag{6.12}$$

7. Model discretization and its numerical solution

An image may be defined as a two-dimensional function, $u(x, y)$, where x and y are spatial (plane) coordinates, and the amplitude of u at any pair of coordinates (x, y) is called the intensity or gray level of the image at that point. We introduce, without loss of generality, a uniform discretization of the 2d domain $(0, 1)^2 \subset \mathbb{R}^2$. In particular, we define the 2d grids $\sqrt{n} \times \sqrt{n}$ of n points:

$$\left\{ \, (x_j, y_k) : \; x_j = j\, h \,, y_k = k\, h \,, \; j, k = 0, \ldots, \sqrt{n} - 1 \,, \; h = 1/(\sqrt{n} - 1) \, \right\}. \tag{7.1}$$

We indicate by $\boldsymbol{u} := (u_0, \ldots, u_i, \ldots, u_{n-1})^{\mathrm{T}} \in \mathbb{R}^n$ the (column) vector containing the values that function u assumes column-wise at all the grid points. First we introduce the two matrices$\mathbf{D}^s_+$, $\mathbf{D}^s_- \in \mathbb{R}^{n \times n}$ which, under right multiplication times the column vector $\boldsymbol{u} \in \mathbb{R}^n$, compute the two vectors of left and right weighted and shifted Grünwald differences – see the definitions in (6.6)–(6.8) – approximating the fractional derivatives of u at those points.

We remark that $\mathbf{D}^s_+$ and $\mathbf{D}^s_-$ are lower and upper Hessenberg matrices, respectively, and $\mathbf{D}^s_+ = (\mathbf{D}^s_-)^T$.

Then denoting by $\otimes$ the Kronecker product operator and by $\mathbf{I}_{\sqrt{n}}$ the identity matrix of order $\sqrt{n}$, the four matrices $\mathbf{D}^s_{x,+}$, $\mathbf{D}^s_{x,-}$, $\mathbf{D}^s_{y,+}$, $\mathbf{D}^s_{y,-} \in \mathbb{R}^{n \times n}$, defined as

$$\begin{aligned} \mathbf{D}^s_{x,+} &= \mathbf{D}^s_+ \otimes \mathbf{I}_{\sqrt{n}} \,, \qquad & \mathbf{D}^s_{y,+} &= \mathbf{I}_{\sqrt{n}} \otimes \mathbf{D}^s_+ \,, \\ \mathbf{D}^s_{x,-} &= \mathbf{D}^s_- \otimes \mathbf{I}_{\sqrt{n}} \,, \qquad & \mathbf{D}^s_{y,-} &= \mathbf{I}_{\sqrt{n}} \otimes \mathbf{D}^s_- \,, \end{aligned} \tag{7.2}$$

with $\mathbf{D}^s_+, \mathbf{D}^s_- \in \mathbb{R}^{\sqrt{n} \times \sqrt{n}}$, allow to compute the left and right, weighted and shifted Grünwald difference approximations of the horizontal and vertical fractional partial derivatives of a function $u : [0, 1]^2 \to \mathbb{R}$ in correspondence of all the 2d grid points in (7.1).

If Dirichlet homogeneous boundary conditions are considered, then the two matrices $\mathbf{D}^s_+$, $\mathbf{D}^s_-$ and the four matrices in (7.2) are Toeplitz and block-Toeplitz with Toeplitz blocks, respectively. If, instead, periodic boundary conditions are adopted, matrices are circulant and block-circulant with circulant blocks.

By exploiting previous definitions, the discretized model reads

$$\widehat{\boldsymbol{u}}(s, \lambda) \;\in\; \mathrm{argmin}_{\boldsymbol{u} \in \mathbb{R}^n} \left\{ \, G(\boldsymbol{u}; s, \lambda) \;:=\; \mathrm{SFV}^s(\boldsymbol{u}) + \lambda \, \|\boldsymbol{u} - \boldsymbol{g}\|_1 \, \right\}, \tag{7.3}$$

with the SFV^s regularizer defined by

$$\mathrm{SFV}^s(\boldsymbol{u}) \;=\; \left\|\mathbf{D}^s_{x,+}\boldsymbol{u}\right\|_1 + \left\|\mathbf{D}^s_{x,-}\boldsymbol{u}\right\|_1 + \left\|\mathbf{D}^s_{y,+}\boldsymbol{u}\right\|_1 + \left\|\mathbf{D}^s_{y,-}\boldsymbol{u}\right\|_1 \,, \tag{7.4}$$

where we made explicit the dependency of the solution $\widehat{\boldsymbol{u}}$ on the parameter pair (s, λ). From now on, $\|\mathbf{a}\|_p := \left(h\sum_j |a_j|^p\right)^{1/p}$ is the weighted ℓ^p norm of the vector $\mathbf{a}$, $p = 1, 2$.

Referring to (4.2), (4.3), we emphasize that $\left\|\mathbf{D}^s_{\pm}\mathbf{u}\right\|_1$ is an approximation of $\left\|D^s_{\pm}u\right\|_{T_\pm}$.

