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Dual Equilibrium Problems in Reflexive Banach Spaces

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The aim of this paper is to investigate the dual problem of a generalized equilibrium problem in the framework of a reflexive Banach space. By means of the Fenchel duality, we introduce the dual problem defined via the Fitzpatrick transform of the bifunction involved, that turns out to be an equilibrium problem itself in the dual space. We present conditions which entail the solvability of both primal and dual problems.

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1. Introduction

The classical generalized equilibrium problem (EP_f) is the problem of finding a point $\bar{x}$ in C such that

$$f(\bar{x}, y) + h(y) \geq h(\bar{x}), \quad \forall y \in C \qquad (\mathrm{EP}_f)$$

where C is a nonempty, closed, and convex subset of a reflexive Banach space X, f is a bifunction on $X \times X$, real-valued on $C \times C$, and h is a proper function defined on X.

These problems have been extensively studied in the literature, including in more general settings, as they encompass many well-known problems as special cases. Such problems often arise in areas like optimization and variational inequalities.

Alongside the equilibrium problem (EP_f), the so-called Minty problem has also been considered in the literature, which consists in finding a point $\overline{x} \in C$ such that

$$-f(y, \overline{x}) + h(y) \geq h(\overline{x}), \quad \forall y \in C \qquad (\mathrm{MEP}_f)$$

(see, for instance, [14]). Minty problems are sometimes referred to as "dual problems", but this terminology can be misleading, as they do not involve any of the standard duality operations. The connections between equilibrium problems and their

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corresponding Minty problems have often been studied under assumptions of generalized monotonicity of the bifunction f. In this work, monotonicity plays no role.

The goal of our study is to formulate a meaningful notion of duality for (EP_f) by interpreting the equilibrium problem as a particular optimization problem. This perspective allows us to apply Fenchel duality theory to derive a corresponding dual formulation.

Although similar approaches have been proposed by other authors (see, for instance, [7] and [16]), a key feature of their formulations is that the resulting dual problems are not themselves equilibrium problems.

In our investigation, starting from (EP_f), we construct a dual equilibrium problem (DEP_f) in the dual space X^*. This construction is based on reformulating the original equilibrium problem as an optimization problem. By suitably applying Fenchel conjugation in an iterative manner, we derive a dual problem that also takes the form of an equilibrium problem, where the associated bifunction is defined using the Fenchel transform of the Fitzpatrick function of f.

The paper is organized as follows: in Section 2 we introduce the mathematical setting, and we recall several technical tools and properties of convex functions and bifunctions useful to support the main constructions later. Section 3 provides solvability conditions for both the equilibrium problem (EP_f) and its Minty counterpart (MEP_f), and we prove existence and inclusion results between the respective solution sets. Section 4 presents the main theoretical contribution of the paper, that is a rigorous formulation of the dual equilibrium problem (DEP_f) using convex duality. We establish strong duality results and we also explore the relationship between solutions of EP_f, DEP_f, and a joint minimization problem involving the Fitzpatrick transform. Section 5 extends the duality analysis to include the Minty dual equilibrium problem (MDEP_f), based on the mirror bifunction. We establish conditions under which the Minty and standard dual problems coincide and provide further solvability results.

2. Preliminaries

Let X be a reflexive Banach space and denote by X^* its dual.

Let us recall that a function h from X to $]-\infty,+\infty]$ is said to be *coercive* on a set $C \subseteq X$ if

$$\lim_{\|x\|\to+\infty,\, x\in C} h(x) = +\infty, \tag{1}$$

while h is said to be *supercoercive* on C if

$$\lim_{\|x\|\to+\infty,\, x\in C} \frac{h(x)}{\|x\|} = +\infty. \tag{2}$$

Remark 2.1. Let C be a nonempty, closed and convex subset of X. If the function $h : X \to]-\infty,+\infty]$ is proper, convex, lower semicontinuous and supercoercive on C, and $h_1 : X \to]-\infty,+\infty]$ is proper, convex and lower semicontinuous on C, then the function $h + h_1$, if it is proper on C, has a minimum on C, that is there exists a point x_0 in C such that $h(x_0) + h_1(x_0)$ is real, and, for every x in C,

$$h(x_0) + h_1(x_0) \le h(x) + h_1(x).$$

This follows from the existence of an affine function $\langle x^*, \cdot \rangle + \alpha$ minorizing h_1 [20, Corollary 2.3.2] so it is easy to see that $h + h_1$ is lower semicontinuous and coercive. Thus, it has a minimum [12, Cor. III.20]. □

Given a function $\psi : X \to]-\infty, +\infty]$ the *Fenchel conjugate* $\psi^* : X^* \to [-\infty, +\infty]$ is defined as

$$\psi^*(x^*) = \sup_{x \in X}(\langle x^*, x\rangle - \psi(x)),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality product. For any function ψ, the conjugate ψ^* is always a lower semicontinuous and convex function. If ψ is proper, that is, $\psi(z) < +\infty$ for some $z \in X$, then ψ^* never takes the value $-\infty$. Moreover, if ψ is bounded from below by an affine function $\langle x_0^*, \cdot \rangle + \alpha$, then $\psi^*(x_0^*) < +\infty$ and therefore ψ^* is proper.

It is worth noting that if ψ is convex, lower semicontinuous, and supercoercive, then the Remark 2.1 implies that the Fenchel transform of ψ is real-valued on the whole space X^*.

In the sequel, we will use the biconjugate $\psi^{**} : X \to [-\infty, +\infty]$ defined as follows:

$$\psi^{**}(x) = \sup_{x^* \in X^*} (\langle x^*, x\rangle - \psi^*(x^*)).$$

It well known that, in case ψ is proper, convex and lower semicontinuous on X, then, by the Fenchel-Moreau Theorem, ψ^* is proper as well, and the equality $\psi \equiv \psi^{**}$ holds true.

Proposition 2.2. (see Proposition 13.47 in [3]) *Let $\{\psi_i\}_{i \in I}$ be a family of proper, lower semicontinuous and convex functions defined on X. Then*

$$\left(\sup_{i \in I} \psi_i\right)^* = \left(\inf_{i \in I} \psi_i^*\right)^{**}.$$

If $\sup_{i \in I} \psi_i$ is not identically $+\infty$, then

$$\left(\sup_{i \in I} \psi_i\right)^* = \left(\inf_{i \in I} \psi_i^*\right)^{\smile}$$

where $\psi^{\smile}$ denotes the lower semicontinuous and convex envelope of ψ.

We recall that a bifunction $f : X \times X \to [-\infty, +\infty]$ is said to be *saddle* if $f(x, \cdot)$ is convex for every $x \in X$, and $f(\cdot, y)$ is concave for every $y \in X$.

In the sequel we will mainly (but not always) consider bifunctions f that are real-valued on $C \times C \subseteq X \times X$, where C is a nonempty, closed and convex set, and satisfy the following conditions:

$$f(x, y) = \begin{cases} -\infty & x \notin C \\ +\infty & x \in C,\ y \notin C \end{cases}$$

Combining notions in [1, Remark 2] and [18, pg. 349], bifunctions with the properties above will be called *lower normal bifunctions* on C.

