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Projection from 4D to 3D

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Abstract. The aim of this paper is to give a survey on analytic representations of central and orthographic projections from $\mathbb{R}^4$ to $\mathbb{R}^3$ or $\mathbb{R}^2$. There are discussed various aspects of these projections, whereby some special relations were revealed, e.g., the fact that homogeneous coordinates or barycentric coordinates in $\mathbb{R}^3$ can be obtained by applying particular projections on a point with given cartesian coordinates in $\mathbb{R}^4$. We would also like to demonstrate that by projecting curves or 2-surfaces of $\mathbb{R}^4$ interesting shapes in $\mathbb{R}^3$ and $\mathbb{R}^2$ can be obtained.

Key Words: geometry in 4D, projections, quaternions

MSC 1994: 51N20, 51N05.

1. Introduction

Geometric objects in the space $\mathbb{R}^4$ can be projected first into the space $\mathbb{R}^3$ and then into the plane $\mathbb{R}^2$. We prefer orthographic projections against other parallel projections, as they are an approximation of the central projection with large distance. The advantage of the orthographic projection is a rather good realism in visualization of unknown geometric objects. The basic aspect of a realistic view of smooth surfaces after a projection $\mathbb{R}^3 \to \mathbb{R}^2$ is to find the outline curve; an algorithm is described in [10].

In the projection $\mathcal{P} : \mathbb{R}^4 \to \mathbb{R}^3$ those 3-dimensional objects are visible, that are in the case of the *central projection* close to the centre of projection (we restrict our consideration on the projection of only one of the open semi-spaces determined by the hyperplane parallel to the projection plane and incident to the centre of projection). In the case of a *parallel projection* visible objects are in larger distance from the 3-dimensional projection plane, if these distances are oriented opposite to the rays of sight. *Visibility* defined in this way will be denoted by $\mathcal{W}^4$.

The visibility $\mathcal{W}^4$ in the projection $\mathcal{P}$ is different from the visibility $\mathcal{W}^3$ in the space $\mathbb{R}^3$ that is applied on any projection $\mathbb{R}^3 \to \mathbb{R}^2$. E.g., in the projection of a simplex $\mathcal{S}^4 \subset \mathbb{R}^4$

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we suppose that the centre of projection is an exterior point (for parallel projections this condition is always satisfied). A simplex $\mathcal{S}^4$ with vertices $A_1, \ldots, A_5$ is projected onto the convex hull of the five image points $A_1^*, \ldots, A_5^*$. If one of these points A_i^* is an interior point of the tetrahedron formed by the remaining four image points A_k^*, then the four edges passing through A_i^* are non-visible in the visibility $\mathcal{W}^3$, but these points are visible in the visibility $\mathcal{W}^4$. If any point A_i^*, $i = 1, \ldots, 5$ is an exterior point of the tetrahedron determined by the remaining four points, then one and only one edge is non-visible in both visibilities $\mathcal{W}^4$ and $\mathcal{W}^3$. If one point A_i^* is located on one edge of the tetrahedron of the remaining A_k^*, then all edges are visible in the visibility $\mathcal{W}^3$.

Generally, there is no chance to define a visibility when any $\mathbb{R}^4$-object is projected into $\mathbb{R}^2$. The reason is that the "rays of sight" are planes, and for any two points in a plane one cannot define that one point "hides" the other. Such a remark can be found also in [6].

Visibility $\mathcal{W}^4$ is suitable for enlightening the space $\mathbb{R}^4$. In the figures included in the paper the visibility $\mathcal{W}^3$ was applied, as it is easier to realize in the projection plane $\mathbb{R}^2$.

2. Central projections, modelling curves and surfaces

2.1. Central projections

Let $\mathcal{V}$ be a curve or a 2-surface in the space $\mathbb{R}^{n+1}$, $n > 1$. There is a central projection of $\mathbb{R}^{n+1}$ from the origin O of the coordinate system onto any hyperplane $\mathcal{R}$. Under this projection any point $B = (x_1, \ldots, x_{n+1})$ of the figure $\mathcal{V} \setminus \{O\}$ can be connected with O by the line $b = OB$ intersecting the hyperplane $\mathcal{R}$ in the image $(x_1^*, \ldots, x_n^*)$ of B.

When the equation of the hyperplane $\mathcal{R}$ is in the form $x_{n+1} = 1$, the we get the same relation as between *homogeneous coordinates* $(x_1, \ldots, x_{n+1})$ and cartesian coordinates

$$\left(\frac{x_1}{x_{n+1}}, \ \ldots \ \frac{x_n}{x_{n+1}}\right), \qquad x_{n+1} \neq 0$$

of points of the projective extension $\mathbf{P}^n$ of the Euclidean space $\mathbb{R}^n$.

When the equation of the hyperplane $\mathcal{R}$ is in the form

$$x_1 + x_2 + \ldots + x_n + x_{n+1} = 1,$$

we speak about *barycentric coordinates* in $\mathbb{R}^n$

$$x_i^b := \frac{x_i}{x_1 + \ldots + x_{n+1}}, \qquad i = 1, \ldots, n,$$

and we even have to assume that no point of the figure $\mathcal{V}$ is located in the hyperplane $x_1 + \ldots + x_{n+1} = 0$ parallel to $\mathcal{R}$. In this situation we do not speak of the projective space $\mathbf{P}^n$, but of barycentric coordinates in $\mathbb{R}^n$.

2.2. Modelling curves and surfaces

Any *parabola* in $\mathbb{R}^{n+1}$, $n > 1$, can be easily determined by parametric equations using quadratic polynomials like

$$x_i(t) := a_{i,2}t^2 + a_{i,1}t + a_{i,0}\,, \quad -\infty < t < \infty.$$

It is clear that generally a *conic section* in the projective extension of $\mathbb{R}^n$ can be determined by the ratio of quadratic polynomials $x_i(t)$ in the form

$$x_i^*(t) = \frac{x_i(t)}{x_{n+1}(t)}, \quad i = 1, \ldots, n.$$

In addition to the presented polynomial representation of a parabola, any ellipse in $\mathbb{R}^{n+1}$ can be represented by the equations

$$x_i(t) = a_i \cos t + b_i \sin t + c_i,\ 0 \le t < 2\pi,\ a_i, b_i, c_i \in \mathbb{R} \ \text{ for } i = 1, \ldots, n+1,$$

any hyperbola by

$$x_i(t) = \pm a_i \cosh t + b_i \sinh t + c_i,\ -\infty < t < \infty, \ \text{ for } i = 1, \ldots, n+1$$

or an ellipse in the exponential form

$$x_i(t) = a_i \exp(\mathrm{i}t) + \overline{a_i} \exp(-\mathrm{i}t) + c_i,\ -\infty < t < \infty,\ a_i \in \mathbb{C},\ c_i \in \mathbb{R} \ \text{ for } i = 1, \ldots, n+1,$$

or a hyperbola in the exponential form

$$x_i(t) = \pm a_i \exp t + b_i \exp(-t) + c_i,\ -\infty < t < \infty,\ a_i, b_i, c_i \in \mathbb{R} \ \text{ for } i = 1, \ldots, n+1.$$

