On the Bisectors of Weakly Separable Sets

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Abstract. A bisector of two sets is the set of points equidistant to them. Bisectors arise naturally in several areas of computational geometry. We show that bisectors of weakly linearly separable sets in \( \mathbb{E}^d \) share many properties with separating lines. Among these, the bisector of a restricted class of linearly separated sets is a homeomorphic image of the linear separator. We also give necessary and sufficient conditions for the existence of a particular continuous map from a portion of any linear separator to the bisector.

Key Words: Bisector, symmetric axis, linearly separable sets

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1. Introduction

Bisectors which are defined as the set of points equidistant to two given sets (see Fig. 10), arise naturally in computing the symmetric axis transform [3] – [6]. A common means of computing the symmetric axis is the following recursive method from computational geometry¹: Compute separately the symmetric axis of two subproblems and then merge them together. During this merge the bisector of the two subproblems is used to trim the symmetric axes of the subproblems. Certain properties of the bisector are the key to an efficient merge process. For example: One algorithm for computing symmetric axis of point sites in \( \mathbb{E}^2 \) using a divide-and-conquer strategy partitions the points into two almost equal-sized sets separated by a line. The bisector between these two sets is connected and is a single-valued map of the dividing line. In the case of computing the symmetric axis of multiply connected polygonal domains [6], certain bisectors are simple, closed curves. In each of these two examples, properties of the bisector allow a linear-time merge. Moreover, knowing the topology of the bisector helps in choosing an appropriate data structure for the bisector. For example: If the bisector of two

¹A descriptive geometry approach to the construction of bisectors is presented in [1] and [7]
sets in $\mathbb{E}^3$ is known to be a 2-manifold, then the QuadEdge data structure [2] can be used to represent and manipulate it.

In this research we show that bisectors of linearly separable sets have many properties of interest. The results presented here are for general sets in $\mathbb{E}^d$. There are several reasons for this. First, many of the results and proofs are simpler when the details of a particular class of sets do not intrude. More importantly, we want to broaden the study of bisectors beyond their use in algorithms for computing the symmetric axis of point sets in $\mathbb{E}^2$ because we believe that proximity properties of more general geometric elements in higher dimensions have important applications.

2. Linearly Separable Sets

We denote the closure of a set $S$ by $\text{cl} S$, the interior by $\text{int} S$, the boundary by $\partial S$, and the closure of the convex hull of $S$ by $\text{ch}(S)$. Uppercase characters denote points in $\mathbb{E}^d$ and $p_i$ denotes the $i$th coordinate of the point $P$.

The Euclidean distance between two points $P$ and $Q$ is denoted by $d(P, Q)$. The distance between a point $P$ and a nonempty set $S$ is $d(P, S) = \inf\{d(P, Q) \mid Q \in S\}$. The nearness of $S_1$ and $S_2$ is $d(S_1, S_2) = \inf\{d(P, Q) \mid P \in S_1, Q \in S_2\}$.

Let $\text{sl}(\pi_1, \pi_2)$ denote the open slab between the two distinct parallel hyperplanes $\pi_1$ and $\pi_2$. The sets $S_1$ and $S_2$ are separated by a slab $\text{sl}(\pi_1, \pi_2)$ if $S_1$ and $S_2$ lie in different components of $\mathbb{E}^d \setminus \text{sl}(\pi_1, \pi_2)$. $S_1$ and $S_2$ are strongly linearly separable if there exists an open slab that separates $\text{cl} S_1$ and $\text{cl} S_2$. A hyperplane contained in such an open slab is called a strong linear separator. $S_1$ and $S_2$ are linearly separable if there exists a hyperplane $\pi$, called a linear separator, such that $\text{cl} S_1$ and $\text{cl} S_2$ are in different components of $\mathbb{E}^d \setminus \pi$. Similarly, $S_1$ and $S_2$ are weakly separable if there exists a hyperplane $\pi_w$, called a weak linear separator, such that $\text{cl} S_1$ and $\text{cl} S_2$ lie in the closures of different components of $\mathbb{E}^d \setminus \pi_w$. Fig. 1 shows some examples of separable sets and separators in $\mathbb{E}^2$.

