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A Generalization of a Classical Geometric Extremum Problem

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Let $\partial\mathcal{C}$ be the boundary of a compact convex body $\mathcal{C}$ in $\mathbb{R}^n$, $n \geq 2$, and O be an interior point of $\mathcal{C}$. Every straight line l containing O cuts from $\mathcal{C}$ a segment $[AB]$ with end-points on $\partial\mathcal{C}$. It is shown that if $[AB]$ is the shortest such segment, then $\partial\mathcal{C}$ is smooth at the points A and B (i.e. at both of them there is only one supporting hyperplane for $\mathcal{C}$) and, something more, the normals to the unique supporting hyperplanes at the points A and B intersect at a point belonging to the hyperplane through O which is orthogonal to $[AB]$.

If $\mathcal{C}$ is a smooth compact convex body in $\mathbb{R}^n$, $n \geq 2$, the above property holds also when $[AB]$ is the longest such segment. Similar results are also valid when O is outside the set $\mathcal{C}$. The "local versions" of these results (when the length $|AB|$ of the segment $[AB]$ is locally maximal or locally minimal) are valid as well. More specific results are obtained in the particular case when $\mathcal{C}$ is a convex polytope.

Keywords: Convex set, extremal problem, Philo's line, critical point, convex polytope, concurrent lines.

2020 Mathematics Subject Classification: 51M16, 52A20, 52A40, 49K05, 49K10.

1. Introduction

This article's starting point was the following well-known geometric extremum problem:

Given an angle in the plane and an interior point O, find a line l through O which cuts from the angle a segment $[AB]$ of shortest length $|AB|$.

It is known that this line exists and is uniquely determined by the following property $(*)$ (see Figure 1a):

$(*)$ *the foot E' of the perpendicular from the angle vertex E to l belongs to $[AB]$ and $|BE'| = |AO|$*

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This characterization of the optimal line l suggests a way for its construction, described in the articles of Eves [5] and Coxeter and van de Craats [4]. It is mentioned in [4] that this procedure for constructing l generalizes a construction "published in 1647 by Gregory of St. Vincent" for the particular case when the angle $\angle El_1l_2$ is right.

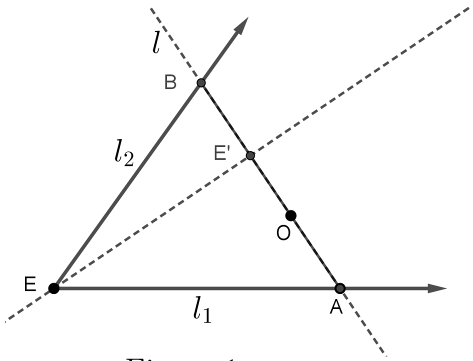

Figure 1a

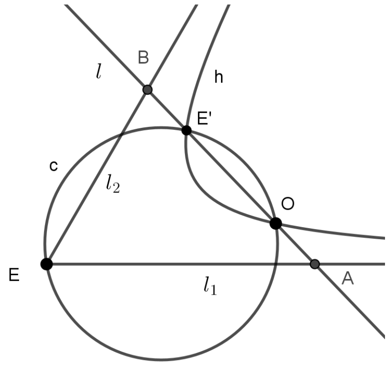

Figure 1b

Given an angle $\angle El_1l_2$ and an interior point O in it, construct first the uniquely determined hyperbola h passing through O and having as asymptotes the lines containing the arms l_1 and l_2 of the angle. Next, construct the circle c with $[EO]$ as diameter (see Figure 1b) and denote by E' the second intersection point of c and h (the first common point being O itself).

The line l determined by the points O and E' intersecting the arms of the angle at A and B, respectively, has the property $(*)$. Indeed, the equality $|BE'| = |OA|$ follows from the well-known fact (see, for instance, Coxeter [3] 8.92 on p. 134) that any hyperbola secant (in particular, the secant $l = OE'$) meets the asymptotes in such a way that $|OA| = |BE'|$. Note also that the angle $\angle EE'O$ is right by construction.

It is interesting that, for the particular case when angle $\angle El_1l_2$ is right, the line l with the property $(*)$ appeared very early in connection with a completely different problem. The famous Philo (or Philon) from Byzantium who lived in the years 280–220 BC, used this line to provide one more solution to the problem of cube doubling which occupied the minds of the ancient philosophers for a long time (see [12]). For this reason, the line l providing the solution to the optimization problem formulated at the beginning of this article is frequently called the "line of Philon" (or "Philo's line").

With the appearance of the infinitesimals and calculus, which are well suited for solving optimization problems, the interest in the above optimization problem increased significantly. Neuberg [10] notes that the problem is mentioned in Newton's *Opuscules*, vol. I, p. 87. In two papers (from the years 1795 [7] and 1811 [8]) S. L'Huilier published the results of his studies on this problem. In a footnote to the first page of [8] the editor of the journal mentions that the same problem is considered (with different means) by M. Puissant. An elementary proof of property $(*)$ was published in the fourth edition of J. Casey's book [2] (p. 39, Proposition 18). In 1887 E. Neovius [9] generalized the problem by allowing the point O to be outside the angle, and has found places for the point O such that the length $|AB|$ of the segment $[AB]$ has one local maximum and one local minimum when the line AB rotates around the point O. From the more recent studies of this problem, we mention the articles of H. Eves

[5], of O. Bottema [1] who used polar coordinates and trigonometry to simplify some of the results of Neovius, and the article of H. S. M. Coxeter and J. van De Craats [4] mentioned above.

In the sequel, we focus our attention on another characterization of the optimal line l, which is somehow less popular nowadays:

The perpendicular to the line l at the point O and the perpendiculars to the arms of the angle at the points A and B, respectively (dashed lines in Figure 1c), meet at a point.

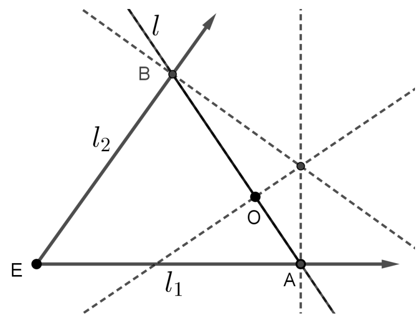

Figure 1c

A simple proof that this property is equivalent to the property (*) can be found in the article of Eves [5]. Without proof and as “communicated by a friendly side” Neovius mentions, at the end of his paper [9], that this property remains valid also in the case when the point O is outside the angle and the length $|AB|$ of the segment $[AB]$ cut from the angle by the line l is locally maximal or locally minimal.

We call this property *Concurrent Perpendiculars Property* (CPP for short) and prefer to work with it because it does not involve the vertex E of the angle and is valid (for trivial reasons) also when the lines l_1 and l_2 are parallel. Besides, this property is more amenable to generalizations, which is the main goal of this paper.

It turns out that CPP is a necessary (but not sufficient!) optimality condition in some other geometric extremal problems. For instance, let $\mathcal{C}$ be any compact convex set in the plane $\mathbb{R}^2$ with a fixed interior point O. Consider all segments $[AB]$ cut from $\mathcal{C}$ by a line $l(\varphi)$ rotating around O with rotation angle φ. The length $|AB|$ of the segment $[AB]$, considered as a function $\tilde{f}(\varphi)$ of the angle φ, may have several local minima (maxima) relative to φ. We prove in Section 2 that for every local minimum φ_* of $\tilde{f}(\varphi)$ with corresponding segment $[A^*B^*]$ the following statements are valid:

(a) $\mathcal{C}$ is *smooth* at A^* and B^* (i.e. there is only one supporting line l_1 for $\mathcal{C}$ at A^* and only one supporting line l_2 for $\mathcal{C}$ at B^*);

(b) CPP holds for the lines l_1, l_2, A^*B^* and the points A^*, B^*, O, respectively.

(c) $\tilde{f}(\varphi)$ is differentiable at φ_* and $\tilde{f}'(\varphi_*) = 0$.

This result cannot be extended to local maxima of $\tilde{f}(\varphi)$. For example, if $\mathcal{C}$ is a polygon and φ^* is a local maximum for $\tilde{f}(\varphi)$, then at least one of the end-points A^* and B^* of the corresponding segment $[A^*B^*]$ must be a vertex of $\mathcal{C}$.

However, if $\mathcal{C}$ is a smooth compact convex set in the plane with interior point O, then (b) and (c) remain true for local maxima as well.

In Section 2, we show that similar results are also true when the point O is outside the compact convex set $\mathcal{C} \subset \mathbb{R}^2$.

In Section 3, we show how the results from Section 2 for sets in $\mathbb{R}^2$ imply results for sets in $\mathbb{R}^n, n > 2$, which are more general than the results announced in the abstract of the paper.

In Section 4 we consider the special case when $\mathcal{C}$ is a convex polytope and line out some specific properties of these sets.

The practical relevance of the problems considered here is discussed briefly in the Appendix to the article.

In what follows, we denote by $\angle ABC$ both the angle as a geometric object and the number representing its measure in radians.

2. The case when the set $\mathcal{C}$ is in $\mathbb{R}^2$

We start with a simple observation that reveals where CPP comes from.

Let A be a point from a line p and O be a point outside p. Denote by O' the orthogonal projection of O on p and put $\varphi = \angle AOO'$ (Figure 2). Depending on the position of A, the angle φ varies between $-\pi/2$ and $\pi/2$. If A is to the left of O', φ is negative. If A is to the right of O', then φ is positive. Denote by A' the intersection point of the line through A perpendicular to p and the line through O perpendicular to OA (the dashed lines in Figure 2).

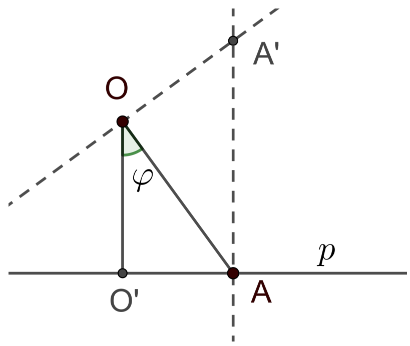

Figure 2

Lemma 2.1. *The following relations hold for the first derivative $|OA|'_\varphi$ and for the second derivative $|OA|''_\varphi$ of $|OA|$ with respect to the angle φ :*

(a) $|OA|'_\varphi = |OA| \tan\varphi = \operatorname{sign}(\varphi)|OA'|$;

(b) $|OA|''_\varphi = |OA|(\tan^2\varphi + \cos^{-2}\varphi)$.