Starting from definitions of $\mathbf{D}^s_+$, $\mathbf{D}^s_-$ and (7.2), and introducing the matrix

$$\mathbf{D}^s := \left(\, \mathbf{D}^s_{x,+} \quad \mathbf{D}^s_{x,-} \quad \mathbf{D}^s_{y,+} \quad \mathbf{D}^s_{y,-}\,\right)^T \in \mathbb{R}^{4n\times n} \,, \tag{7.5}$$

the SFV^s regularizer in (7.4) can be compactly rewritten as

$$\mathrm{SFV}^s(\boldsymbol{u}) \;=\; \|\mathbf{D}^s\boldsymbol{u}\|_1 \;. \tag{7.6}$$

The following result provides the main properties of cost function G which lead to the existence of solutions of the minimization problem in (7.3).

Proposition 7.1. *For every $s>0$, $\lambda>0$ and data $\boldsymbol{g}\in\mathbb{R}^n$, the cost function G defined in* (7.3) *is proper, continuous, convex and coercive in $\boldsymbol{u}$, hence the minimization problem in* (7.3) *admits a compact convex set of solutions.*

Proof. The function G in (7.3) is proper, continuous, convex and coercive as it is given by the sum of two functions which, for any $s \in \mathbb{R}_{++}$, $\lambda \in \mathbb{R}_{++}$, $g \in \mathbb{R}^n$, are both clearly proper, continuous, convex and bounded below by zero, with the second being coercive. It is then a classical result of convex analysis that a function having the properties of G admits a compact and convex set of global minimizers. □

For the solution of the finite-dimensional minimization problem we exploited an efficient iterative algorithm based on the Alternating Direction Method of Multipliers (ADMM), see [8] for details. After introducing the auxiliary variable $\boldsymbol{y}$ defined by

$$\boldsymbol{y} \;=\; \begin{pmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2\end{pmatrix} = \begin{pmatrix}\mathbf{D}^s\boldsymbol{u}\\ \boldsymbol{u}\end{pmatrix} \in \mathbb{R}^m, \quad \text{with } m = 5n \tag{7.7}$$

it is easy to verify that our discrete model (7.3) with the SFV^s regularization term defined in (7.5)–(7.6) can be equivalently reformulated as a standard two-blocks (additively) separable minimization problem with linear constraints,

$$\{\widehat{\boldsymbol{u}}(s,\lambda), \widehat{\boldsymbol{y}}(s,\lambda)\} \in \mathrm{argmin}_{\boldsymbol{u},\boldsymbol{y}} \{U(\boldsymbol{u}) + Y(\boldsymbol{y};\lambda)\} \text{ subject to } \mathbf{U}^s\boldsymbol{u} + \mathbf{Y}\boldsymbol{y} = \mathbf{0}, \tag{7.8}$$

with the two cost functions U and Y defined by

$$U(\boldsymbol{u}) \equiv 0\,, \qquad Y(\boldsymbol{y};\lambda) = \|\boldsymbol{y}_1\|_1 + \lambda\,\|\boldsymbol{y}_2 - \boldsymbol{g}\|_1 \,, \tag{7.9}$$

and with matrices $\mathbf{U}^s$, $\mathbf{Y}$ given by

$$\mathbf{U}^s = \begin{pmatrix}\mathbf{D}^s\\ \mathbf{I}_n\end{pmatrix} \in \mathbb{R}^{m\times n}, \quad \text{with } m = 5n\,. \tag{7.10}$$

The Lagrangian function $\mathcal{L}$ and augmented Lagrangian function $\mathcal{L}_\beta$ associated with problem (7.8) read

$$\mathcal{L}(\boldsymbol{u},\boldsymbol{y},\boldsymbol{\rho};s,\lambda) \;=\; U(\boldsymbol{u}) + Y(\boldsymbol{y};\lambda) + \boldsymbol{\rho}^{\mathrm{T}}\left(\mathbf{U}^s\boldsymbol{u} + \mathbf{Y}\boldsymbol{y}\right), \tag{7.11}$$

$$\mathcal{L}_\beta(\boldsymbol{u},\boldsymbol{y},\boldsymbol{\rho};s,\lambda) \;=\; \mathcal{L}(\boldsymbol{u},\boldsymbol{y},\boldsymbol{\rho};\lambda) + \frac{\beta}{2}\|\mathbf{U}^s\boldsymbol{u} + \mathbf{Y}\boldsymbol{y}\|_2^2, \tag{7.12}$$

where $\beta \in \mathbb{R}_{++}$ is a penalty parameter and $\boldsymbol{\rho}$ is the vector of Lagrange multipliers associated to the system of linear constraints in (7.8), with $\boldsymbol{\rho} \in \mathbb{R}^{5n}$.

Solving problem (7.8) amounts to seek the saddle point(s) of the augmented Lagrangian function $\mathcal{L}_\beta$ in (7.12); in formula,

$$\{\widehat{\boldsymbol{u}}(s,\lambda), \widehat{\boldsymbol{y}}(s,\lambda), \widehat{\boldsymbol{\rho}}(s,\lambda)\} \in \arg\min_{\boldsymbol{u},\boldsymbol{y}} \max_{\boldsymbol{\rho}} \mathcal{L}_\beta(\boldsymbol{u},\boldsymbol{y},\boldsymbol{\rho};s,\lambda). \tag{7.13}$$