Generally, a conic section in $\mathbb{R}^{n+1}$ can be represented by a linear vector combination of different basic functions

$$\{1, t, t^2\}, \quad \{1, \sin t, \cos t\}, \quad \{1, \exp t, \exp(-t)\},$$

and so on. In the space $\mathbb{R}^n$ we get the corresponding "rational" functions.

Quite a wide variety of curves in $\mathbb{R}^3$ that are useful for technical applications can be determined with the basis $\{1,\ t,\ t^2,\ t^3\}$. These are curves generated from cubic curves in $\mathbb{R}^4$, while the vector coefficients can be four linearly independent vectors in $\mathbb{R}^4$.

Any affine transformation $\mathbb{R}^n \to \mathbb{R}^n$ or parallel projection $\mathbb{R}^n \to \mathbb{R}^{n-1}$ transforms the control polygon (or net) to the control polygon (or net). An affine transformation $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ or the central projection from the origin O, $\mathbb{R}^{n+1} \to \mathbf{P}^n$ (that is the extension of $\mathbb{R}^n$ by the hyperplane $x_{n+1} = 0$), transforms the control polygon $\{Q_j \,|\, j = 1, \ldots, k\} \subset \mathbb{R}^{n+1}$ onto the polygon $\{Q_j^* \,|\, j = 1, \ldots, k\} \subset \mathbf{P}^n$. If all vertices Q_j^* of the polygon are real points of the projective extension of the space $\mathbb{R}^n$, then the function coefficient at the vertex Q_j^* will be of the form

$$f_j^* = \frac{f_j}{\sum\limits_j f_j\, Q_{j,n+1}}\,,$$

provided the function coefficients f_j, $j = 1, \ldots, k$, are linear combinations of polynomial functions in the basis $\{1,\ t,\ t^2,\ t^3\}$, and the coordinates of the control point Q_j are denoted as

$$Q_j = (Q_{j,1},\ \ldots, Q_{j,n+1}).$$

The situation is a bit more complicated at the transition to barycentric coordinates: The representation of the point $B \in \mathbb{R}^{n+1}$ in terms of the control polygon $\{Q_j \,|\, j = 1, \ldots, k\}$

$$B = \sum_{j=1}^{k} f_j Q_j \ \text{ will be replaced by } \ B_i^* = \frac{\sum\limits_j f_j Q_{j,i}}{\sum\limits_{j,m} f_j Q_{j,m}}\,,\ i = 1, \ldots, n+1.$$

Let the control polygon $\{Q_1, \ldots, Q_k\}$ be a simplex in $\mathbb{R}^{n+1}$, i.e., $k = n+1$, then the vertex Q_1 can be associated to the $(n+1)$-tuple $(1, 0, \ldots, 0)$, and vertex Q_{n+1} to the $(n+1)$-tuple $(0, \ldots, 0, 1)$. For $k > n+1$ the vertices $Q_i \in \mathbb{R}^{n+1}$, $i = 1, \ldots, k$, can be regarded as the parallel views of the vertices of any simplex in $\mathbb{R}^{k-1}$.

The practical advantage of the determination of barycentric coordinates for the control polygons is that all barycentric coordinates of any point B are positive numbers, if and only if the point is located inside the simplex. In connection to the parallel projection of the simplex, the following statement is valid: If all coefficients determining the point B with respect to the control polygon are positive and the sum of them equals 1, then B is located inside the convex hull of the control polygon.

2-surfaces in $\mathbb{R}^4$ can sometimes be modelled as the graph of any complex function $f(z)$ in one complex variable $z = x + \mathrm{i}y$. This gives for $f(z) = u(x,y) + \mathrm{i}v(x,y)$

$$x_1 = \operatorname{Re} z, \quad x_2 = \operatorname{Im} z, \quad x_3 = u = \operatorname{Re} f(z), \quad x_4 = v = \operatorname{Im} f(z).$$

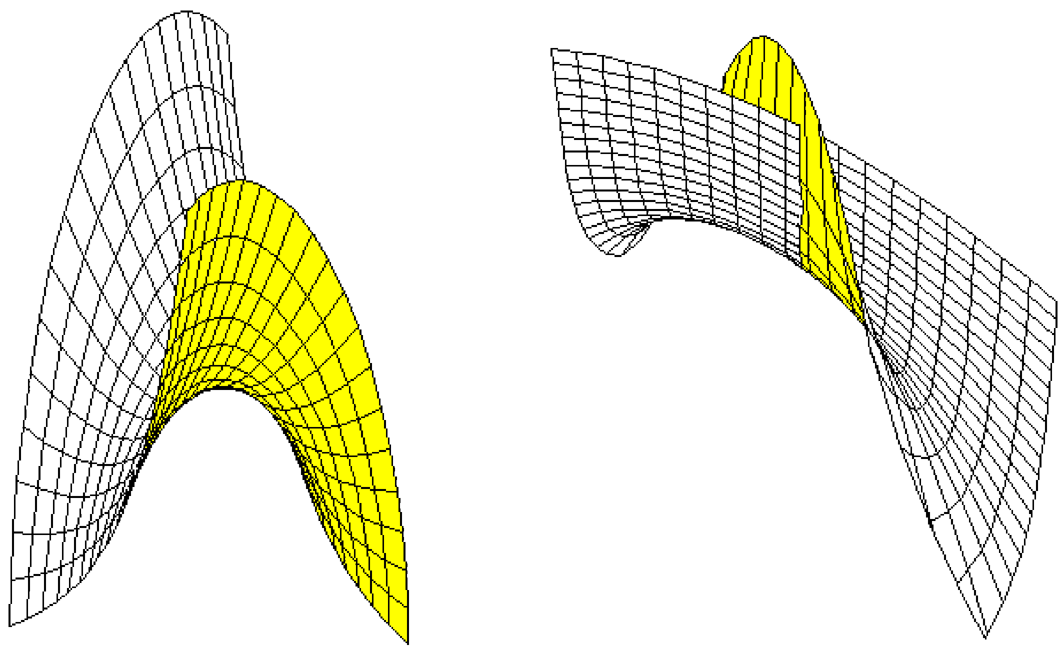

Figure 1: Central views of 2-surfaces defined by complex functions

In Fig. 1 central views of graphs of the functions $f(z) = 4 + \mathrm{i} + z^2$ (on the left) and $f(z) = 4 + z^2$ (on the right) under the central projection from the origin $(0,0,0,0)$ onto the hyperplane $x_4 = 1$ are displayed.