Here, as usual, $\operatorname{sign}(\varphi) = 1$ if $\varphi > 0$, $\operatorname{sign}(\varphi) = -1$ if $\varphi < 0$, and $\operatorname{sign}(\varphi) = 0$ if $\varphi = 0$.

Proof. (a) Clearly, $|OA| = |OO'| \cos^{-1}\varphi$. Hence

$$|OA|'_\varphi = |OO'| \cos^{-1}\varphi \tan\varphi = |OA| \tan\varphi = \operatorname{sign}(\tan\varphi)|OA'| = \operatorname{sign}(\varphi)|OA'|.$$

(b) Differentiating $|OA|'_\varphi = |OA| \tan\varphi$ we get

$$|OA|''_\varphi = |OA|'_\varphi \tan\varphi + |OA| \cos^{-2}\varphi = |OA|(\tan^2\varphi + \cos^{-2}\varphi). \qquad \square$$

2.1. Point O is in the interior of $\mathcal{C}$

We first prove the validity of CPP for angles and then derive this property for more general sets $\mathcal{C}$ in the plane. Consider an angle $\angle El_1l_2$ with vertex E, arms l_1 and l_2, and angle measure θ, $0 < \theta < \pi$, (Figure 3). Let O be an interior point for this angle and let O' and O'' be the orthogonal projections of O onto the lines containing l_1 and l_2 , respectively. Let $[AB]$ be the segment cut from this angle by a line l passing through O where $A \in l_1$ and $B \in l_2$. Put $\varphi := \angle AOO'$ if O' belongs to the ray $\overrightarrow{AE}$ and $\varphi := -\angle AOO'$ otherwise. Clearly, the segment $[AB]$ exists only if $\theta - \pi/2 < \varphi < \pi/2$. Note that the angle φ determines the line l. We use the notation $l = l(\varphi)$ to reflect this fact.

Denote by A' the intersection point of the line through O perpendicular to $l(\varphi)$ and the line perpendicular to l_1 at A (dashed lines in Figure 3).

Analogously, put $\psi := \angle O''OB$ if O'' belongs to the ray $\overrightarrow{BE}$ and $\psi := -\angle O''OB$ otherwise. Denote by B' the intersection point of the line through O perpendicular to $l(\varphi)$ and the line through B perpendicular to l_2.

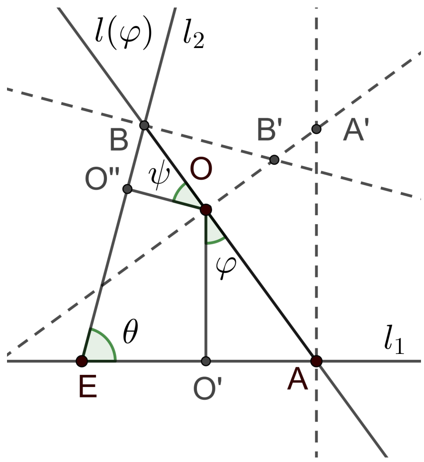

Figure 3

Proposition 2.2. *The function $f(\varphi) := |OA| + |OB|$ is differentiable and strictly convex. For its derivative we have $f'(\varphi) = \operatorname{sign}(\varphi)|OA'| - \operatorname{sign}(\theta - \varphi)|OB'|$.*

Proof. As $\theta - \pi/2 < \varphi < \pi/2$, we have $\theta - \pi/2 < \theta - \varphi < \pi/2$ as well. It is not difficult to realize that, for every possible φ, $\psi = \theta - \varphi$. Evidently, $|OB| = |OO''| \cos^{-1}(\theta - \varphi)$ and we can calculate, as in Lemma 2.1, that

(a') $|OB|'_\varphi = -|OB| \tan(\theta - \varphi) = -\operatorname{sign}(\tan(\theta - \varphi))|OB'| = -\operatorname{sign}(\theta - \varphi)|OB'|$ and

(b') $|OB|''_\varphi = |OB|(\tan^2(\theta - \varphi) + \cos^{-2}(\theta - \varphi))$.

Strict convexity of f follows from property (b) of Lemma 2.1 and (b'). The expression for the derivative follows from property (a) of Lemma 2.1 and (a'). □

Proposition 2.3. (CPP for angles) *Let O be an interior point for an angle $\angle El_1l_2$ with angle measure $\theta, 0 < \theta < \pi$. Consider all segments $[AB]$ cut from this angle by lines $l(\varphi)$, $\theta - \pi/2 < \varphi < \pi/2$. For any angle φ_0, $\theta - \pi/2 < \varphi_0 < \pi/2$, the following statements are equivalent:*

(i) *φ_0 is a critical point of the function $f(\varphi) = |AB|$ (i.e. $f'(\varphi_0) = 0$).*

(ii) *The segment $[A_0B_0]$ corresponding to the line $l(\varphi_0)$ is the shortest among all segments $[AB]$ cut from the angle by lines passing through O.*

(iii) (CPP) *The perpendiculars to l_1 at A_0, to l_2 at B_0 and to the line A_0B_0 at O meet at a point.*

There exists only one φ_0, $\theta - \pi/2 < \varphi_0 < \pi/2$, for which one (and then all) of these three equivalent statements holds.

Proof. We consider separately the cases when $0 < \theta < \pi/2$, and when $\pi/2 \le \theta < \pi$.

Assume first that $0 < \theta < \pi/2$. This case is depicted in Figure 3. For $\varphi \le 0$, $f'(\varphi) = -|OA'| - |OB'| < 0$. If $\varphi \ge \theta$, then $f'(\varphi) = |OA'| + |OB'| > 0$. Strict convexity of the function $f(\varphi)$ implies that it has a unique minimizer φ_0 which belongs to the open interval $(0, \theta)$ and is characterized by the relation $f'(\varphi_0) = 0$. This shows that (i) and (ii) are equivalent.

For $\varphi \in (0, \theta)$ we have $f'(\varphi) = |OA'| - |OB'|$. Note also that in this case $O \notin [A'B']$. Hence, $f'(\varphi) = |OA'| - |OB'| = 0$ if and only if $A' = B'$. This means that (ii) and (iii) are equivalent which completes the proof of Proposition 2.3 for the case when $0 < \theta < \pi/2$.

The case $\pi/2 \le \theta < \pi$ is even simpler. From $\varphi > \theta - \pi/2 \ge 0$ and $\theta - \varphi > \theta - \pi/2 \ge 0$ we derive that $f'(\varphi) = |OA'| - |OB'|$. The strictly convex function $f(\varphi)$ tends to $+\infty$ when φ tends to $\theta - \pi/2$ from above and to $\pi/2$ from below. This implies $f(\varphi)$ has a unique minimizer φ_0 characterized by the relation $f'(\varphi_0) = 0$. Therefore we conclude that (i) and (ii) are equivalent.

Note also that in this case $O \notin [A'B']$. This means that $f'(\varphi) = |OA'| - |OB'| = 0$ if and only if $A' = B'$. Therefore (i) and (iii) are equivalent. □

Remark 2.4. The above proposition remains true also in the degenerate case of a strip bounded by two different parallel lines. Moreover, in this case, CPP holds for every point $O \in [A_0B_0]$ and the line A_0B_0 is perpendicular to the lines bounding the strip. □

Recall that a line p is said to be *supporting* for a set $\mathcal{C} \subset \mathbb{R}^2$ at some point $A \in \mathcal{C}$ if $A \in p$ and $\mathcal{C}$ lies entirely in one of the two closed half-planes determined by p. It is well-known that for every point A from the boundary $\partial\mathcal{C}$ of a closed convex set $\mathcal{C}$ with nonempty interior there exists at least one supporting line at this point.

Let $\mathcal{C}$ be a compact convex set $\mathcal{C}$ in $\mathbb{R}^2$ and O be an interior point of it. Consider a line $l(\varphi)$ rotating around O with φ as angle of rotation. Put $\tilde{f}(\varphi) = |AB|$ where $[AB] = l(\varphi) \cap \mathcal{C}$. Numerical experiments with GeoGebra show that the function $\tilde{f}(\varphi)$ may have several local minima and local maxima. For instance, consider the ellipse

$$\left(\frac{x}{12}\right)^2 + \left(\frac{y}{1.6}\right)^2 = 1$$

and the point $O(-1, 1.4)$ which is interior for the ellipse. The graph of the function $\tilde{f}(\varphi)$ is exhibited in Figure 4.

Figure 4: Ellipse with axes a =12, b = 1.6, and point O(-1,1.4)

The next Proposition gives some information about the local minima of $\tilde{f}(\varphi)$.

Proposition 2.5. *Let $\mathcal{C} \subset \mathbb{R}^2$ be a compact convex set, O be an interior point of it, and $\tilde{f}(\varphi) := |\tilde{A}(\varphi)\tilde{B}(\varphi)|$ where $[\tilde{A}(\varphi)\tilde{B}(\varphi)] = l(\varphi) \cap \mathcal{C}$. Suppose φ_* is a local minimum for $\tilde{f}(\varphi)$ and let $[A^*B^*] = l(\varphi_*) \cap \mathcal{C}$. Then:*

(i) CPP *holds for every pair of supporting for $\mathcal{C}$ lines l_1 at A^*, l_2 at B^* and the point $O \in [A^*B^*]$.*

(ii) *At each of the end points A^* and B^* there exists only one supporting line for $\mathcal{C}$.*

(iii) *$\tilde{f}(\varphi)$ is differentiable at φ_* and $\tilde{f}'(\varphi_*) = 0$.*

Proof. (i) Let $A(\varphi) = l(\varphi) \cap l_1$ and $B(\varphi) = l(\varphi) \cap l_2$. For every φ such that $l(\varphi)$ intersects both l_1 and l_2 we have $[\tilde{A}(\varphi)\tilde{B}(\varphi)] \subseteq [A(\varphi)B(\varphi)] \cap \mathcal{C}$. In particular, $A(\varphi_*) = A^*$ and $B(\varphi_*) = B^*$. Obviously, $|A(\varphi)B(\varphi)| \geq |\tilde{A}(\varphi)\tilde{B}(\varphi)|$. If φ is close enough to φ_* we have

$$|A(\varphi)B(\varphi)| \geq |\tilde{A}(\varphi)\tilde{B}(\varphi)| \geq |A^*B^*| = |A(\varphi_*)B(\varphi_*)|.$$

Hence φ_* is a local minimum for the function $f(\varphi) = |A(\varphi)B(\varphi)|$ which, as we have seen in Proposition 2.3, is strictly convex. Therefore, φ_* is a global minimum for $f(\varphi)$ and the CPP we want to prove follows from property (iii) of Proposition 2.3.

