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Classical Geometry and Computers

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Abstract. We show that the present stage of development of computer hardware and software enables to solve many elementary and non-elementary problems of classical geometry, which in the past could not be solved for the complexity of involved equations or the degree of the problem. Demonstrative examples are given, including corresponding MAPLE session record.

Key Words: Classical geometry, Computer algebra.

MSC 1994: 51N20

1. Introduction

Quick development of computer hardware and software in last years has influenced almost all human activities, including mathematics and geometry as its part as well. Computers in geometric applications are usually understood as high performance plotting machines producing large scale of geometric models from simple planar drawings to real life scenes.

In this paper we want to show how computers can influence geometry as a part of mathematics. We demonstrate how computer equipped with corresponding software can help to solve geometric problems, which were several years ago either unsolvable or solvable with great computational effort.

We shall concentrate on classical geometry, which is in our context understood as geometry of the Euclidean plane or space. It studies properties and relations between elementary objects (points, lines, planes, elementary surfaces and solids) or objects given by equations (explicit or parametric).

A large part of classical geometry is devoted to the solution of problems. Problems in classical geometry can be divided into two classes: Elementary problems are problems solvable by ruler and compass (they are described by linear and quadratic equations, the solution can be expressed by elementary operations including square roots). Non-elementary problems are either algebraic (to solve them we have to solve algebraic equations of higher degree) or transcendental (solution of non-algebraic equations is necessary).

The intersection of a line with a circle in the plane is an example of an elementary problem; the intersection of two conics in the plane is in general a non-elementary algebraic problem

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of degree four; the intersection of a helix with a plane is a transcendental problem. The last example shows that even transcendental problems appear in geometric applications. Some non-elementary problems are very old, for instance the duplication of a cube is algebraic of degree 3, the rectification of an arc of a circle is a transcendental problem, both two thousand years old.

For a long time classical geometry was devoted to solutions of elementary problems, restriction due to available means. In plane geometry we can find a large quantity of elementary problems, constructions of triangles, circles, regular polygons and similar ones and they can be solved effectively by classical means. The situation in space is different. We have elementary problems obtained by extending planar problems to the space – for instance the construction of a sphere from four points as the spatial analogy of the circumscribed circle of a triangle.

The synthetic method works as well, but the classical means of elementary geometry are not optimal for practical solution of such problems. As an illustrative example we shall discuss the spatial analogy of the problem of Apollonius (construction of a sphere contacting four given spheres), which is elementary in any dimension. This means that there exists a sequence of elementary planar constructions in Monge projection (for instance) which leads from the given four spheres to the center and radius of a solution. It is hard to imagine anybody to perform the whole sequence of these steps to a result, which is of no practical use because the construction error will be probably of the order of given data. On the other side this problem has found practical application in global positioning systems (GPS), where it is solved almost immediately with exactness of order about 10^{-6}, see for instance [8] or [4].

This only example justifies the question, whether the approach to classical geometry should not be changed. Moreover, even in the geometry of a triangle (which usually means to construct a triangle from any three of given elements – sides, angles, heights, medians, radius of inscribed or circumscribed circle, circumference) we find many non-elementary problems. Non-elementary problems are considered as uninteresting and we can ask whether modern means can change this approach. We would like to show that the answer is yes to both questions.

During the historical development of classical geometry two solution methods were developed – synthetic and analytic. (We often speak about analytical and synthetic geometry, which is not quite correct.) In this paper we shall not discuss the synthetic method, because we do not see, how computers could influence it (not saying that it is not possible).

The analytical method in geometry was founded by R. Descartes (1596–1650) in his book *"La Géométrie"* (1637), where he used the idea of the correspondence between pairs of numbers and points in plane through two systems of parallel lines. This is well known. More interesting is to notice the problem which inspired Descartes to his discovery (see [2]). It is an old problem of Pappus (about 320 A.D.), which was solved for the first time by Descartes by his new coordinate method. We shall present briefly this problem, because it is instructive and it demonstrates how the choice of the right method can make a difficult problem easy and vice versa, see Fig. 1:

We are given four lines a, b, c, d in the plane and four angles $\alpha, \beta, \gamma, \delta$. To each point P in the plane we construct four lines passing through P such that the angle between them and a, b, c, d is $\alpha, \beta, \gamma, \delta$, respectively. The intersection points are denoted by A, B, C, D, respectively. We look for the set of points P satisfying the condition

$$\overline{AP} \cdot \overline{BP} = k \cdot \overline{CP} \cdot \overline{DP}$$

for a given constant k.

