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Almost Curvature Continuous Fitting of B-Spline Surfaces

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Abstract. In this paper an algorithm is presented for the fitting of a tube shaped B-spline surface of (3,2) degrees to another surface along a given connection curve. Satisfying prescribed first and second order boundary conditions at a finite number of interpolation points on the connection curve several vertices of the control net of the bordering patches are computed from a fairness condition. The resulting B-spline surface joins the second surface with almost curvature continuity at those common points in which the tangent planes and the normal curvatures in one prescribed direction of both surfaces coincide.

Keywords: Computer-aided design, B-spline surface, boundary conditions, fairing.

1. Introduction

Methods for shaping surfaces along their boundary curves are required in several applications of CAGD, for example at joining surfaces like a kettle and a spout or a handle. However, instead of modifying one of the surfaces in order to obtain a smooth composite surface, the technique of blending surfaces is usually applied. Blending surfaces form a smooth transition between two distinct intersecting or disconnected surfaces generated by prescribed continuity conditions. References to related literature can be found in [2].

In this paper a solution for the smooth connection of a B-spline surface and an other surface is presented by the modification of the B-spline surface according to prescribed boundary data. The boundary data are interpolation points of a connection (linkage) curve given as a continuous surface curve on the underlying (second) surface, moreover conditions for first and second order derivatives at the interpolation points. They ensure normal continuity (coinciding tangent planes) and equal normal curvatures in one direction not tangent to the connection curve of both surfaces at those common points. In this case we say that the two surfaces join with almost G^2 (curvature) continuity. The conditions for G^2 connection of two surfaces are given by the linkage curve theorem [3]. The cited theorem says that a pair of patches meet with G^2 continuity if they are tangent continuous along a smooth connection

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curve and have equal normal curvatures in a single transverse direction at every point of the connection curve. In the presented algorithm the G^2 continuity along a connection curve is not guaranteed, because only at a finite number of points of the linkage curve the first and second order derivatives of the underlying surface provide the boundary conditions for the fitting of the B-spline surface.

2. Initial conditions

The B-spline surface representing the spout or the handle of a kettle is presented by a parametric polynomial B-spline vector function of (3,2) degrees. The equation of the $(i,j)^{\text{th}}$ patch in the matrix form is

$$\mathsf{r}_{i,j}(u,v) = \begin{bmatrix}1 & u & u^2 & u^3\end{bmatrix} \left[B_i^{(3)}(\mathsf{t})\right] \left[\mathsf{G}_{i,j}\right] \left[B_j^{(2)}(\mathsf{s})\right]^T \begin{bmatrix}1 & v & v^2\end{bmatrix}^T, \tag{1}$$
$$(u,v) \in [0,1]\times[0,1] \text{ and } i = 1,\dots,n-2, \quad j = 1,\dots,m-1$$

where $\mathsf{t} = \{t_i\}_{i=-2}^{n+2}$ and $\mathsf{s} = \{s_j\}_{j=-1}^{m+2}$ are given knot vectors. Both of the knot vectors are periodical, namely, open knot vectors do not allow to change the first and second derivatives at the end points of the corresponding curves. $\left[B_i^{(3)}(t)\right]$ is the coefficient matrix of the i^{th} cubic B-spline basis with the support $[t_{i-3}, t_{i+1}]$, $(i = 1,\dots,n+1)$. Similarly, $\left[B_j^{(2)}\right]$ is the coefficient matrix of the j^{th} quadratic B-spline basis with the support $[s_{j-2}, s_{j+1}]$, $(j = 1,\dots,m+1)$. In the non-uniform case the elements of these matrices are expressed by the intervals of the knot vectors t and s, respectively [1]. Further, $u = (t - t_i)/(t_{i+1} - t_i)$, $v = (s - s_j)/(s_{j+1} - s_j)$. The matrix $[\mathsf{G}_{\mathsf{i,j}}]$ contains the geometric data of the $(i,j)^{\text{th}}$ patch, which are $(n+1)\times(m+1)$ control points.

