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# On the Theorems of Central Axonometry

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**Abstract.** One of the important questions of central axonometry is to give a condition under which a central axonometric mapping is a central projection. The aim of this paper is to prove that the well-known STIEFEL's condition can be considered as a limiting case of a recent theorem proved by SZABÓ, STACHEL and VOGEL.

## 1. Introduction

Central axonometry, as a projective generalization of the classical axonometric mapping, plays an important role in descriptive geometry and computer graphics. A *central axonometric mapping* of the projective space onto the projective plane $\pi$ can be defined as follows: Let a point $O$ in the plane $\pi$ be given as the image of the spatial origin. Three arbitrary lines $x, y$ and $z$ through $O$ are in $\pi$ the given images of the axes of a spatial cartesian coordinate system. Furthermore, let two distinct points on each axis, $E_x, U_x, E_y, U_y$ and $E_z, U_z$ be the images of the unit points and the vanishing points of the spatial axes, respectively (cf. Figure 2). Now the image of a spatial point $P = (p_x, p_y, p_z)$ is obtained by completing the projective scales on the axes. This means that the $i$-th coordinate $p_i$ of $P$ equals the cross ratio $p_i = (P_i E_i O U_i)$ on the axis, when $P_i$ denotes the orthogonal projection of point $P$ onto the $i$-th axis. The seven points $O, E_x, E_y, E_z, U_x, U_y, U_z$ form the so-called *reference system* of the central axonometric mapping.

One of the central questions in this field is the following: Under which conditions is a central axonometric mapping of an object a central projection of this object? Or — in other words — in which way can the reference system of central projection be characterized? The first (rather complicated) result has been given by KRUPPA [1], but this characterization is synthetic, so it cannot be applied immediately in computer graphics. An analytical condition for a particular case has been given by STIEFEL [2] (Theorem 1 of this paper), while finally SZABÓ et al. [3] proved a general theorem (Theorem 2 of this paper) with a simple analytical condition. In this paper we prove that STIEFEL's condition can be considered as a limiting case of this general theorem.

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## 2. The Theorems

With the notations introduced above, STIEFEL's condition can be expressed as follows:

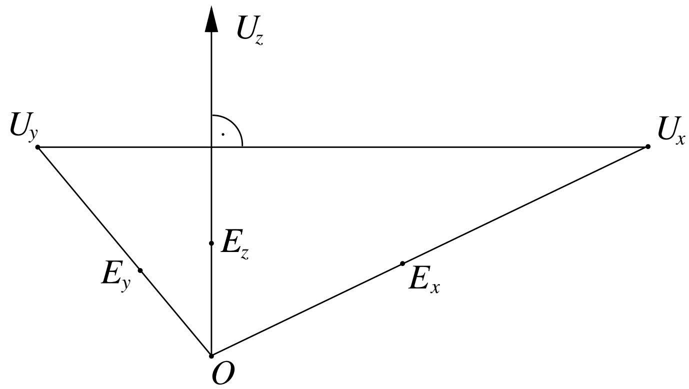


Figure 1: STIEFEL's particular case

**Theorem 1** (STIEFEL)**:** *If the vanishing point $U_z$ of the central axonometric reference system $O, E_x, E_y, E_z, U_x, U_y, U_z$ is infinite and perpendicular to the line joining the vanishing points $U_x, U_y$ of the $x$- and $y$-axis (see Figure 1), then this central axonometric mapping is a central projection if and only if the mutual distances obey*

$$\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2 = \left(\frac{U_xU_y}{OE_z}\right)^2 . \tag{1}$$

**Theorem** (SZABÓ, STACHEL, VOGEL)**:** *When all points $O, E_x, E_y, E_z, U_x, U_y, U_z$ of a central axonometric reference system are finite, then the defined mapping is a central projection if and only if*

$$\left(\frac{OE_x}{E_xU_x}\right)^2 : \left(\frac{OE_y}{E_yU_y}\right)^2 : \left(\frac{OE_z}{E_zU_z}\right)^2 = \tan\alpha : \tan\beta : \tan\gamma\,, \tag{2}$$

*where $\alpha = \angle U_zU_xU_y, \beta = \angle U_xU_yU_z, \gamma = \angle U_yU_zU_x$ (see Figure 2).*

## 3. The Limiting Process

Now we will prove that the general Theorem 2 implies Theorem 1 as its limiting case. First a lemma will be stated.