2-surfaces in $\mathbb{R}^4$ can also be determined by basic functions

$$\{1,\ \sin u,\ \cos u,\ \sin v,\ \cos v\}, \quad 0 \le u, v \le 2\pi.$$

When $x_4(u,v)$ is sufficiently far from zero, we receive in $\mathbb{R}^3$ a closed torus-like surface. Some examples are shown in the Figures 2 and 3. The orthographic view of the surface defined by the parametric equations

$$w = 8 + \cos u + 2\sin u + \cos v + \sin v$$

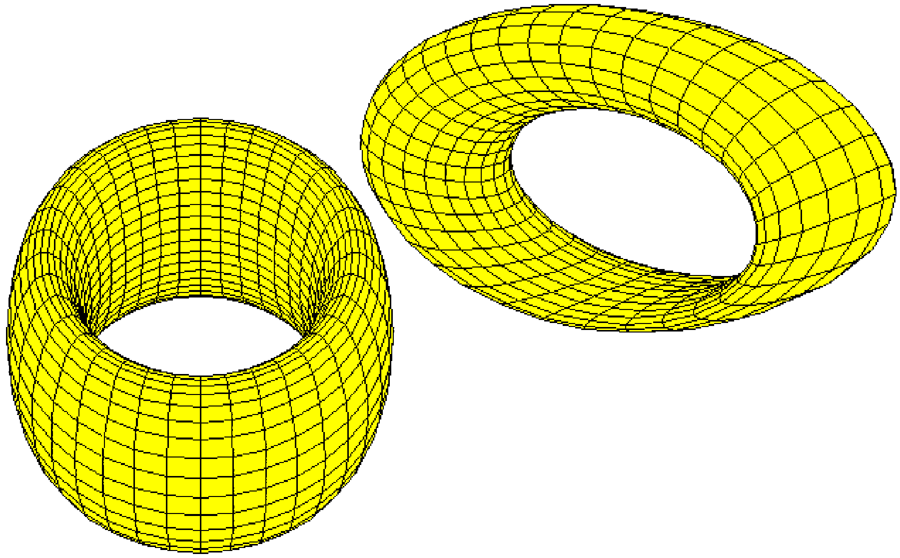

Figure 2: Central views of torus-like surfaces

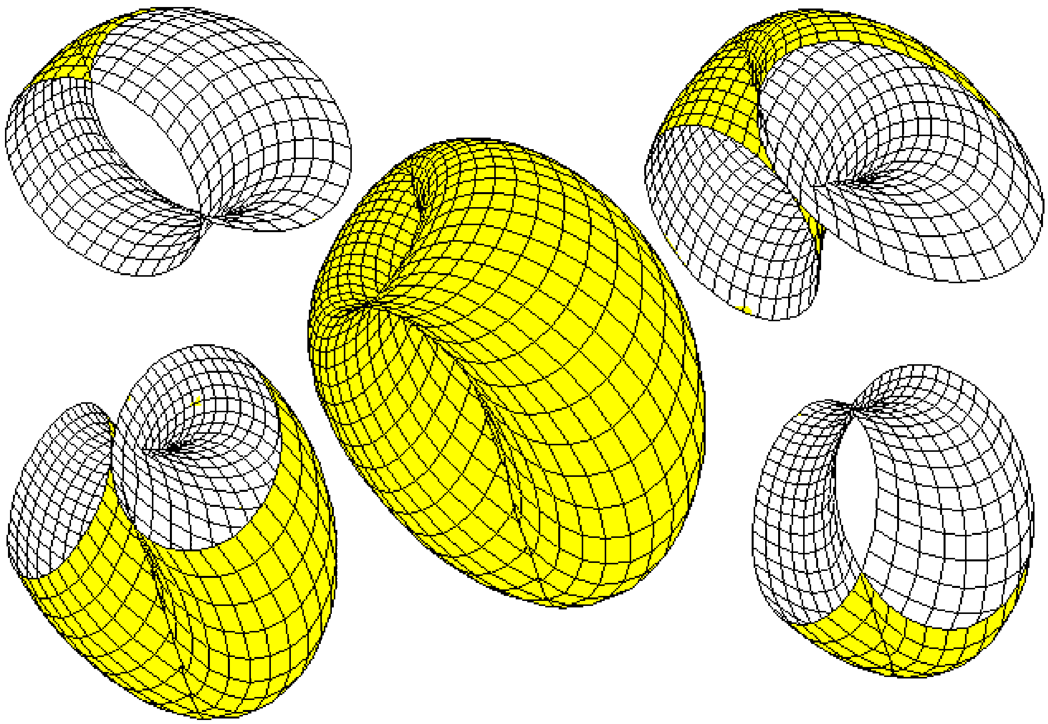

Figure 3: Central views of 2-surfaces defined by linear rational functions

$$x = \frac{1}{w}, \quad y = \frac{(2 + \sin u)}{w}, \quad z = \frac{0.5 \cos v}{w}, \quad 0 \leq u, v < 2\pi$$

is presented in Fig. 4. The outline of the orthographic view is shown on the left, the net of isoparametric curves is displayed on the right.

In the space $\mathbb{R}^4$ with the coordinates x_1, x_2, x_3, x_4 two tori can share no more than two meridian circles. Let one of them be located in the hyperplane $x_4 = 0$. Both tori in $\mathbb{R}^4$ can be projected from the point $(0, 0, 0, 0)$ to the hyperplane $x_4 = 0$ (see Fig. 5).