The *Fitzpatrick transform* associated to a lower normal bifunction f is the bifunction $\varphi_f : X \times X^* \to]-\infty, +\infty]$ given by

$$\varphi_f(y, x^*) = (-f(\cdot, y))^*(x^*) = \sup_{x \in X}(\langle x^*, x\rangle + f(x, y)) = \sup_{x \in C}(\langle x^*, x\rangle + f(x, y))$$

(see [5], [11]). Note that, by construction, the Fitzpatrick transform of a lower normal bifunction has the following properties:

P1 It satisfies the inequality

$$\varphi_f(y, x^*) \geq \langle x^*, x\rangle + f(x, y) \quad \forall x, y \in X. \tag{3}$$

In particular, if $f(\tilde{x}, \cdot)$ is (super)coercive for some $\tilde{x} \in C$, then $\varphi_f(\cdot, x^*)$ is (super)coercive for every $x^* \in X^*$;

P2 if $f(x, x) \geq 0$ for every $x \in C$, from (3) we have $\varphi_f(y, x^*) \geq \langle x^*, y\rangle$ for every $y \in C, x^* \in X^*$. In addition, $\varphi_f(y, x^*) = +\infty$ for every $x^* \in X^*$ whenever $y \notin C$. Thus, $\varphi_f(y, x^*) \geq \langle x^*, y\rangle$ holds for every $(y, x^*) \in X \times X^*$;

P3 if $f(x, \cdot)$ is convex (respectively, lower semicontinuous) for every $x \in C$, then φ_f is convex (respectively, lower semicontinuous) on $X \times X^*$;

P4 the bifunction φ_f is proper if and only if for some $y \in C$, the function $f(\cdot, y)$ is majorized by an affine function $\langle x^*, \cdot\rangle + \alpha$, where $x^* \in X^*$. In this case, $\varphi_f(y, -x^*) \in \mathbb{R}$. For example, this happens if for some $y \in C$ the function $f(\cdot, y)$ is concave and upper semicontinuous.

The following theorem (actually, a version of Berge's maximal theorem [4]) gives a basic property of the marginal function of a bifunction under a technical condition, that is fulfilled in some important cases of interest:

Theorem 2.3. *Let X be a reflexive Banach space, and $\varphi : X^* \times X \to]-\infty, +\infty]$ be a convex and lower semicontinuous function. Assume further that the following condition holds:*

(C) *if $\{x_n^*\} \subset X^*$ is a bounded sequence and $\{y_n\} \subset X$ is a sequence such that $\|y_n\| \to +\infty$, then $\varphi(x_n^*, y_n) \to +\infty$.*

Then the marginal function m defined by

$$m(x^*) = \inf_{y \in X} \varphi(x^*, y)$$

is convex, does not take the value $-\infty$, and it is lower semicontinuous. In particular, $m^{**} = m$.

Proof. It is well known that the function m is convex (see, for instance, Th. 2.1.3(v) in [20]). Let us now show that $m(x^*) > -\infty$ for every $x^* \in X^*$. Fix $x^* \in X^*$. If $\varphi(x^*, y) = +\infty$ for every $y \in X$, then $m(x^*) = +\infty$. Otherwise, suppose that $\varphi(x^*, \cdot)$ is proper. By taking in **(C)** $x_n^* = x^*$ for every n, we deduce that the function $y \mapsto \varphi(x^*, y)$ is coercive. Since it is also convex and lower semicontinuous, it attains a minimum on X. In particular, m never takes the value $-\infty$. To prove the lower semicontinuity, we follow Aubin's proof with some adjustements (see Prop. 1.7 in [2]). If m is identically $+\infty$, then there is nothing to prove.

Otherwise, if m is proper, then it is sufficient to show that for every $\lambda \in \mathbb{R}$, the set

$$S_{m,\lambda} := \{x^* \in X^* : m(x^*) \leq \lambda\}$$

is closed. Since X is reflexive and $S_{m,\lambda}$ is convex, we will use the norm topology. Let $\{x_n^*\}$ be a sequence in $S_{m,\lambda}$, converging to x_0^*. For each $n \in \mathbb{N}$, since we have $m(x_n^*) \leq \lambda < +\infty$ and, as said before, the function $y \mapsto \varphi(x_n^*, y)$ attains a minimum, there exists $y_n \in X$ such that $\varphi(x_n^*, y_n) = m(x_n^*)$. We deduce that

$$\lambda \geq m(x_n^*) = \varphi(x_n^*, y_n),$$

and therefore, by condition (**C**), the sequence $\{y_n\}$ must be bounded. By the reflexivity of the space X, there exists a subsequence $\{y_{n'}\}$ converging weakly to some $y_0 \in C$. Then $(x_{n'}^*, y_{n'})$, norm$\times w$-converges to (x_0^*, y_0), and since $\varphi(\cdot,\cdot)$ is also norm$\times w$-lower semicontinuous, $\varphi(x_0^*, y_0) \leq \lambda$ so $m(x_0^*) \leq \lambda$. This proves that $S_{m,\lambda}$ is closed. Finally, the equality $m^{**} = m$ trivially holds if $m(x) = +\infty$ for all $x \in X$, and it follows from classical results in case m is proper. □

Later on, we will apply Theorem 2.3 to the bifunction φ defined by

$$\varphi(x^*, y) = \varphi_f(y, x^*) + h(y) \tag{4}$$

where $f : X \times X \to [-\infty, +\infty]$ is a lower normal bifunction on C such that $f(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$, and $h : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous.

Let us list some cases where condition (**C**) holds for the particular bifunction in (4). Let $\{x_n^*\} \subset X^*$ be a bounded sequence such that $\|x_n^*\| \leq K$, and $\{y_n\} \subset X$ with $\|y_n\| \to +\infty$. Note that, from Property P2 of the Fitzpatrick transform, if $y_n \notin C$ then $\varphi_f(y_n, x_n^*) = +\infty$. Thus without loss of generality, we can take $\{y_n\} \subset C$.

(a) Suppose that h is supercoercive on C. Take any $\bar{x} \in C$. Then $f(\bar{x},\cdot)$ is minorized by an affine function of the form $\langle y^*, \cdot\rangle + \alpha$, so

$$\begin{aligned}
\varphi_f(y_n, x_n^*) + h(y_n) &\geq \langle x_n^*, \bar{x}\rangle + f(\bar{x}, y_n) + h(y_n) \\
&\geq -\|\bar{x}\|\,\|x_n^*\| + \langle y^*, y_n\rangle + \alpha + h(y_n) \\
&\geq -K\|\bar{x}\| - \|y^*\|\,\|y_n\| + \alpha + h(y_n) \\
&= \|y_n\| \left(\frac{-K\|\bar{x}\| + \alpha}{\|y_n\|} - \|y^*\| + \frac{h(y_n)}{\|y_n\|} \right) \to +\infty.
\end{aligned}$$

(b) Suppose that $f(x, y)$ has the form $f_1(y) - f_1(x)$, where $f_1 : X \to \mathbb{R}$ is convex, lower semicontinuous, coercive, and h is bounded from below. In this case f is real-valued in $X \times X$, and $\varphi_f(y, x^*) = f_1(y) + f_1^*(x^*)$. In particular f_1^* is proper, convex, lower semicontinuous so it is minorized by an affine function. Thus, it is bounded from below on bounded sets. It follows that:

$$\lim_n(\varphi_f(y_n, x_n^*) + h(y_n)) = \lim_n(f_1(y_n) + f_1^*(x_n^*) + h(y_n)) = +\infty.$$

(c) Suppose that there exists $\bar{x} \in X$ such that $f(\bar{x},\cdot)$ is coercive and h is bounded from below. Under these assumptions,

$$\varphi_f(y_n, x_n^*) + h(y_n) \geq -\|x_n^*\|\,\|\bar{x}\| + f(\bar{x}, y_n) + h(y_n) \to +\infty$$

as $\|y_n\| \to +\infty$. Cases like $h \equiv 0$, as well as case (b) are included.