**Lemma:** *When the vertex $C$ of a triangle $ABC$ tends to infinity along the fixed altitude of $C$, then according to the notation in Figure 3*

$$\lim_{C\to\infty} m_c \sin\gamma = c\,.$$

*Proof of the Lemma:* From the area of the triangle $ABC$ in Figure 3 we obtain

$$ab\sin\gamma = m_c\, c \quad \text{or} \quad m_c \sin\gamma = c\,\frac{m_c^2}{ab}\,.$$

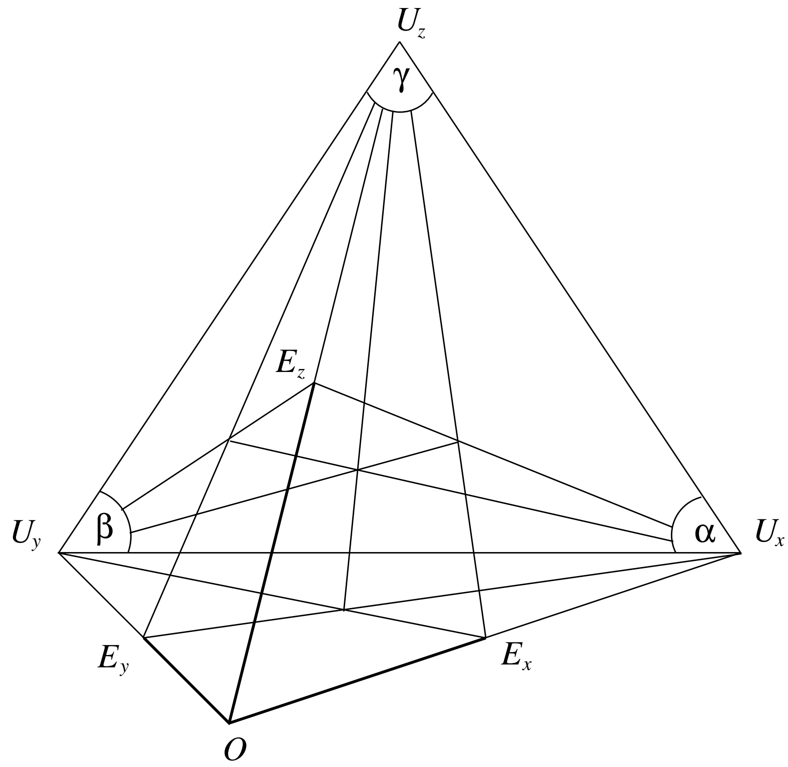


Figure 2: Central axonometric reference system $O, E_x, E_y, E_z, U_x, U_y, U_z$

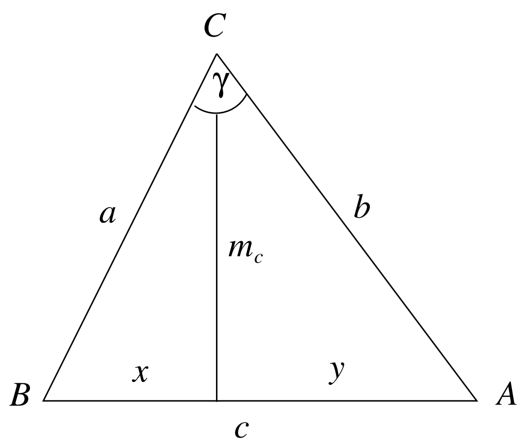


Figure 3:

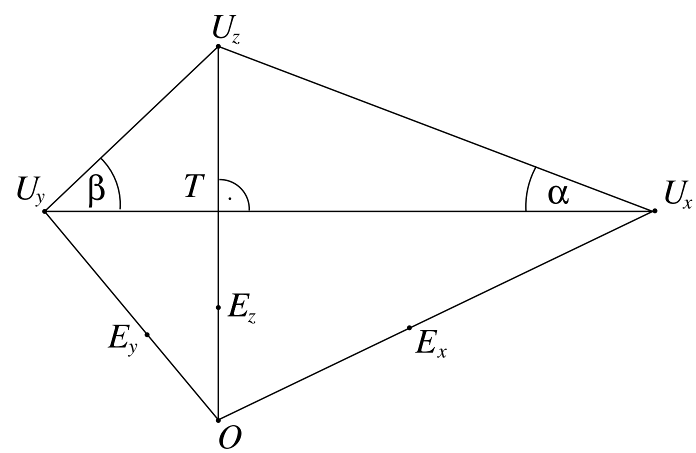


Figure 4:

Using this equation and the fact that $C \to \infty$ implies $m_c \to \infty$ we obtain

$$\lim_{C\to\infty} m_c \sin\gamma = \lim_{C\to\infty} c\,\frac{m_c^2}{ab} = c \lim_{C\to\infty} \frac{{m_c}^2}{ab} = c \lim_{C\to\infty} \frac{1}{\frac{\sqrt{m_c^2+x^2}}{m_c}\,\frac{\sqrt{m_c^2+y^2}}{m_c}} = c\cdot 1 = c$$

where $x$ and $y$ according to Figure 3 denote the lengths of the sections of $AB$ separated by the altitude of $C$. $\square$

Now we can prove the main result:

*Proof (Theorem 2 ⇒ Theorem 1):* From formula (2) we deduce

$$\cot\alpha = \left(\frac{E_xU_x}{OE_x}\right)^2 \left(\frac{OE_z}{E_zU_z}\right)^2 \cot\gamma \quad \text{and} \quad \cot\beta = \left(\frac{E_yU_y}{OE_y}\right)^2 \left(\frac{OE_z}{E_zU_z}\right)^2 \cot\gamma$$

which imply

$$\cot\alpha + \cot\beta = \left[\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2\right] \left(\frac{OE_z}{E_zU_z}\right)^2 \cot\gamma\,. \tag{3}$$

On the other hand, let us consider a special case of the coordinate system, which is partially equal to STIEFEL's case: Let the $z$-axis be perpendicular to the line $U_xU_y$, but let the point $U_z$ be finite (see Figure 4). Then the point $T$ of intersection between the $z$-axis and the line $U_xU_y$ obeys

$$\cot\alpha = \frac{U_xT}{U_zT} \quad \text{and} \quad \cot\beta = \frac{U_yT}{U_zT}\,,$$

therefore

$$\cot\alpha + \cot\beta = \frac{U_xU_y}{U_zT}\,.$$

The right side of this equation must be equal to the right side of (3):

$$\frac{U_xU_y}{U_zT} = \left[\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2\right] \left(\frac{OE_z}{E_zU_z}\right)^2 \cot\gamma.$$

Multiply both sides of the equation with $\dfrac{U_xU_y \cdot U_zT}{{OE_z}^2}$:

$$\left(\frac{U_xU_y}{OE_z}\right)^2 = \left[\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2\right] \frac{U_xU_y \cdot U_zT \cdot \cos\gamma}{{E_zU_z}^2 \cdot \sin\gamma}\,.$$

Note that this latter equation is the same as (1) apart from the last factor of the right side. Hence it is sufficient to prove that if the point $U_z$ tends to infinity along the fixed $z$-axis, then this factor tends to 1, that is

$$\lim_{U_z\to\infty} \frac{U_xU_y \cdot U_zT \cdot \cos\gamma}{{E_zU_z}^2 \cdot \sin\gamma} = 1\,.$$

From Figure 4 one can deduce that $E_zU_z = U_zT \pm TE_z$. So this formula can be written as

$$\frac{U_zT}{E_zU_z} \cdot \frac{U_xU_y \cdot \cos\gamma}{U_zT \cdot \sin\gamma \pm TE_z \cdot \sin\gamma}\,.$$

Now in order to compute this limit we decompose the term into the following parts:

1. $\lim\limits_{U\to\infty} \dfrac{U_zT}{E_zU_z} = 1$, since $E_zU_z = U_zT \pm TE_z$ and $TE_z$ is constant.
2. $\lim\limits_{U\to\infty} U_xU_y \cdot \cos\gamma = U_xU_y$, since $\lim\limits_{U\to\infty} \gamma = 0$.
3. $\lim\limits_{U\to\infty} TE_z \cdot \sin\gamma = 0$.
4. $\lim\limits_{U\to\infty} U_zT \cdot \sin\gamma = U_xU_y$ due to the Lemma.

Finally 1–4 imply

$$\lim_{U\to\infty} \frac{U_xU_y \cdot U_zT \cdot \cos\gamma}{{E_zU_z}^2 \cdot \sin\gamma} = \frac{U_xU_y}{U_xU_y} = 1$$

which completes the proof. □

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