Note that strongly linearly separable sets are also linearly separable and linearly separable sets are also weakly linearly separable. Likewise, a strong linear separator is also a linear separator and a linear separator is also a weak linear separator. Most results presented in this research work are formulated in terms of weakly linearly separable sets and weak linear separators.

A practically useful characterization of strong linear separability is given in

**Theorem 1.** If at least one of the sets $S_1$ and $S_2$ is bounded, then $S_1$ and $S_2$ are strongly linearly separable if and only if $\text{ch}(S_1) \cap \text{ch}(S_2) = \emptyset$.

**Proof:** Since at least one of $S_1$ and $S_2$ is bounded, $\text{ch}(S_1) \cap \text{ch}(S_2) = \emptyset$ implies that $d((\text{ch}(S_1), \text{ch}(S_2)) > 0$. Since the nearness is positive, there exists a slab that separates $\text{ch}(S_1)$ and $\text{ch}(S_2)$. Therefore $S_1$ and $S_2$ are strongly linearly separable. The converse is trivial. $\square$

3. Partitions

In this section we investigate properties of the bisector of two weakly linearly separable sets $S_1$ and $S_2$. These are of use in Section 4 in studying the topology of the bisector. Hereafter, we assume that $S_1$ and $S_2$ are nonempty sets in $\mathbb{E}^d$ such that $\text{cl} S_1 \cap \text{cl} S_2 = \emptyset$. 
Figure 1: Three types of linear separability:

(a) $S_1$ and $S_2$ are strongly linearly separable since they are separated by the cross-hatched slab.

(b) $S_1 : x_2 = x_1^{-1}$ and $S_2 : x_2 = -x_1^{-1}, x_1 > 0$, are linearly separable (but not strongly separable) and $x_2 = 0$ is the linear separator.

(c) $S_1$ and $S_2$ are weakly linearly separable (but not linearly separable) and $\pi_w$ is the weak linear separator.

The bisector $B(S_1, S_2)$ of two sets $S_1$ and $S_2$ is the set of points equidistant to $S_1$ and $S_2$, i.e., $B(S_1, S_2) = \{ P \in \mathbb{E}^d \mid d(P, S_1) = d(P, S_2) \}$. Assume that $Q$ is a point of $B(S_1, S_2)$ and we define the maximal ball $\beta$ to be the open ball centered at $Q$ with radius $r = d(Q, S_1) = d(Q, S_2)$. We also define the maximal sphere $\sigma := \partial \beta$.

We observe that by definition $\beta$ does not contain points of $S_1$ or $S_2$ and that $\sigma$ contains at least one point $P_1$ from $\text{cl} \ S_1$ and at least one point $P_2 \in \text{cl} S_2$. The points $P_1$ and $P_2$ are called touching points of $\sigma$ with $S_1$ and $S_2$, respectively, and $\sigma$ is said to touch $S_1$ and $S_2$. Notice that $P_1 \neq P_2$ as $\text{cl} \ S_1 \cap \text{cl} S_2 = \{\}$. One maximal ball cannot contain another one because the included maximal sphere would lack touching points.

Without loss of generality, we assume that the hyperplane $x_d = 0$ is a weak linear separator $\pi_w$ and that $S_1$ is contained in the closure of the open half-space $\pi_w^{-1} : x_d > 0$, and that $S_2$ is contained in the closure of the open half-space $\pi_w^2 : x_d < 0$. In the sequel we consider the mapping which takes a point on a weak linear separator orthogonally to $\pi_w$ onto a point of the bisector. We first show in Section 3.1 that if a line perpendicular to $\pi_w$ intersects $B(S_1, S_2)$, then the intersection is connected. Then, in Section 3.2, we establish the necessary and sufficient conditions under which all lines perpendicular to $\pi_w$ intersect $B(S_1, S_2)$. Finally, in Section 3.3 we give necessary and sufficient conditions for a specific line perpendicular to $\pi_w$ to intersect $B(S_1, S_2)$. 
3.1. Connectedness of the intersection with a perpendicular

**Theorem 2.** If a line \( \ell \) perpendicular to a weak linear separator \( \pi_w \) of \( S_1 \) and \( S_2 \) intersects \( B(S_1, S_2) \), then it does so in a connected subset of \( \ell \). Moreover, if a point \( Q \) is in the relative interior of \( \ell \cap B(S_1, S_2) \), then all the touching points of the maximal sphere centered at \( Q \) are in \( \pi_w \).