Remark 2.4. Suppose that h is supercoercive on C, $\mathrm{dom}(h)\cap C \neq \emptyset$, and $f(\cdot,y)$ is concave and upper semicontinuous for every $y \in C$. Then, under the assumptions of Theorem 2.3, the marginal function m of (4) is proper. In fact, let $\bar{y} \in C$ be such that $h(\overline{y}) \in \mathbb{R}$. By property P4 of the Fitzpatrick transform, there exists x^* such that $\varphi_f(\overline{y}, x^*) \in \mathbb{R}$. Hence, $m(x^*) \in \mathbb{R}$.

3. Existence results for EP and MEP

In this section we consider the (generalized) equilibrium problem (EP_f): Given a nonempty, closed and convex subset C of X, find a point $\bar{x}$ in C such that

$$f(\bar{x}, y) + h(y) \geq h(\bar{x}), \quad \forall y \in C, \qquad (\mathrm{EP}_f)$$

where f is a bifunction real-valued on $C\times C$, and h is a proper function defined in X.

Besides to (EP_f) one can consider the Minty (generalized) equilibrium problem (MEP_f): find a point $\bar{x}$ in C such that

$$-f(y, \bar{x}) + h(y) \geq h(\bar{x}), \quad \forall y \in C. \qquad (\mathrm{MEP}_f)$$

We will denote by S^f_{EP} and S^f_{MEP} the solution sets of (EP_f) and (MEP_f), respectively. Note that if we define the bifunction $\hat{f} : X \times X \to \mathbb{R}$ by

$$\hat{f}(x, y) = -f(y, x),$$

then (MEP_f) is nothing but ($\mathrm{EP}_{\hat{f}}$), and

$$S^{\hat{f}}_{\mathrm{MEP}} = S^f_{\mathrm{EP}}, \quad S^{\hat{f}}_{\mathrm{EP}} = S^f_{\mathrm{MEP}}. \qquad (5)$$

We call $\hat{f}$ the *mirror bifunction* of f. Note that if f is lower normal with respect to C, then $\hat{f}$ is not lower normal; in the language of [18, pg. 349] it is the upper simple extension of the bifunction $(x, y) \in C \times C \mapsto -f(y, x)$.

Let us recall the following existence result for an equilibrium problem (see [17, Theorem 2.1]):

Theorem 3.1. *Let E be a Hausdorff topological vector space, and C be a nonempty, closed and convex subset of E. Suppose that the function $\psi_1 : C \to \,]-\infty, +\infty]$ is proper, lower semicontinuous and convex, and that the function $\psi : C \to \mathbb{R}$ satisfies the following assumptions:*

(A_1) $\psi(v, v) \leq 0$ *for every* $v \in C$,

(A_2) $\psi(v, \cdot)$ *is concave for each* $v \in C$,

(A_3) $\psi(\cdot, w)$ *is lower semicontinuous on* C *for each* $w \in C$.

If there exists a compact subset B of E, and a vector $w_0 \in B \cap C$ such that

$$\psi_1(w_0) < +\infty \quad \textit{and} \quad \psi_1(v) + \psi(v, w_0) > \psi_1(w_0) \qquad (6)$$

for all v in $C \setminus B$, then the set of solutions to the problem: find a point $\overline{v}$ in C with

$$\psi_1(\overline{v}) + \psi(\overline{v}, w) \leq \psi_1(w) \quad \forall w \in C$$

is a nonempty and compact subset of $B \cap C$.

The following conditions on f and h will be taken into account in the sequel:

(**A**) f is a saddle bifunction, real-valued on $C \times C$, such that for each x in C, $f(x,x) \geq 0$, $f(x,\cdot)$ is lower semicontinuous, and $f(\cdot,x)$ is upper semicontinuous;

(**B**) $h : X \to]-\infty, +\infty]$ is convex, lower semicontinuous, and $\mathrm{dom}(h) \cap C \neq \emptyset$.

Proposition 3.2. *Suppose that conditions* (**A**) *and* (**B**) *hold. Then* $S^f_{\mathrm{MEP}} \subseteq S^f_{\mathrm{EP}}$. *Furthermore, if* h *is supercoercive, the set* S^f_{EP} *is nonempty.*

Proof. Set $C_1 := \mathrm{dom}(h) \cap C$. By the assumptions, C_1 is nonempty and convex. To show that $S^f_{\mathrm{MEP}} \subseteq S^f_{\mathrm{EP}}$, define the bifunction $F : C_1 \times C_1 \to \mathbb{R}$ by

$$F(x,y) = f(x,y) + h(y) - h(x).$$

We assume that $S^f_{\mathrm{MEP}} \neq \emptyset$, otherwise the inclusion is trivial. Let $\bar{x} \in S^f_{\mathrm{MEP}}$; then $\bar{x} \in C_1$ and $F(x,\bar{x}) \leq 0$ for all $x \in C_1$. For every $x \in C_1$ the function $F(x,\cdot)$ is convex, so setting $x_t = (1-t)\overline{x} + tx,\ 0 \leq t \leq 1$, we find

$$0 \leq F(x_t, x_t) \leq (1-t)F(x_t,\overline{x}) + tF(x_t, x). \tag{7}$$

Since $F(x_t,\overline{x}) \leq 0$, (7) implies that $F(x_t,x) \geq 0$, $0 < t \leq 1$. By upper semicontinuity of $F(\cdot,x)$ we have

$$F(\bar{x},x) \geq \limsup_{t\to 0^+} F(x_t,x) \geq 0.$$

This means that $f(\bar{x},x) + h(x) \geq h(\bar{x})$, i.e., $\bar{x} \in S^f_{\mathrm{EP}}$ and $S^f_{\mathrm{MEP}} \subseteq S^f_{\mathrm{EP}}$. Let us show that (EP) is solvable. Setting $\psi_1 := h$ and $\psi(v,w) := -f(v,w)$, it is easy to check that assumptions (A_1-A_3) in Theorem 3.1 are fulfilled when we consider the weak topology on X. Choose $\overline{x} \in C_1$, and for each $r > 0$ define the weakly compact set

$$B_r = \{x \in X : \|x - \overline{x}\| \leq r\}.$$

Since h is supercoercive and $-f(\cdot,\overline{x})$ is convex and lower semicontinuous, there exists a point x^* in X^* and a real number α such that for all $x \in C$,

$$h(x) - f(x,\overline{x}) \geq h(x) + \langle x^*, x\rangle + \alpha \geq \|x\| \left(\frac{h(x)}{\|x\|} - \|x^*\| + \frac{\alpha}{\|x\|} \right) \underset{\|x\|\to+\infty}{\longrightarrow} +\infty. \tag{8}$$

Therefore, there exists a positive number $\hat{r}$ such that

$$h(x) - f(x,\overline{x}) > h(\overline{x})$$

for all $x \in C$ satisfying the inequality $\|x - \overline{x}\| > \hat{r}$. Since $B_{\hat{r}}$ is weakly compact, by Theorem 3.1, $S^f_{\mathrm{EP}} \neq \emptyset$. □

Proposition 3.3. *Suppose that conditions* (**A**) *and* (**B**) *hold. Furthermore, assume that* $f(x,x) = 0$ *for every* $x \in C$*, and one of the following conditions holds:*

(a) *for some* $\tilde{x} \in \mathrm{dom}(h) \cap C$*,* $f(\tilde{x},\cdot)$ *is coercive and* h *is bounded from below;*

(b) *for some* $\tilde{x} \in \mathrm{dom}(h) \cap C$*,* $-f(\cdot,\tilde{x})$ *is coercive and* h *is bounded from below;*

(c) h *is supercoercive.*

Then $S^f_{\mathrm{EP}} = S^f_{\mathrm{MEP}}$*, both sets are nonempty, and they are included in* $\mathrm{dom}(h) \cap C$.