(ii) Let l_1 be some supporting for $\mathcal{C}$ line at A^*. Suppose $\mathcal{C}$ has two supporting lines l_2' and l_2'' at B^*. Denote by $A^{*\prime}$ the intersection point of the line perpendicular to l_1 at A^* and the line perpendicular to the line A^*B^* at O. It follows from (i) that $A^{*\prime}B^*$ is perpendicular to both l_2' and l_2''. This is possible only if l_2' and l_2'' coincide. Thus, we have shown that there is only one supporting line for the set $\mathcal{C}$ at the point B^*. Similarly, one shows that the supporting line for $\mathcal{C}$ at A^* is unique as well.

Property (iii) follows from Lemma 2.9, which will be proved later. □

The proof of Proposition 2.5 is also valid, with minor changes, for unbounded closed convex sets $\mathcal{C}$ with nonempty interiors. For such a set, the function $\tilde{f}$ may be equal to $+\infty$ at some φ.

Corollary 2.6. *Let $\mathcal{C}$ be a closed convex set and O be an interior point of it. If some φ_* belonging to the set $\{\varphi : \tilde{f}(\varphi) < +\infty\}$ is local minimizer for $\tilde{f}(\varphi)$, then* (i), (ii) *and* (iii) *from Proposition* 2.5 *are valid.*

As mentioned in the introduction, Proposition 2.5 is not valid for local maximizers.

Proposition 2.7. *Let $\mathcal{C}$ be a closed convex set in $\mathbb{R}^2$, O an interior point of it. Let φ^* be a local maximizer for the function $\tilde{f}(\varphi)$ with $l(\varphi^*) \cap \mathcal{C} = [A^*B^*]$. Then at least one of the end-points A^* and B^* is an extreme point of $\mathcal{C}$.*

Proof. Suppose both A^* and B^* are not extreme points of $\mathcal{C}$. Then they are interior points of some sides $[A'A'']$ and $[B'B'']$ from the boundary of $\mathcal{C}$. If the lines $l_1 = A'A''$ and $l_2 = B'B''$ intersect at some point E, then O is an interior point for the angle $\angle El_1l_2$ and the strictly convex function $f(\varphi)$ from Proposition 2.2 can not have local maximizers, a contradiction. The same argument also holds if the lines l_1 and l_2 are parallel (see Remark 2.4). $\square$

Our next immediate goal is to extend the validity of CPP by replacing the lines l_1, l_2 with arbitrary smooth curves. This idea is rather old. Botema [1] notes that it was mentioned in M. L'Huilier's book [7] from 1795 (but not mentioned in the later article by L'Huilier [8] from 1811). H. Eves says in [5] "This generalization seems to be due to Isaac Newton". Both O. Botema and H. Eves express concerns that some of the proofs of the results related to the replacement of straight lines by curves do not correspond to the contemporary requirements for rigor. To make this article self-contained and to remove the mentioned concerns we provide proofs of the facts needed for our considetations.

Definition 2.8. Let p be a straight line and γ a curve in $\mathbb{R}^2$. Let the point P_0 belongs to both p and γ. We say that p is a tangent to $\gamma \subset \mathbb{R}^2$ at the point P_0 if there exists an open circle U in $\mathbb{R}^2$ centered at P_0 such that for a rectangular coordinate system with p as abscissa axis, the set $\gamma \cap U$ is the graph of a continuous real-valued function which is differentiable at P_0 and its derivative at this point vanishes.

We say that the curve γ is smooth if it has a tangent at every point. $\square$

The curves γ we mostly use in this article are parts of the boundaries $\partial\mathcal{C}$ of closed convex sets $\mathcal{C} \subset \mathbb{R}^2$ with nonempty interior. Given a point O (inside or outside the set $\mathcal{C}$), a point $\tilde{A}(\varphi)$ of such a curve is determined by its distance $|O\tilde{A}(\varphi)|$ from the point O and by the angle φ the ray $\overrightarrow{O\tilde{A}(\varphi)}$ concludes with a suitably chosen direction. Such curves are, therefore, represented by a function $\tilde{\rho}(\varphi) := |O\tilde{A}(\varphi)|$ where φ runs over some interval.

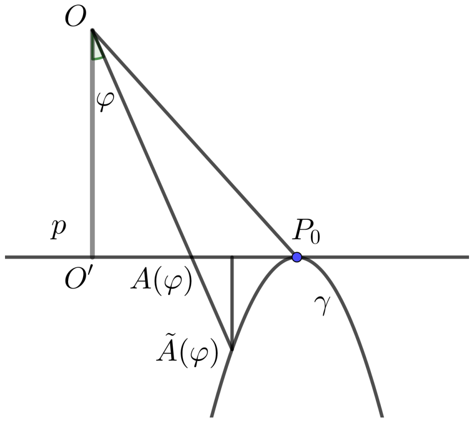

Figure 5

Suppose a straight line p is tangent at some point P_0 to a curve γ (given by some function $\tilde{\rho}(\varphi) := |O\tilde{A}(\varphi)|$) and O is a point outside $p \cup \gamma$. Denote, as above, by O' the orthogonal projection of O to p. Without loss of generality we may assume that the ray $\overrightarrow{OO'}$ is the "suitably chosen direction" and $\varphi = \angle \tilde{A}(\varphi)OO'$ (Figure 5). Put $\varphi_0 := \angle P_0OO'$.

When the angle φ is close enough to φ_0 the ray $\overrightarrow{O\tilde{A}(\varphi)}$ intersects the line p at some point $A(\varphi)$. Put $\rho(\varphi) := |OA(\varphi)|$. Clearly, $\tilde{\rho}(\varphi_0) = \rho(\varphi_0) = |OP_0|$.

Lemma 2.9. *The functions $\tilde{\rho}(\varphi)$ and $\rho(\varphi)$ are differentiable at φ_0 and we have $\tilde{\rho}'(\varphi_0) = \rho'(\varphi_0)$.*

Proof. Consider the rectangular coordinate system with origin O' and abscissa axis p. Since p is tangent to γ at P_0, we may assume that in small open circle centered at P_0 the curve γ is the graph of a continuous real-valued function $h(t)$, defined in an open interval $\Delta \subset p$ containing $P_0 = (p_0, 0)$ and such that $h(p_0) = h'(p_0) = 0$.

Then $\tilde{A}(\varphi) = (t, h(t))$, $t \in \Delta$, and we may assume without loss of generality that $|OO'| = 1$. Then $\tilde{\rho}(\varphi) = (1 - h(t))\rho(\varphi)$ and we get

$$\frac{\tilde{\rho}(\varphi) - \tilde{\rho}(\varphi_0)}{\varphi - \varphi_0} = (1 - h(t)) \cdot \frac{\rho(\varphi) - \rho(\varphi_0)}{\varphi - \varphi_0} - \frac{h(t) - h(p_0)}{\varphi - \varphi_0} \cdot \rho(\varphi_0).$$

Note that $lim_{\varphi \to \varphi_0} h(t) = lim_{t \to p_0} h(t) = h(p_0) = 0$ and we have to prove that

$$lim_{\varphi \to \varphi_0} \frac{h(t) - h(p_0)}{\varphi - \varphi_0} = 0.$$

To do this observe that $\tan(\varphi) = \dfrac{t}{1 - h(t)}$ which implies the identity

$$\frac{h(t) - h(p_0)}{\varphi - \varphi_0} = \frac{\tan(\varphi) - \tan(\varphi_0)}{\varphi - \varphi_0} \cdot \frac{\dfrac{h(t) - h(p_0)}{t - p_0}}{1 + \tan(\varphi) \cdot \dfrac{h(t) - h(p_0)}{t - p_0}}.$$

Having in mind that

$$lim_{\varphi \to \varphi_0} \frac{\tan(\varphi) - \tan(\varphi_0)}{\varphi - \varphi_0} = \cos^{-2}(\varphi_0) \quad \text{and} \quad lim_{t \to p_0} \frac{h(t) - h(p_0)}{t - p_0} = h'(p_0) = 0$$

we obtain from the above identity that

$$lim_{\varphi \to \varphi_0} \frac{h(t) - h(p_0)}{\varphi - \varphi_0} = 0$$

and we are done. □

It is known (cf. [11, Theorem 25.1]) that a convex set has a tangent p at some $P_0 \in \partial \mathcal{C}$ if and only if p is the only supporting line for $\mathcal{C}$ at P_0.

We are now ready to prove item (iii) of Proposition 2.5:

Proof of Proposition 2.5(iii): The unique line t_1 supporting $\mathcal{C}$ at A^* is tangent to ∂C at A^*. Lemma 2.9 implies that the function $|O\tilde{A}(\varphi)|$ is differentiable at φ_*.

Using similar arguments, we see that the function $|O\tilde{B}(\varphi)|$ is also differentiable at φ_*. This proves the differentiability of the function

$$\tilde{f}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)| = |O\tilde{A}(\varphi)| + |O\tilde{B}(\varphi)|.$$

As φ_* is a local minimizer of $\tilde{f}(\varphi)$ we get that $\tilde{f}'(\varphi_*) = 0$ $\square$

We call a convex set $\mathcal{C} \subset \mathbb{R}^2$ *smooth* if there exists a tangent at every point of its boundary ∂C.