**Proof:** To show that the intersection is connected, assume that there are two distinct points \( Q_a \) and \( Q_b \) in \( \ell \cap B(S_1, S_2) \). Assume that \( B_a, S_a, B_b, \) and \( S_b \) are the corresponding maximal balls and spheres. Since \( Q_a \) and \( Q_b \) are distinct, \( B_a \neq B_b \). Since one maximal ball cannot be contained in another, there are three remaining cases:

*Case 1,* \( S_a \) and \( S_b \) are disjoint:
Since \( Q_a \) and \( Q_b \) are in \( \ell \), there exists a hyperplane \( \lambda \) perpendicular to \( \ell \) which is a strong linear separator of \( S_a \) and \( S_b \) (see Fig. 2a). Since \( \lambda \) is parallel to \( \pi_w \), \( S_a \) cannot touch \( S_2 \) and/or \( S_b \) cannot touch \( S_1 \) — a contradiction.

*Case 2,* \( S_a \) and \( S_b \) intersect at one point \( P \):
Assume that \( \lambda \) is the hyperplane perpendicular to \( \ell \) through \( P \) (see Fig. 2b). \( S_a \) and \( S_b \) can touch both \( S_1 \) and \( S_2 \) only if \( P \in \text{cl}\, S_1 \cap \text{cl}\, S_2 \), which violates the assumption stated at the beginning of Section 3 that \( \text{cl}\, S_1 \cap \text{cl}\, S_2 = \emptyset \).

*Case 3,* \( S_a \) and \( S_b \) intersect in a nondegenerate lower-dimensional sphere:
Assume that \( \lambda \) is the hyperplane through the sphere of intersection. Note that \( \lambda \) is parallel to \( \pi_w \). If \( \lambda \neq \pi_w \), the sets of touching points of \( S_a \) and \( S_b \) lie in opposite closed half-spaces bounded by \( \lambda \) (see Fig. 2c). Therefore, since \( \lambda \) is parallel to \( \pi_w \), \( S_a \) cannot touch \( S_2 \) and/or \( S_b \) cannot touch \( S_1 \), which is a contradiction. Thus we have \( \lambda = \pi_w \). \( S_a \) and \( S_b \) can touch both \( S_1 \) and \( S_2 \) only if one of the touching points on \( S_1 \) and one of the touching points on \( S_2 \) are in \( \pi_w \). These touching points are contained in \( S_a \cap S_b \). Assume that \( Q \) is strictly between \( Q_a \) and \( Q_b \), and that \( S_Q \) is the sphere centered at \( Q \) and passing through \( S_a \cap S_b \) (see Fig. 2d). Clearly the open ball \( B \) defined by \( S_Q \) is contained in \( B_a \cap B_b \) and hence is free of points of \( S_1 \cup S_2 \). Thus \( S_Q \) is a maximal sphere touching both \( S_1 \) and \( S_2 \) in \( \pi_w \), and \( Q \in B(S_1, S_2) \).

The theorem follows directly. \( \square \)

3.2. Intersection with every perpendicular

In Section 3.1 we proved that if the intersection between the bisector and a line perpendicular to a weak linear separator exists, then it is connected. In this section we give necessary and sufficient conditions for every line perpendicular to the weak linear separator to intersect the bisector.

Assume that

\[
I := \{ X = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} \mid \alpha_i \leq x_i \leq \beta_i \quad \text{for} \quad i = 1, \ldots, d-1 \},
\]

for real numbers \( \alpha_i \leq \beta_i \) is a closed \((d - 1)\)-cell in \( \pi_w \).