Proof. The mirror bifunction $\hat{f}(x,y) = -f(y,x)$ satisfies the assumptions of the first part of Proposition 3.2. Thus, $S^{\hat{f}}_{\text{MEP}} \subseteq S^{\hat{f}}_{\text{EP}}$. It follows immediately from (5) that $S^f_{\text{EP}} \subseteq S^f_{\text{MEP}}$, so finally $S^f_{\text{MEP}} = S^f_{\text{EP}}$. To finish the proof, it is enough to show that at least one of the sets S^f_{EP}, S^f_{MEP} is nonempty.

Assume that (a) holds, i.e., $f(\tilde{x},\cdot)$ is coercive for some $\tilde{x} \in C_1$, and h is bounded from below. We set $\psi_1 := h$ and $\psi(v,w) := f(w,v)$. The assumptions (A_1)–(A_3) in Theorem 3.1 are fulfilled for the functions ψ_1 and ψ when we consider the weak topology on X. Since $f(\tilde{x},\cdot) + h(\cdot)$ is coercive, we deduce that there exists $\hat{r} > 0$ such that

$$f(\tilde{x},x) + h(x) > h(\tilde{x})$$

for all $x \in C \backslash B_{\hat{r}}$. This means that all conditions of Theorem 3.1 are fulfilled and $S^f_{\text{MEP}} \neq \emptyset$.

Let us now consider condition (b), and observe that this is equivalent to say that $\hat{f}$ satisfies (a). Thus, following the same steps, we obtain $S^{\hat{f}}_{\text{MEP}} = S^f_{\text{EP}} \neq \emptyset$.

As for condition (c), we know already from Proposition 3.2 that $S^f_{\text{EP}} \neq \emptyset$. Thus, the solution sets are nonempty in all cases. □

4. Dual equilibrium problem

In this section, we introduce and investigate a suitable notion of the dual problem for (EP_f), which is itself an equilibrium problem. Our study builds on results obtained in finite-dimensional settings (see [6]), while also extending them in several directions. The bifunction f may take extended real values, the function h is not necessarily the indicator function of a compact convex set C; instead, C is now assumed to be convex and closed, rather than convex and compact.

In the sequel we will assume that f is lower normal on C, that h is proper and that $C_1 := \text{dom}(h) \cap C \neq \emptyset$.

It is well-known that equilibrium problems are strictly related to optimization problems. To this purpose, set

$$g(x) = \sup_{y \in C_1} \left(-f(x,y) - h(y)\right), \quad x \in X. \tag{9}$$

Under our assumptions, the function g takes values within $]-\infty, +\infty]$.

Let us consider the following optimization problem:

$$\inf_{x \in X} (g(x) + h(x)). \tag{P}$$

Note that a point $\bar{x}$ is a solution to (EP_f) if and only if $g(\bar{x}) + h(\bar{x}) \leq 0$. In fact, if $\bar{x}$ is a solution, then it belongs to C_1 and

$$\sup_{y \in C}(-f(\bar{x},y) - h(y)) \leq -h(\bar{x}).$$

Since we have $\sup_{y \in C}(-f(\bar{x},y) - h(y)) = \sup_{y \in C_1}(-f(\bar{x},y) - h(y))$, we conclude $g(\bar{x}) + h(\bar{x}) \leq 0$. The converse works in a similar way. In addition, if $f(x,x) = 0$ for every $x \in C$, then $g(x) + h(x) \geq 0$; in fact, if $x \notin C_1$ the inequality is trivial.

Otherwise, if $x \in C_1$, we have

$$g(x) = \sup_{y \in C_1} (-f(x, y) - h(y)) \geq -f(x, x) - h(x) = -h(x).$$

We summarize the above discussion in the following proposition:

Proposition 4.1. *Let f be a lower normal bifunction such that $f(x, x) = 0$ for all $x \in C$, and h be proper and such that $C_1 \neq \emptyset$. Then $\bar{x}$ is a solution of* (EP$_f$) *if and only if it is a solution of* (P). *In this case, the infimum of $g + h$ equals* 0 *and is achieved at $\bar{x}$.*

Remark 4.2. Some properties of f affect g :

(i) g is convex if $f(\cdot, y)$ is concave, for all $y \in C$;

(ii) g is real-valued on C under one of the following conditions:

   (a) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$, and h is proper, convex, lower semicontinuous and supercoercive on C;

   (b) $f(x, \cdot)$ is convex, lower semicontinuous and coercive for all $x \in C$, and h is proper, convex, lower semicontinuous and bounded from below.

   In fact, in both cases, the supremum in (9) is a maximum; this follows because $f(x, \cdot) + h(\cdot)$ is proper, convex, lower semicontinuous and coercive. □

Let us now consider the bifunction $\Phi_f : X^* \times X^* \to [-\infty, +\infty]$ associated to a bifunction f and defined as follows:

$$\begin{aligned} \Phi_f(x^*, y^*) &= \sup_{y \in X} \inf_{x \in X} (\langle y^*, y\rangle - \langle x^*, x\rangle - f(x, y)) \\ &= \sup_{y \in X} (\langle y^*, y\rangle - \varphi_f(y, x^*)) = (\varphi_f(\cdot, x^*))^* (y^*). \end{aligned} \tag{10}$$

We first investigate some properties of Φ_f. We will assume that f is lower normal with respect to a nonempty, closed and convex set C, and we will add further properties on f when needed.

(a) $\Phi_f(x^*, \cdot)$ is convex on X^* for every $x^* \in X^*$, since it is the Fenchel transform of a function. Moreover, if $f(x, \cdot)$ is convex for every $x \in X$, then $\Phi_f(\cdot, y^*)$ is concave for every $y^* \in X^*$, and thus Φ_f is a saddle function.

(b) $\Phi_f(x^*, \cdot)$ is lower semicontinuous on X^* for every $x^* \in X^*$, since it is the Fenchel transform of a function.

Furthermore, if $f(x, \cdot)$ is lower semicontinuous and convex for every $x \in X$, and there exists $\tilde{x} \in X$ such that $f(\tilde{x}, \cdot)$ is supercoercive, then

(i) $\Phi_f(\cdot, y^*)$ is upper semicontinuous for every y^* in X^*;

(ii) Φ_f is real-valued on $X^* \times X^*$.

In fact, by relation (10), we get $-\Phi_f(x^*, y^*) = \inf_{x \in X}(-\langle y^*, x\rangle + \varphi_f(x, x^*))$.

We will apply Theorem 2.3 to the bifunction $\varphi(x^*, x) = \varphi_f(x, x^*) - \langle y^*, x\rangle$, for every $y^* \in X^*$. By the properties of the Fitzpatrick transform, we know that φ is convex, lower semicontinuous, and never takes the value $-\infty$. Let us prove that condition

(**C**) is fulfilled. Take any bounded sequence $\{x_n^*\} \subset X^*$, and $\{x_n\} \subset X$ such that $\|x_n\| \to +\infty$. Then

$$\begin{aligned}\varphi(x_n^*, x_n) &\geq \langle x_n^*, \tilde{x}\rangle + f(\tilde{x}, x_n) - \langle y^*, x_n\rangle \\ &\geq -\|x_n^*\| \cdot \|\tilde{x}\| + f(\tilde{x}, x_n) - \|y^*\| \cdot \|x_n\| \\ &= -\|x_n^*\| \cdot \|\tilde{x}\| + \|x_n\| \left(\frac{f(\tilde{x}, x_n)}{\|x_n\|} - \|y^*\|\right),\end{aligned}$$

and thus $\varphi(x_n^*, x_n) \to +\infty$. Application of Theorem 2.3 now gives point (i).

