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New Applications of Geometry[1]

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Abstract. Two problems from different areas are presented in order to demonstrate the applicability of geometry. In both cases the solutions are based on results that are beyond the topics we usually teach engineering students.
(i) For a compliance element to be used in robotics a mechanism has been developed which produces a "centerless rotation". The presented solution consists of an infinitesimally movable structure.
(ii) The geometry behind panoramic radiographies is analyzed. The aim is to make measurements on this important diagnostic tool in dentistry and to use panoramic X-rays in medical imaging for a image fusion with a live video image.

Key Words: Applied geometry, compliance element, panoramic X-ray.

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1. Introduction

The statement *"everything is geometry"* is obviously an exaggeration, but there is a true essence in it: Almost every object, real or virtual, has a *geometric* component, e.g. its shape or its dimensions. Frequently the development of a theory goes hand in hand with the development of a *geometric model*.[2] And sometimes problems in technical sciences are of pure *geometric nature*. Why not train consequently the students' ability of geometric reasoning (cf. [8][3])?

As an example let me focus on a dramatic progress in biochemistry: It has been well known for a long time that the shape of crystals reflects symmetry properties of the molecules. But recently it turned out that also the ability of proteins to form specific stable complexes depends in the majority of cases on *geometric features* of the molecular structure only. The docking mostly takes place at cavities or pockets of the bounding surface of the protein. In view of the complexity of these surfaces, algorithms from the field of Computational Geometry

[1]lecture presented at the Plenary Session of the Eighth International Conference on Engineering Design Graphics and Descriptive Geometry, July 31 to August 3, 1998 in Austin, Texas, USA

[2]Galileo: *"Nobody can philosophize without using geometry as a guide".*

[3]Presented as a plenary paper at the 7th ICECGDG in Cracow (Poland).

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(concerning e.g. Delaunay triangulation) have been successfully applied to the automatic search of binding sites (cf. [5]).

In the sequel two problems are presented where the "geometric component" has been really substantial for obtaining a reasonable solution.

2. A Compliance Element

One of the simplest operations at automatic assembly is to place a part P in a hole or cavity. Even small misalignments can require high mechanical forces at this peg-in-hole process (see Fig. 1). Such lateral and angular errors can be ruled out either "actively" by sensors that figure out the errors in positioning, or in a "passive" way by a *compliance element* mounted between the end effector E and the part P to be placed. Compliance elements have a certain flexibility, e.g. because of their sandwich structure with inserted elastomers. This together with an additional force should compensate the errors – without any electronic feedback.