**Theorem 3.** Every line perpendicular to the weak linear separator \( \pi_w \) intersects \( B(S_1, S_2) \) if and only if

1. \( S_1, S_2 \not\subset \pi_w \), or
2. \( S_2 \subset \pi_w \) and \( \text{cl}\, S_1 \cap \pi_w = \emptyset \), or vice versa.
Moreover, if every line perpendicular to $\pi_w$ intersects $B(S_1, S_2)$, then for any $(d-1)$-cell $I$ in $\pi_w$ the set $B(S_1, S_2) \cap (I \times \mathbb{R})$ is bounded.

We prove Theorem 3 by showing the sufficient conditions in Lemma 1 and 2 and the necessary condition in Lemma 3.

**Lemma 1.** If $S_1, S_2 \not\subset \pi_w$, then every line perpendicular to $\pi_w$ intersects $B(S_1, S_2)$. Moreover, $B(S_1, S_2) \cap (I \times \mathbb{R})$ is bounded.

**Proof:** Assume that $Q$ is a point on the line
$$\ell := \{X = (x_1, \ldots, x_d) \mid x_i = q_i, \; i = 1, \ldots, d-1\}$$
that is perpendicular to $\pi_w$. We define $f(Q) = d^2(Q, S_1) - d^2(Q, S_2)$, and we observe that $Q \in B(S_1, S_2)$ if and only if $f(Q) = 0$. First we assume that $q_d > 0$. Let $P \in S_1$, $p_d > 0$. Such a point must exist since $S_1 \not\subset \pi_w$ (see Fig. 3).

Since $d^2(Q, S_1) \leq d^2(Q, P) = \sum_{i=1}^d (p_i - q_i)^2$ and $d^2(Q, S_2) \geq d^2(Q, \pi_w) = q_d^2$,
$$f(Q) \leq \sum_{i=1}^d (p_i - q_i)^2 - q_d^2 = \sum_{i=1}^{d-1} (p_i - q_i)^2 + p_d^2 - 2p_dq_d.$$ 

For sufficiently large $q_d$ we get $f(Q) < 0$. By a symmetry argument, for sufficiently small $q_d < 0$ we get $f(Q) > 0$. But $d(Q, S_i)$ is a continuous function of $P$ (as assumed in Theorem 3). Therefore, since $f$ changes sign, it must have at least one zero, which implies that the line $\ell$ intersects $B(S_1, S_2)$.

To prove that $B(S_1, S_2) \cap (I \times \mathbb{R})$ is bounded, we observe that for
$$q_d > \left( \sum_{i=1}^d (p_i - q_i)^2 + p_d^2 \right) / 2p_d$$
we get \( f(Q) < 0 \). Thus the intersection of \( B(S_1, S_2) \) with \( \ell \) is bounded in the positive \( x_d \)-direction by a continuous function of \( q_1, \ldots, q_d \). It is similarly bounded in the negative \( x_d \)-direction. In particular, as \( I \) is compact, \( B(S_1, S_2) \cap (I \times \mathbb{R}) \) is bounded. \( \square \)

**Lemma 2.** If \( S_2 \subset \pi_w \) and \( \text{cl} S_1 \cap \pi_w = \{\} \), then every line perpendicular to \( \pi_w \) intersects \( B(S_1, S_2) \). Moreover, \( B(S_1, S_2) \cap (I \times \mathbb{R}) \) is bounded.

*Proof:* Let \( Q \) be a point on a line \( \ell \) perpendicular to \( \pi_w \) and vary \( q_d \) so that \( Q \) moves along \( \ell \). By the arguments used in the proof of Lemma 1 we have \( f(Q) = d^2(Q, S_1) - d^2(Q, S_2) < 0 \) for sufficiently large \( q_d > 0 \).

We now show that there exists a \( q_d < 0 \) such that \( f(Q) \geq 0 \). Assume that \( U := \ell \cap \pi_w \) and \( P \) is a point of \( \text{cl} S_2 \) closest to \( U \). If \( P = U \), then \( d^2(Q, S_2) = q_d^2 \). Therefore, since \( d^2(Q, S_1) > d^2(Q, \pi_w) = q_d^2 \), we get \( q_d \leq 0 \).