Point (ii) easily follows from Properties P1 and P3 of the Fitzpatrick transform. Indeed, under our assumptions, $\varphi_f(\cdot, x^*)$ is supercoercive and lower semicontinuous, for every $x^* \in X^*$, and therefore the supremum is achieved in some point.

(c) To analyze the behaviour of Φ_f on the diagonal, note that

$$\Phi_f(x^*, x^*) = \sup_{x\in X} \inf_{y\in X} \left(\langle x^*, x - y\rangle - f(y, x)\right).$$

Since f is lower normal with respect to the set C, we have

$$\inf_{y\in X} \left(\langle x^*, x - y\rangle - f(y, x)\right) = \inf_{y\in C} \left(\langle x^*, x - y\rangle - f(y, x)\right),$$

for every $x \in C$, while, if $x \notin C$,

$$\inf_{y\in X} \left(\langle x^*, x - y\rangle - f(y, x)\right) = -\infty.$$

Therefore, if $f(x, x) \geq 0$ for all $x \in C$, it follows for all $x^* \in X^*$ that

$$\Phi_f(x^*, x^*) = \sup_{x\in C} \inf_{y\in C} \left(\langle x^*, x - y\rangle - f(y, x)\right) \leq \sup_{x\in C}(-f(x, x)) \leq 0.$$

In order to satisfy the reverse inequality $\Phi_f(x^*, x^*) \geq 0$, it is enough to show that there exists $x = x(x^*)$ in X such that

$$\langle x^*, x - y\rangle - f(y, x) \geq 0, \quad \forall y \in X. \tag{11}$$

Suppose that f is null on the diagonal, $f(\cdot, y)$ is concave for every y in X, $f(x, \cdot)$ is lower semicontinuous for every x in X, and there exists $\tilde{x}$ in X such that the function $f(\tilde{x}, \cdot)$ is supercoercive. Then, if we set $\psi(x, y) = f(y, x)$ and $\psi_1(x) = -\langle x^*, x\rangle$, we can apply Theorem 3.1 to get (11) fulfilled for a suitable $x = x(x^*)$.

Our aim is to relate the bifunction Φ_f to an equilibrium problem that can be considered to be the dual of the original one.

Let h be a function on X such that h^* is real-valued on X^*. We define the dual equilibrium problem to (EP_f) to be the problem of finding a point $\overline{x}^*$ in X^* such that

$$\Phi_f(\overline{x}^*, y^*) + h^*(-y^*) \geq h^*(-\overline{x}^*) \quad \forall y^* \in X^*. \tag{DEP$_f$}$$

The solutions to the dual problem will be denoted by S^f_{DEP}.

Note that for every $x^* \in S^f_{\mathrm{DEP}}$, (DEP_f) implies $\Phi_f(x^*, x^*) \geq 0$. From property (c) above, we deduce:

$$\Phi_f(x^*, x^*) = 0, \quad \forall x^* \in S^f_{\mathrm{DEP}} \tag{12}$$

provided that $f(x, x) \geq 0$ for all $x \in C$.

In order to state our main results, let us consider the optimization problem (P) with value

$$p = \inf_{x\in X}(g(x) + h(x)),$$

and denote by d the value of the dual concave optimization problem according to Fenchel (see, for instance, [10], Chapter 4):

$$d = \sup_{x^*\in X^*} (-g^*(x^*) - h^*(-x^*)). \tag{D}$$

Then the following result holds:

Theorem 4.3. *Let f, h satisfy the assumptions* (**A**) *and* (**B**)*, with $f(x, x) = 0$ on C. In addition, suppose that h is supercoercive on C. Then*

$$0 = p = d = \sup_{x^*\in X^*} \inf_{y^*\in X^*} (\Phi_f(x^*, y^*) + h^*(-y^*) - h^*(-x^*)). \tag{13}$$

Proof. First of all note that the supercoercivity of h implies that h^* is real-valued. By Proposition 3.2, $S^f_{\mathrm{EP}} \neq \emptyset$, hence by Proposition 4.1, $p = 0$. By Remark 4.2, g is convex and real-valued on C. By the assumption on $f(\cdot, y)$, g is lower semicontinuous on X. Hence, $g^{**} = g$ and $h^{**} = h$.

If we define $h_1(x) = h(-x)$, then $h_1^*(x^*) = h^*(-x^*)$. Now we apply strong Fenchel duality (see for instance [10, Theorem 4.4.3]) to the functions g^* and h_1^*, defined on the space X^*. We deduce that

$$\inf_{x^*\in X^*} (g^*(x^*) + h_1^*(x^*)) = \sup_{x\in X} (-g(x) - h_1(-x)) \tag{14}$$

which immediately gives $p = d$. It remains to show the last equality in (13). We first calculate $g^*(x^*)$. Since $C_1 = C \cap \mathrm{dom}(h)$ we find, using Proposition 2.2:

$$\begin{aligned} g^*(x^*) &= \Big(\sup_{y\in C_1} (-f(\cdot, y) - h(y))\Big)^*(x^*) \\ &= \Big(\inf_{y\in C_1} (-f(\cdot, y) - h(y))^*\Big)^{**}(x^*) \end{aligned}$$

Now we note that

$$\begin{aligned} \inf_{y\in C_1} (-f(\cdot, y) - h(y))^* (x^*) &= \inf_{y\in C_1} ((-f(\cdot, y))^* (x^*) + h(y)) \\ &= \inf_{y\in C_1} (\varphi_f(y, x^*) + h(y)). \end{aligned}$$

From Theorem 2.3 and the discussion that follows it, we know that the function

$$m(x^*) := \inf_{y\in C_1} (\varphi_f(y, x^*) + h(y)) = \inf_{y\in X} (\varphi_f(y, x^*) + h(y))$$

is convex and lower semicontinuous, so $m = m^{**}$. Thus,

$$g^*(x^*) = m(x^*) = \inf_{y\in X} (\varphi_f(y, x^*) + h(y)) \tag{15}$$

Now given x^*, we apply again strong Fenchel duality to the functions $(\varphi_f(\cdot, x^*))^*$ and $h^*(-\cdot)$. This is possible because they are both convex and lower semicontinuous, h^* is real valued, and $(\varphi_f(\cdot, x^*))^*$ is proper by property **P4**.