The solution(s) of the saddle-point problem (7.13) can be computed as the limit point(s) of the standard two-blocks ADMM algorithm [8]. Upon suitable initialization, and for any $k \geq 0$, the k-th iteration of the algorithm reads as follows:

$$\boldsymbol{u}^{(k+1)} = \mathrm{argmin}_{\boldsymbol{u}}\; \mathcal{L}_\beta\big(\boldsymbol{u},\boldsymbol{y}^{(k)},\boldsymbol{\rho}^{(k)};s,\lambda\big) \tag{7.14}$$

$$= \mathrm{argmin}_{\boldsymbol{u}} \left\{ U(\boldsymbol{u}) + \frac{\beta}{2}\left\|\mathbf{U}^s\boldsymbol{u} + \mathbf{Y}\boldsymbol{y}^{(k)} + \frac{1}{\beta}\boldsymbol{\rho}^{(k)}\right\|_2^2 \right\} \tag{7.15}$$

$$= \mathrm{argmin}_{\boldsymbol{u}} \frac{1}{2}\left\|\mathbf{U}^s\boldsymbol{u} - \boldsymbol{q}^{(k)}\right\|_2^2, \quad \boldsymbol{q}^{(k)} = \boldsymbol{y}^{(\boldsymbol{k})} - \frac{1}{\beta}\boldsymbol{\rho}^{(k)}, \tag{7.16}$$

$$\boldsymbol{y}^{(k+1)} = \mathrm{argmin}_{\boldsymbol{y}}\; \mathcal{L}_\beta\big(\boldsymbol{u}^{(k+1)},\boldsymbol{y},\boldsymbol{\rho}^{(k)};s,\lambda\big) \tag{7.17}$$

$$= \mathrm{argmin}_{\boldsymbol{y}} \left\{ Y(\boldsymbol{y}) + \frac{\beta}{2}\left\|\mathbf{U}^s\boldsymbol{u}^{(k+1)} + \mathbf{Y}\boldsymbol{y} + \frac{1}{\beta}\boldsymbol{\rho}^{(k)}\right\|_2^2 \right\} \tag{7.18}$$

$$= \mathrm{argmin}_{\boldsymbol{y}} \left\{ Y(\boldsymbol{y}) + \frac{\beta}{2}\left\|\boldsymbol{y} - \boldsymbol{v}^{(k)}\right\|_2^2 \right\}, \quad \boldsymbol{v}^{(k)} = \mathbf{U}^s\boldsymbol{u}^{(k+1)} + \frac{1}{\beta}\boldsymbol{\rho}^{(k)}, \tag{7.19}$$

$$\boldsymbol{\rho}^{(k+1)} = \boldsymbol{\rho}^{(k)} + \beta\left(\mathbf{U}^s\boldsymbol{u}^{(k+1)} + \mathbf{Y}\boldsymbol{y}^{(k+1)}\right), \tag{7.20}$$

$$= \boldsymbol{\rho}^{(k)} + \beta\left(\mathbf{U}^s\boldsymbol{u}^{(k+1)} - \boldsymbol{y}^{(k+1)}\right), \tag{7.21}$$

where (7.15) and (7.18) follow easily from (7.14) and (7.17) by recalling the definition of $\mathcal{L}_\beta$ in (7.11), dropping the constant terms and carrying out some algebraic manipulations, whereas (7.16), (7.19) and (7.21) come immediately from (7.15), (7.18) and (7.20) after recalling from (7.9)-(7.10) that for our specific two-blocks separable minimization problem we have $U(\boldsymbol{u}) \equiv 0$ and $\mathbf{Y}$ equal to the negative identity matrix.

We refer the reader to [33] where, for the 1-dimensional case, we illustrated how the minimization of the subproblems (7.16) and (7.19) for primal variables $\boldsymbol{u}$ and $\boldsymbol{y}$ respectively, can be efficiently solved. For the 2-dimensional case the computation steps differ in the Hessian matrix $\mathbf{H}^s$ of the quadratic cost function in (7.16) which is given by

$$\mathbf{H}^s = \left(\mathbf{U}^s\right)^{\mathrm{T}}\mathbf{U}^s \tag{7.22}$$

$$= \mathbf{I}_n + \left(\mathbf{D}^s_{x,+}\right)^{\mathrm{T}}\mathbf{D}^s_{x,+} + \left(\mathbf{D}^s_{x,-}\right)^{\mathrm{T}}\mathbf{D}^s_{x,-} + \left(\mathbf{D}^s_{y,+}\right)^{\mathrm{T}}\mathbf{D}^s_{y,+} + \left(\mathbf{D}^s_{y,-}\right)^{\mathrm{T}}\mathbf{D}^s_{y,-}.$$

The primal and dual residuals at iteration $k+1$ are defined, respectively, as

$$\boldsymbol{r}^{(k+1)} = \mathbf{U}^s \boldsymbol{u}^{(k+1)} - \boldsymbol{y}^{(k+1)}, \qquad \boldsymbol{s}^{(k+1)} = -\beta \left(\mathbf{U}^s\right)^{\mathrm{T}} \left(\boldsymbol{y}^{(k+1)} - \boldsymbol{y}^{(k)}\right). \tag{7.23}$$

Hence, we stop ADMM iterations as soon as both the primal and dual relative residual norms drop below some fixed thresholds:

$$\frac{\left\|\boldsymbol{r}^{(k+1)}\right\|_2}{\max\left\{\left\|\mathbf{U}^s \boldsymbol{u}^{(k)}\right\|_2, \left\|\boldsymbol{y}^{(k)}\right\|_2\right\}} < \epsilon_{\text{primal}}, \qquad \frac{\left\|\boldsymbol{s}^{(k+1)}\right\|_2}{\left\|\left(\mathbf{U}^s\right)^{\mathrm{T}} \boldsymbol{\rho}^{(k)}\right\|_2} < \epsilon_{\text{dual}}. \tag{7.24}$$

8. Numerical simulations

In this section we carry out some numerical simulations aimed to provide evidence for the effectiveness of the proposed variational denoising approach. We applied the SFV-L^1 model to denoise four synthetically generated images as well as two more realistic images (a magnetic resonance medical image and a satellite image).