$$\mathsf{G}_{i,j} = \begin{bmatrix} \mathsf{V}_{i,j} & \mathsf{V}_{i,j+1} & \mathsf{V}_{i,j+2} \\ \mathsf{V}_{i+1,j} & \mathsf{V}_{i+1,j+1} & \mathsf{V}_{i+1,j+2} \\ \mathsf{V}_{i+2,j} & \mathsf{V}_{i+2,j+1} & \mathsf{V}_{i+2,j+2} \\ \mathsf{V}_{i+3,j} & \mathsf{V}_{i+3,j+1} & \mathsf{V}_{i+3,j+2} \end{bmatrix}, \quad (i = 1,\dots,n-2, \quad j = 1,\dots,m-1)$$

In the uniform case, when $t_i = i$ and $s_j = j$, the coefficient matrices are the well-known constant matrices

$$B^{(3)} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad B^{(2)} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

The u parameter lines of the surface are cubic longitudinal curves, and the v parameter lines are quadratic cross section curves which are closed under the assumption $\mathsf{V}_{i,m} = \mathsf{V}_{i,1}$, $\mathsf{V}_{i,m+1} = \mathsf{V}_{i,2}$, $(i = 1,\dots,n+1)$.

The underlying surface representing, for example, a piece of a kettle is an arbitrary smooth surface described by a parametric vector function $\rho(\xi,\eta)$, $\xi_1 \le \xi \le \xi_2$, $\eta_1 \le \eta \le \eta_2$. The linkage curve is given on this surface as a C^1-continuous closed surface curve generated, for example, by mapping a circle onto the surface. The input data of the shaping algorithm are given along the linkage curve in the form of interpolation points, tangent vectors, twist vectors and normal curvatures of the underlying surface.

Each interpolation point P_k, $(k = 1, \ldots, d)$ of the linkage curve will coincide with the corner point $(u, v) = (0, 0)$ of a boundary patch $\mathsf{r}_{1,j}(u, v)$, $(j = 1, 3, \ldots, m-1)$ of the B-spline surface. We prescribe boundary conditions for each second patch along the borderline. Therefore the number of patches in the cross sectional direction will be set to $2d$, i.e., $m-1 = 2d$ is assumed. The tangent vector of the linkage curve will be the derivative in the v parameter direction of the boundary patch. Further, the tangent vector and the normal curvature of a surface curve of the underlying surface orthogonal to the linkage curve at the interpolation point determine the tangent direction and the normal curvature of the u parameter line $\mathsf{r}_{1,j}(u, 0)$, $(j = 1, 3, \ldots, m-1)$ at the starting point $u = 0$, respectively. Also, the twist vectors of the two surfaces are supposed to be equal at the interpolation points. These boundary conditions are expressed by the following equations:

$$\mathsf{P}_k = \mathsf{r}_{1,j}(0,0), \tag{2}$$

$$\lambda_k \mathsf{T}_{u,k} = \frac{\partial}{\partial u}\mathsf{r}_{1,j}(u,v)|_{(u,v)=(0,0)}, \tag{3}$$

$$\mathsf{T}_{v,k} = \frac{\partial}{\partial v}\mathsf{r}_{1,j}(u,v)|_{(u,v)=(0,0)}, \tag{4}$$

$$\mathsf{T}_{uv,k} = \frac{\partial^2}{\partial u \partial v}\mathsf{r}_{1,j}(u,v)|_{(u,v)=(0,0)}, \tag{5}$$

$$\kappa_k = \frac{L_{1,j}}{E_{1,j}}|_{(u,v)=(0,0)} \tag{6}$$

for $k = 1, \ldots, d$, $j = 1, 3, \ldots, m-1$, $2d = m-1$, where

$$E_{1,j} = \langle \frac{\partial \mathsf{r}_{1,j}(u,v)}{\partial u}, \frac{\partial \mathsf{r}_{1,j}(u,v)}{\partial u} \rangle, \quad L_{1,j} = \langle \frac{\partial^2 \mathsf{r}_{1,j}(u,v)}{\partial u^2}, \mathsf{n} \rangle$$

are coefficients in the first and second fundamental forms, respectively, and n is the unit vector of the surface normal.