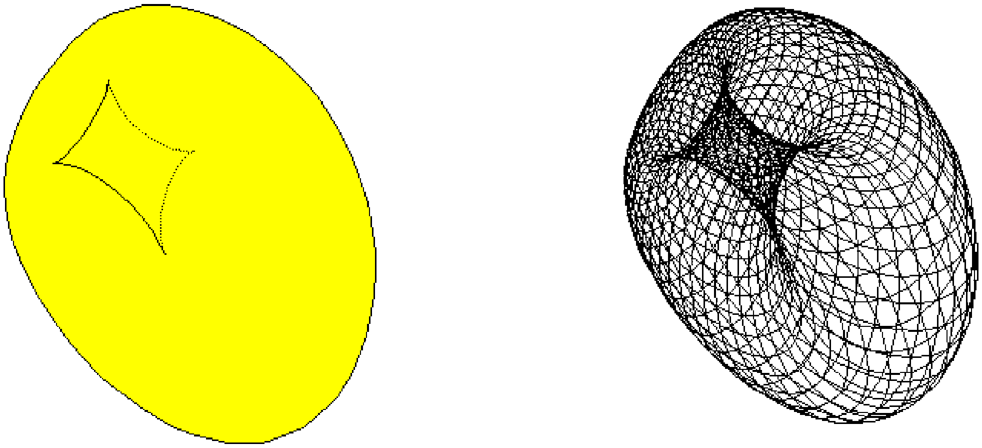

Figure 4: Orthographic view of a 2-surface defined parametrically

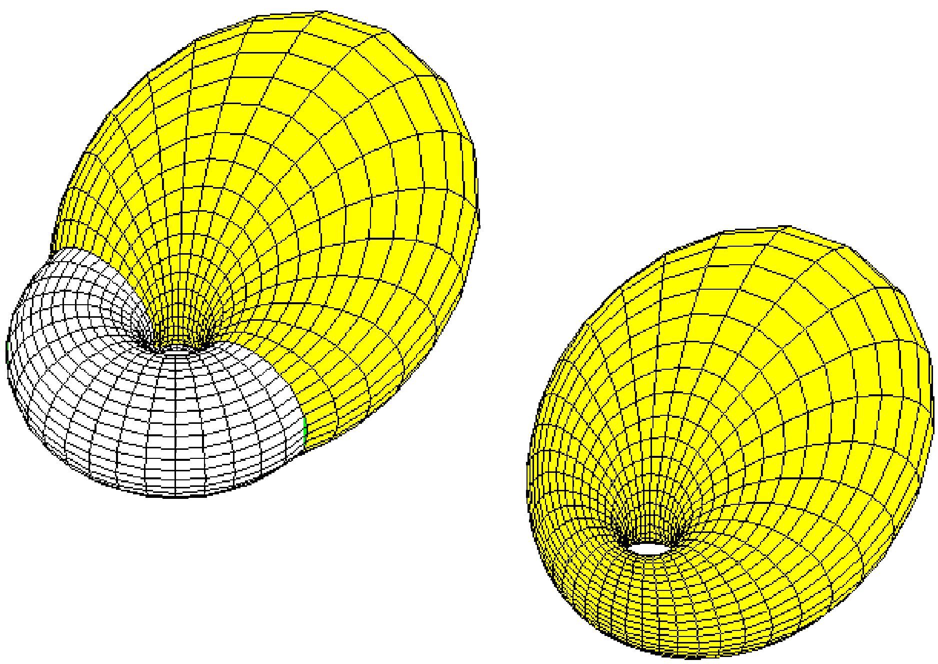

Figure 5: Central views of two tori sharing two meridian circles

3. Spherical coordinates and orthographic projections

In $\mathbb{R}^4$ the norm of a vector is the l^2-norm

$$\|(x, y, z, w)\| := \sqrt{x^2 + y^2 + z^2 + w^2}.$$

The *hypersphere* S^3 with the implicit equation $x^2 + y^2 + z^2 + w^2 = r^2$ can be parametrized in many ways, from which that one will be chosen that is the extension of the geographic spherical coordinates from $\mathbb{R}^3$ to $\mathbb{R}^4$:

$$\begin{aligned} x(r,t,u) &= r\cos t\cos u \\ y(r,t,u) &= r\sin t\cos u \\ z(r,u) &= r\sin u \end{aligned} \quad \mapsto \quad \begin{aligned} x(r,t,u,v) &= r\cos t\cos u\cos v \\ y(r,t,u,v) &= r\sin t\cos u\cos v \\ z(r,u,v) &= r\sin u\cos v \\ w(r,v) &= r\sin v \end{aligned}$$

where $r = 1$ is the radius, t the longitude obeying $0 \le t < 2\pi$; u is the classical latitude with $-\pi/2 \le u \le \pi/2$, v the additional new latitude with $-\pi/2 \le v \le \pi/2$.

The parametrization of the sphere $S^2 \in \mathbb{R}^2$ has a singular subsphere S^0, i.e., the two poles $v = \pm\pi/2$. The parametrization of the hypersphere S^3 has a singular subsphere S^1, i.e., the circle $\kappa: \ x = y = 0, \ z^2 + w^2 = 1$. Excluding this singularity of our parametrization, we receive:

$$v = \arcsin w\,, \qquad u = \arcsin \frac{z}{\sqrt{1-w^2}}\,, \qquad t = \arg(x + \mathrm{i}y).$$

A *spherical motion* $\mathcal{O}$ in $\mathbb{R}^4$ keeping invariant the origin $O = (0,0,0,0)$ is represented by an orthogonal matrix Q with $\det \mathrm{Q} = 1$. This matrix Q is an element of the group $O^+(4)$ (see [1]). Similarly, the group of revolutions in $\mathbb{R}^3$ is represented by the group of orthogonal matrices of degree 3 with determinant 1 and denoted by $O^+(3)$.

An orthogonal matrix Q can be obtained from the Jacobian matrix of the transformation $(r,t,u,v) \mapsto (x,y,z,w)$ in $\mathbb{R}^4$, i.e., from the partial derivatives of the vector

$$[r\cos t\cos u\cos v,\ r\sin t\cos u\cos v,\ r\sin u\cos v,\ r\sin v]$$

by normalizing:

$$\mathrm{Q} = \begin{pmatrix} \cos t\cos u\cos v & \sin t\cos u\cos v & \sin u\cos v & \sin v \\ -\sin t & \cos t & 0 & 0 \\ -\cos t\sin u & -\sin t\sin u & \cos u & 0 \\ -\cos t\cos u\sin v & -\sin t\cos u\sin v & -\sin u\sin v & \cos v \end{pmatrix}.$$

It is easy to prove, that $\mathrm{Q}^T = \mathrm{Q}^{-1}$ and $\mathrm{Q} \in O^+(4)$.