If \( P \neq U \), then we consider an open ball \( B_u \) of radius \( d(U, P) \) centered at \( U \). If \( \text{cl} S_1 \cap \text{cl} B_u = \{\} \), then \( d^2(U, S_1) > d^2(U, S_2) \), which leads to \( f(U) > 0 \). Otherwise, since \( \text{cl} S_1 \cap \text{cl} B_u \) is compact, there exists a point \( T \in (\text{cl} S_1 \cap \text{cl} B_u) \) with smallest \( x_d \)-coordinate. Moreover, \( t_d > 0 \) because \( \text{cl} S_1 \cap \pi_w = \{\} \) (see Fig. 4).

**Figure 3:** Selecting points \( P \) and \( Q \) for the proof of Lemma 1.

**Figure 4:** Construction for proof of Lemma 2.
For \( q_d \leq 0 \), \( d^2(Q, S_1) \geq (t_d - q_d)^2 \). Therefore, since \( d^2(Q, S_2) = \sum_{i=1}^{d} (p_i - q_i)^2 \),

\[
f(Q) \geq t_d^2 - 2t_d q_d - \sum_{i=1}^{d-1} (p_i - q_i)^2
\]

if \( q_d \leq 0 \). For sufficiently small \( q_d \leq 0 \) is \( f(Q) \geq 0 \). Thus \( f(Q) \) has a zero and the intersection result follows. Boundedness is guaranteed, since \( T \) is confined to a bounded set as long as \( U \in I \). \( \Box \)

To complete proof of Theorem 3, it remains to prove the necessary condition.

**Lemma 3.** If \( S_2 \subset \pi_w \) and \( \cl S_1 \cap \pi_w \neq \{\} \), then there exist lines perpendicular to \( \pi_w \) that do not intersect \( B(S_1, S_2) \).

**Proof:** Consider the line \( \ell \) perpendicular to \( \pi_w \) through a point \( P \in \cl S_1 \cap \pi_w \) (see Fig. 5). For any \( Q \in \ell \) we obtain \( d^2(Q, S_1) \leq d^2(Q, P) = q_d^2 \). Since \( P \notin \cl S_2 \) and \( S_2 \in \pi_w \), \( d^2(Q, S_2) > q_d^2 \). Therefore \( d^2(Q, S_1) < d^2(Q, S_2) \) for all \( Q \in \ell \), so \( \ell \) cannot intersect \( B(S_1, S_2) \). \( \Box \)

![Figure 5: Construction for proof of Lemma 3.](image-url)

Fig. 6 shows two examples as an illustration of Lemma 3. Let \( S_1 := \{P1, P2, P3\} \) and \( S_2 := \{P4, P5\} \). We have:

1. Both \( S_1 \) and \( S_2 \) are completely contained in their weak linear separator \( \pi_w \). \( B(S_1, S_2) \) consists of four lines (shown dashed), all of which are perpendicular to \( \pi_w \).

2. \( S_1 \) is only partially contained in \( \pi_w \). In both cases there exist lines perpendicular to \( \pi_w \) that do not intersect \( B(S_1, S_2) \).

![Figure 6: Sets illustrating the condition of Lemma 3.](image-url)
Corollary 1. If $S_1$ and $S_2$ are weakly linearly separated by $\pi_w$ and $\text{cl} S_1 \subset \pi_w$, then every line perpendicular to $\pi_w$ intersects $B(S_1, S_2)$ in a single point.

Corollary 2. If a hyperplane $\pi$ is a linear separator of $S_1$ and $S_2$, then every line perpendicular to $\pi$ intersects $B(S_1, S_2)$ in a single point.

3.3. Intersection with a specific perpendicular

Theorem 3 gave the necessary and sufficient conditions under which every line perpendicular to $\pi_w$ intersects $B(S_1, S_2)$. Even if such global conditions do not hold, it is still possible to obtain local results. This section gives necessary and sufficient conditions under which specific line perpendicular to $\pi_w$ intersects $B(S_1, S_2)$.