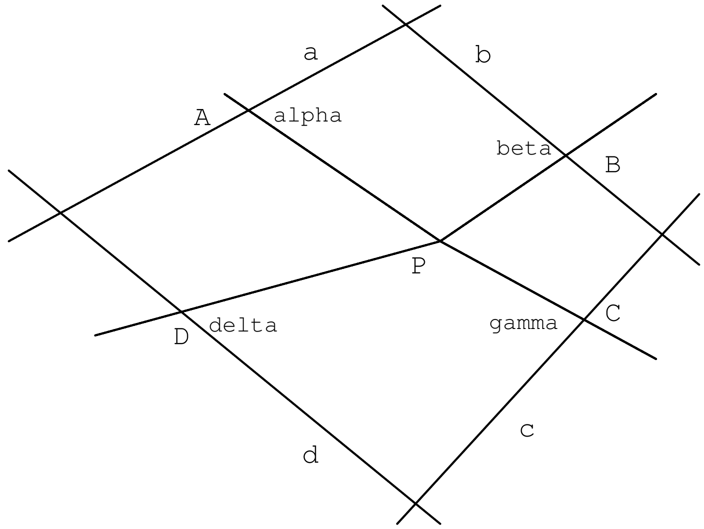

Figure 1: An old problem of Pappus

The solution is as follows: We introduce Cartesian coordinates x, y in the plane. We see immediately that we can replace distances $\overline{AP}, \overline{BP}, \overline{CP}, \overline{DP}$ (up to a fixed factor) by distances of P from a, b, c, d and these are obtained (again up to a fixed factor) by substitution of coordinates x, y of point P into equations of lines. If $a_i x + b_i y + c_i = 0$, $i = 1, \ldots, 4$ are equations of lines a, b, c, d, the solution set of the problem is given by the equation

$$(a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) \pm K(a_3 x + b_3 y + c_3)(a_4 x + b_4 y + c_4) = 0,$$

where K is some other constant, derived from k. This shows that the solution of the Pappus problem is a conic (or parts of conics if we do not consider orientation). Any synthetic solution of this problem is not known to the author.

The coordinate method was extended to the space by A. Clairaut (1731) and led to a great development of the analytical method with its culmination at the end of the 19th century. Two of the most famous results from this time are: Cayley and Simon have shown that every cubic surface contains 27 straight lines and gave an example, where all these lines are real. Kummer has found an algebraic surface of degree four with maximal number (16) of singular points. It would be interesting to have computer models of these surfaces, so far as I know, only solid models are known.

The solution of geometric problems by the analytical method is usually reduced to the problem of solution of systems of equations for several unknowns. Transcendental equations in most cases have to be solved numerically from the very beginning and therefore we shall restrict ourselves to geometric problems leading to systems of algebraic equations. Theoretical background for solution of such systems lies in algebra and it is very well developed (theory of ideals, Gröbner bases and so on). The difficulty lies in the fact that even for not very complicated problems we can obtain long and complicated equations, sometimes it is impossible to write them down. These were the boundaries of the analytical method in classical geometry at the end of the last century.

This situation has now changed by the existence of formal manipulation systems for computers (for instance MAPLE or MATHEMATICA) which allow handling of equations, which were of no use several years ago. The use of computer manipulation systems opens new possibilities for classical geometry (and also new problems and difficulties, as should be expected). This statement deserves at least partial explanation; we use an example: It is easy to formulate problems from the classical geometry of the space which are not elementary, but their solution makes sense – sometimes even for applications. To name just a few of them – the construction of a sphere, cylinder or cone of revolution from a suitable number of tangent lines, to construct a cylinder or cone of revolution from a suitable number of points and combined problems, together with tangent planes or radii. Not all of them are non-elementary, but the majority is. Such problems were studied and many properties in this respect are known (see for instance [1], [5], [6], [7] to name only a few). To bring a solution of such a problem to a successful end – which the author understands as to produce a drawing of the solution – a numerical solution of an algebraic equation of a high order is needed. Moreover, considered equations can be very long and the computer is a useful tool for handling them.