At parallel projections from $\mathbb{R}^3$ to $\mathbb{R}^2$ usually the image of the last coordinate axis is specified as a vertical line in the projection plane. This can be assured by excluding the row with the partial derivatives with respect to r in the matrix of the projection. After erasing the first row in Q the matrix

$$\mathrm{P} = \begin{pmatrix} -\sin t & \cos t & 0 & 0 \\ -\cos t\ \sin u & -\sin t\ \sin u & \cos u & 0 \\ -\cos t\ \cos u\ \sin v & -\sin t\ \cos u\ \sin v & -\sin u\ \sin v & \cos v \end{pmatrix}$$

of an orthographic projection $\mathbb{R}^4 \to \mathbb{R}^3$ onto the hyperplane passing through the origin O is obtained.

The rows of the matrix P are basic vectors of the tangent space of the hypersphere S^3 at an arbitrary point determined by parameters (t,u,v). The image hyperplane $\mathcal{R}$ is in general position, parallel to this tangent hyperplane of S^3. Coordinates of the orthographic views will be denoted by upper asterixes,

$$[x^*, y^*, z^*]^T = \mathrm{P} \cdot [x, y, z, w]^T\,.$$

The sum of squared norms of the columns in the matrix Q equals 4. The columns in the matrix P determine the orthographic views of four unit vectors along the coordinate axes $+x$, $+y$, $+z$, $+w$. The sum of squared norms of the columns of P is 3. This is a well-known property when the unit points of a cartesian frame are mapped under an orthogonal projection; this result can be found in [8, 5, 7]. Using the matrix Q instead of P, the first additional coordinate will determine the oriented distance of the projected point in $\mathbb{R}^4$ to the image hyperplane $\mathcal{R}$ passing through O.

Omitting the visibility, the parametrized hypersphere S^3 is projected orthogonally to the ball in $\mathbb{R}^3$ with two-parametric nets of space curves. Iso-parametric t-curves on S^3 are circles or points (for $u = \pm\pi/2$). These points are located in the plane (z, w) on the unit circle κ with the centre in the origin. Iso-parametric u-curves are semi-circles or the points S, J (for $v = \pm\pi/2$), the North and South Pole located on the axis w. The planes of the arcs are perpendicular to the axis w. Iso-parametric v-curves are semi-circles with the end points in S and J.

The 2-surface $v = const.$, a sphere S^2, is projected to the ellipsoid, that can be also contracted to the point S^* or J^*. The 2-surface $u = const.$ is projected to the "garlic pod" (closed surface) with point S^*, J^* that can appear in the form of an ellipsoid ($u = 0$) transformed contractively up to a semi-ellipse ($u = \pm\pi/2$). The 2-surface $t = const.$, a semi-sphere, is projected to a semi-ellipsoid.

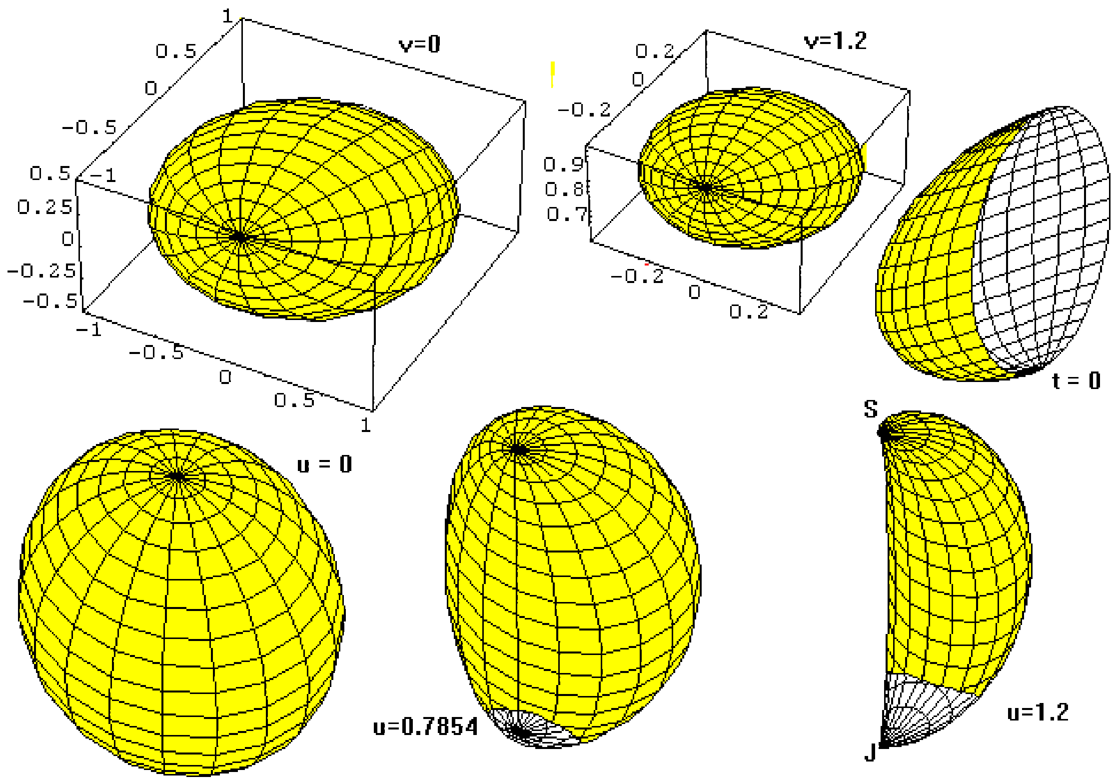

Figure 6: Orthographic views of parametric surfaces on the hypersphere S^3

In Fig. 6 orthographic views of 2-surfaces $\subset S^3$ defined by the constant parameters $v = 0$ and $v = 1.2$, $t = 0$ and $u = 0, u = 0.7854, u = 1.2$ are displayed. Orthographic views of a 2-surface patch determined by parametric equations in the form

$$x = \cos u \cos v,\ y = \sin u \cos v,\ z = \sin v,\ w = v, \quad -\pi/2 \le u, v \le \pi/2,$$

are presented in Fig. 7. Different orthographic projections were derived from the spherical motion determined by omitting one row in the randomly defined matrix $\mathrm{Q} \in O^+(4)$,

$$\mathrm{Q} = \begin{pmatrix} -0.699441 & -0.232487 & -0.657456 & 0.156472 \\ -0.628437 & 0.029543 & 0.520789 & -0.577038 \\ -0.317978 & 0.707540 & 0.228128 & 0.588417 \\ -0.121403 & -0.666681 & 0.494458 & 0.544343 \end{pmatrix}.$$