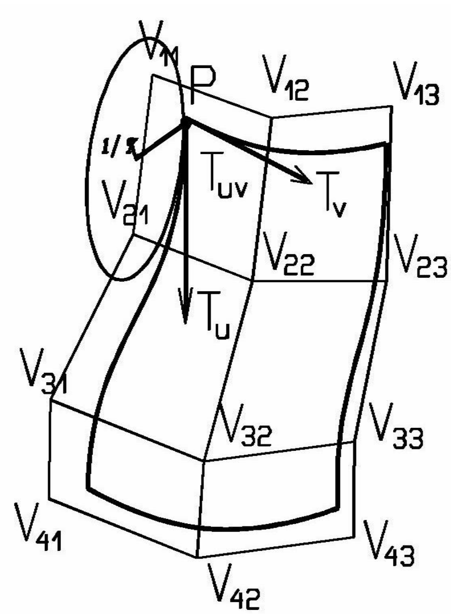

Figure 1: Patch, control net and boundary conditions

In Eq. (3) a scalar variable λ_k is introduced as shape parameter for changing the length of the tangent vector of the longitudinal u parameter curve. The expression on the right-hand side of Eq. (6) gives the normal curvature in the direction $(\dot{u}, \dot{v}) = (1, 0)$ of the patch $\mathsf{r}_{1,j}(u, v)$ at the corner point $(u, v) = (0, 0)$.

Under these conditions the connection at the interpolation points will be almost curvature continuous.

Now consider a concrete case and to simplify notation substitute the derivatives of the B-spline function given in (1) into the equations (2)–(6). Then they will be expressed by the geometric data of the patches. We assume that $k = j = 1$, and omit the index k of the interpolation point. According to this, the boundary conditions at the first interpolation point and the control points $\mathsf{V}_{i,j}$ $(i = 1, \ldots, 4,\ j = 1, 2, 3)$ of the patch $\mathsf{r}_{1,1}(u, v)$ will appear on the left and right-hand side of the equations, respectively (Fig. 1).

$$
\begin{aligned}
12\mathsf{P} &= \mathsf{V}_{11} + 4\mathsf{V}_{21} + \mathsf{V}_{31} + \mathsf{V}_{12} + 4\mathsf{V}_{22} + \mathsf{V}_{32}, && (7)\\
12\lambda\mathsf{T}_u &= -3\mathsf{V}_{11} + 3\mathsf{V}_{31} - 3\mathsf{V}_{12} + 3\mathsf{V}_{32}, && (8)\\
12\mathsf{T}_v &= -2\mathsf{V}_{11} - 8\mathsf{V}_{21} - 2\mathsf{V}_{31} + 2\mathsf{V}_{12} + 8\mathsf{V}_{22} + 2\mathsf{V}_{32}, && (9)\\
6\mathsf{T}_{uv} &= -\mathsf{V}_{11} + \mathsf{V}_{31} + \mathsf{V}_{12} - \mathsf{V}_{32}, && (10)\\
\kappa &= \langle \mathsf{r}_{uu}, \mathsf{n}\rangle / \langle \lambda\mathsf{T}_u, \lambda\mathsf{T}_u\rangle = && (11)\\
&= \langle \frac{1}{12}(6\mathsf{V}_{11} - 12\mathsf{V}_{21} + 6\mathsf{V}_{31} + 6\mathsf{V}_{12} - 12\mathsf{V}_{22} + 6\mathsf{V}_{32}), \mathsf{n}\rangle / \lambda^2,
\end{aligned}
$$

where

$$
\mathsf{r}_{uu} = \frac{\partial^2 \mathsf{r}_{1,1}(u, v)}{\partial u^2}\Big|_{(u,v)=(0,0)} \quad \text{and} \quad \mathsf{n} = \frac{\mathsf{T}_u \times \mathsf{T}_v}{|\mathsf{T}_u \times \mathsf{T}_v|}.
$$

The boundary data on the left-hand sides of the equations are computed from the underlying surface $\rho(\xi, \eta)$ and the connection curve on this surface. P is the first interpolation point on the linkage curve, T_v is the tangent vector of this curve at P. T_u is the unit vector of the appropriate directional derivative, κ is the normal curvature in the direction of T_u of the underlying surface at the point P. The corresponding equations can be written at the other interpolation points for each second boundary patch similarly.

As the equations do not contain the control vertices $\mathsf{V}_{i,3}$ $(i = 1, \ldots, 4)$, the system of equations written at the interpolation points do not overlap. Consequently, we may consider the tubular B-spline surface cut into stripes along the borderline of every second patch in the longitudinal direction, and we shall look for a solution for each stripe separately.