Proposition 2.10. *Let $\mathcal{D}$ be a smooth compact convex subset of $\mathbb{R}^2$ and O an interior point of it. As above, consider all straight lines $l(\varphi)$ through O and the function $\tilde{f}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)|$ where $[\tilde{A}(\varphi)\tilde{B}(\varphi)] = l(\varphi) \cap \mathcal{D}$. Denote by $t_1(\varphi)$ and $t_2(\varphi)$ the tangents to $\partial\mathcal{D}$ at $\tilde{A}(\varphi)$ and $\tilde{B}(\varphi)$, respectively. Then:*

(i) *The function $\tilde{f}(\varphi)$ is everywhere differentiable.*

(ii) *$\tilde{f}'(\varphi) = 0$ if and only if* CPP *holds for the lines $t_1(\varphi)$, $t_2(\varphi)$, $l(\varphi)$ and the points $\tilde{A}(\varphi)$, $\tilde{B}(\varphi)$, O.*

Proof. Take an arbitrary φ_0 and put $A(\varphi) := l(\varphi) \cap t_1(\varphi_0)$, $B(\varphi) := l(\varphi) \cap t_2(\varphi_0)$. Consider the function $f(\varphi) := |A(\varphi)B(\varphi)| = |OA(\varphi)| + |OB(\varphi)|$. It is well defined and differentiable whenever $l(\varphi)$ is parallel neither to $t_1(\varphi_0)$ nor to $t_2(\varphi_0)$. In particular, this is so for $\varphi = \varphi_0$. By Lemma 2.9, the function

$$\tilde{f}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)| = |\tilde{A}(\varphi)O| + |O\tilde{B}(\varphi)|$$

is differentiable at φ_0 (this proves the validity of (i)), and its derivative coincides with the derivative of the function $f(\varphi) = |A(\varphi)B(\varphi)|$ at φ_0. Hence, the condition $\tilde{f}'(\varphi_0) = 0$ is equivalent to the requirement $f'(\varphi_0) = 0$. In view of Proposition 2.3, this is precisely the case when CPP holds for the lines $t_1(\varphi_0)$, $t_2(\varphi_0)$, $l(\varphi_0)$ and the points $\tilde{A}(\varphi_0) = A(\varphi_0)$, $\tilde{B}(\varphi_0) = B(\varphi_0), O$. $\square$

Corollary 2.11. *Let $\mathcal{D}$ and $\tilde{f}(\varphi)$ be as in Proposition* 2.10. *Then* CPP *holds for every local maximizer φ^* and every local minimizer φ_* of this function.*

Remark 2.12. (i) Pay attention to the seemingly strange phenomenon connected with the proof of Proposition 2.10: If φ_0 is a local maximizer for the function $\tilde{f}(\varphi)$, it is also a global minimizer for the closely related function $f(\varphi)$ which, in a sense, is a kind of "twofold linearization" of $\tilde{f}(\varphi)$ at φ_0 ($|OA(\varphi)|$ is a linearization of $|O\tilde{A}(\varphi)|$ and $|OB(\varphi)|$ is a linearization of $|O\tilde{B}(\varphi)|$).

(ii) Items (i) and (ii) of Proposition 2.10 remain valid also for unbounded smooth closed convex sets $\mathcal{D}$ with nonempty interiors whenever φ satisfies $\tilde{f}(\varphi) < +\infty$. $\square$

At the end of this subsection we prove a lemma that will be used in the next section.

Lemma 2.13. *Let $\mathcal{D}$ and $\tilde{f}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)|$ be as in Proposition 2.10. Let φ^* be a local maximizer for $\tilde{f}(\varphi)$ and let the tangent $t_1(\varphi^*)$ to $\partial\mathcal{D}$ at $\tilde{A}(\varphi^*)$ be parallel to the tangent $t_2(\varphi^*)$ to $\partial\mathcal{D}$ at $\tilde{B}(\varphi^*)$. Then the line $\tilde{A}(\varphi^*)\tilde{B}(\varphi^*)$ is perpendicular to the tangents.*

Proof. As in the proof of Proposition 2.10, we define $A(\varphi) := l(\varphi) \cap t_1(\varphi^*)$ and $B(\varphi) := l(\varphi) \cap t_2(\varphi^*)$. We know that the function

$$f(\varphi) := |A(\varphi)B(\varphi)| = |OA(\varphi)| + |OB(\varphi)|$$

is strictly convex and differentiable. It attains its global minimum at the point where its derivative is zero and this happens precisely when $l(\varphi)$ is perpendicular to the lines $t_1(\varphi^*)$ and $t_1(\varphi^*)$. Note that the function

$$\tilde{f}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)| = |\tilde{A}(\varphi)O| + |O\tilde{B}(\varphi)|$$

is differentiable at φ^* and its derivative at this point coincides with the derivative of the function $f(\varphi) = |A(\varphi)B(\varphi)|$ at the same point. As φ^* is a local maximizer for $\tilde{f}(\varphi)$ we get $f'(\varphi^*) = \tilde{f}'(\varphi^*) = 0$. $\square$

As seen from its proof, this lemma is also valide for unbounded smooth closed convex sets $\mathcal{C}$ with nonempty interior.

2.2. Point O is outside the set $\mathcal{C}$

We consider first the case when $\mathcal{C}$ is an angle $\angle El_1l_2$ with $0 < \theta := \angle El_1l_2 < \pi/2$, and O is outside the angle and its opposite angle. This case is presented in Figure 6 where O' and O'' are the orthogonal projections of O onto the lines containing the arms l_1 and l_2, respectively. A line $l(\varphi)$ (the angle φ will be specified next) passes through O and intersects the arms of the angle at the points $A = A(\varphi)$ and $B = B(\varphi)$ ($B \in [OA]$). We put $\varphi := \angle AOO'$ if O' belongs to the ray $\overrightarrow{AE}$ and $\varphi := -\angle AOO'$ otherwise.

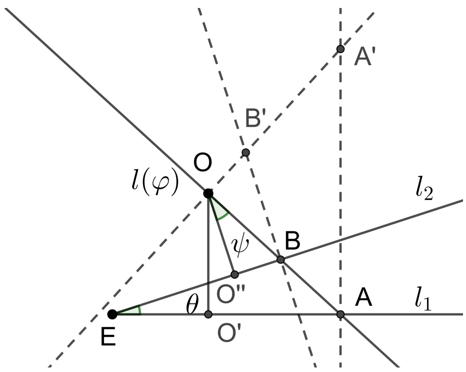

Figure 6

We are interested in the behavior of the function

$$g(\varphi) := |OA(\varphi)| - |OB(\varphi)| = |A(\varphi)B(\varphi)|$$

which is well-defined and differentiable in its domain (which is the open interval $(-\pi/2 + \angle OEO', \pi/2)$).

As mentioned in the introduction, Neovius [9] has shown that depending on the angle measure of the angle $\angle OEO'$ the function $g(\varphi)$ may have $0, 1$, or 2 critical points in the interval $(-\pi/2 + \angle OEO', 0)$. In the third case, the two critical points correspond to a local maximum and a local minimum of $g(\varphi)$.

Proposition 2.14. (CPP for angles) *Let φ be from the domain of the function g. Then $g'(\varphi) = 0$ if and only if* CPP *holds for the line $l(\varphi)$, the arms l_1, and l_2 and the points O, $A(\varphi)$, and $B(\varphi)$, respectively.*

In particular, CPP *holds for some φ whenever the function g has a local minimum or a local maximum at φ.*

Proof. The dashed lines in Figure 6 are orthogonal to: l_1 at $A = A(\varphi)$, l_2 at $B = B(\varphi)$, and $l(\varphi)$ at O. Point $A' = A'(\varphi)$ is the intersection of the dashed line orthogonal to $l(\varphi)$ and the dashed line orthogonal to l_1. Similarly, point $B' = B'(\varphi)$ is the intersection point of the dashed lines orthogonal to $l(\varphi)$ and l_2.

Consider the angle $\psi = \varphi - \theta$ concluded between $l(\varphi)$ and the line OO''. Clearly, $|OB| = |OO''| \cos^{-1} \psi$. Hence, by Lemma 2.1,

$$g'(\varphi) = \operatorname{sign}(\varphi)|OA'| - \operatorname{sign}(\varphi - \theta)|OB'|.$$

Note that $\theta = \angle O''OO'$. If $0 \le \varphi \le \theta$, then $g'(\varphi) = |OA'| + |OB'| > 0$. If $\varphi > \theta$ we have

$$g'(\varphi) = |OA'| - |OB'| = |OA| \tan\varphi - |OB| \tan(\varphi - \theta) > |OB|(\tan\varphi - \tan(\varphi - \theta)) > 0.$$

This shows that $g(\varphi)$ increases strictly when $\varphi > 0$. If we take φ from the interval $(-\pi/2 + \angle OEO, 0)$ we have

$$g'(\varphi) = -|OA'| + |OB'|.$$

Note also that in this case the segment $[A'B']$ does not contain the point O. This means $g'(\varphi) = 0$ if and only if $A' = B'$ which is the CPP. $\square$

Remark 2.15. It is easy to realize that, when $\pi/2 \le \theta < \pi$, the behavior of the function $g(\varphi)$ is not interesting. It is strictly increasing in its domain; therefore, $g'(\varphi)$ does not have zeros. $\square$

Let now $\mathcal{C}$ be a closed convex subset (not necessarily bounded) of $\mathbb{R}^2$ with a nonempty interior and O be a point outside $\mathcal{C}$. Consider all straight lines $l(\varphi)$ through O and put $D := \{\varphi : l(\varphi) \cap \mathcal{C} = [\tilde{A}(\varphi)\tilde{B}(\varphi)] \neq \emptyset$ and $\tilde{B}(\varphi) \in [O\tilde{A}(\varphi)]\}$. The set D is the domain of the function $\tilde{g}(\varphi) := |O\tilde{A}(\varphi)| - |O\tilde{B}(\varphi)| = |\tilde{A}(\varphi)\tilde{B}(\varphi)|$.