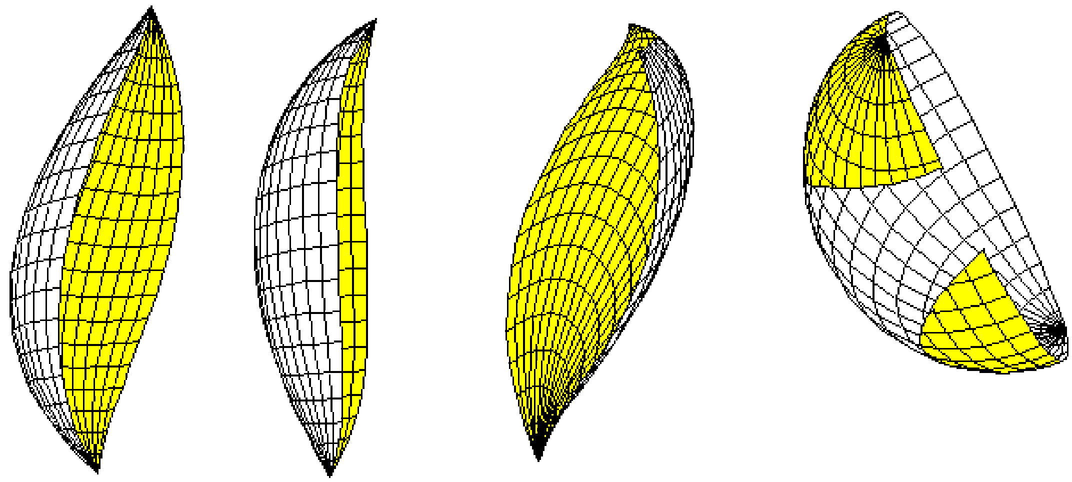

Figure 7: Different orthographic views of the same surface patch

The views in Fig. 7 from left to right are obtained from Q by cancelling the first, the second, the third, or finally the last row, respectively.

Any helix in $\mathbb{R}^3$ is a non-self-intersecting curve, but its orthographic view to the plane is a prolonged cycloidal curve which can be self-intersecting. Similarly, a 2-surface in $\mathbb{R}^4$ which is not self-intersecting can have a self-intersecting orthogonal view in $\mathbb{R}^3$.

Let us inscribe into the hypersphere S^3 a regular *cross-polytope* with vertices $(\pm 1, 0, 0, 0)$, $\ldots, (0, 0, 0, \pm 1)$, the dual to the hypercube. There are four triples of axes, and any triple generates a regular octahedron that is inscribed into a sphere $S_i^2 \subset S^3$, $i = 1, \ldots, 4$. In this way we obtain on the hypersphere S^3 four congruent concentric spheres S_i^2, $i = 1, \ldots, 4$, any two of them intersect orthogonally in a concentric circle with the same radius. The spheres S_i^2 intersect the hypersphere S^3 in 16 curve-like tetrahedra, precisely in 16 homeomorphic images of a tetrahedron that compose the hypersphere S^3. Four "principal" spheres S_i^2 are projected by the orthographic projection $\mathcal{P}\colon \mathbb{R}^4 \to \mathbb{R}^3$ onto four ellipsoids that are inscribed into the "outline" (i.e. a sphere S^2) in $\mathbb{R}^3$, and then projected by a perspective, as it can be seen in Fig. 8. Conjugate diameters of these ellipsoids are three lines from four diagonals of the regular cross-polytope.

The central projection from the origin O onto the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ results in the determination of well-known barycentric coordinates in $\mathbb{R}^3$. This hyperplane with the normal vector $\mathbf{n} = (1, 1, 1, 1)$ can be regarded as the image plane for an orthogonal axonometry. The unit normal vector is

$$\mathbf{n}_1 = (1/2,\ 1/2,\ 1/2,\ 1/2), \text{ while} \quad \sin v = \frac{1}{2}, \quad \cos v = \sqrt{\frac{3}{2}}.$$

The unit vector of the orthographic view of $\mathbf{n}_1$ in the plane (x_1, x_2, x_3) is

$$\mathbf{n}_{11} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), \text{ and then} \quad \sin u = 1/\sqrt{3}, \quad \cos u = 2/\sqrt{3}.$$

The unit vector of the orthographic view of $\mathbf{n}_{11}$ in the plane (x_1, x_2) is

$$\mathbf{n}_{111} = (1/\sqrt{2}, 1/\sqrt{2}), \text{ and then} \quad \sin t = 1/\sqrt{2}, \quad \cos t = \sqrt{3}/2.$$