Denoting by $\chi_{\mathcal{S}} : \mathbb{R}^2 \to \{0,1\}$ the characteristic function of a set $\mathcal{S} \subset \mathbb{R}^2$, with $\chi_{\mathcal{S}}(x,y) = 1$ for $(x,y) \in \mathcal{S}$ and $\chi_{\mathcal{S}}(x,y) = 0$ otherwise, we consider the four functions

$$f_1(x,y) = m_1\, \chi_{\mathcal{S}_1}(x,y), \tag{8.1}$$

$$f_2(x,y) = m_2\, \left(0.4 + 0.6\chi_{\mathcal{S}_2}(x,y)\,(1 - \|(x,y) - (0.5,0.5)\|_\infty/0.3)\right), \tag{8.2}$$

$$f_3(x,y) = m_3\, \chi_{\mathcal{S}_3}(x,y)\, r\left(\|(x,y) - (0.5,0.5)\|_2/0.4\right), \;\; r(t) = (t+1)^2(t-1)^2, \tag{8.3}$$

$$f_4(x,y) = m_4\, \sin(4\pi x)\, \sin(2\pi y), \tag{8.4}$$

with parameter values $m_1 = m_2 = m_3 = 0.8$, $m_4 = 0.4$, and with function supports

$$\begin{aligned} \mathcal{S}_1 = \mathcal{S}_2 &= \{\, (x,y) \in \mathbb{R}^2 : \;\; \|(x,y) - (0.5,0.5)\|_\infty \le 0.3 \,\}, \\ \mathcal{S}_3 &= \{\, (x,y) \in \mathbb{R}^2 : \;\; \|(x,y) - (0.5,0.5)\|_2 \le 0.4 \,\}, \\ \mathcal{S}_4 &= [0,1]^2. \end{aligned} \tag{8.5}$$

The four functions f_1, f_2, f_3, f_4, whose graphs are shown in the first row of Figure 1, are piecewise constant, piecewise affine with jump discontinuities and smooth compactly supported functions, respectively. We generate the associated noiseless (or ground truth) discrete images $\overline{\mathbf{U}}_1, \overline{\mathbf{U}}_2, \overline{\mathbf{U}}_3, \overline{\mathbf{U}}_4 \in \mathbb{R}^{31\times 31}$, shown in the second row of Figure 1, by sampling the functions f_1, f_2, f_3, f_4 at the nodes of a uniform 31×31 domain grid, defined in accordance with the domain discretization strategy outlined in Section 7; namely, for any of the functions - say for a generic function $f : [0,1]^2 \to \mathbb{R}$ - the associated noiseless image $\overline{\mathbf{U}}$ reads

$$\overline{\mathbf{U}} = \{\overline{u}_{i,j}\}_{i,j=1}^{31}, \quad \text{with } \; \overline{u}_{i,j} = f\left(x_i, y_j\right). \tag{8.6}$$

Then, the Laplace noise-corrupted images $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4 \in \mathbb{R}^{31\times 31}$, shown in the third row of Figure 1, are generated by adding to each pixel intensity $\overline{u}_{i,j}$ of the noiseless images independent realizations $\varepsilon_{i,j}$ of a Laplace-distributed scalar random variable $\mathcal{E}$ with zero-mean and standard deviation σ, denoted $\mathcal{E} \sim L(0,\sigma)$. In formula,

$$\mathbf{G} = \{g_{i,j}\}_{i,j=1}^{31}, \quad \text{with } \; g_{i,j} = \overline{u}_{i,j} + \varepsilon_{i,j}, \;\; \varepsilon_{i,j} \text{ realization of } \mathcal{E}. \tag{8.7}$$

In particular, in order to test the proposed denoising approach under the action of noises of different intensities, for the four test images we added Laplace noises with different standard deviations $\sigma_1 = 0.04$, $\sigma_2 = 0.06$, $\sigma_3 = 0.08$, and $\sigma = 0.10$ respectively.