We deduce that

$$\inf_{y^*\in X^*}((\varphi_f(\cdot,x^*))^*(y^*)+h^*(-y^*)) = \sup_{y\in X}(-\varphi_f(y,x^*)-h(y)). \tag{16}$$

Combining with (15) we obtain

$$\begin{aligned} g^*(x^*) &= -\inf_{y^*\in X^*}((\varphi_f(\cdot,x^*))^*(y^*)+h^*(-y^*)) \\ &= -\inf_{y^*\in X^*}(\Phi_f(x^*,y^*)+h^*(-y^*)). \end{aligned}$$

Substituting into (D) we find

$$\begin{aligned} d &= \sup_{x^*\in X^*}\left(\inf_{y^*\in X^*}(\Phi_f(x^*,y^*)+h^*(-y^*)) - h^*(-x^*)\right) \\ &= \sup_{x^*\in X^*}\inf_{y^*\in X^*}(\Phi_f(x^*,y^*)+h^*(-y^*)-h^*(-x^*)) \end{aligned} \tag{17}$$

thereby obtaining the desired representation for the dual optimal value d. □

Theorem 4.4. *Suppose that the assumptions of Theorem* 4.3 *are fulfilled. If, in addition,* $\operatorname{int}\operatorname{dom}(h)\cap C \neq \emptyset$*, then the dual equilibrium problem* (DEP$_f$) *has a solution.*

Proof. Note that, under the assumptions, strong Fenchel duality can be applied directly to problem (P). We deduce that $p = d$ and in addition, given that $p = 0$, the supremum in (D) is attained [10, Theorem 4.4.3]. From the proof of Theorem 4.3 one sees that the supremum in (17) is attained. Since $d = 0$, this means that there exists $x^* \in X^*$ such that

$$\Phi_f(x^*,y^*)+h^*(-y^*)-h^*(-x^*) \geq 0 \quad \forall y^* \in X^*.$$

Hence, (DEP$_f$) has a solution. □

Remark 4.5. Under the assumptions of Theorem 4.3,

(i) for every $x^* \in X^*$ the infimum in (15) is effectively a minimum, since the function $y \mapsto \varphi_f(y,x^*)+h(y)$ is proper, lower semicontinuous, convex and coercive on the nonempty, closed and convex set C;

(ii) from the last part of the proof of Theorem 4.4 we obtain that a point z^* in X^* is a solution to (DEP$_f$) if and only if the equality $g^*(z^*)+h^*(-z^*) = 0$ is satisfied. □

We will now investigate the relationships between the set

$$S = \{(x,x^*) \in X \times X^* : \varphi_f(x,x^*)+h(x)+h^*(-x^*) = 0\}$$

and the solution sets of the primal equilibrium problem (EP$_f$) and of the dual equilibrium problem (DEP$_f$). We define the function $\xi : X \times X^* \to]-\infty,+\infty]$ by

$$\xi(x,x^*) := \varphi_f(x,x^*)+h(x)+h^*(-x^*).$$

Note that, assuming only that $f(x,x) \geq 0$ for every $x \in C$, property **P2** of φ_f gives that for all $(x,x^*) \in X \times X^*$,

$$\varphi_f(x,x^*) \geq \langle x^*, x\rangle.$$

Since we also have $h(x) + h^*(-x^*) \geq -\langle x^*, x\rangle$ we deduce that $\xi(x, x^*) \geq 0$ for all $(x, x^*) \in X \times X^*$. Thus, if S is nonempty, then $S = \operatorname{argmin} \xi$ is a closed and convex subset of $X \times X^*$, since ξ is convex and lower semicontinuous. Also, by Proposition 4.1, $S^f_{\mathrm{EP}} = \operatorname{argmin}(g + h)$, so this set is also closed and convex. Finally, from Theorem 4.3 we see that

$$S^f_{\mathrm{DEP}} = \operatorname{argsup}\Big(\inf_{y^*\in X^*}(\Phi_f(\cdot, y^*) + h(-y^*) - h^*(-\cdot))\Big).$$

Assuming that f is lower normal and $f(x, \cdot)$ is convex for every x, we know that Φ_f is a saddle function (see property (a) of Φ_f). It follows easily that S^f_{DEP}, being the argsup of a concave function, is convex. It is also closed, if we assume in addition that $f(x, \cdot)$ is lower semicontinuous for every $x \in C$, and supercoercive for some $\hat{x} \in C$ (see property (b) of Φ_f).

According to the definitions above, we have

$$(x, x^*) \in S \iff \xi(x, x^*) = 0.$$

Let us now relate S^f_{EP} and S^f_{DEP} with the function ξ.

Proposition 4.6. *Suppose that the assumptions of Theorem* 4.3 *are fulfilled and, in addition,* $\operatorname{int}\operatorname{dom}(h) \cap C \neq \emptyset$. *Then*

$$x \in S^f_{\mathrm{EP}} \iff \min_{x^*\in X^*} \xi(x, x^*) = 0 \tag{18}$$

$$x^* \in S^f_{\mathrm{DEP}} \iff \min_{x\in X} \xi(x, x^*) = 0. \tag{19}$$

Proof. We have the equivalences:

$$\begin{aligned} x \in S^f_{\mathrm{EP}} = S^f_{\mathrm{MEP}} &\iff -f(y, x) + h(y) \geq h(x), \quad \forall y \in X \\ &\iff \inf_{y\in X}(-f(y, x) + h(y)) = h(x) \end{aligned} \tag{20}$$

where we used $f(x, x) = 0$. The infimum in (20) is attained (in particular, it is finite) because h is assumed supercoercive. Now we apply strong Fenchel duality to the convex functions $-f(\cdot, x)$ and h, to find:

$$x \in S^f_{\mathrm{EP}} \iff \max_{x^*\in X^*}\left(-\left(-f(\cdot, x)\right)^*(x^*) - h^*(-x^*)\right) = h(x) \tag{21}$$

$$\iff \min_{x^*\in X^*}(\varphi_f(x, x^*) + h^*(-x^*)) + h(x) = 0$$

$$\iff \min_{x^*\in X^*} \xi(x, x^*) = 0$$

where the maximum in (21) is attained by [10, Theorem 4.4.3]. This proves (18). As for (19), if $x^* \in S^f_{DEP}$, then we know that $\Phi_f(x^*, x^*) = 0$, so the following equivalences hold:

$$x^* \in S^f_{\mathrm{DEP}} \iff \inf_{y^*\in X^*}(\Phi_f(x^*, y^*) + h^*(-y^*)) = h^*(-x^*)$$

$$\iff \inf_{y^*\in X^*}\left(\left(\varphi_f(\cdot, x^*)\right)^*(y^*) + h^*(-y^*)\right) = h^*(-x^*)$$

$$\iff \min_{x\in X}(\varphi_f(x, x^*) + h(x)) + h^*(-x^*) = 0$$

where we have used relation (16) (i.e., Fenchel duality again) and Remark 4.5. This shows (19). □

We deduce a relation between S, S^f_{EP} and S^f_{DEP}:

Proposition 4.7. *Suppose that the assumptions of Theorem* 4.3 *are fulfilled and, in addition,* $\operatorname{int}\operatorname{dom}(h) \cap C \neq \emptyset$. *Then:*

(i) $\emptyset \neq S \subseteq S^f_{\mathrm{EP}} \times S^f_{\mathrm{DEP}}$;

(ii) *for every* $x \in S^f_{\mathrm{EP}}$ *there exists* $x^* \in S^f_{\mathrm{DEP}}$ *such that* $(x, x^*) \in S$;

(iii) *for every* $x^* \in S^f_{\mathrm{DEP}}$ *there exists* $x \in S^f_{\mathrm{EP}}$ *such that* $(x, x^*) \in S$.

Proof. (ii) is an immediate consequences of Proposition 4.6. Indeed, if $x \in S^f_{\mathrm{EP}}$, then by (18) there exists $x^* \in X^*$ such that $\xi(x, x^*) = 0$. This means that $(x, x^*) \in S$. Likewise for (iii).

Since by Proposition 6, $S^f_{\mathrm{EP}} \neq \emptyset$, we see that (ii) implies that $S \neq \emptyset$.