Lemma 4. Let $S_1, S_2 \subset \pi_w$ and let $U$ be a point in $\pi_w$. A line $\ell$ perpendicular to $\pi_w$ and passing through $U$ intersects $B(S_1, S_2)$ if and only if $d(U, S_1) = d(U, S_2)$. Moreover, if $\ell$ intersects $B(S_1, S_2)$, then $\ell \subset B(S_1, S_2)$.

Proof: Let $Q \in \ell$. Then $d^2(Q, S_i) = d^2(U, S_i) + q^2_i$ for $i = 1, 2$. Therefore, if $d(U, S_1) = d(U, S_2)$, then $d(Q, S_1) = d(Q, S_2)$, which implies that $\ell \subset B(S_1, S_2)$. Conversely if $Q \subset B(S_1, S_2)$, then $d^2(Q, S_1) = d^2(Q, S_2)$ and hence $d(U, S_1) = d(U, S_2)$. \hfill $\Box$

Lemma 5. Let $S_2 \subset \pi_w$, $S_1 \not\subset \pi_w$, $\text{cl} S_1 \cap \pi_w \neq \emptyset$, and $U \in \pi_w$ such that $d(U, S_2) \neq d(U, \text{cl} S_1 \cap \pi_w)$. A line $\ell$ perpendicular to $\pi_w$ and passing through $U$ intersects $B(S_1, S_2)$ if and only if $d(U, S_2) < d(U, \text{cl} S_1 \cap \pi_w)$. Moreover, if $\ell$ intersects $B(S_1, S_2)$, then it does so in a single point.

Figure 7: Construction for proof of Lemma 5.

Proof: Let $Q$ be a point in $\ell$ (see Fig. 7). Sufficiency follows from arguments similar to those of the proof Lemma 2. To show the necessity, assume that $Q \in B(S_1, S_2)$. Since $d^2(Q, S_1) = d^2(Q, S_2)$, $d^2(Q, S_2) = d^2(U, S_2) + q^2_d$, $d^2(Q, \text{cl} S_1 \cap \pi_w) = d^2(U, \text{cl} S_1 \cap \pi_w) + q^2_d$, and $d^2(Q, S_1) \leq d^2(Q, \text{cl} S_1 \cap \pi_w)$, we have

$$d^2(U, S_2) + q^2_d \leq d^2(U, \text{cl} S_1 \cap \pi_w) + q^2_d.$$ 

Necessity follows since, by hypothesis, $d(U, S_2) \neq d(U, \text{cl} S_1 \cap \pi_w)$.

To see that the intersection is a single point, assume the contrary. Then, by Theorem 2, there exists $Q \in \text{int} [\ell \cap B(S_1, S_2)]$ such that the touching points of the maximal sphere centered at $Q$ are contained in $\pi_w$. This implies that the touching points in $S_1$ are in $\text{cl} S_1 \cap \pi_w$ and that $d^2(Q, S_2) = d^2(Q, \text{cl} S_1 \cap \pi_w)$ — a contradiction. \hfill $\Box$
Let $B$ be an open $(d-1)$-dimensional ball and let $V$ be a point on a line perpendicular to the hyperplane that contains $B$, and passing through the center of $B$. The truncated semicone $C(V, B)$ is defined to be $\text{int ch}(V \cup B)$.

**Lemma 6.** Let $S_2 \subset \pi_w$, $S_1 \not\subset \pi_w$, $\text{cl } S_1 \cap \pi_w \neq \emptyset$ and $U \in \pi_w$ such that $d(U, S_2) = d(U, \text{cl } S_1 \cap \pi_w)$. Let $B \subset \pi_w$ be the $(d-1)$-dimensional ball of radius $d(U, S_2)$ centered at $U$. A line $\ell$ perpendicular to $\pi_w$ and passing through $U$ intersects $B(S_1, S_2)$ if and only if there exists a point $Q \in (\ell \cap \pi^1_w)$ such that $C(Q, B) \cap S_1 = \emptyset$. Furthermore, if $\ell$ intersects $B(S_1, S_2)$, then a half-line of $\ell$ is contained in $B(S_1, S_2)$.