The next two propositions (and their proofs) are similar to Propositions 2.5 and 2.10.

Proposition 2.16. *If the function $\tilde{g}(\varphi)$ has a local minimum at some φ_* from the interior of its domain D and $l(\varphi_*) \cap \mathcal{C} = [A^*B^*]$, then*

(i) *$\mathcal{C}$ has a unique supporting line t_1 at A^* and a unique supporting line t_2 at B^*. I.e., t_1 is a tangent to ∂C at A^* and t_2 is a tangent to ∂C at B^*.*

(ii) CPP *holds for the lines $t_1, t_2, l(\varphi_*)$ and the points A^*, B^* and O.*

Proof. Let l_1 and l_2 be any supporting lines for $\mathcal{C}$ at the points A^* and B^* , respectively. For $\varphi \in D$ put $A(\varphi) := l(\varphi) \cap l_1$ and $B(\varphi) := l(\varphi) \cap l_2$. In particular, $A(\varphi_*) = A^*$ and $B(\varphi_*) = B^*$. Also, we have $[\tilde{A}(\varphi)\tilde{B}(\varphi)] = [A(\varphi)B(\varphi)] \cap \mathcal{C}$ and, therefore, $|A(\varphi)B(\varphi)| \ge |\tilde{A}(\varphi)\tilde{B}(\varphi)|$.

If φ is close enough to φ_* we have

$$|A(\varphi)B(\varphi)| \geq |\tilde{A}(\varphi)\tilde{B}(\varphi)| \geq |A^*B^*| = |A(\varphi_*)B(\varphi_*)|.$$

I.e., φ_* is a local minimum for the function $g(\varphi) = |A(\varphi)B(\varphi)|$ from Proposition 2.14. It follows that CPP holds for the lines $l_1, l_2, l(\varphi_*)$ and the points A^*, B^* and O. This implies, precisely as in the proof of Proposition 2.5, that $\mathcal{C}$ has unique supporting lines at A^* and B^*. □

Let $\mathcal{D}$ be a smooth closed convex subset of $\mathbb{R}^2$ with nonempty interior and O a point outside it. Consider as above all straight lines $l(\varphi)$ through O and put

$$D := \{\varphi : l(\varphi) \cap \mathcal{D} = [\tilde{A}(\varphi)\tilde{B}(\varphi)] \neq \emptyset \quad \text{and} \quad \tilde{B}(\varphi) \in [O\tilde{A}(\varphi)]\}.$$

The set D is the domain of the function $\tilde{g}(\varphi) := |O\tilde{A}(\varphi)| - |O\tilde{B}(\varphi)| = |\tilde{A}(\varphi)\tilde{B}(\varphi)|$. For every $\varphi \in D$ denote by $t_1(\varphi)$ and $t_2(\varphi)$ the the tangents to $\partial\mathcal{D}$ at $\tilde{A}(\varphi)$ and $\tilde{B}(\varphi)$, respectively

Proposition 2.17. *The following statements hold for every φ from the interior of D:*

(i) *The function $\tilde{g}(\varphi)$ is differentiable at φ.*

(ii) *$\tilde{g}'(\varphi) = 0$ if and only if* CPP *holds for the lines $t_1(\varphi)$, $t_2(\varphi)$, $l(\varphi)$, and the points $\tilde{A}(\varphi), \tilde{B}(\varphi), O$, respectively.*

In particular, CPP *holds for every local minimizer φ_* and every local maximizer φ^* of the function $\tilde{g}$ (whenever φ_* and φ^* are from the interior of the domain D of $\tilde{g}$).*

Proof. Take an arbitrary φ_0 from the interior of D and put $A(\varphi) := l(\varphi) \cap t_1(\varphi_0)$, $B(\varphi) := l(\varphi) \cap t_2(\varphi_0)$. Consider the function

$$g(\varphi) := |A(\varphi)B(\varphi)| = |OA(\varphi)| - |OB(\varphi)|.$$

It is well defined and differentiable at $\varphi = \varphi_0$. By Lemma 2.9, the function $\tilde{g}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)| = |\tilde{A}(\varphi)O| - |O\tilde{B}(\varphi)|$ is differentiable at φ_0 (this proves the validity of (i)), and its derivative coincides with the derivative of the function $g(\varphi) = |A(\varphi)B(\varphi)|$ at φ_0. Hence, the condition $\tilde{g}'(\varphi_0) = 0$ is equivalent to the requirement $g'(\varphi_0) = 0$. In view of Proposition 2.14, this is precisely the case when CPP holds for the lines $t_1(\varphi_0)$, $t_2(\varphi_0)$, $l(\varphi_0)$, and the points $\tilde{A}(\varphi_0) = A(\varphi_0)$, $\tilde{B}(\varphi_0) = B(\varphi_0)$, and O. □

An analog of Lemma 2.13 is valid in the case when O is outside the set $\mathcal{D}$.

Lemma 2.18. *Let $\mathcal{D}$ and $\tilde{g}(\varphi) = |\tilde{A}(\varphi)\tilde{B}(\varphi)|$ be as in Proposition* 2.17. *Let φ^* from the interior of the domain of $\tilde{g}$ be a local maximizer for $\tilde{g}(\varphi)$. Suppose the tangent $t_1(\varphi^*)$ to $\partial\mathcal{D}$ at $\tilde{A}(\varphi^*)$ is parallel to the tangent $t_2(\varphi^*)$ to $\partial\mathcal{D}$ at $\tilde{B}(\varphi^*)$. Then the line $\tilde{A}(\varphi^*)\tilde{B}(\varphi^*)$ is perpendicular to the tangents.*

The proof of this statement is omitted because it is quite similar to the proof of Lemma 2.13.

Remark 2.19. The assumption about the convexity of the set $\mathcal{D}$ in Proposition 2.17 and in Proposition 2.10 is not essential for the validity of the statements. It suffices to consider two smooth curves γ_1 and γ_2 and a point O as in Figure 7.

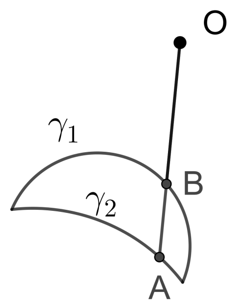

Figure 7: Arbitrary smooth curves γ_1 and γ_2

Then, CPP holds for every local maximizer φ^* or a local minimizer φ_* of the function $\tilde{g}(\varphi) = |O\tilde{A}(\varphi)| - |O\tilde{B}(\varphi)|$ whenever φ^* and φ_* belong to the interior of the domain of $\tilde{g}$. Similar is the situation with the function $\tilde{f}(\varphi) = |O\tilde{A}(\varphi)| + |O\tilde{B}(\varphi)|$. □

3. The case when the set $\mathcal{C}$ is in $\mathbb{R}^n, n > 2$

In this section, we show how the 2-dimensional statements established in the previous section imply their higher dimensional generalizations, including the results announced in the paper's abstract.

Let $\mathcal{C} \subset \mathbb{R}^n$, $n > 2$, be a closed convex set and O be a point from its interior. Consider all segments $[AB]$ containing O with end-points A and B from the boundary $\partial\mathcal{C}$, and define for such segments the function $F([AB]) = |AB|$.

Definition 3.1. A segment $[A^*B^*]$ containing O, with end points on $\partial\mathcal{C}$, is said to be a *local minimizer (maximizer)* for F if there are open sets $U \ni A^*$ and $V \ni B^*$ such that $F([AB]) = |AB| \geq |A^*B^*|$ $(F([AB]) = |AB| \leq |A^*B^*|)$ whenever we have $O \in [AB]$, $A \in \partial\mathcal{C} \cap U$ and $B \in \partial\mathcal{C} \cap V$. □

This definition agrees with the definitions of a local minimizer (maximizer) for the function $f(\varphi)$ used in Section 2.

Recall that a hyperplane Π is *supporting a set $\mathcal{C}$ at a point $A \in \mathcal{C}$* if $A \in \Pi$ and $\mathcal{C}$ is contained in one of the two closed half-spaces determined by Π. It is well known that for every point A from the boundary of a closed convex set $\mathcal{C} \subset \mathbb{R}^n$, there exists at least one hyperplane that supports $\mathcal{C}$ at A. We call a convex set $\mathcal{C}$ with non-empty interior *smooth* if there is only one supporting hyperplane at every point of its boundary.

Theorem 3.2. *Let O be an interior point of a closed convex set $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, and let a segment $[A^*B^*]$ be a local minimizer for F. Then:*

(i) *There exists only one hyperplane Π_1 supporting $\mathcal{C}$ at A^* and only one hyperplane Π_2 supporting $\mathcal{C}$ at B^*;*

(ii) *The normal line to Π_1 at A^* and the normal line to Π_2 at B^* intersect at a point belonging to the hyperplane through O which is orthogonal to the line A^*B^*.*

Proof. The particular case of this theorem when $n = 2$ follows from Proposition 2.5. Therefore, we will assume that $n > 2$. Our first step is to show that (ii) holds for arbitrary hyperplanes Π_1 and Π_2 supporting $\mathcal{C}$ at A^* and B^*, respectively.