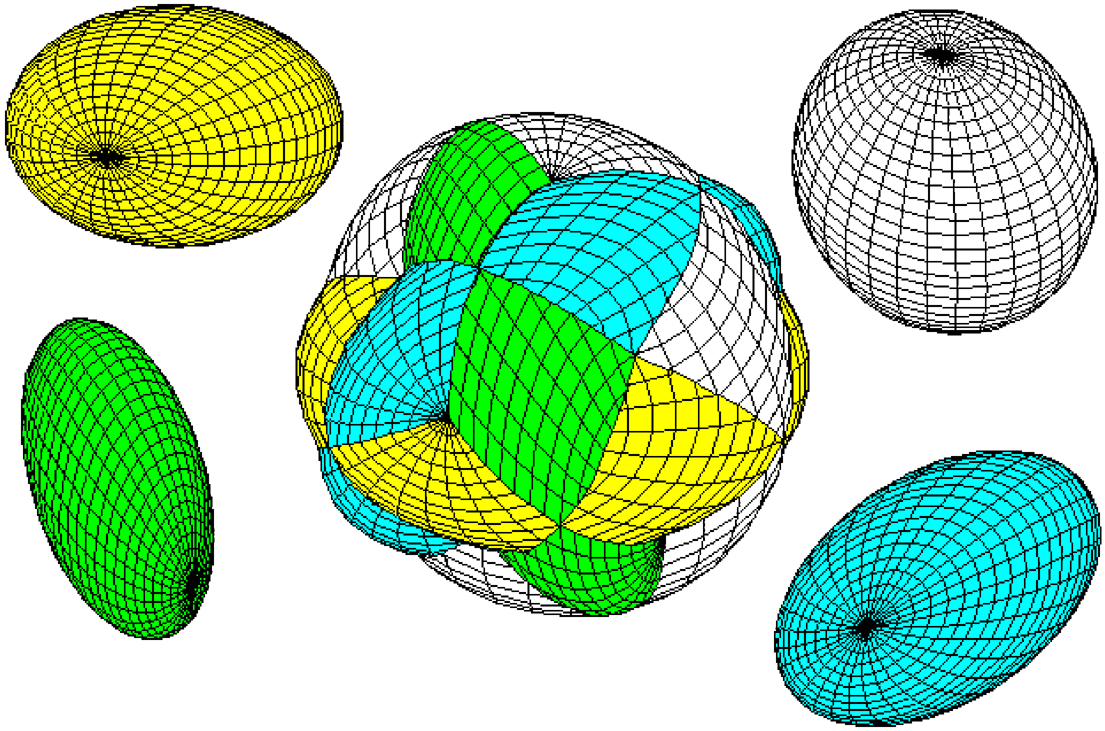

Figure 8: Four concentric spheres pairwise intersecting in circles

The matrix of the orthogonal isometry $\mathbb{R}^4 \to \mathbb{R}^3$ is therefore in the form

$$\mathrm{P} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2/3} & 0 \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & \sqrt{3}/2 \end{pmatrix}.$$

Isometric views of the n-space have also been studied in [4].

Another parametrization of the hypersphere S^3 of radius r,

$$\overline{x} = r\cos u \cos t, \quad \overline{y} = r\cos u \sin t, \quad \overline{z} = r \sin u \cos v, \quad \overline{w} = r \sin u \sin v$$

for $0 \le t < 2\pi$, $0 \le u \le \pi/2$, $0 \le v < 2\pi$ produces a matrix

$$\overline{\mathrm{Q}} = \begin{pmatrix} \cos u \cos t & \cos u \sin t & \sin u \cos v & \sin u \sin v \\ -\sin t & \cos t & 0 & 0 \\ -\sin u \cos t & -\sin u \sin t & \cos u \cos v & \cos u \sin v \\ 0 & 0 & -\sin v & \cos v \end{pmatrix}.$$

4. Quaternions

The space $\mathbb{R}^4 = \mathbb{C}^2$ can be regarded as the set $\mathbb{H}$ of *quaternions*. Quaternions were defined by W. Hamilton in the year 1843. They are a generalization of complex numbers $\mathbb{R}^2 = \mathbb{C}$, where the real (scalar) part $\mathrm{Re}\,\mathbb{H}$ remains in $\mathbb{R}$ and a new imaginary part (vector part, pure quaternion) $\mathrm{Im}\,\mathbb{H}$ in $\{\mathbb{R}^3 \setminus \mathbf{O}\}$ with three axes $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is introduced, i.e. for $a_1, \dots, a_4 \in \mathbb{R}$

$$\mathbf{a} := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k} \in \mathbb{H}, \quad \mathrm{Re}\,\mathbf{a} = a_1, \quad \mathrm{Im}\,\mathbf{a} = a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}.$$

The sum of two quaternions $\mathbf{a}+\mathbf{b}$ can be regarded as the standard vector sum. The multiplication of quaternions is associative and satisfies the distributive laws and the relations

$$\mathbf{ii}=\mathbf{jj}=\mathbf{kk}=-1, \quad \mathbf{ij}=-\mathbf{ji}=\mathbf{k}, \quad \mathbf{jk}=-\mathbf{kj}=\mathbf{i}, \quad \mathbf{ki}=-\mathbf{ik}=\mathbf{j}.$$

Hence

$$\mathbf{ab}=(a_1+a_2\mathbf{i}+a_3\mathbf{j}+a_4\mathbf{k})(b_1+b_2\mathbf{i}+b_3\mathbf{j}+b_4\mathbf{k})=$$

$$(a_1b_1-a_2b_2-a_3b_3-a_4b_4)+(a_2b_1+a_1b_2-a_4b_3+a_3b_4)\,\mathbf{i}+$$
$$+(a_3b_1+a_4b_2+a_1b_3-a_2b_4)\,\mathbf{j}+(a_4b_1-a_3b_2+a_2b_3+a_1b_4)\,\mathbf{k}.$$

It follows from these properties that the product of two pure quaternions from $\operatorname{Im}\mathbb{H}$, i.e. $a_1=b_1=0$, can be expressed in the cartesian coordinates as the difference of the vector product $\mathbf{a}\times\mathbf{b}$ minus the scalar product $\mathbf{a}\cdot\mathbf{b}$:

$$\begin{gathered}(a_2\mathbf{i}+a_3\mathbf{j}+a_4\mathbf{k})\,(b_2\mathbf{i}+b_3\mathbf{j}+b_4\mathbf{k})=\\ =-a_2b_2-a_3b_3-a_4b_4+(a_3b_4-a_4b_3)\mathbf{i}-(a_2b_4-a_4b_2)\mathbf{j}+(a_2b_3-a_3b_2)\mathbf{k}=\\ =(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\cdot\mathbf{b}).\end{gathered}$$

If $\mathbf{c}\in\operatorname{Im}\mathbb{H}$, then $\mathbf{c}^2<0$.

In the terms of algebra, the quaternions form an associative non-commutative field. With respect to the Frobenius theorem (1877), quaternions form the unique associative non-commutative finitely dimensional algebra with a unit element and without zero divisors. It is interesting to point out that both (scalar and vector) products $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ were born historically in the theory of quaternions.