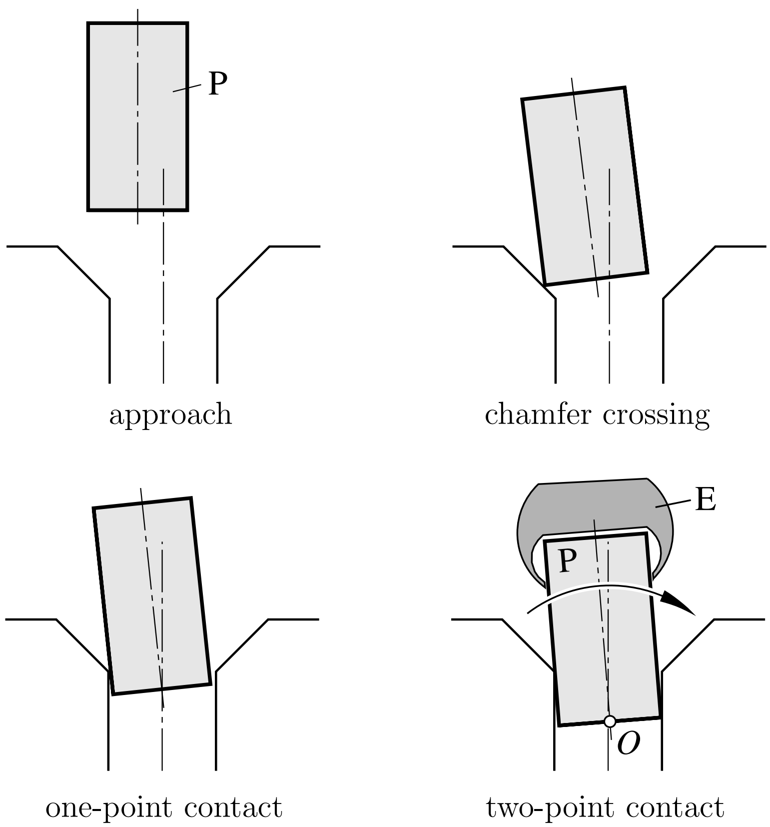

Figure 1: The four stages of assembly

The Austrian Aerospace Company ORS started an industrial project which gave rise to the thesis [1]. Here it was proved that the compliance elements available on the market do not really flex as they should. Therefore a new concept was developed: As soon as at the peg-in-hole process (Fig. 1) the two-point contact is reached, then the part P to be placed should perform a so-called *"centerless rotation"*. This is a spherical motion about a center O which is far distant to the end effector. The correction of the misalignment should then be caused by reaction forces acting on P during the two-point contact.

This was the point where the Institute of Geometry has been involved. How to find a simply structured mechanism which produces a spherical motion with two degrees of freedom in order to eliminate an angular error of less than 2°? After this correction the position of the moving system Σ_{P} (representing the part P) with respect to the frame Σ_{E} (representing the end effector E) should be locked so that a pure translation is sufficient to finish the assembly.

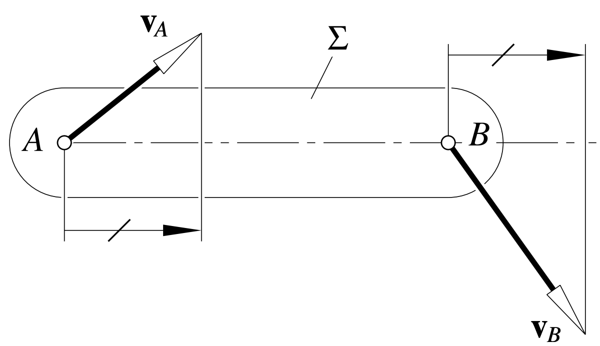

Figure 2: Projection Theorem

A finitely movable mechanism which produces a proper spherical motion about O would be rather complex. So the idea was born to find a structure which is only *infinitesimally movable*. At such *shaky* structures it is possible to associate to each moving point X a velocity vector $\mathbf{v}_X$ against the frame link such that for each two points A, B of the same link Σ the distance is instantaneously preserved. This means that due to the *Projection Theorem* the components of $\mathbf{v}_A$ and $\mathbf{v}_B$ in direction of the line AB must be equal (see Fig. 2). So these shaky structures behave like finitely movable ones, however up to the first order approximation only. Examples of shaky structures can e.g. be found in [9] or [7].