Suppose first that Π_1 and Π_2 are not parallel. Then there exist a linear subspace $L \subset \mathbb{R}^n$ of co-dimension two and a point $T \in \Pi_1 \cap \Pi_2$ such that

$$\Pi_1 \cap \Pi_2 = \{T + h : h \in L\} = T + L.$$

Take an arbitrary non-zero vector $h \in L$ and consider the two-dimensional plane $\Pi(h)$ containing the lines $l_1(h) = \{A^* + th : t \in R\}$ and $l_2(h) = \{B^* + th : t \in R\}$. It is clear that the parallel lines $l_1(h)$ and $l_2(h)$ are supporting for the set $\mathcal{C} \cap \Pi(h)$ in the plane $\Pi(h)$. Moreover, $[A^*B^*]$ is a local minimizer for the the function $\tilde{f}$ corresponding to the set $\Pi(h) \cap \mathcal{C}$. Then, as we have seen in the previous section, the segment $[A^*B^*]$ is a local minimizer for the function f corresponding to the strip bounded by the lines $l_1(h)$ and $l_2(h)$. This means that A^*B^* is perpendicular to $l_1(h)$ and $l_2(h)$. Since h was an arbitrary vector from L we conclude that the line A^*B^* is orthogonal to L (and to $T+L$). Hence there exists a two dimensional plane Π which is orthogonal to $T + L$ and contains the line A^*B^*. Put $l_1 = \Pi_1 \cap \Pi$ and $l_2 = \Pi_2 \cap \Pi$. The lines l_1 and l_2 are supporting in Π the set $\mathcal{C} \cap \Pi$ at the points A^* and B^*, respectively, and $[A^*B^*]$ is a local minimizer for the set $\mathcal{C} \cap \Pi$ in Π. Proposition 2.5 gives us three concurrent lines p_0, p_1, p_2 in Π (with common point P) which are perpendicular, respectively, to the line A^*B^* at O, to l_1 at A^* and to l_2 at B^*. As p_1 lies in Π, it is orthogonal to $T + L$. It is also perpendicular to l_1. This implies that p_1 coincides with the normal line to Π_1 at A^*. Similarly, p_2 coincides with the normal line to Π_2 at B^*. Clearly, the line p_0, together with its point P lies in the hyperplane through O which is orthogonal to the line A^*B^*. Thus (ii) holds for any nonparallel hyperplanes Π_1 and Π_2 supporting $\mathcal{C}$ at A^* and B^*, respectively.

Note that the line p_1, the normal to Π_1 at A^*, does not depend on the hyperplane Π_2 supporting $\mathcal{C}$ at B^*. Hence, the point P, as the intersection of p_1 and the hyperplane through O orthogonal to the line A^*B^* is the same for every hyperplane Π_2 supporting $\mathcal{C}$ at B^* and nonparallel to Π_1. This, in turn, means that any hyperplane Π_2' supporting $\mathcal{C}$ at B^* and not parallel to Π_1 is orthogonal to the line PB^* and, therefore coincides with Π_2.

To complete the proof, we have to discuss also the case where, along with Π_2, there is another hyperplane Π' which supports $\mathcal{C}$ at B^* and is parallel to Π_1. Suppose such a hyperplane Π' exists. In this case, there is a linear subspace $L \subset \mathbb{R}^n$ of codimension one such that $\Pi_1 = A^* + L$ and $\Pi' = B^* + L$. As above we can see that for every non zero vector $h \in L$ the line A^*B^* is perpendicular to the lines $l_1(h) = \{A^* + th : t \in \mathbb{R}\}$ and $l_2(h) = \{B^* + th : t \in \mathbb{R}\}$. This means that A^*B^* is normal to both Π_1 and Π' at A^* and B^* , respectively. Thus $p_1 = A^*B^*$ and, therefore, $P \in A^*B^*$. Since PB^* is orthogonal to Π_2 we get $p_2 = A^*B^*$. This contradicts the assumption that Π_1 and Π_2 are not parallel. □

As we have seen in the previous section (Proposition 2.7) one cannot expect a similar result for local maximizers. If, however, the set $\mathcal{C}$ is smooth, then CPP holds for local maximizers as well.

Theorem 3.3. *Let O be an interior point for a smooth closed convex set $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 2$, and let the segment $[A^*B^*]$ be a local maximizer for F. Let Π_1 and Π_2 be the only supporting hyperplanes at A^* and B^*, respectively. Then the normal line to Π_1 at A^* and the normal line to Π_2 at B^* intersect at a point belonging to the hyperplane through O which is orthogonal to the line A^*B^*.*

The proof of this theorem is omitted. It almost coincides with the proof of Theorem 3.2. One uses Lemma 2.13 to show that the lineA^*B^* is perpendicular to the parallel lines $l_1(h)$ and $l_2(h)$. Instead of Proposition 2.5, one now uses Proposition 2.10. The possibility to apply Proposition 2.10 is based on the following simple observation, which is easily proved by a separation theorem argument.

Lemma 3.4. *Let $\mathcal{D}$ be a smooth closed convex subset of $\mathbb{R}^n$, $n > 2$, with nonempty interior, and let a two dimensional plane Π intersect its interior. Then the intersection $\mathcal{C}' := \mathcal{C} \cap \Pi$ is a smooth convex subset of Π (i.e. there is a unique supporting for $\mathcal{C}'$ line in Π at every point of the boundary $\partial\mathcal{C}'$ in Π).*

Minor changes are needed to formulate the corresponding results for the case when the point O is outside the set $\mathcal{C}$.

Let $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, be a closed convex set with nonempty interior and let O be a point outside it. Consider all segments $[AB]$ with end-points A and B on the boundary $\partial\mathcal{C}$, and such that the line AB passes through O and $B \in [OA]$. Define for such segments the function $G([AB]) = |AB|$. The definition of "local minimizer"and "local maximizer"for G are introduced as in Definition 3.1.

Theorem 3.5. *Let O be a point outside a closed convex set $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, with nonempty interior, and let the segment $[A^*B^*]$ be a local minimizer for G. Suppose that $[A^*B^*]$ intersects the interior of $\mathcal{C}$. Then:*

(i) *There exists only one hyperplane Π_1 supporting $\mathcal{C}$ at A^* and only one hyperplane Π_2 supporting $\mathcal{C}$ at B^*;*

(ii) *The normal line to Π_1 at A^* and the normal line to Π_2 at B^* intersect at a point in the hyperplane through O which is orthogonal to the line A^*B^*.*

The proof of this theorem follows step-by-step the proof of Theorem 3.2. Instead of Proposition 2.5 one uses Proposition 2.16.

Theorem 3.6. *Let O be a point outside a smooth closed convex set $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 2$ with nonempty interior and suppose the segment $[A^*B^*]$ is a local maximizer for G which intersects the interior of $\mathcal{D}$. Let Π_1 and Π_2 be the only supporting for $\mathcal{D}$ hyperplanes at A^* and B^*, respectively. Then the normal line to Π_1 at A^* and the normal line to Π_2 at B^* intersect at a point belonging to the hyperplane through O which is orthogonal to the line A^*B^*.*

The proof of this theorem follows the steps in the proof of Theorem 3.2. One uses Lemma 2.18 to show that the lineA^*B^* is perpendicular to the parallel lines $l_1(h)$ and $l_2(h)$. Instead of using Proposition 2.5 one can apply Lemma 3.4 and Proposition 2.17.

Remark 3.7. Using the method of Lagrange multipliers, one may show that the theorems proved above for convex sets in $\mathbb{R}^n, n \geq 2$, remain true for any C^1-smooth hypersurface in $\mathbb{R}^n$ (allowing the intersection point of the normals to be an infinite point). Details will appear in a forthcoming paper. □

At the end of this section, we present a generalization of Proposition 2.7. This generalization will be used in the next section.

Definition 3.8. Let $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, be a closed convex set with nonempty interior and $A \in \mathcal{C}$. Denote by $d_{\mathcal{C}}(A)$ the maximal integer k for which there exists a k-dimensional affine subspace $H \subset \mathbb{R}^n$ such that A belongs to the interior of the set $H \cap \mathcal{C}$ in H. □

In particular, A is an interior point of $\mathcal{C}$ if and only if $d_{\mathcal{C}}(A) = n$. Also, A is an extreme point of $\mathcal{C}$ if and only if $d_{\mathcal{C}}(A) = 0$. The most important case for us is when $A \in \partial\mathcal{C}$. In this case, the affine subspace H in the above definition does not intersect the interior of $\mathcal{C}$. Further, it is easy to realize that H is uniquely determined.

Proposition 3.9. *Let $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, be a closed convex set with nonempty interior and O a point in $\mathbb{R}^n$. Consider all lines through O, each of which intersects $\mathcal{C}$ in some (possibly degenerated) segment $[AB]$.*

(i) *if $O \notin \mathcal{C}$ and the segement $[A^*B^*]$ is a local maximizer for the function $G([AB])$ defined above, then $d_{\mathcal{C}}(A^*) + d_{\mathcal{C}}(B^*) \leq n$.*

(ii) *if O is an interior point of $\mathcal{C}$ and the segment $[A^*B^*]$ is a local maximizer for the function $F([AB])$ defined above, then $d_{\mathcal{C}}(A^*) + d_{\mathcal{C}}(B^*) \leq n - 1$.*

Proof. First note that in both cases $A^* \neq B^*$. For the case (ii), this is obvious because O is an interior point for $[A^*B^*]$. In the case (i), this follows from the assumptions that $\mathcal{C}$ has nonempty interior and that $[A^*B^*]$ is a local maximizer. Let H_1, H_2 be the affine subspaces from the definition of $d_{\mathcal{C}}(A^*)$ and $d_{\mathcal{C}}(B^*)$ respectively. Since A^* is an interior point (relative to H_1) of the set $H_1 \cap \mathcal{C}$, we derive that $B^* \notin H_1$ (otherwise there would exist a poit $A \in \mathcal{C}$, different from A^*, and such that $A^* \in [AB^*]$; this would contradict the local maximality of $[A^*B^*]$). Similarly, one derives that H_2 does not contain A^*. Further, there are linear subspaces L_1, L_2 such that $H_1 = A^* + L_1$ and $H_2 = B^* + L_2$. We show next that the intersection $L_1 \cap L_2$ contains only the zero element of $\mathbb{R}^n$. Indeed, suppose there exists a non-zero vector $h \in L_1 \cap L_2$. Consider the two-dimensional plane Π containing the strip bounded by the two different parallel lines $\{A^* + th : t \in \mathbb{R}\} \subset H_1$ and $\{B^* + th : t \in \mathbb{R}\} \subset H_2$. Π contains the line A^*B^* and, therefore, the point O as well. As A^* is from the interior of $\mathcal{C} \cap H_1$ relative to H_1, there exists some $t_1 > 0$ such that $[A^* - t_1 h, A^* + t_1 h] \subset \mathcal{C}$.