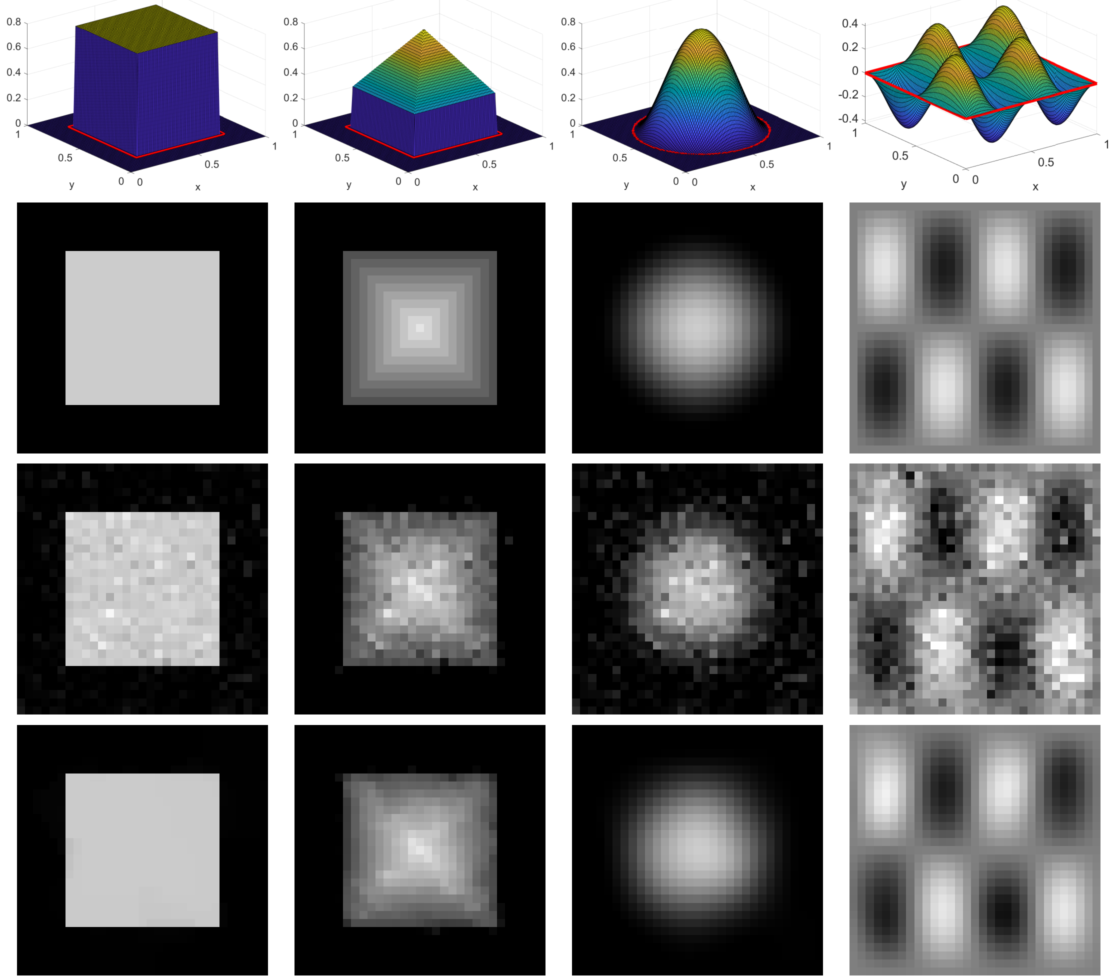

Figure 1: First row: original images described by the functions f_1, f_2, f_3, f_4 as defined in (8.1),(8.2),(8.3),(8.4). Second row: associated discretized noiseless images $\overline{\mathbf{U}}_1, \overline{\mathbf{U}}_2, \overline{\mathbf{U}}_3, \overline{\mathbf{U}}_4 \in \mathbb{R}^{31\times 31}$ with support boundaries depicted in red. Third row: Laplace noise-corrupted images $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4 \in \mathbb{R}^{31\times 31}$. Fourth row: denoised images $\widehat{\mathbf{U}}_1, \widehat{\mathbf{U}}_2, \widehat{\mathbf{U}}_3, \widehat{\mathbf{U}}_4 \in \mathbb{R}^{31\times 31}$ evaluated by the SFV-L^1 approach developed here.

The four noisy images (actually, their vectorized versions) are then fed as input to the proposed approach, and the obtained denoised images $\widehat{\mathbf{U}}_1, \widehat{\mathbf{U}}_2, \widehat{\mathbf{U}}_3, \widehat{\mathbf{U}}_4 \in \mathbb{R}^{31\times 31}$ are reported in the last row of Figure 1.

In Figure 2 two noise-free images $\overline{\mathbf{U}}_{5,6} = \{\bar{u}_{i,j}\}_{i,j=1}^{64}$ are illustrated (left), together with the images $\mathbf{G}_5, \mathbf{G}_6$ corrupted by an additive Laplace noise realization with standard variation $\sigma = 0.08$ (middle), and denoised images $\widehat{\mathbf{U}}_5$, $\widehat{\mathbf{U}}_6$.

We measure the quality of the obtained results by means of the restored relative error with respect to the ground truth images, defined by

$$RRE := \frac{\|\widehat{\mathbf{U}} - \overline{\mathbf{U}}\|_F}{\|\overline{\mathbf{U}}\|_F}, \tag{8.8}$$

with $\|\cdot\|_F$ denoting the Frobenius norm.