Finally, for every $(x, x^*) \in S$, since $\xi(x, x^*) = 0$, (18) and (19) imply that $x \in S^f_{\mathrm{EP}}$ and $x^* \in S^f_{\mathrm{DEP}}$, i.e., $(x, x^*) \in S^f_{\mathrm{EP}} \times S^f_{\mathrm{DEP}}$. This finishes the proof. □

We deduce an immediate corollary:

Corollary 4.8. *Assumptions as in Proposition* 4.7. *If* S^f_{EP} *or* S^f_{DEP} *is a singleton, then* $S = S^f_{\mathrm{EP}} \times S^f_{\mathrm{DEP}}$.

Proof. Assume, for instance, that S^f_{EP} is a singleton, say $\{x\}$. For every $x^* \in S^f_{\mathrm{DEP}}$, Proposition 4.7(iii) implies the existence of an element $y \in X$ such that $(y, x^*) \in S$. Obviously, $y = x$, so $(x, x^*) \in S$ and $S^f_{\mathrm{EP}} \times S^f_{\mathrm{DEP}} \subseteq S$. The case S^f_{DEP} is a singleton is similar. □

The equality $S = S^f_{\mathrm{EP}} \times S^f_{\mathrm{DEP}}$ also holds in case the bifunction f actually stems from an optimization problem:

Example 4.9. Let F, h be proper convex lower semicontinuous functions and set $C = \operatorname{dom}(F)$. Assume that $C \cap \operatorname{dom}(h) \neq \emptyset$ and h is supercoercive. Define the lower normal bifunction f by

$$f(x, y) = F(y) - F(x), \quad (x, y) \in X \times X$$

where we use the convention $+\infty - \infty = -\infty$. A standard computation (using the same convention) gives

$$\begin{aligned}\varphi_f(y, x^*) &= F(y) + F^*(x^*),\\ \Phi_f(x^*, y^*) &= (\varphi_f(\cdot, x^*))^*(y^*) = F^*(y^*) - F^*(x^*).\end{aligned}$$

Therefore, the dual equilibrium problem is exactly the dual optimization problem according to Fenchel: find $\overline{x}^* \in X^*$ such that

$$-F^*(\overline{x}^*) - h^*(-\overline{x}^*) \geq -F^*(y^*) - h^*(-y^*) \quad \text{for all } y^* \in X^*.$$

Thus, the set S has the following characterization:

$$\begin{aligned}S &= \{(x, x^*) \in X \times X^* : F(x) + h(x) + F(x^*) + h^*(-x^*) = 0\}\\ &= \operatorname{argmin}_{X \times X^*}(F(x) + h(x) + F(x^*) + h^*(-x^*))\\ &= (\operatorname{argmin}_X(F(x) + h(x))) \times (\operatorname{argmin}_{X^*}(F(x^*) + h^*(-x^*)))\\ &= S^f_{\mathrm{EP}} \times S^f_{\mathrm{DEP}}\end{aligned}$$

Example 4.10. A particular example of the above, where both S^f_{EP} and S^f_{DEP} are not singletons, is the following. Define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) = \max\{0, x^2 - 1\}$. Then ψ is supercoercive, thus ψ^* is real-valued, and in fact

$$\psi^*(y) = \begin{cases} \frac{y^2}{4} + 1, & |y| \geq 2 \\ |y|, & |y| < 2. \end{cases}$$

Now define $F, h : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) = h(x,y) = \psi(x) + \psi^*(y)$$
$$f(x,y,x_1,y_1) = F(x_1,y_1) - F(x,y).$$

Since h is separable with respect to its variables, it is easy to see that $h^*(u,v) = \psi^*(u) + \psi(v)$ (that is, $h^*(u,v) = h(v,u)$) so

$$h^*(u,v) = \max\left\{0, v^2 - 1\right\} + \begin{cases} \frac{u^2}{4} + 1, & |u| \geq 2 \\ |u|, & |u| < 2. \end{cases}$$

According to the calculations above and $F = h$,

$$\varphi_f(x_1,y_1,u,v) = F(x_1,y_1) + F^*(u,v) = h(x_1,y_1) + h^*(u,v)$$
$$\Phi_f(u,v,u_1,v_1) = F^*(u_1,v_1) - F^*(u,v) = h^*(u_1,v_1) - h^*(u,v).$$

Now consider (EP_f) and (DEP_f): For (EP_f),

$$f(x,y,x_1,y_1) + h(x_1,y_1) - h(x,y) = 2\left(h(x_1,y_1) - h(x,y)\right)$$

so $S^f_{\text{EP}} = \operatorname{argmin} h = [-1,1] \times \{0\}$. As for ($\text{DEP}_f$), noting that ψ, ψ^* are even functions (and thus the same is true for h and h^*) we find

$$\begin{aligned} &\Phi_f(u,v,u_1,v_1) + h^*(-u_1,-v_1) - h^*(-u,-v) \\ &= h^*(u_1,v_1) - h^*(u,v) + h^*(-u_1,-v_1) - h^*(-u,-v) \\ &= 2(h^*(u_1,v_1) - h^*(u,v)). \end{aligned}$$

Thus, $S^f_{\text{DEP}} = \operatorname{argmin} h^* = \{0\} \times [-1,1]$.

So $S^f_{\text{EP}}, S^f_{\text{DEP}}$ are not singletons. Yet, according to Example 4.9, $S = S^f_{\text{EP}} \times S^f_{\text{DEP}}$.

5. The Minty dual equilibrium problem

In the previous section we introduced the dual equilibrium problem (DEP_f) and demonstrated its solvability. We now aim to establish the solvability of the dual equilibrium problem directly by applying Theorem 3.1 under different conditions.

As we can associate a Minty counterpart with any equilibrium problem related to a bifunction f through the mirror bifunction $\hat{f}$, we can similarly associate the Minty dual equilibrium problem (MDEP_f) with the dual equilibrium problem (DEP_f) through the mirror bifunction. If f is lower normal, then

$$\begin{aligned} \widehat{\Phi_f}(x^*,y^*) = -\Phi_f(y^*,x^*) &= -\left(\varphi_f(\cdot,y^*)\right)^*(x^*) \\ &= \inf_{x \in X} \sup_{y \in X} \left(\langle y^*, y\rangle - \langle x^*, x\rangle + f(y,x)\right) \\ &= \inf_{x \in C} \sup_{y \in C} \left(\langle y^*, y\rangle - \langle x^*, x\rangle + f(y,x)\right) \qquad (22) \end{aligned}$$

Then, the (MDEP$_f$) consists in finding a point $\overline{x}^*$ in X^* such that

$$\widehat{\Phi_f}(\overline{x}^*, y^*) + h^*(-y^*) \geq h^*(-\overline{x}^*) \quad \forall y^* \in X^*. \qquad (\text{MDEP}_f)$$

The solutions to the Minty dual problem will be denoted by S^f_{MDEP}.