Above mentioned problems lead often to a solution of quadratical equations for many unknowns. The geometrical meaning of such a problem can be expressed as follows: We have n quadrics in the n-dimensional space, which have some special property depending on the problem in question and we want to determine the set of their common points. For this problem the following questions are basic:

1. What is the degree of the problem – what is the degree of the last polynomial we have to solve to obtain all solution?
2. What is the maximal number of real solutions?
3. Under which conditions the problem has infinitely many solutions and what is the geometrical meaning for it?

These problems are difficult problems of algebraic geometry and there exist sophisticated approaches to it (for example intersection theory), but very little is known. From the theoretical point of view this problem can be solved by resultants using MAPLE if given data are rational numbers, provided the last equation is not too large. In the general case the algorithm is limited by the size of appearing equations.

Some problems in classical geometry have already been solved with the help of computer algebra, mainly in connection with applications of geometry in practice, see for instance [3]. In what follows we shall present three geometric problems of such nature that the computer is a helpful tool for their solution. Solutions are illustrated by corresponding MAPLE sessions, which appear in abbreviated form, just to show how to proceed in similar cases.

2. Examples

2.1. Example 1

Construct an isosceles triangle from its circumference $2m$ and radius r of the inscribed circle.

We take Cartesian system of coordinates x, y in the plane; the base is placed on x axis, the corresponding altitude lies on the y axis. Let $2a$ denote the length of the base and b be the altitude. We have $a > 0$, $b > 0$ and we obtain the equation

$$m = a + \sqrt{a^2 + b^2},$$

which yields $(m - a)^2 = a^2 + b^2$.

The computation is shown in Appendix 1. In Fig. 2 the cubic curve of the connection between the parameter m/r and the value of b/r is displayed. It is easy to find out for which values of m/r we have two, one or no real solutions of the problem. In Fig. 2 the unit of measurement is changed to have $r = 1$.

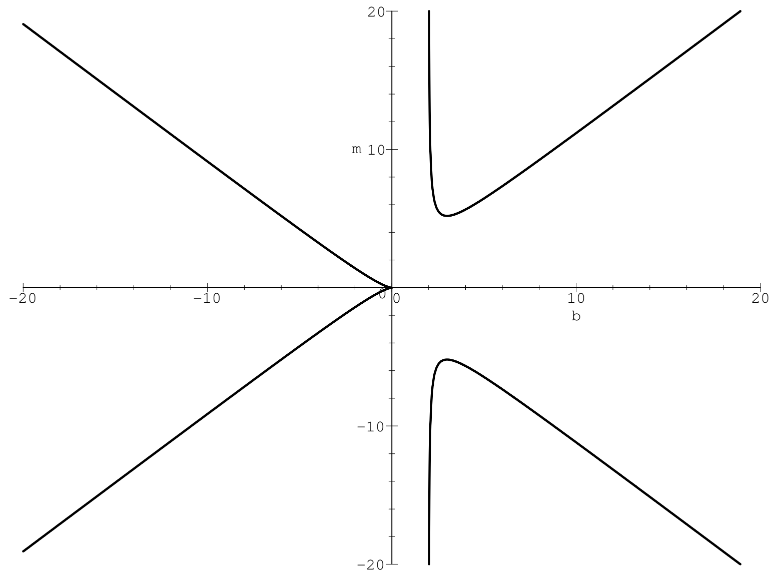

Figure 2: Example 1, plot of the cubic curve $(b-2)m^2 - b^3 = 0$

2.2. Example 2

Four algebraic curves $p_1, \ldots, p_4$ in the plane are given. We want to construct a square $A_1, \ldots, A_4$ such that $A_i \in p_i$ for all $i = 1, \ldots, 4$.