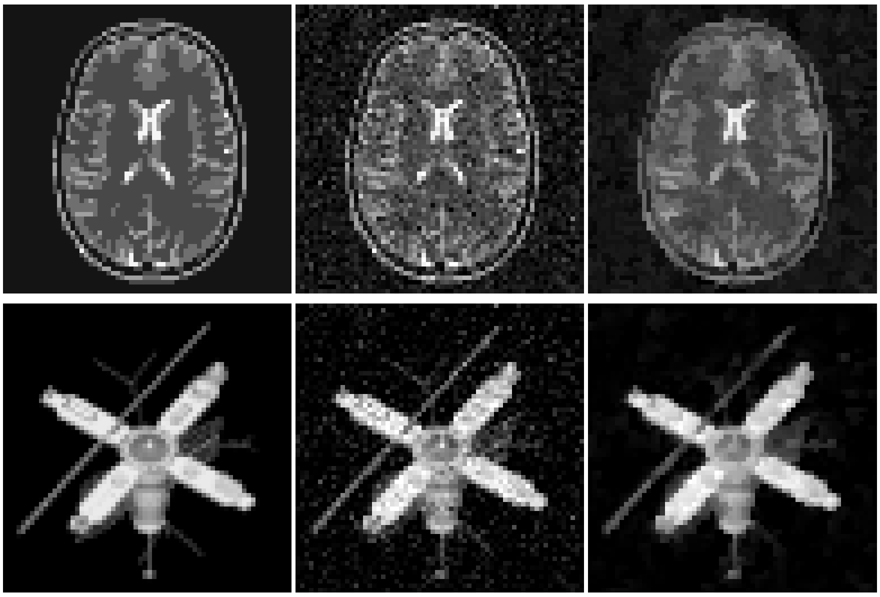

Figure 2: Left: noiseless images $\overline{\mathbf{U}}_5, \overline{\mathbf{U}}_6 \in \mathbb{R}^{64\times 64}$. Centre: Laplace noise-corrupted images $\mathbf{G}_5, \mathbf{G}_6$. Right: denoised images $\widehat{\mathbf{U}}_5, \widehat{\mathbf{U}}_6$ estimated by the proposed approach.

In Table 1 we report all quantitative data and results of the simulation. In particular, for each test image, we report the standard deviation σ of the synthetically added Laplace noise (second column), the model parameters $\widehat{s}$ and $\widehat{\lambda}$ selected by our approach (third and fourth columns), and the restored relative error RRE associated to the estimated denoised image.

image	σ	$\widehat{s}$	$\widehat{\lambda}$	RRE
$\mathbf{G}_1$	0.04	1.0	2.0	0.0148
$\mathbf{G}_2$	0.06	1.5	5.0	0.0858
$\mathbf{G}_3$	0.08	4.5	1.2	0.0455
$\mathbf{G}_4$	0.10	4.5	1.5	0.1064
$\mathbf{G}_5$	0.08	1.4	6.7	0.2317
$\mathbf{G}_6$	0.08	1.2	5.6	0.1481

Table 1: Quantitative data and results of the simulation.

The results in Table 1 indicate that the SFV-L^1 fractional denoising model has good performance, and more specifically, it adapts automatically the fractional order s to the regularity of the analytical functions discretized in the first three test images. Both the MRI image and the satellite image present a mixture of constant and linear details and thus the best denoising result is obtained for $s = 1.4$ and $s = 1.2$, respectively.

Although generally the choice of an appropriate order s value for the fractional model may be difficult to decide on, the whiteness principle applied for the free parameter selection of both s and λ allowed us to obtain optimal results.

Finally, we performed an additional numerical experiment aimed to provide evidence for usefulness of symmetrisation in the proposed SFV-L^1 fractional variational model in (7.3). In particular, for the Laplace noise-corrupted smooth image $\mathbf{G_3}$ shown in the third column of Figure 1, using the same optimal parameter values $\widehat{s} = 4.5$, $\widehat{\lambda} = 1.2$ previously estimated for the SFV-L^1 model, we ran the two un-symmetrised versions of the model – namely, the models obtained from (7.3) by dropping in the SFV regularizer defined in (7.4) the 1-norms of the two right-side partial derivatives or of the two left-side partial derivatives.

The relative errors associated to the three restored images are $RRE = 0.0455$ for the symmetrised model, $RRE = 0.0523$ and $RRE = 0.0549$ for the two un-symmetrised models: they show that symmetrisation in our proposed fractional variational model SFV provides better denoising in numerical simulations than the unsymmetrised fractional approach.

9. Conclusion

We presented and analyzed SFV-L^1, a fractional-order model that leverages Riemann-Liouville fractional derivatives of arbitrary positive order for signal denoising. The theoretical results encompasses both one-dimensional and two-dimensional scenarios: we introduced the symmetrised fractional variation spaces BV^s_* of any real order $s > 0$, we showed the weak sequential compactness property of bounded subsets of BV^s_* and that these spaces provide a suitable environment for well-posedness of Abel integral equations in the distributional framework.

The proposed image denoising model employs an L^1 data fidelity term, which is robust to impulse noise, combined with both right and left Riemann-Liouville fractional derivatives as regularizers. This dual regularization strategy enables orientation-independent analysis, capturing fine-scale details while preserving edges. To demonstrate the effectiveness of our model we presented a numerical optimization method based on alternating energy minimizations. The Riemann-Liouville fractional derivatives are discretized by a second-order accurate Grünwald-Letnikov formula. While integer-order derivatives in image denoising regularize specific morphological details, such as piecewise constant, linear or smooth features, fractional-order derivatives offer an additional degree of freedom, enabling superior denoising results.

Funding. The authors Antonio Leaci and Franco Tomarelli are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Serena Morigi is member of Gruppo Nazionale di Calcolo Scientifico (GNCS)-INDAM.

Antonio Leaci is partially funded by the Italian M.U.R. PRIN grant n. 20223L2NWK "Elliptic and parabolic problems, heat kernel estimates and spectral theory".

Franco Tomarelli is partially funded from the INdAM - GNAMPA 2023 Project Codice CUP_E53C22001930001, "Modelli variazionali ed evolutivi per problemi di adesione e di contatto" and acknowledges support from PRIN 2022 (Project no. 2022J4FYNJ) funded by MUR, Italy, and the European Union – Next Generation EU, Mission 4 Component 1 CUP F53D23002760006.