Proposition 5.1. *Let f be a saddle, lower normal bifunction with respect to a nonempty, closed and convex set C, such that $f(x,x) = 0$ and $f(x,\cdot)$ is lower semicontinuous for all $x \in C$. Let h be a proper and convex function. If $f(\tilde{x},\cdot)$ is supercoercive for some $\tilde{x} \in C$, then $S^f_{\text{DEP}} = S^f_{\text{MDEP}}$.*

Proof. We apply the first part of Proposition 3.2 to the space X^*, the function $h_1(x^*) = h^*(-x^*)$, and the bifunction $f_1(x^*, y^*) := \Phi_f(x^*, y^*)$. To avoid confusion, we denote by S^{f_1,h_1}_{EP} and S^{f_1,h_1}_{MEP} the solutions of the equilibrium problem and the Minty equilibrium problem for f_1 and h_1 in the space X^*. According to Properties (a)–(c) of the bifunction Φ_f, f_1 is a saddle function, real-valued on X^*, such that $f_1(x^*, x^*) = 0$ for all $x^* \in X^*$. Also, $f_1(x^*,\cdot)$ is lower semicontinuous and $f_1(\cdot, y^*)$ is upper semicontinuous. Finally, h_1 is proper, convex and lower semicontinuous. Hence, the first part of Proposition 3.2 implies that $S^{f_1,h_1}_{\text{MEP}} \subseteq S^{f_1,h_1}_{\text{EP}}$. Since $S^{f_1,h_1}_{\text{MEP}} = S^f_{\text{MDEP}}$ and $S^{f_1,h_1}_{\text{EP}} = S^f_{\text{DEP}}$, we have $S^f_{\text{MDEP}} \subseteq S^f_{\text{DEP}}$.
Now we apply Proposition 3.2 to the bifunction $f_2(x^*, y^*) := \widehat{\Phi_f}(x^*, y^*)$ and the same function h_1. Again f_2 and h_1 satisfy the assumptions of the first part of Proposition 3.2, so $S^{f_2,h_1}_{\text{MEP}} \subseteq S^{f_2,h_1}_{\text{EP}}$. Since $S^{f_2,h_1}_{\text{MEP}} = S^f_{\text{DEP}}$ and $S^{f_2,h_1}_{\text{EP}} = S^f_{\text{MDEP}}$, we obtain the reverse inclusion $S^f_{\text{DEP}} \subseteq S^f_{\text{MDEP}}$. □

Theorem 5.2. *Let f be a saddle, lower normal bifunction with respect to a nonempty, closed and convex set C, such that $f(x,x) = 0$ and $f(x,\cdot)$ is lower semicontinuous for all $x \in C$. Let h be proper, convex, lower semicontinuous. Assume that there exists a point $y_0 \in \operatorname{int}\operatorname{dom}(h) \cap C$ such that $f(\cdot, y_0)$ is bounded from above by a continuous affine function $\langle y_0^*, \cdot\rangle + \alpha$. If $f(\tilde{x},\cdot)$ is supercoercive for some $\tilde{x} \in C$, then the dual equilibrium problem is solvable.*

Proof. By Proposition 5.1 it suffices to prove that S^f_{MDEP} is nonempty. We will apply Theorem 3.1 to the space X^*, the bifunction $-\widehat{\Phi_f}$, and the function $h^*(-\cdot)$. All assumptions (A_1-A_3) are satisfied. The remaining task is to check the coercivity condition (6). To do so, it is enough to show that

$$\lim_{\|x^*\|\to+\infty} (h^*(-x^*) - \widehat{\Phi_f}(x^*, -y_0^*)) = +\infty. \tag{23}$$

Since h is continuous at y_0, we know that $h^*(-\cdot) + \langle \cdot, y_0\rangle$ is coercive (see Corollary 4.4.11 in [9]). Thus,

$$\lim_{\|x^*\|\to+\infty} (h^*(-x^*) + \langle x^*, y_0\rangle) = +\infty. \tag{24}$$

On the other hand,

$$\begin{aligned} -\langle x^*, y_0\rangle - \widehat{\Phi_f}(x^*, -y_0^*) &= -\langle x^*, y_0\rangle + \Phi_f(-y_0^*, x^*) \\ &= -\langle x^*, y_0\rangle + \sup_{y\in X}\inf_{x\in X} (\langle x^*, y\rangle - \langle -y_0^*, x\rangle - f(x,y)) \\ &\geq -\langle x^*, y_0\rangle + \inf_{x\in X} (\langle x^*, y_0\rangle - \langle -y_0^*, x\rangle - f(x,y_0)) \geq -\alpha. \end{aligned} \tag{25}$$

From (24) and (25) we obtain (23). Thus, Theorem 3.1 applies, so (MDEP_f) is solvable. $\square$

Remark 5.3. The assumptions of the two existence results for the dual equilibrium problem in Theorem 4.4 and 5.2 can be compared: if we drop the condition of upper semicontinuity of $f(\cdot, x)$ and supercoercivity of h, then we need to require some boundedness from above for $f(\cdot, y)$ together with a supercoercivity of $f(x, \cdot)$ for suitable x, y. $\square$

In this section, we considered the Minty problem of the dual equilibrium problem (MDEP_f). What if we consider the dual of the original Minty problem? The Minty problem is simply the equilibrium problem corresponding to the bifunction $\hat{f}$. Hence its dual, that we denote by (DMEP_f), consists in finding a point $\overline{x}^*$ in X^* such that

$$\Phi_{\hat{f}}(\overline{x}^*, y^*) + h^*(-y^*) \geq h^*(-\overline{x}^*) \quad \forall y^* \in X^*. \tag{DMEP_f}$$

Here, assuming that f is lower normal on C,

$$\begin{aligned} \Phi_{\hat{f}}(x^*, y^*) = \left(\varphi_f\left(\cdot, y^*\right)\right)^*(x^*) &= \sup_{y \in X} \inf_{x \in X} \left(\langle y^*, y\rangle - \langle x^*, x\rangle + f\left(y, x\right)\right) \\ &= \sup_{y \in C} \inf_{x \in C} \left(\langle y^*, y\rangle - \langle x^*, x\rangle + f\left(y, x\right)\right) \qquad (26) \end{aligned}$$

Comparing with (22) we see that

$$\Phi_{\hat{f}}(x^*, y^*) \leq \widehat{\Phi_f}(x^*, y^*) \quad \forall (x^*, y^*) \in X^* \times X^*.$$

To obtain the equality, we can apply a minimax result due to Tuy (see Theorem 3 in [19]):

Proposition 5.4. *Let f be a saddle, lower normal bifunction on C, such that for each x in C, $f(x, \cdot)$ is lower semicontinuous, and $f(\cdot, x)$ is upper semicontinuous. If there exists $\tilde{x} \in X$ such that $f(\tilde{x}, \cdot)$ is supercoercive, then we have $\Phi_{\hat{f}} = \widehat{\Phi_f}$ and $S^f_{\text{DMEP}} = S^f_{\text{MDEP}}$.*

Proof. Take any $(x^*, y^*) \in X^* \times X^*$. We apply Theorem 3 in [19] taking $M = \{\tilde{x}\}$ and $F(x, y) = -\langle y^*, x\rangle + \langle x^*, y\rangle - f(x, y)$, $(x, y) \in C \times C$. This F satisfies all the requirements, so

$$\inf_{x \in C} \sup_{y \in C} \left(-\langle y^*, x\rangle + \langle x^*, y\rangle - f(x, y)\right) = \sup_{y \in C} \inf_{x \in C} \left(-\langle y^*, x\rangle + \langle x^*, y\rangle - f(x, y)\right).$$

Now we exchange x and y and then change the sign, to find

$$\sup_{y \in C} \inf_{x \in C} \left(\langle y^*, y\rangle - \langle x^*, x\rangle + f\left(y, x\right)\right) = \inf_{x \in C} \sup_{y \in C} \left(\langle y^*, y\rangle - \langle x^*, x\rangle + f\left(y, x\right)\right),$$

that is, $\Phi_{\hat{f}}(x^*, y^*) = \widehat{\Phi_f}(x^*, y^*)$. It follows immediately that $S^f_{\text{DMEP}} = S^f_{\text{MDEP}}$. $\square$

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