The equations (7)–(10) form a linear system of vector equations, from which we express the control points V_{11}, V_{12}, V_{21} and V_{22} situated at the corner of the control net of the patch.

$$
\begin{aligned}
\mathsf{V}_{11} &= -2\lambda\mathsf{T}_u - 3\mathsf{T}_{uv} + \mathsf{V}_{31} && (12)\\
\mathsf{V}_{12} &= -2\lambda\mathsf{T}_u + 3\mathsf{T}_{uv} + \mathsf{V}_{32} && (13)\\
\mathsf{V}_{21} &= \frac{1}{4}(6\mathsf{P} + 2\lambda\mathsf{T}_u + 3\mathsf{T}_{uv} - 3\mathsf{T}_v - 2\mathsf{V}_{31}) && (14)\\
\mathsf{V}_{22} &= \frac{1}{4}(6\mathsf{P} + 2\lambda\mathsf{T}_u - 3\mathsf{T}_{uv} + 3\mathsf{T}_v - 2\mathsf{V}_{32}) && (15)
\end{aligned}
$$

This solution expresses the tangential continuity of the B-spline surface and the underlying surface at the interpolation point P, and has been investigated in [5] and [6]. The value of the scalar shape parameter λ has been considered as free user input, or has been computed from a fairness condition.

The scalar equation (11) given for the normal curvature can be written in the form

$$\kappa\lambda^2 - \langle \mathsf{r}_{uu}, \mathsf{n} \rangle = 0.$$

We substitute here the solutions (12)–(15) and get

$$\kappa\lambda^2 - \langle \frac{3}{2}(-2\mathsf{P} - 2\lambda\mathsf{T}_u + \mathsf{V}_{31} + \mathsf{V}_{32}), \mathsf{n} \rangle = 0. \tag{16}$$

Now we want to get a solution of the system of equations (12)–(16). As variables we have the position vectors of the control points V_{31} and V_{32}, moreover the length λ of the tangent vector of the cubic u parameter line on the boundary $v = 0$. The scalar equation (16) obviously does not have a unique solution for the two vectorial and one scalar variables. Consequently, additional equations are necessary, if we want to ensure uniqueness.

3. Shape preserving and fairing conditions

We are going to analyse the second order boundary conditions in some concrete situations. First, we suppose that the twist vector T_{uv} of the underlying surface is zero, what is true for cylindrical and ruled surfaces chosen as underlying surfaces in our examples. This assumption makes the expressions and the computation shorter but does not influence the structure of the equations, because r_{uu} is not influenced by T_{uv}.

We will not consider all the control points fixed which are not included in the equations (12)–(16), because in this case the solution leads to a wavy shaped surface in the cross sectional direction [7]. The control points $\mathsf{V}_{4,j}$, $(j = 1,2,3)$ in the fourth row of the control net are independent of the boundary conditions. Consequently, they do not influence the phantom vertices V_{11}, V_{12}, V_{21} and V_{22} having the forms (12)–(15). Therefore, we shall keep the fourth row of the control net fixed.

Analysing the situation of the control vertices in the third row we see that the points $\mathsf{V}_{i,3}$, $(i = 1,2,3)$ along the borderline $u = 1$ will appear in the equations written at the next interpolation point as $\mathsf{V}_{i,1}$, $(i = 1,2,3)$ of the patch $\mathsf{r}_{1,3}(u,v)$. According to this, we require additional equations for these control points similarly to the equations (12) and (14).

$$\mathsf{V}_{13} = -2\lambda\mathsf{T}_{uc} + \mathsf{V}_{33}, \tag{17}$$

$$\mathsf{V}_{23} = \frac{1}{4}(6\mathsf{P}_c + 2\lambda\mathsf{T}_{uc} - 3\mathsf{T}_{vc} - 2\mathsf{V}_{33}), \tag{18}$$

where P_c, T_{uc} and T_{vc} are the next interpolation point, the unit tangent vector in u direction and the tangent vector in v direction at this point, respectively. However, the points V_{13} and V_{23} computed for the patch $\mathsf{r}_{1,1}$ will be overwritten by the solution obtained for the patch $\mathsf{r}_{1,3}$, the equations (17) and (18) express tangential continuity of the patch $\mathsf{r}_{1,1}$ and the underlying surface at the interpolation point P_c. Going along the connection curve from stripe to stripe, the replacement of the control points by their recomputed locations leads to an iterative improvement of the control net.