We choose an equiform transformation to solve the problem. The curves p_i, $i = 1, \ldots, 4$, are expressed by their equations in Cartesian coordinates x, y. We take another copy of the plane with Cartesian coordinates and place the unknown square symmetrically with respect to axes,

$$A_{1,3} = [\pm 1, 0], \quad A_{2,4} = [0, \pm 1].$$

We have to find an equiform transformation of the plane given by equations

$$x_0 = xk + yl + m, \quad y_0 = -xl + yk + p$$

in such a way that $A_i \in p_i$ for $i = 1, \ldots, 4$. We substitute the transformed vertices $A_1, \ldots, A_4$ into equations of curves $p_1, \ldots, p_4$ and we obtain four equations for four unknowns k, l, m, p.

We see immediately that this substitution preserves the degree of curves $p_1, \ldots, p_4$. This means that in general the degree of the problem will be given as the product of the degrees of the given curves p_i. If p_i are straight lines, the problem is linear and the solution is obtained by solving a system of linear equations. The synthetic solution is based on a theorem from equiform kinematics, which says that if during an equiform motion in the plane three points have straight trajectories, all points have straight trajectories, see [9], Chapter 2, Theorem 3, page 71. (By the way, appropriate changes in Appendix 2 yield a demonstration of this property.)

Interesting situation appears if curves p_i are conic sections. In this case the problem is of degree at most 16 and it leads to the problem of intersection of four quadrics in the 4-dimensional space. This can be done by the computer, but formulas become large. For an illustration we have chosen two straight lines and two conic sections. Appendix 2 shows the MAPLE session for this case, only one solution is given, we see it at Fig. 3.

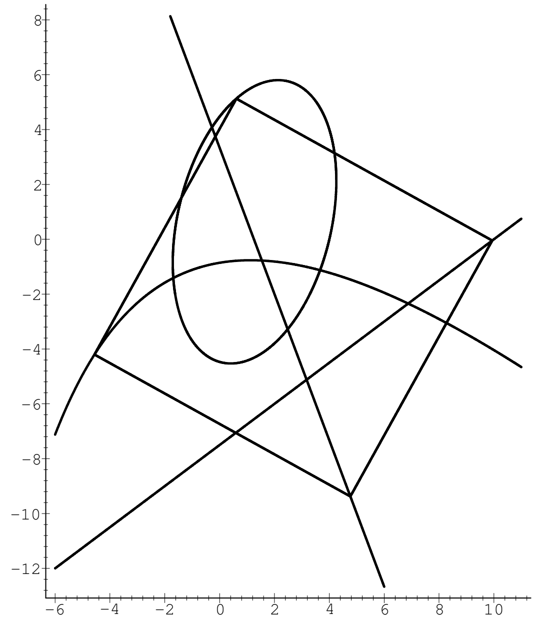

Figure 3: Example 2

2.3. Example 3

We want to find a cylinder of revolution of given radius r which passes through four given points in the space.

We introduce Cartesian coordinates $[x, y, z]$ in the space, let the axis o of the unknown cylinder be given by equations $x = a + mz$, $y = b + nz$, we suppose $o \not\perp z$. Let $A = [p, q, s]$ be a point. Then A lies on the cylinder with axis o and radius r iff the distance from a to o is equal to r. Evaluation for given points $A_i = [p_i, q_i, s_i]$, $i = 1, \ldots, 4$, yields four equations, which are relatively long and they are not displayed here. The rest is done on the computer (see Appendix 3). We obtain four equations for four unknowns a, b, m, n.

They are linear in a and b, we express a and b from them and substitute into remaining equations. They are simplified by a suitable choice of system of coordinates. All operations can be performed to the end. As the result we obtain an algebraic equation of degree 12 of lengths 17232. For any choice of starting data we can obtain the numerical result, we give one choice which leads to 8 real solutions. If more then 8 real solutions are possible I do not know.