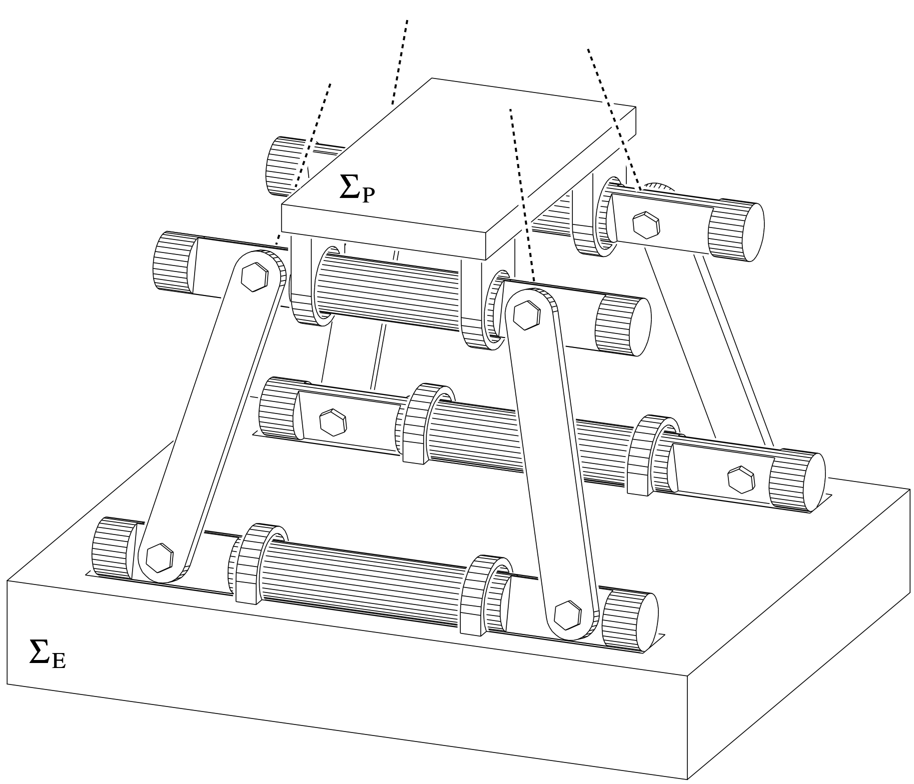

Figure 3: The infinitesimally movable structure of the centerless rotation module CRM

For obtaining an infinitesimal centerless rotation, one could proceed as follows: Take a Stewart-Gough platform in a position where the axes of all six legs are passing through O.[4] If then the lengths of all legs are fixed, the structure is still shaky. The infinitesimal movability is of degree 3. The system Σ_{P} can rotate infinitesimally about each line passing through O. However, this is too much mobility since the legs could even twist around the axis of the hole. Therefore we restricted it in the following way:

The prototype of the centerless rotation module consists of two independent four-bar mechanisms in inclined planes (see the simplified geometric model in Fig. 3). These planes are connected by revolute joints with the frame Σ_{E} as well as with the moving system Σ_{P}. In the initial position the axes of these four revolute joints are parallel and supposed to be horizontal. The midaxes of the four arms (dotted lines in Fig. 3) meet at the given center O. Now the permitted infinitesimal motions are rotations about horizontal axes through O.

An elementary computation for a supposed arm length of 100 mm at the four-bar mechanisms reveals the following: For rotations of Σ_{P} against Σ_{E} through 2° (1°) it is necessary that the joint clearances admit a variation of the arm lengths through ±0.061 mm (±0.015 mm).