The investigated B-spline surface has a tubular shape whose control net near the end curve is rotational symmetric. We assume that after fitting the surface to the underlying surface the location of the control points in the third row remains almost symmetric. This shape preserving condition can be written for the coordinates of V_{31} and V_{32} as follows:

$$\mathsf{V}_{31} = (a\cos\varphi_1, a\sin\varphi_1, b), \tag{19}$$
$$\mathsf{V}_{32} = (a\cos\varphi_2, a\sin\varphi_2, c). \tag{20}$$

Here the coordinate axis z has been chosen for the rotational axis, a is the radius of the control points and φ_1, φ_2 are the rotational angles measured from the coordinate plane $[x, z]$.

Further, we require a fair resulting surface, i.e., the variables will get their values at a minimum location of a fairness function expressed by the area integral

$$A(a, b, c, \lambda) = \int_{u=0}^{1}\int_{v=0}^{1} (\mathsf{r}_{uu}^2 + \mathsf{r}_{vv}^2)\,du\,dv, \tag{21}$$

which is frequently used for the approximation of the thin plate energy function. As the second derivatives $\mathsf{r}_{uu}(u, v)$ and $\mathsf{r}_{vv}(u, v)$ are expressed by the control vertices $\mathsf{V}_{i,j}$, $(i = 1, \ldots, 4,$ $j = 1, \ldots, 3)$, after integration and substitution of the expressions (12)–(15) and (17)–(20) the energy function becomes a quadratic function of the scalar variables a, b, c and λ.

Considering the equation (16) as a prescribed condition for the normal curvature, the required solution leads to a nonlinear conditional extremum problem. A solution can be found by the Lagrange method, where the objective function is

$$F(a, b, c, \lambda, \mu) = A(a, b, c, \lambda) + \mu(\kappa\lambda^2 - \langle \mathsf{r}_{uu}, \mathsf{n}\rangle), \tag{22}$$

and μ is the Lagrange multiplicator. For a minimum point of the function $F(a, b, c, \lambda, \mu)$ the system of equations

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = \frac{\partial F}{\partial c} = \frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial \mu} = 0$$

must hold. The first three equations are linear in a, b and c, the other two lead to a cubic equation in λ. A solution can be found by a symbolical algebra program, and the extremal property of the solution can be investigated numerically.

4. Examples

In the modelling problems chosen for the illustration of our algorithm we have striven for minimizing the set of data. In order to minimize the computational time only a small number of interpolation points have been given on the connection curve, what magnifies the shaping effect of the prescribed boundary conditions in the neighbourhood of those points. The generation of the surfaces and the computation have been carried out by Mathematica [8].

The input B-spline surface and its control net are shown in Fig. 2 and Fig. 3, respectively. The control net has 6×14 points from which, according to the closed cross section curves, the 13^{th} coincide with the first and the 14^{th} coincide with the second in each row. Consequently, the number of patches is 3×12.

First, the upper end curve of the tube shaped surface will be replaced by the closed smooth surface curve generated by mapping a circle onto the cylindrical surface (Fig. 4). The boundary conditions are given at 6 points of the connection curve. The tangents determining the tangent planes at the interpolation points are drawn by heavy lines, and are computed as