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> APPENDIX 1 - Construction of a triangle

Equation for the circumference. Equation of the side. Equation of the inscribed circle.

```
> f:=expand((m-a)^2-(a^2+b^2)):g:=x/a+y/b-1:h:=expand(x^2+(y-r)^2-r^2):
> x:=solve(g,x):h:=normal(h*b^2);A:=coeff(h,y,2):B:=coeff(h,y,1):C:=coeff(h,y,0):
```

$$h := a^2\, y^2 - 2\, a^2\, y\, b + a^2\, b^2 + y^2\, b^2 - 2\, y\, r\, b^2$$

Condition for contact of the side with the inscribed circle:

```
> di:=factor(B^2-4*A*C);
```

$$di := 4\, b^3\, (2\, a^2\, r + b\, r^2 - a^2\, b)$$

```
> k:=op(3,"):a:=solve(f,a):k:=factor(4*k*m^2);k1:=op(2,k);k2:=op(2,k):
> k:=factor(4*k*m^2);
```

$$k := -(-2\, m^2\, r + b\, m^2 - b^3)\, (2\, b\, r + m^2 - b^2)$$

$$k1 := -2\, m^2\, r + b\, m^2 - b^3$$

Solution for k2=0:We substitute into a:

```
> a1:=subs(b^2=m^2+2*b*r,a);
```

$$a1 := -\frac{b\, r}{m}$$

This shows that we have no solution in this case. Solution for k2=0: We change the unit of measurement to have r=1
and plot k1=0. We see that we have at most two solutions.

```
> with(plots):r:=1;implicitplot(k1=0,b=-8..8,m=-10..10,numpoints=5000,color=bla
  ck,thickness=3);
```

We see that we have at most two solutions.

Figure 4: Appendix 1

> APPENDIX 2-CONSTRUCTION OF A SQUARE WITH VERTICES ON GIVEN FOUR CURVES

Equations of curves

```
> p1:=3*x^2-x*y+1*y^2-7*x-20:p2:=-2*x^2-2*x*y+3*x-28*y-24:p3:=3*y+8*x-10:p4:=
  -3*x+4*y+30:
```

Equiform transformation of the square from the basic position

```
> x0:=x*k+y*l+m:y0:=-x*l+y*k+p:
```

Vertices of the transformed square

```
> x1:=subs(x=1,y=0,x0):y1:=subs(x=1,y=0,y0):x3:=subs(x=-1,y=0,x0):y3:=subs(x
  =-1,y=0,y0):
> x2:=subs(x=0,y=1,x0):y2:=subs(x=0,y=1,y0):x4:=subs(x=0,y=-1,x0):y4:=subs(x
  =0,y=-1,y0):
```

Incidence conditions

```
> q1:=expand(subs(x=x1,y=y1,p1));q2:=subs(x=x2,y=y2,p2):q3:=subs(x=x3,y=y3,
  p3):q4:=subs(x=x4,y=y4,p4):
```

$$q1 := 3\,k^2 + 6\,k\,m + 3\,m^2 + k\,l - k\,p + m\,l - m\,p + l^2 - 2\,l\,p + p^2 - 7\,k - 7\,m - 20$$

We express the translation part

```
> solve({q3,q4},{p,m});assign("):T:=factor(resultant(q1,q2,l));
```

$$\left\{m = -\frac{3}{41}\,l + \frac{20}{41}\,k + \frac{130}{41},\ p = -\frac{33}{41}\,l + \frac{56}{41}\,k - \frac{210}{41}\right\}$$

$$T := \frac{2281778112}{2825761}\,k^4 + \frac{32240944}{2825761}\,k^3 + \frac{42732463700}{2825761}\,k^2 + \frac{81540198132}{2825761}\,k - \frac{57417887792}{2825761}$$

```
> k:=fsolve(T,k)[1];
```