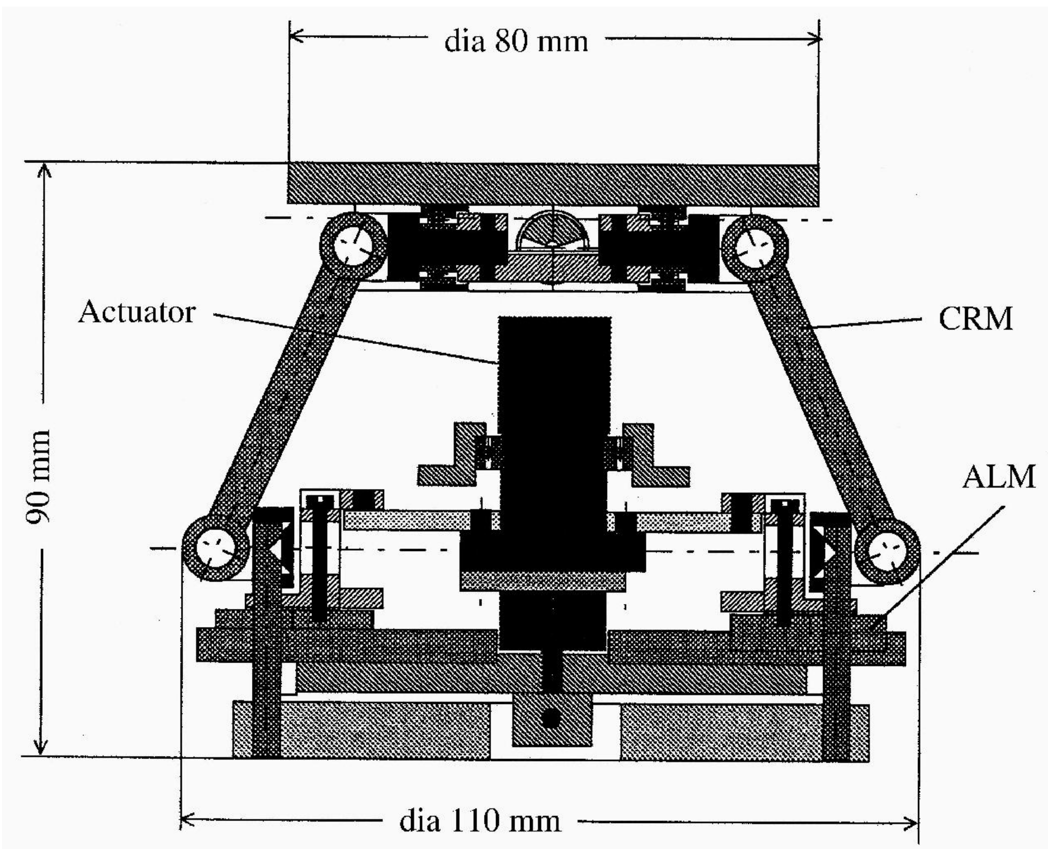

Figure 4: The new compliance element ACRC

In order to adjust the compliance element to different positions of the rotation center O with respect to the end effector, the distance between the revolute axes in Σ_{E} and the lengths of the frame links of the four-bar mechanisms must be made variable. Therefore this "centerless rotation module" CRM was combined with the "adjustment and locking module" ALM. The final form of this prototype depicted in Fig. 4 was called "Adjustable Centerless Rotation Compliance" ACRC. The company ORS took a patent for it.

[4]This is a *singular posture* of the platform (see e.g. [2], [3] or [6]).

Unfortunately, economic reasons caused a break in this project, and until now it is open whether this new kind of element does really passively correct small angular errors so that the assembly needs much smaller mechanical forces than with standard compliance elements. This would finally enable to use smaller dimensioned robots – a substantial improvement for the usage in orbital stations.

3. Panoramic Radiography

Panoramic X-rays (see Fig. 5) have become an almost indispensable tool for dentists in oral and maxillofacial surgery. Such overview radiographies provide a comprehensive basis for early diagnostics, for treatment planning and evaluation of therapy. They offer a systematic and economically favorable method for data collection which protects the patient from unnecessary radiation exposure. Conventional dental X-rays serve only as a completion in exceptional situations.

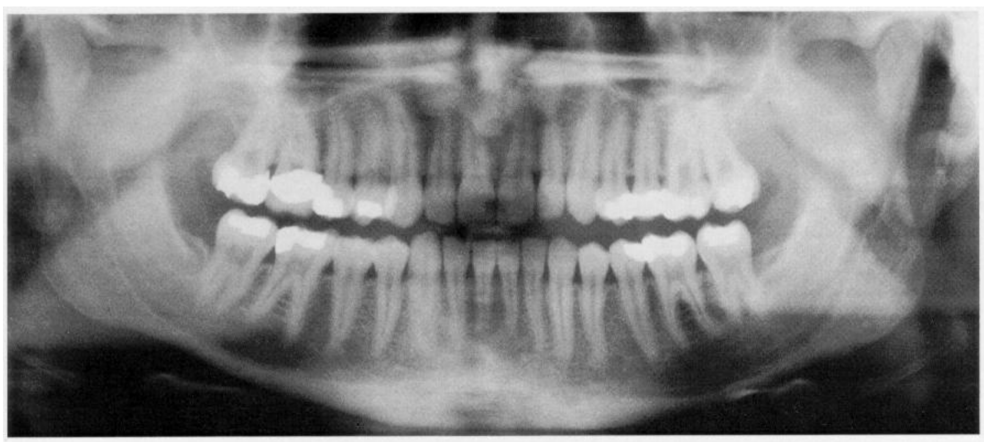

Figure 5: Panoramic X-ray

The fundamentals of panoramic radiography were developed in the late forties by a person named Paatero[5]. He modified the classical tomography which worked as follows (Fig. 6): The X-ray source Z and the film are moving in opposite directions thus defining a planar in-focus layer like a cross section (unshaded bar in Fig. 6) while structures outside of this layer are blurred and therefore eliminated.

In panoramic radiography the in-focus layer is bent, due to the following procedure (Fig. 7): X-ray source and film holder rotate clockwise – if seen from above – around the parabolically shaped dental arche of the patient. Simultaneously the film moves in the casette. The vertical, millimeter-wide ray r meets the film only at a vertical slit S. Such a type of projection seems to be new for geometers.