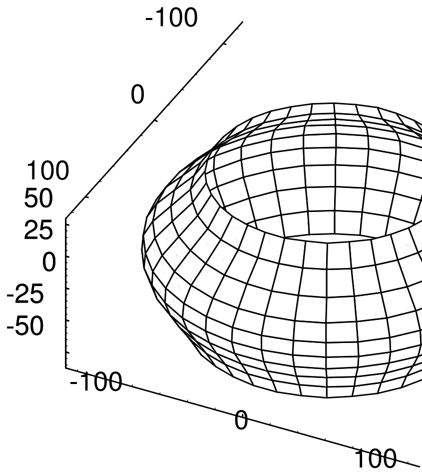

Figure 2: B-spline surface of (3,2) degrees

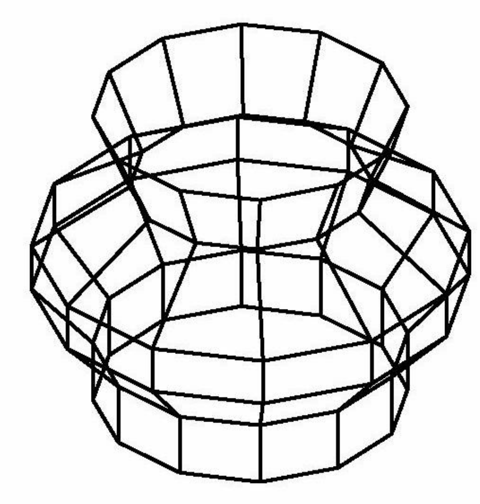

Figure 3: Control net of the surface shown in Fig. 2

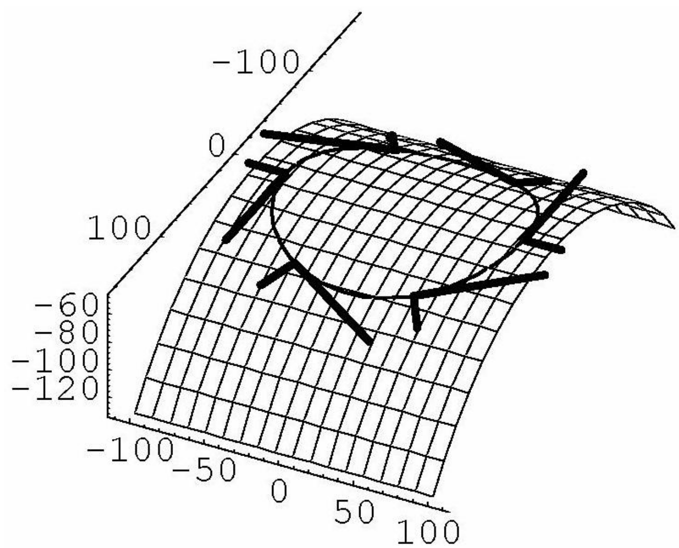

Figure 4: Underlying surface, connection curves and tangent vectors

directional derivatives of the cylindrical surface in the appropriate directions. The tangents of the connection curve give the prescribed $\mathsf{T}_{v,k}$ v-derivatives and the tangents in the radial directions determine the directions of the $\mathsf{T}_{u,k}$ u-derivatives of the B-spline surface at the interpolation points P_k, $(k = 1, \ldots, d)$. The twist vectors are zero vectors and the normal curvatures κ_k are computed in the radial directions as well. The modified surface generated according to the prescribed boundary conditions is shown in Fig. 5. In the control net (Fig. 6) the first three rows of vertices have been recomputed as the solution of the conditional extremum problem formulated in (22). The vertices in the 4–6 rows are unchanged. The resulting surface joins the cylindrical surface at the interpolation points with almost curvature continuity. The reason is that the tangent planes of the two surfaces coincide and the normal curvature of the B-spline surface in the direction of the u parameter line equals the normal curvature of the underlying surface in the radial direction at each interpolation point. The boundary region of the end curve of the generated surface has a satisfactorily smooth shape despite of the small number of the interpolation points.

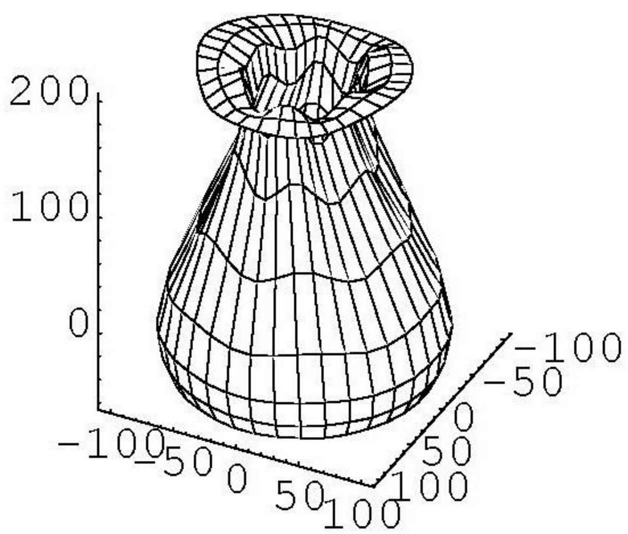

Figure 5: Modified surface joining to the cylindrical surface shown in Fig. 4

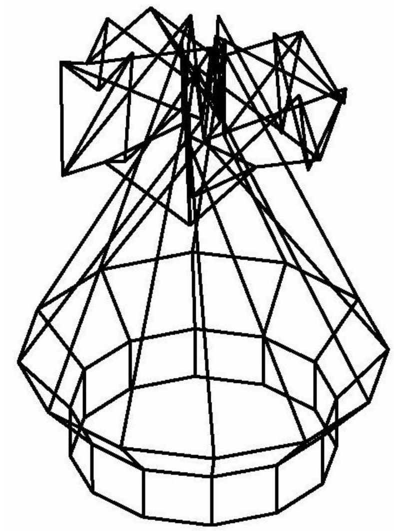

Figure 6: Control net of the surface with recomputed vertices in the first three rows