$$k := -2.078448572$$

```
> fsolve(q1,l);fsolve(q2,l);
```

$$-7.250549201,\ -1.908326410$$

$$-7.250549201,\ 181.5538503$$

```
> l:="[1];
```

$$l := -7.250549201$$

```
> with(plots):
> implicitplot({p1,p2,p3,p4},x=-5..5,y=-10..10);
> solve(p1,y):Y1:="[1]:Y2:=""[2]:Y3:=solve(p2,y):Y4:=solve(p3,y):Y5:=solve(p4,y)
  :solve(p1,x):X1:="[1]:X2:=""[2]:
> Y6:=(y2-y1)/(x2-x1)*(x-x1)+y1:Y7:=(y3-y2)/(x3-x2)*(x-x2)+y2:Y8:=(y4-y3)/(x4-x3)*
  (x-x3)+y3:Y9:=(y4-y1)/(x4-x1)*(x-x4)+y4:
> plot({[x,Y1,x=-4..6],[x,Y2,x=-4..4],[x,Y3,x=-6..11],[x,Y4,x=-1.8..6],[x,Y5,x=-6..11],[
  x,Y6,x=x1..x2],[x,Y7,x=x2..x3],[x,Y8,x=x3..x4],[x,Y9,x=x1..x4],      [X1,y,y=-2..5],[
  X2,y,y=-2..5] },color=black,thickness=3,numpoints=4000);
```

Figure 5: Appendix 2

```
> APPENDIX 3- CONSTRUCTION OF CYLINDER
  Equation of axis of the cylinder, plane perpendicular to axis
> X:=A+M*z:Y:=B+N*z:rho:=M*(x-p)+N*(y-q)+(z-s):rho:=subs(x=X,y=Y,rho):z1:=so
  lve(rho,z):
> d:=(X-p)^2+(Y-q)^2+(z-s)^2:f:=normal(subs(z=z1,d-r)):f:=op(1,f):f1:=subs(p=p1,
  q=q1,s=s1,f);
```

$$f1 := -2\,A\,M\,N\,B + A^2\,N^2 + A^2 + B^2\,M^2 + B^2 - r\,M^2 - r\,N^2 + p1^2 - 2\,A\,p1 + p1^2\,N^2 - 2\,B\,q1 + q1^2\,M^2 + M^2\,s1^2 + N^2\,s1^2 + 2\,M\,N\,B\,p1 - 2\,A\,N^2\,p1 - 2\,B\,M^2\,q1 + 2\,A\,M\,s1 + 2\,B\,N\,s1 - 2\,M\,s1\,p1 - 2\,N\,s1\,q1 + 2\,A\,M\,N\,q1 + q1^2 - 2\,M\,N\,q1\,p1 - r$$

```
  r is the square of the radius of the cylinder, other equations are similar and not displayed
> f2:=subs(p=p2,q=q2,s=s2,f):f3:=subs(p=p3,q=q3,s=s3,f):f4:=subs(p=p4,q=q4,s
  =s4,f):
  We choose a special system of coordinates and unit of measurement
> s1:=0:p1:=0:q1:=0:q2:=0:s2:=0:s3:=0:p2:=1:
> u2:=factor(f1-f2):u3:=factor(f1-f3):u4:=factor(f1-f4):solve({u2=0,u3=0},{A,B}):as
  sign(");
> A:=factor(A);B:=factor(B):
```

$$A := \frac{1}{2}\,(N^3\,M\,p3^2 + N\,M^3\,q3^2 + N\,M\,p3^2 + N\,M\,q3^2 - 2\,N^2\,M^2\,q3\,p3 - M\,N\,p3 + q3\,N^2 + q3 - M\,N^3\,p3 + M^2\,q3\,N^2 + M^2\,q3)\Big/(q3\,(M^2 + N^2 + 1))$$

```
  We substitute into remaining equations, no display
> f1:=factor(f1):f1:=op(2,f1):u4:=factor(u4):u4:=op(1,u4):
  We eliminate M from u4=0, f1=0, resulting equation is a very large equation of degree 12
  in N
> T:=resultant(f1,u4,M):nops(T);degree(T,N);
                                   17232

                                    12

  For example we choose numerical values
> p3:=3:q3:=2:p4:=-1:q4:=4:s4:=12/5:r:=13/2:g:=fsolve(T,N);
       g := -2.347273427, -1.709436008, -1.282620199, -.4794124872, -.3719465229,
            -.1186604371, .6760208649, 1.047551979
```

Figure 6: Appendix 3

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