**Proof:** To show sufficiency, assume that there is a point $Q_1 \in (\ell \cap \pi^1_w)$ such that $C(Q_1, B) \cap S_1 = \emptyset$ (see Fig. 8).

Now we consider the one-parameter family of $d$-dimensional balls (and the associated boundary spheres) that intersect $\pi_w$ in $B$ and whose centers lie on $\ell$. Some of these boundary spheres must intersect $\pi^1_w$ within $C(Q_1, B)$. Since $B$ intersects neither $\text{cl } S_1$ nor $\text{cl } S_2$, but the boundary of $B$ touches both $\text{cl } S_1$ and $\text{cl } S_2$, such spheres must be maximal spheres. Furthermore, any member of the family whose center has smaller $x_d$-coordinate must also be maximal. Thus, a half-line of $\ell$ is contained in $B(S_1, S_2)$.

To show necessity, we consider a maximal sphere $\sigma$ (and the associated ball $\beta$) centered at $Q \in [\ell \cap B(S_1, S_2)]$. Note that $U \notin \text{cl } (S_1 \cup S_2)$ because $d(U, S_2) = d(U, \text{cl } S_1 \cap \pi_w)$ and $\text{cl } S_1 \cap \text{cl } S_2 = \sigma$. Therefore exists $Q_2 \in (\pi^1_w \cap \ell \cap \sigma)$ (see Fig. 8). Since $C(Q_2, B) \subset \beta$, $C(Q_2, B) \cap S_2 = \emptyset$. □

An example where the conditions of Lemma 6 do not hold is shown in Fig. 9. $S_1$ is an arc of a circle, $S_2 \subset \pi_w$ and $d(U, S_2) = d(U, \text{cl } S_1 \cap \pi_w)$. The line $\ell$ will not intersect $B(S_1, S_2)$.

Necessary and sufficient conditions for any specific line perpendicular to $\pi_w$ to intersect $B(S_1, S_2)$ follow directly from Theorem 3 and Lemmas 4 - 6.

From the above study, we can conclude the following Theorem:

**Theorem 4.** Assume that $U \in \pi_w$ and let $\ell$ be the line perpendicular to $\pi_w$ passing through $U$. The line $\ell$ intersects $B(S_1, S_2)$ if and only if (up to a switch of $S_1$ and $S_2$):
4. Continuous mapping from linear separator to bisector

We have thus far shown exactly when a line perpendicular to \( \pi_w \) intersects \( B(S_1, S_2) \) at a single point. This defines a mapping which lifts points of the separator up to the bisector. We now show that wherever such a mapping exists, it is continuous. Notice that this map will automatically be a homeomorphism, as its inverse is the orthogonal projection onto the separator a well-defined continuous map.

**Theorem 5.** Let the hyperplane \( \pi_w : x_d = 0 \) be a weak linear separator of \( S_1 \) and \( S_2 \) and let \( M \) be a open subset of \( \pi_w \) such that for all \( P \in M \) the line through \( P \) perpendicular to \( \pi_w \) intersects \( B(S_1, S_2) \) at a single point. Then the mapping \( b : M \to \mathbb{R} \) defined by \( (x_1, \ldots, x_{d-1}, b(x_1, \ldots, x_{d-1})) \in B(S_1, S_2) \) is continuous.

To prove this we use the known computational geometry principle: Let \( E \) and \( F \) be topological spaces. A function \( g : E \to F \) is said to have a closed graph if its graph \( \{(x, y) \mid y = g(x), x \in E\} \) in the product space \( E \times F \) is a closed set. If \( g \) has a closed graph and \( F \) is compact, then \( g \) is continuous.

**Proof:** The mapping \( b \) is a function by hypothesis. We show that \( b \) is continuous at an arbitrary \( P \in M \):

Let \( I := \{X \in \mathbb{R}^{d-1} \mid \alpha_i \leq x_i \leq \beta_i, \ i = 1, \ldots, d-1\} \), for real numbers \( \alpha_i < \beta_i \), \( i = 1, \ldots, d-1 \) be a closed \( (d-1) \)-cell in \( M \) containing \( P \) in its interior and let \( b_1 \) denote the restriction of \( b \) to \( I \). We claim that \( b_1 \) has a compact graph and thus, by the previous principle, \( b \) is continuous at \( P \).