Serena Morigi has been supported in part by the GNCS-INDAM, Research Projects 2024.

The work of Serena Morigi and Alessandro Lanza was supported by PRIN2022_MORIGI, titled "Inverse Problems in the Imaging Sciences (IPIS)" 2022 ANC8HL - CUP J53D23003670006, and PRIN2022_PNRR_CRESCENTINI CUP J53D23014080001. Serena Morigi is supported by MICS (Made in Italy – Circular and Sustainable) Extended Partnership and received funding from the European Union Next-Generation-EU (PNRR) – MISSIONE 4 COMPONENTE 2, INVESTIMENTO 1.3 – D.D. 1551.11-10-2022, PE00000004).

References

[1] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford University Press, Oxford (2000).

[2] L. Ambrosio, T. Ikonen, D. Lucic, E. Pasqualetto: *Metric Sobolev spaces. I: Equivalence of definitions*, Milan J. Math. 92/2 (2024) 255–347.

[3] S. Arora, T. Mathur, S. Agarwal, K. Tiwari, P. Gupta: *Applications of fractional calculus in computer vision: A survey*, Neurocomputing 489 (2022) 407–428.

[4] H. Attouch, G. Buttazzo, G. Michaille: *Variational Analysis in Sobolev and BV Spaces*, SIAM Series on Optimization, SIAM and MPS, Philadelphia (2006).

[5] M. Bergounioux, A. Leaci, G. Nardi, F. Tomarelli: *Fractional Sobolev spaces and functions of bounded variation of one variable*, Fract. Calc. Appl. Anal. 20/4 (2017) 936–962.

[6] F. Bevilacqua, A. Lanza, M. Pragliola, F. Sgallari: *A general framework for whiteness-based parameters selection in variational models*, Comp. Optim. Appl. 91/2 (2025) 457–489.

[7] V. I. Bogachev: *Measure theory. Vol. I, II*, Springer, Berlin (2007).

[8] S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein: *Distributed optimization and statistical learning via the alternating direction method of multipliers*, Found. Trends Machine Learning 3/1 (2011) 1–122.

[9] K. Bredies, K. Kunisch, T. Pock: *Total generalized variation*, SIAM J. Imaging Sci. 3/3 (2010) 492–526.

[10] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York (2011).

[11] M. Cantarini, D. Costarelli, G. Vinti: *Approximation results in Sobolev and fractional Sobolev spaces by sampling Kantorovich operators*, Fract. Calc. Appl. Anal. 26/6 (2023) 2493–2521.

[12] M. Carriero, A. Leaci, F. Tomarelli: *Special bounded Hessian and elastic-plastic plate*, Rend. Accad. Naz. Sci. XL, V. Ser., Mem. Mat. 16/1 (1992) 223–258.

[13] M. Carriero, A. Leaci, F. Tomarelli: *A second order model in image segmentation: Blake & Zisserman functional*, in: *Variational Methods for Discontinuous Structures*, R. Serapioni et al. (eds.), Progress in Nonlinear Differential Equations and their Applications 25, Birkäuser, Basel (1996) 57–72.

[14] M. Carriero, A. Leaci, F. Tomarelli: *Necessary conditions for extremals of Blake & Zisserman functional*, C. R. Math. Acad. Sci. Paris 334/4 (2002) 343–348.

[15] M. Carriero, A. Leaci, F. Tomarelli: *Euler equations for Blake & Zisserman functional*, Calc. Var. Partial Diff. Equations 32 (2008) 81–110.

[16] M. Carriero, A. Leaci, F. Tomarelli: *A candidate local minimizer of Blake & Zisserman functional*, J. Math. Pures Appl. 96 (2011) 58–87.

[17] M. Carriero, A. Leaci, F. Tomarelli: *Uniform density estimates for Blake & Zisserman functional*, Discrete Contin. Dyn. Syst., Ser. A 31/4 (2011) 1129–1150.

[18] M. Carriero, A. Leaci, F. Tomarelli: *A survey on Blake and Zisserman functional*, Milan J. Math. 83/2 (2015) 397–420.

[19] M. Carriero, A. Leaci, F. Tomarelli: *Segmentation and inpainting of color images*, J. Convex Analysis 25/2 (2018) 435–458.

[20] M. Carriero, A. Leaci, F. Tomarelli: *Almansi decomposition and expansion of a polyharmonic function near a crack-tip*, J. Convex Analysis 31/2 (2024) 379–409.

[21] R. H. Chan, A. Lanza, S. Morigi, F. Sgallari: *An adaptive strategy for the restoration of textured images using fractional order regularization*, Numer. Math. Theory Methods Appl. 6/1 (2013) 276–296.

[22] D. Chen, Y. Chen, D. Xue: *Fractional-order total variation image denoising based on proximity algorithm*, Appl. Math. Comput. 257 (2015) 537–545.

[23] G. E. Comi, D. Spector, G. Stefani: *The fractional variation and the precise representative of $BV^{\alpha,p}$ functions*, Fract. Calc. Appl. Anal. 25/2 (2022) 520–558.

[24] P. M. de Carvalho-Neto, R. Fehlberg Júnior: *The Riemann-Liouville fractional integral in Bochner-Lebesgue spaces III*, J. Math. Anal. Appl. 543/2, Part 3 (2025), art.no. 129023, 15p.

[25] F. Demengel: *Fonctions a hessien borné*, Ann. Inst. Fourier 34 (1984) 155–190.