In section 3.1 the geometrical analysis of this projection will reveal why only the teeth are depicted on the radiography (Fig. 5), though the X-ray penetrates the whole skull. In addition, it will be explained in which way the location and thickness of the in-focus layer (unshaded area in Fig. 7) depends on the motion of the X-ray source, on the velocity of the film, and on the location of the instantaneous center P of rotation, which in standard textbooks on dental radiography (e.g. [4]) is described as an "imaginary column" (Fig. 8).

[5]Unfortunately no reference is available to the author.

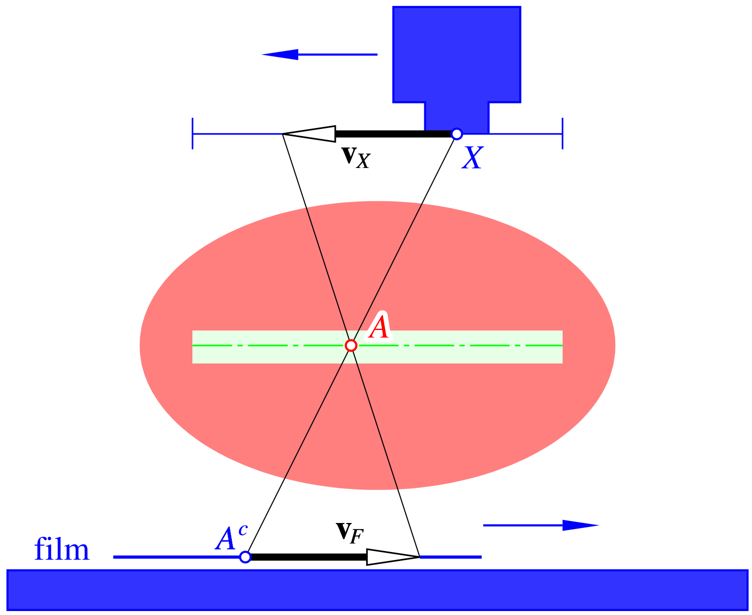

Figure 6: Classical tomography with a planar in-focus layer

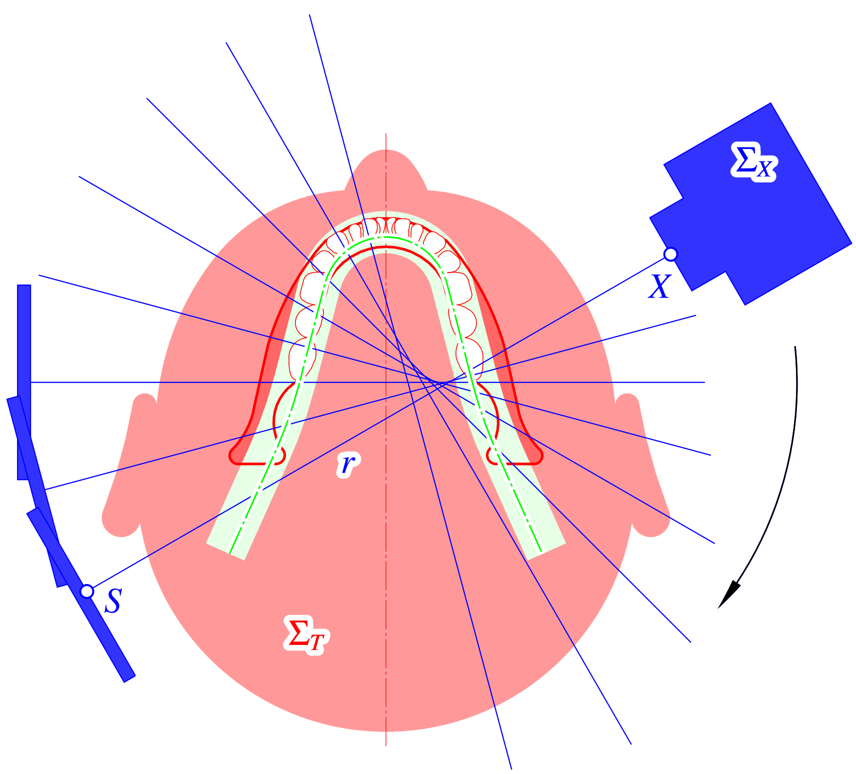

Figure 7: Panoramic X-ray projection, the bent in-focus layer and the motion of X-ray source and film holder

In [4], p. 11, it is stated that precise measurements on panoramic X-rays are not possible. This is no longer true! As soon as the motion of the X-ray source and film holder with respect to the patient and the motion of the film against the casette are analyzed, the mapping of the dental arche onto the film can be described mathematically. However, the equations are nonlinear and implicit.