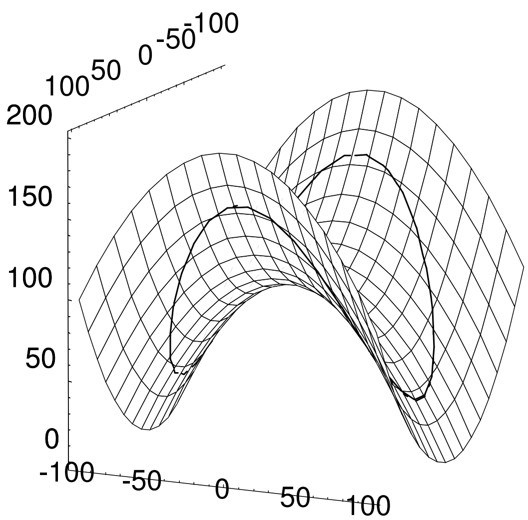

Figure 7: Underlying surface

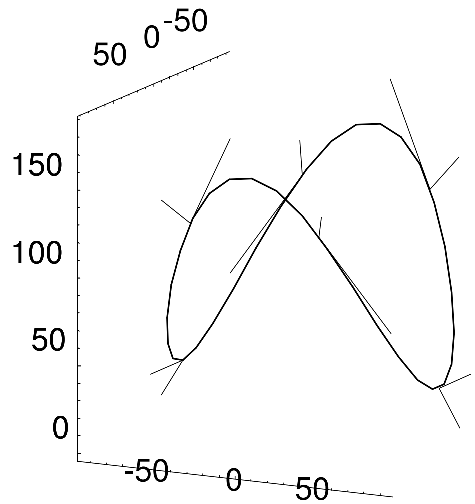

Figure 8: Connection curve on the surface shown in Fig. 7

In the solution the undesired side effects, as wiggling cross sectional v parameter lines in the second row of patches are enlarged due to the large distances between the interpolation points. In practice the interpolation points are positioned more densely along the connection curve.

In the next example the underlying surface is a hyperboloid (Fig. 7), the connection curve is a mapped circle again shown with the directional derivatives in the appropriate directions in Fig. 8. The B-spline surface joining the hyperboloid with almost G^2-continuity at the interpolation points is shown in Fig. 9.

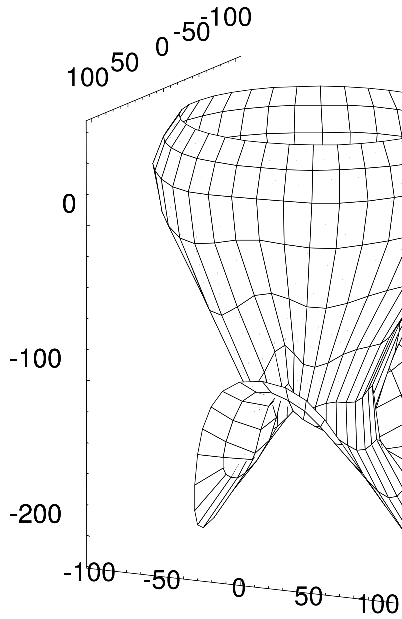

Figure 9: Fitted surface by shaping the surface shown in Fig. 2

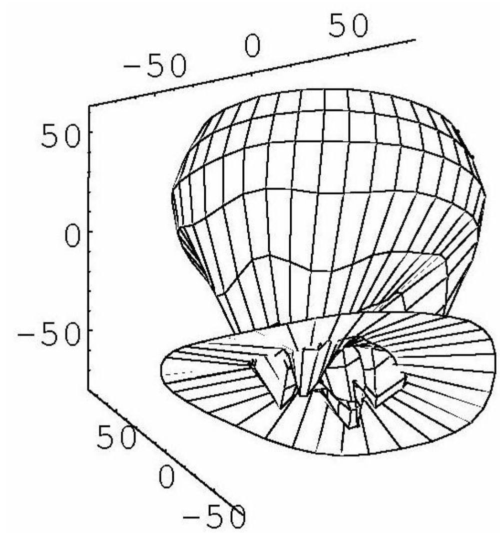

Figure 10: Fitting the surface to the cylindrical one with fairing functional A^v

In the next three solutions (Figures 10, 11, 12) different fairing functions are chosen, but the underlying surfaces and connection curves are the same (Fig. 4).