Similarly, there exists $t_2 > 0$ such that $[B^* - t_2 h, B^* + t_2 h] \subset \mathcal{C}$. This means that the segment $[A^*B^*]$ is a local maximizer for the function g from Section 2 corresponding to this strip when $O \notin \mathcal{C}$ and local maximizer for the function f from Section 2 corresponding to this strip when O is from the interior of $\mathcal{C}$. This contradicts the strict convexity of the functions f and g. Therefore, both in the case (i) and in the case (ii) we have

$$d_{\mathcal{C}}(A^*) + d_{\mathcal{C}}(B^*) = dim L_1 + dim L_2 = dim(L_1 + L_2) \leq n.$$

This completes the proof of (i).

Let now O be an interior point of $\mathcal{C}$. We prove first that $H_1 \cap H_2 = \emptyset$. Indeed, if there is a point $E \in H_1 \cap H_2$, then $E \neq A^*$ and $E \neq B^*$. Put $v_1 = \overrightarrow{A^*E}$ and $v_2 = \overrightarrow{B^*E}$. Since A^* is in the relative interior of $\mathcal{C} \cap H_1$in H_1, there is some $r_1 > 0$ such that $[A^* - r_1 v_1, A^* + r_1 v_1] \subset \mathcal{C}$. Similarly, there exists $r_2 > 0$ with the property $[B^* - r_2 v_2, B^* + r_2 v_2] \subset \mathcal{C}$. Consider the two-dimensional plane Π' containing the angle $\angle A^*EB^*$. The segment $[A^*B^*]$ being a local maximizer for the function F will be a local maximizer for the function f from Section 2 corresponding to this angle. This contradicts the strict convexity of the function f.

It is now easy to realize that $L_1 + L_2$ does not contain the vector $\overrightarrow{A^*B^*}$. Indeed, if $\overrightarrow{A^*B^*} = v_1 + v_2$ where $v_1 \in L_1$ and $v_2 \in L_2$ then $E := A^* + v_1 = B^* - v_2 \in H_1 \cap H_2$. This implies $d_{\mathcal{C}}(A^*) + d_{\mathcal{C}}(B^*) = dimL_1 + dimL_2 = dim(L_1 + L_2) < n$. □

Corollary 3.10. *Let $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, be a closed convex set and O be an interior point of it. Let $[A^*B^*]$ be a local maximizer for the function $F([AB])$ and let there exist a hyperplane H_{A^*} supporting $\mathcal{C}$ at A^* and such that A^* is an interior point (relative to H_{A^*}) of the set $\mathcal{C} \cap H_{A^*}$. Then B^* is an extreme point of $\mathcal{C}$.*

Proof. Since $d_{\mathcal{C}}(A^*) = n - 1$, we derive that $d_{\mathcal{C}}(B^*) = 0$. □

Similarly, if $d_{\mathcal{C}}(A^*) = n - 2$, then B^* is either an extreme point of $\mathcal{C}$ or lies on some segment contained in $\partial\mathcal{C}$. Note also that in the case when $\mathcal{C}$ is a subset of $\mathbb{R}^2$, this corollary reduces to Proposition 2.7.

4. Convex polytopes

This section presents some results for compact convex sets $\mathcal{C} \subset \mathbb{R}^n, n \geq 2$, with nonempty interior which are polytopes (convex hulls of finitely many points or, equivalently, bounded sets which can be presented as the intersections of finitely many closed half-spaces determined by hyperplanes).

As was mentioned above, the planar version of Proposition 3.9 generalizes Proposition 2.7. It follows that if $[A^*B^*]$ is a local maximizer for a convex polygon $\mathcal{C}$ in the plane and O is a point from its interior, then at least one of the points A^* and B^* is a vertex of the polygon. The following example of a tetrahedron shows that this is not true in higher dimensions.

Example 4.1. Let $\mathcal{C}$ be the convex hull of the points $A_1(1, -1, 0)$, $A_2(1, 1, 0)$, $A_3(-1, 0, 1)$, $A_4(-1, 0, -1)$. Let O be the point $(0, 0, 0)$ (Figure 8).

Every line through O cuts a non-degenerate segment from $\mathcal{C}$. Consider the segment $[AB]$ with end-points $A(1, 0, 0)$ and $B(-1, 0, 0)$. Then it is easy to check that the four segments cut from the tetrahedron by lines OA_i, $i = 1, 2, 3, 4$ have the same length equal to $4\sqrt{2}/3 < 2 = |AB|$. So, the endpoints of any global maximizer $[A^*B^*]$ for the function F are not among the vertices of the tetrahedron. Proposition 3.9(ii) implies now that the end points of $[A^*B^*]$ are also not internal points of the triangles on the surface of the tetrahedron. The only remaining options for A^* and B^* are to belong to the opposite edges $[A_1A_2]$ and $[A_3A_4]$ (as does $[AB]$) or to belong to another pair of opposite skew edges. For symmetry reasons, and because $O \in [A^*B^*]$, the points A^* and B^* must be the mid-points of the corresponding opposite skew edges which are of equal length. There are only two pairs of such skew edges. Consider the segment $[CD]$ where $C = (0, -0.5, -0.5)$ (middle of $[A_1A_4]$) and $D = (0, 0.5, 0.5)$ (middle of $[A_2A_3]$). Its length is $\sqrt{2} < 2$. Similarly, the distance between the middles of the segments $[A_2A_4]$ and $[A_1A_3]$ is less than 2. All this shows that $[AB]$ is the only global maximizer $[A^*B^*]$ for the function F corresponding to the tetrahedron $\mathcal{C}$ and the point O. □

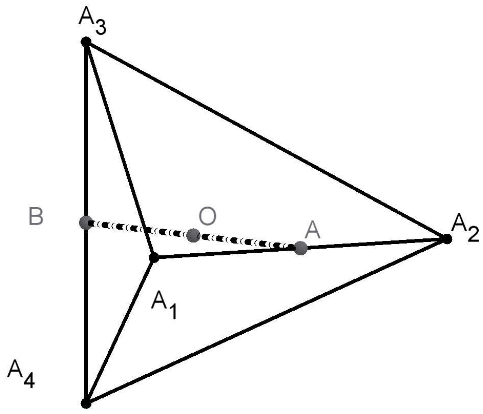

Figure 8

Let us consider another example which shows that Proposition 2.7 is not valid if O is outside the set $\mathcal{C}$.

Example 4.2. Let $\mathcal{C} \subset \mathbb{R}^2$ be the right-angled triangle PQR where $P = (0,0)$, $Q = (6,0)$, $R = (0,2)$, and let $O = (0,3)$ (Figure 9). Consider the line OA where $A = (3,0)$. Then $B = (1.5, 1.5)$ and $|AB| = \sqrt{4.50} > 2 = |PR|$. This suffices to conclude that the end-points of a global maximizer $[A^*B^*]$ for the function g corresponding to this problem are not among vertices of $\mathcal{C}$. □

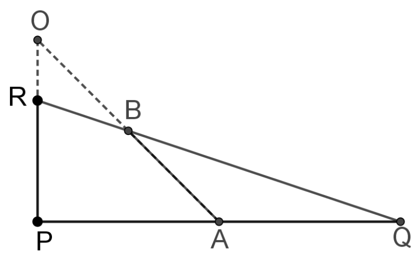

Figure 9

Our next goal is to show that, when the point O is far from a polygon $\mathcal{C} \subset \mathbb{R}^2$, then the local maximizers do contain a vertex of it. More precisely, we will show that the inequality in Proposition 3.9(ii) is valid also in the case when O is at a sufficiently large distance from the convex polytope $\mathcal{C} \subset \mathbb{R}^n, n \geq 2$.

We use the term *"exposed face of a polytope $\mathcal{C}$"* for any set $\mathcal{C} \cap \Pi$ where Π is a supporting hyperplane for $\mathcal{C}$. An exposed face $\mathcal{C} \cap \Pi$ that has nonempty interior relative to Π is called *"facet"*. By *"dimension of an exposed face"* we have in mind the dimension of the affine hull of the exposed face. This affine hull is a translation of some linear subspace L of the same dimension. In particular, the dimension of a facet is $n - 1$. Exposed faces of dimension 1 are also called *"edges"*. The vertices of the polytope are exposed faces of dimension 0.

Let further A be a point from the boundary of the polytope $\mathcal{C}$ and H be the affine subspace from Definition 3.8 which determines the number $d_{\mathcal{C}}(A)$. It is easy to see that H does not intersect the interior of $\mathcal{C}$. Also, it is possible to prove that $H \cap \mathcal{C}$ is an exposed face of the polytope. Finally, it is well-known that any polytope has finitely many exposed faces.

Proposition 4.3. *For any convex polytope $\mathcal{C}$ in $\mathbb{R}^n, n \geq 2$, with nonempty interior there exists a bounded set U such that if $O \notin U$, and $[A^*B^*]$ is a local maximizer for the function G then*

$$d_{\mathcal{C}}(A^*) + d_{\mathcal{C}}(B^*) \leq n - 1.$$

In particular, when $n = 2$, at least one of the points A^* and B^* is a vertex of the polygon $\mathcal{C}$.

Proof. We start with the following

Lemma 4.4. *Let A, B, O, E be points in the plane such that: O is outside the angle $\angle AEB$, $B \in [AO]$, and $0 < \theta := \angle AEB < \pi/2$ (Figure 10). Suppose* CPP *holds for the points A, B, O and the angle $\angle AEB$. Then*

$$\frac{2|OA|}{|AB|} \leq 1 + \frac{1}{\sin\theta}.$$

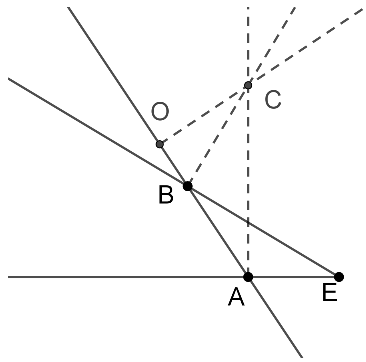

Figure 10

Proof of Lemma 4.4. Denote by C the intersection point of the perpendiculars to EA, EB and AB at A, B and O, respectively (Figure 10). Then $EABC$ is a convex quadrilateral inscribed in the circle with diameter EC. Set $\psi = \angle BAC$. Then

$$AB = EC\sin\theta, \quad OA = AC\cos\psi, \quad AC = EC\sin(\theta + \psi)$$

and hence

$$\frac{2OA}{AB} = \frac{2\sin(\theta+\psi)\cos\psi}{\sin\theta} = 1 + \frac{\sin(\theta + 2\psi)}{\sin\theta}$$

which implies the desired inequality and completes the proof of the lemma.