These equations are the basis for a current joint project with Artma Biomedical Inc. (Vienna): Panoramic X-rays should be used in medical imaging for an image fusion with a live video image. For this practical use it will however be inevitable that also the problem of calibration is solved, separately for the upper and lower jaw. In addition, it will be substantial to estimate the deviations caused by the patient's incorrect positioning as well as by other disturbances like movements or swallowing while the apparatus is operating.

3.1. Geometric Analysis

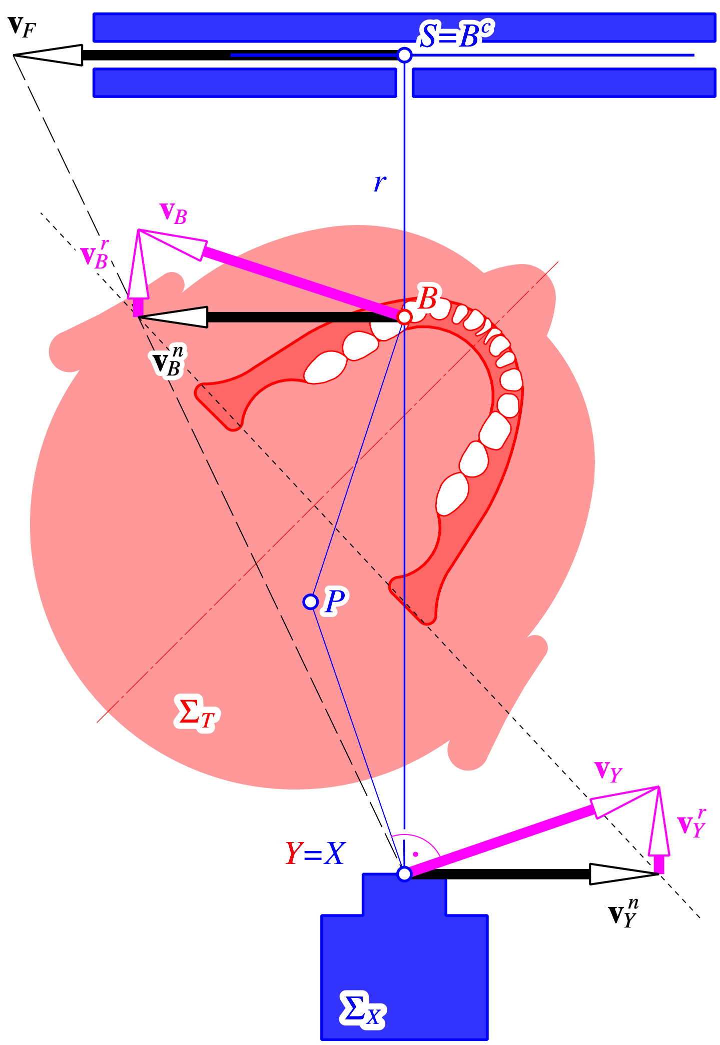

Figure 8: Kinematic analysis of the panoramic X-ray projection

In the sequel only the basic geometry will be addressed: Following the terminology of kinematics, let Σ_X denote the system represented by the X-ray source and the film holder. Let Σ_T be the fixed system representing the patient's head (Figs. 7 and 8). The motion $\mu: \Sigma_\mathrm{X}/\Sigma_\mathrm{T}$ is performed by the apparatus according to the selected program. Usually there are several programs offered, at least one for adults and one for children. $\Sigma_\mathrm{X}/\Sigma_\mathrm{T}$ is a planar motion. At the machine we observed[6] it was the composition of a pure rotation of the camera-

[6] Siemens 257/82 Ro-Typ X 1426

film-system Σ_X against a main bar and a translatory and rotational motion with standstills of this bar with respect to the frame in Σ_T.