[26] E. De Giorgi, M. Carriero, A. Leaci: *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rational Mech. Anal. 108 (1989) 195–218.

[27] F. Ferrari: *Weyl and Marchaud derivatives: a forgotten history*, Mathematics 6/6 (2018) 155–190.

[28] P. Ghamisi, M. S. Couceiro, J. A. Benediktsson, N. M. F. Ferreira: *An efficient method for segmentation of images based on fractional calculus and natural selection*, Expert Syst. Appl. 39/16 (2012) 12404–12417.

[29] W. Hinterberger, O. Scherzer: *Variational methods on the space of functions of bounded Hessian for convexification and denoising*, Computing 76 (2006) 109–133.

[30] D. Idczak: *Fractional Sobolev type spaces of functions of two variables via Riemann-Liouville derivatives*, Fract. Calc. Appl. Anal. 27/6 (2024) 2892–2947.

[31] H. Jalalinejad, A. Tavakoli, F. Zarmehi: *A simple and flexible modification of Grünwald–Letnikov fractional derivative in image processing*, Math. Sci. 12 (2018) 205–210.

[32] F. Kazemi Golbaghi, R. Eslahchi, M. Rezghi: *Image denoising by a novel variable-order total fractional variation model*, Math. Meth. Appl. Sci. 44 (2021) 7250–7261.

[33] A. Lanza, A. Leaci, S. Morigi, F. Tomarelli: *Symmetrized fractional variation with L^1 fidelity for signal denoising via Grünwald-Letnikov scheme*, Appl. Math. Comput. 500 (2025), art.no. 129429, 24p.

[34] A. Lanza, S. Morigi, F. Sgallari: *Automatic parameter selection based on residual whiteness for convex non-convex variational restoration*, Springer Proc. Math. Statistics 360 (2021) 95–111.

[35] A. Lanza, M. Pragliola, F. Sgallari: *Residual whiteness principle for parameter-free image restoration*, Electronic Trans. Num. Analysis 53 (2020) 329–351.

[36] A. Lanza, M. Pragliola, F. Sgallari: *Parameter-free restoration of piecewise smooth images*, Electronic Trans. Numer. Analysis 59 (2023) 202–229.

[37] A. Leaci, F. Tomarelli: *Bilateral Riemann Liouville fractional Sobolev spaces*, Note Mat. 41/2 (2021) 61–83.

[38] A. Leaci, F. Tomarelli: *Riemann-Liouville fractional Sobolev and bounded variation spaces*, Axioms 11/1 (2022), art.no. 30, 31p.

[39] A. Leaci, F. Tomarelli: *Symmetrized fractional total variation for signal and image analysis*, Adv. Cont. Discrete Models 1/14 (2023) 1–17.

[40] S. Morigi, L. Reichel, F. Sgallari: *Fractional Tikhonov regularization with a nonlinear penalty term*, J. Comp. Appl. Math. 324 (2017) 142–154.

[41] A. Nakib, H. Oulhadj, P. Siarry: *A thresholding method based on two-dimensional fractional differentiation*, Image Vision Comput. 27 (2009) 1343–1357.

[42] K. Oldham, J. Myland, J. Spanier: *An Atlas of Functions*, 2nd ed., Springer, New York (2009).

[43] M. D. Ortigueira: *Power functions and their relationship with the unified fractional derivative*, Mathematics 13 (2025), art.no. 852.

[44] D. Percivale, F. Tomarelli: *A variational principle for plastic hinges in a beam*, Math. Models Methods Appl. Sci. 19/12 (2009) 2263–2297.

[45] Y. Pu, P. Siarry, J. Zhou, N. Zhang: *A fractional partial differential equation based multiscale denoising model for texture image*, Math. Methods Appl. Sci. 37/12 (2014) 1784–1806.

[46] Y. Pu, J. Zhou, P. Siarry, N. Zhang, Y. Liu: *Fractional partial differential equation: Fractional total variation and fractional steepest descent approach-based multiscale denoising model for texture image*, Abstract Appl. Analysis 2013 (2013), art.no. 483791, 19p.

[47] Y. Pu, J. Zhou, X. Yuan: *Fractional differential mask: A fractional differential based approach for multiscale texture enhancement*, IEEE Trans. Image Processing 19/2 (2010) 491–511.

[48] L. I. Rudin, S. Osher, E. Fatemi: *Nonlinear total variation based noise removal algorithms*, Physica D: Nonlinear Phenomena 60/14 (1992) 259–268.

[49] W. Rudin: *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York (1974).

[50] S. Samko, A. Kilbas, O. Marichev: *Fractional Integrals and Derivatives – Theory and Applications*, Gordon and Breach, New York (1993).

[51] G. Savaré, F. Tomarelli: *Superposition and chain rule for Bounded Hessian functions*, Adv. Math. 140/2 (1998) 237–281.

[52] W. Tian, H. Zhou, W. Deng: *A class of second order difference approximations for solving space fractional diffusion equations*, Math. Comput. 84/294 (2015) 1703–1727.

[53] J. You, S. Hungnahally, A. Sattar: *Fractional discrimination for texture image segmentation*, IEEE Proc. Int. Conf. Image Processing 3 (1997) 220–223.

[54] J. Zhang, K. Chen: *A total fractional-order variation model for image restoration with nonhomogeneous boundary conditions and its numerical solution*, SIAM J. Imaging Sci. 8/4 (2015) 2487–2518.