The graph of \( b_1 \) is the intersection of \( B(S_1, S_2) \) and \( I \times \mathbb{R} \). Referring to the proof of Lemma 1, \( B(S_1, S_2) = f^{-1}(0) \). Since the inverse image of a closed set under a continuous map is closed, \( B(S_1, S_2) \) is closed. Therefore, the graph of \( b_1 \) is closed.
It remains to show that the range of \( b \) is also bounded and, hence, compact. Since the line perpendicular to \( \pi_w \) at any point \( P \in M \) intersects \( B(S_1, S_2) \), one of the five conditions of Theorem 4 must hold for each point in \( M \). Furthermore, since by hypothesis, each such line intersects \( B(S_1, S_2) \) in a single point, Lemma 4 and 6 imply that, up to a switch of \( S_1 \) and \( S_2 \), one of the following must hold:

1. \( S_1 \not\subset \pi_w \) and \( S_2 \not\subset \pi_w \), or

2. \( S_2 \subset \pi_w \) and \( \text{cl} \ S_1 \cap \pi_w = \{\} \), or

3. \( S_2 \subset \pi_w \), \( S_1 \not\subset \pi_w \), \( \text{cl} \ S_1 \cap \pi_w \neq \{\} \), and \( d(U, S_2) < d(U, \text{cl} \ S_1 \cap \pi_w) \) for all \( U \in M \).

In the first two cases Theorem 3 establishes that the graph is bounded. In the third case Lemma 5 and the fact that \( I \) is compact establishes that the graph is bounded. Therefore, the graph of \( b_1 \) is compact, which implies that \( b \) is continuous at each point of \( M \). \( \square \)

Thus, when \( \pi \) is a linear separator, we assume that the hyperplane \( \pi : x_d = 0 \) be a linear separator of \( S_1 \) and \( S_2 \). If \( b \) is the mapping \( b : \pi \to \mathbb{R} \) such that \((x_1, \ldots, x_{d-1}, b(x_1, \ldots, x_{d-1})) \in B(S_1, S_2) \), then \( b \) is a continuous function. In fact, the perpendicular projection of \( B(S_1, S_2) \) onto \( \pi \) is a homeomorphism. This result describes that the bisector of linearly separated point sites in \( \mathbb{E}^2 \) is a monotone chain. More importantly, it shows that \( B(S_1, S_2) \) is a \((d-1)\)-manifold in \( \mathbb{E}^2 \).

5. Conclusion

In this paper we have presented some general properties of bisectors of sets in \( \mathbb{E}^d \) that are separated by hyperplanes. We have given necessary and sufficient conditions for the perpendicular projection of the bisector of two weakly linearly separated sets onto a separator to be a homeomorphism. This study needs to be expanded in two major directions:

![Figure 10: A nonhomogeneously two-dimensional bisector. B(S1, S2) is the bisector between two open line-segments S1 := OP and S2 := OQ that do not intersect. However, the closures of the line-segments share the endpoint O.](image-url)

Throughout this paper we have required that \( \text{cl} \ S_1 \cap \pi_w = \{\} \). When the closures of the sets are not disjoint, the bisector need not be a manifold, as illustrated in Fig. 10. This
issue has been addressed in the literature on symmetric axes of point sets and open line segments in \( \mathbb{E}^2 \) by defining the bisectors between individual elements so that they are always homogeneously one dimensional \([3, 5, 6]\). We think that this needs further investigation.

Another direction in which the theory could be generalized is to investigate the general conditions under which the bisector of two sets is a \((d - 1)\)-manifold that subdivides \( \mathbb{E}^d \) into two disjoint regions. In this paper we have shown that if the two sets are linearly separable, then their bisector has this property. Also, some sufficient conditions for the bisector to be a simply closed curve were given in \([4, 6]\) for sets in \( \mathbb{E}^2 \) that are not even weakly linearly separable.

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References


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