For the following it is much more convenient to pay attention to the inverse motion $\mu^{-1}: \Sigma_T/\Sigma_X$. This means that the patient's system Σ_T is moving against the X-ray system Σ_X. The top view in Fig. 8 shows an intermediate position. P denotes the instantaneous pole (rotation center) which mostly is located next to the X-ray $r = ZS$.

Let $\mathbf{v}_A$ in Fig. 8 be the velocity vector of an arbitrary point A of the dental arche under μ^{-1}. The arrow $\mathbf{v}_F$ with initial point S indicates the velocity of the film against the film holder in Σ_X (compare also Fig. 6). This velocity needs not be constant during the process.

Now a point $B \in \Sigma_T$, which is instantaneously located on the ray r, is *mapped without blurring on the film* if and only if the line connecting the moving point B with the X-ray source Z meets the film plane at a point $B^c = S$ whose velocity equals $\mathbf{v}_F$. In order to figure out the velocity of B^c, we decompose the vector $\mathbf{v}_B$ according to

$$\mathbf{v}_B = \mathbf{v}_B^n + \mathbf{v}_B^r$$

into components normal and parallel to r, respectively. Then the condition above is equivalent to the statement that Z *and the endpoints of* $\mathbf{v}_B^n$ *and* $\mathbf{v}_F$ *must be aligned* (dashed line in Fig. 8).

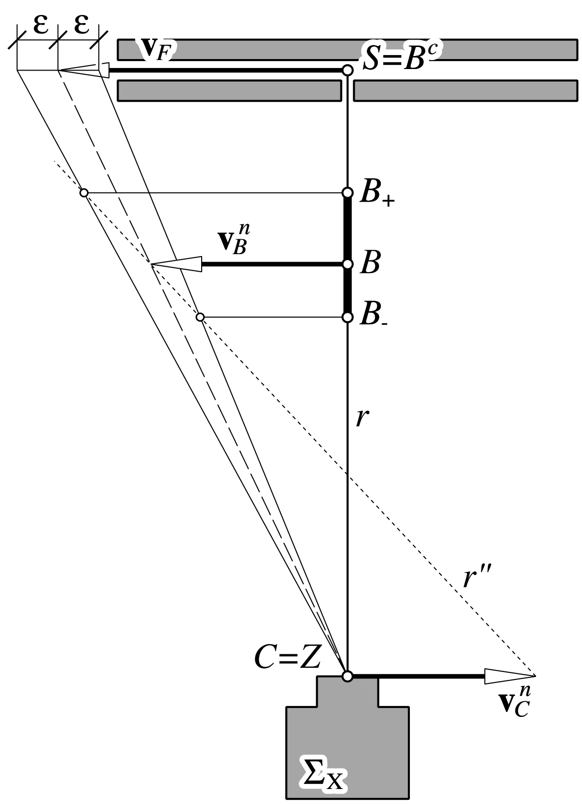

Figure 9: Instantaneous width of the in-focus layer

It is well known that in each moment for all points on line r (see e.g. B and C in Fig. 8) the endpoints of the velocity vectors under Σ_T/Σ_X are located on a line r'. On the other hand – due to the Projection Theorem (Fig. 2) – the components parallel to r are equal, i.e., $\mathbf{v}_B^r = \mathbf{v}_C^r$. Therefore also the endpoints of the components orthogonal to r are aligned. Note in Figs. 8 and 9 the dotted line r'' parallel to r' and connecting the endpoints of $\mathbf{v}_B^n$ and $\mathbf{v}_C^n$.

This proves that because of $P \neq Z$ there is exactly one point $B \in \Sigma_T$ on the line r which is projected without blurring.

Actually the image B^c will still look well-focused as long as the difference between the velocities of B^c and the film is smaller than a certain ε (see Fig. 9). The corresponding interval on r is terminated by two points B_+, B_-, which are harmonic with respect to B and Z. Therefore B is not the midpoint of this interval but nearer to the interior endpoint B_-. The length of the interval B_+B_-, i.e., the instantaneous width of the in-focus layer, depends on the velocity of the film, the angular velocity of Σ_T/Σ_X, and the distances of the instantaneous pole P from r and from the film plane. All these parameters are varying during the process. So it is obvious that also the thickness of the bent in-focus layer varies.

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