Further, let $\mathcal{L}$ be the family of all pairs (L_1, L_2) of linear subspaces of $\mathbb{R}^n$, each being a translation of the affine hull of some exposed face of $\mathcal{C}$, and such that $L_1 \cap L_2$ has just one element (the zero element of $\mathbb{R}^n$). $\mathcal{L}$ is a nonempty family (each vertex of $\mathcal{C}$ belongs to at least two different edges to which correspond different one-dimensional linear subspaces L_1 and L_2). $\mathcal{L}$ is also a finite family. Put

$$c = \max_{(L_1,L_2)\in\mathcal{L}} \max\{\langle u_1, u_2\rangle : u_1 \in L_1, u_2 \in L_2, ||u_1|| = ||u_2|| = 1\}.$$

Compactness of the unit sphere and continuity of the function $\langle u_1, u_2\rangle$ yield that $c \in [0, 1)$.

Suppose now that $[A^*B^*]$ is a local maximizer for the function G which correspond to the polytope $\mathcal{C}$ and a point $O \notin \mathcal{C}$. Let H_{A^*}, H_{B^*} be the affine subspaces from the definition of $d_{\mathcal{C}}(A^*)$ and $d_{\mathcal{C}}(B^*)$, respectively. Denote by L_{A^*} and L_{B^*} the linear spaces such that $H_{A^*} = A^* + L_{A^*}$ and $H_{B^*} = B^* + L_{B^*}$. We know from the proof of Proposition 3.9 that $A^* \neq B^*$ and that $L_{A^*} \cap L_{B^*}$ contains only the zero vector of the space. Hence the pair (L_{A^*}, L_{B^*}) belongs to $\mathcal{L}$. Proposition 3.9(i) says also that $dim(L_{A^*}) + dim(L_{B^*}) \leq n$.

Let $U := \{X \in \mathbb{R}^n : \text{dist}(X, \mathcal{C}) \leq M\text{diam}(\mathcal{C})\}$ where $M := \frac{1}{2}(1 + \frac{1}{\sqrt{1-c^2}})$, and let $O \notin U$. Suppose $d_{\mathcal{C}}(A^*) + d_{\mathcal{C}}(B^*) = dim(L_{A^*}) + dim(L_{B^*}) = n$. Then the non-zero vector $\overrightarrow{A^*B^*} = h' + h''$ where $h' \in L_{A^*}$ and $h'' \in L_{B^*}$. Put $E = A^* + h' = B^* - h''$ and consider the angle $\angle A^*EB^*$. Note that A^* belongs to the relative interior of $H_{A^*} \cap \mathcal{C}$ in H_{A^*} and B^* belongs to the relative interior of $H_{B^*} \cap \mathcal{C}$ in H_{B^*}. Therefore, the segment $[A^*B^*]$ is a local maximizer for the function g corresponding to the angle $\angle A^*EB^*$ and the point O. This implies (see Remark 2.15) that $\angle A^*EB^* < \pi/2$. Then Lemma 4.4 implies that $O \in U$, a contradiction. $\square$

At the end of this section, we show that, for a large class of polytopes, the function G does not have local minimizers intersecting the interior of $\mathcal{C}$ when O is far away from the polytope. In other words, the effect observed by Neovius and mentioned above in Subsection 2.2 (that the function g may have a local minimizer intersecting the interior of the angle) does not apper when O is far away from the polytope.

Proposition 4.5. *Let $\mathcal{C}$ be a polytope with nonempty interior and without parallel facets. Then there exists a bounded set U such that if $O \notin U$, the function G does not have local minimizers intersecting the interior of $\mathcal{C}$. In particular, every local minimizer of G is contained in the boundary of $\mathcal{C}$ when O is at a large distance from the polytope $\mathcal{C}$.*

Proof. Let $O \notin \mathcal{C}$ and $[A^*B^*]$ be a local minimizer for the function G which contains interior points of $\mathcal{C}$. According to Theorem 3.5, only one hyperplane supports the polytope at the point A^*. This means that A^* is an interior point of the facet determined by this hyperplane and, therefore, this hyperplane coincides with H_{A^*}. Similarly, B^* is an interior point of the facet $\mathcal{C} \cap H_{B^*}$ relative to H_{B^*}. Put $H := H_{A^*} \cap H_{B^*}$. Since there are no parallel facets, H is a nonempty $(n-2)$-dimensional affine subspase. Let L be the $(n-2)$-dimensional linear subspace corresponding to H. Take a nonzero vector $h \in L$ and consider the strip bounded by the two dufferent lines $\{A^* + th : t \in R\} \subset H_{A^*}$ and $\{B^* + th : t \in R\} \subset H_{B^*}$. Then $[A^*B^*]$ is a local minimizer for the function g corresponding to this strip. This implies that the line $OA^* = OB^*$ is perpendicular to the vector h and, therefore, to L. Hence, there exists a uniquely determined two-dimensional plane Π' that contains OA^* and is orthogonal to the $(n-2)$-dimensional affine subspace H. Let $E = H \cap \Pi'$. Then $[A^*B^*]$ is a local minimizer for the function g corresponding to the angle $\angle A^*EB^*$. Note that the angle $\angle A^*EB^*$ is, actually, the angle between the hyperplanes H_{A^*} and H_{B^*}, and we consider it to be the angle between the two facets $H_{A^*} \cap \mathcal{C}$ and $H_{B^*} \cap \mathcal{C}$. This implies (see Remark 2.15) that $\angle A^*EB^* < \pi/2$. By Lemma 4.4, we get

$$\frac{2|OA^*|}{|A^*B^*|} \leq 1 + \frac{1}{\sin \angle A^*EB^*}.$$

Denote by Θ the set of all numbers θ which are angles between two facets of $\mathcal{C}$.

Since there are no parallel facets, and the number of facets is finite, the number satisfies $m = min\{|sin\theta| : \theta \in \Theta\} > 0$. It follows that $O \in U$ where

$$U := \{X \in \mathbb{R}^n : \operatorname{dist}(X, \mathcal{C}) \leq M \operatorname{diam}(\mathcal{C})\} \text{ with } M := \frac{1}{2}\Big(1 + \frac{1}{m}\Big). \qquad \square$$

5. Appendix

This section presents briefly two practice-oriented problems in which CPP plays some role.

5.1. A special parking problem from [6]

A car (wheel-chair, baby carriage) is on the street and has to be parked in the basement of a house over a slope (see Figure 11) without damaging the bottom of the vehicle.

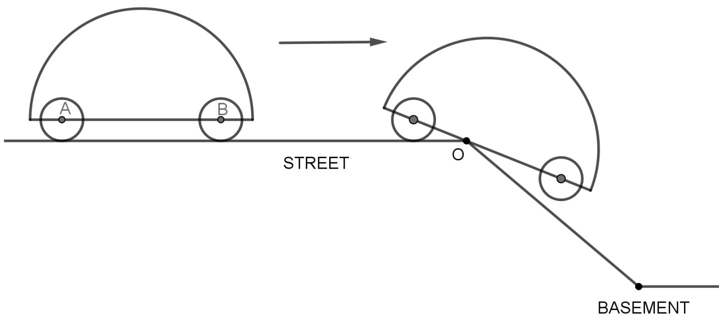

Figure 11

Whether or not this is possible depends on the distance d between the centers of the wheels (points A and B), the radius r of the wheels and the steepness of the slope. Suppose the slope and the radius r are given. Then, a natural question arises:

"What is the maximal d such that a car with $d = |AB|$ can overcome the slope keeping the bottom safe?"

An upper bound for d is given by the mathematical problem considered in this paper. It is depicted in Figure 12:

"Find the shortest segment $[DE]$ passing through O with end points on the dotted line (consisting of points r-distanced from the street or the slope)?".

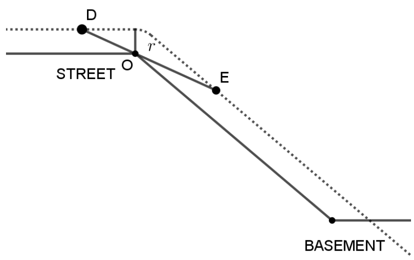

Figure 12

As we know, the solution to this problem is governed by CPP.

5.2. Estimating the distance to a broadcasting object

A point-like source O casts (radiates, sends) signals in all directions. An observer knows the direction the signals are coming from and wants to find, at least approximately, the distance to the source. The observer disposes with a solid object (shaped as in Figure 13) made of material which absorbs the intensity (the strength) of the signals proportionally to the path-length the signals travel through the object.

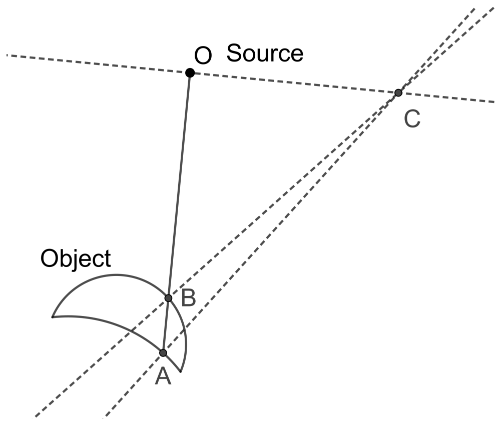

Figure 13

Placing sensors along the smaller arc, one finds the point A where the signal intensity is minimal. Since the direction to the source O is known, one can find the point $B \in [OA]$ on the other arc of the object. The segment $[AB]$ is a local maximum among the segments cut from the object by lines containing point O. Since both arcs are smooth curves, Lemma 2.9 implies that CPP holds. Let C be the intersection of the normals to the object at the points A and B. By CPP, the origin O of the signals must belong to the perpendicular from C to AB.

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