The phantom points according to the minimum locations of the area integral

$$A^v(a,b,c,\lambda) = \int\limits_{u=0}^{1} \int\limits_{v=0}^{1} \mathsf{r}_{vv}^2 du dv$$

under the curvature condition (16) generate a surface, whose u parameter lines wave very strongly (Fig. 10).

The resulting surface computed from the fairing functional

$$A^u(a,b,c,\lambda) = \int\limits_{u=0}^{1} \int\limits_{v=0}^{1} \mathsf{r}_{uuu}^2 du dv$$

and the curvature condition (16) has smooth u parameter lines, but the shape of the surface shows big hollows at the interpolation points (Fig. 11). We remark that using second order partial derivatives instead of third order ones a very bad shaped surface has arisen.

Finally, the solution according to the fairing functional (21) is shown in Fig. 12.

5. Conclusion

In this paper a technique is described for shaping a tubular B-spline surface of $(3,2)$ degrees at its end curve according to prescribed first and second order boundary data. The methods of phantom points and surface fairing were used. The boundary conditions ensured that

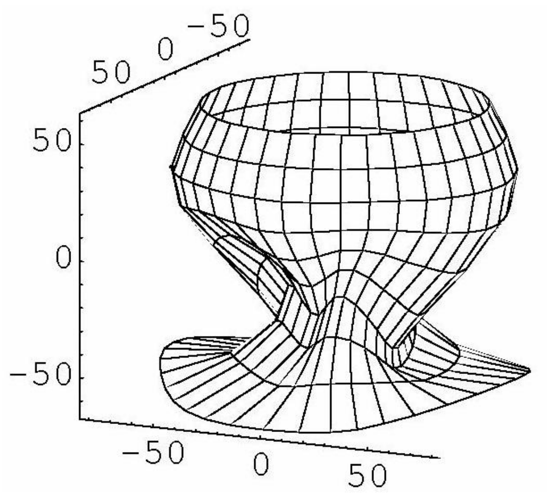

Figure 11: The same fitting with fairing functional A^u

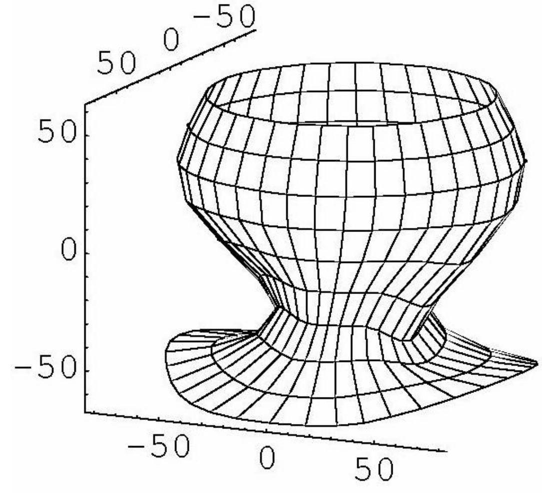

Figure 12: The same fitting with fairing functional A

the B-spline surface joined another surface along a connection curve with almost curvature continuity at the given interpolation points of the connection curve. The smooth shape of the resulting surface was achieved by the minimization of a fairness functional. By appropriate shape preserving conditions and by piecewise computation along the borderline the algorithm gave an exact solution. The method described here gives a solution for surface fitting without generating a third blending surface. It can be applied directly for non uniform B-spline surfaces by calculating with the corresponding coefficient matrices of the patches instead of the constant matrices $B^{(3)}$ and $B^{(2)}$. The algorithm is basically the same in the rational case under the restriction that the weights of the control points of the boundary patches are fixed and the weights of the phantom points are set to one. The programs of the algorithm have been implemented on a 16MB PC. This technical circumstance gave the reason to write the expressions in the simplest form, which made the description shorter and more transparent.

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