Conjugate quaternions and their norms satisfy the following formulae:

$$\bar{\mathbf{a}}:=a_1-a_2\mathbf{i}-a_3\mathbf{j}-a_4\mathbf{k}=\operatorname{Re}\mathbf{a}-\operatorname{Im}\mathbf{a}, \quad \overline{\mathbf{a}\,\mathbf{b}}=\bar{\mathbf{b}}\,\bar{\mathbf{a}},$$

$$\|\mathbf{a}\|=\sqrt{\mathbf{a}\,\bar{\mathbf{a}}}, \quad \mathbf{a}^{-1}=\frac{\bar{\mathbf{a}}}{\|\mathbf{a}\|^2} \text{ for } \mathbf{a}\neq 0.$$

The product $\mathbf{ab}$ of quaternions $\mathbf{a}$, $\mathbf{b}$ can be expressed in several matrix forms:

$$\mathbf{ab}=\begin{pmatrix}1 & \mathbf{i} & \mathbf{j} & \mathbf{k}\end{pmatrix}\begin{pmatrix}a_1 & -a_2 & -a_3 & -a_4\\ a_2 & a_1 & -a_4 & a_3\\ a_3 & a_4 & a_1 & -a_2\\ a_4 & -a_3 & a_2 & a_1\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ b_3\\ b_4\end{pmatrix}=$$

$$=\begin{pmatrix}1 & \mathbf{i} & \mathbf{j} & \mathbf{k}\end{pmatrix}\mathrm{A}^{\triangleright}\begin{pmatrix}b_1\\ b_2\\ b_3\\ b_4\end{pmatrix}=$$

$$=\begin{pmatrix}a_1 & a_2 & a_3 & a_4\end{pmatrix}\begin{pmatrix}b_1 & b_2 & b_3 & b_4\\ -b_2 & b_1 & -b_4 & b_3\\ -b_3 & b_4 & b_1 & -b_2\\ -b_4 & -b_3 & b_2 & b_1\end{pmatrix}\begin{pmatrix}1\\ \mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix}=$$

$$=\begin{pmatrix}a_1 & a_2 & a_3 & a_4\end{pmatrix}{}^{\triangleleft}\mathrm{B}\begin{pmatrix}1\\ \mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix}.$$

For quaternions from $\mathrm{Im}\,\mathbb{H}$ the matrices $\mathrm{A}^{\triangleright}$ and ${}^{\triangleleft}\mathrm{B}$ are antisymmetric. For unit quaternions the matrices $\mathrm{A}^{\triangleright}$ and ${}^{\triangleleft}\mathrm{B}$ are orthogonal and their determinant is equal 1; they are elements of the group $O^+(4)$.

In analogy to the fact that the complex number $b_1 + b_2\mathrm{i}$ corresponds homomorphically to the real matrix

$$\begin{pmatrix} b_1 & b_2 \\ -b_1 & b_1 \end{pmatrix},$$

Cayley showed 1858 that the quaternion $b_1 + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k}$ corresponds to the complex matrix (see [2])

$$\begin{pmatrix} b_1 + b_2\mathrm{i} & b_3 + b_4\mathrm{i} \\ -b_3 + b_4\mathrm{i} & b_1 - b_2\mathrm{i} \end{pmatrix}.$$

In this sense the imaginary unit $\mathrm{i} \in \mathbb{C}$ corresponds to the unit $\mathbf{j} \in \mathbb{H}$.

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \leftrightarrow \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} \leftrightarrow \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

In this model the quaternion $\mathbf{b}$ clearly refers to the real matrix ${}^{\triangleleft}\mathrm{B}$.

J.W. Gibbs is the founder of the vector analysis without quaternions, which is today widely used.

5. Spherical motions and orthographic projections

A spherical motion $\mathcal{O}$ fixing the origin $O \in \mathbf{R}^4$ can be expressed by an orthogonal matrix M with $\det \mathrm{M} = 1$, which is regarded as an element of the group $O^+(4)$. Compared with the group of revolutions $O^+(3)$ that is simple, the group $O^+(4)$ has a non-trivial normal subgroup (see [1]).

Any pair of unit quaternions $(\mathbf{r}, \mathbf{s})$ defines an element $\mathcal{O}$ of the group $O^+(4)$ according to

$$\mathcal{O}\colon\ \mathbb{H} \to \mathbb{H},\ \mathbf{q} \mapsto \mathbf{r}\,\mathbf{q}\,\overline{\mathbf{s}}\,.$$

The converse statement is also valid (see [1]) which means, that for any $\mathcal{O} \in O^+(4)$ there exist two unit quaternions $\mathbf{r}$, $\mathbf{s}$ generating this spherical motion. The pairs $(\mathbf{r}, \mathbf{s}) = (1, 1)$ and $(-1, -1)$ correspond to the identity $\mathrm{Id}_{\mathbb{R}^4}$ in $O^+(4)$.

The multiplication in the group $O^+(4)$ can be expressed by means of the quaternion product: Let two elements from $O^+(4)$ correspond to two pairs $(\mathbf{r}, \mathbf{s})$ and $(\mathbf{r}', \mathbf{s}')$ of unit quaternions, respectively: The product of these pairs is the pair $(\mathbf{r}\,\mathbf{s},\ \mathbf{r}'\mathbf{s}')$, because the following holds:

$$\mathbf{q}\ \mapsto\ \mathbf{r}\mathbf{r}'\mathbf{q}\,\overline{\mathbf{s}\mathbf{s}'} = \mathbf{r}(\mathbf{r}'\mathbf{q}\,\overline{\mathbf{s}'})\overline{\mathbf{s}}.$$

Comparing quaternion and matrix expression of the elements of $O^+(4)$ we have

$$\mathbf{r}(\mathbf{q}\,\overline{\mathbf{s}}) = \begin{pmatrix} 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} \mathrm{R}^{\triangleright}\,{}^{\triangleleft}\overline{\mathrm{S}}^{T} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} \mathrm{M}^{\triangleright} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix},$$

$$\mathrm{M}^{\triangleright} = \mathrm{R}^{\triangleright}\,{}^{\triangleleft}\overline{\mathrm{S}}^{T} = R^{\triangleright}\,{}^{\triangleleft}S =$$

$$= \begin{pmatrix} r_1 & -r_2 & -r_3 & -r_4 \\ r_2 & r_1 & -r_4 & r_3 \\ r_3 & r_4 & r_1 & -r_2 \\ r_4 & -r_3 & r_2 & r_1 \end{pmatrix} \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4 & -s_3 & s_2 & s_1 \end{pmatrix}.$$

For $\mathbf{r} = \mathbf{s}$ we receive the classical revolution in $\operatorname{Im}\mathbb{H}$:

$$\mathrm{M}^{\triangleright} = \mathrm{R}^{\triangleright\triangleleft}\mathrm{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2(r_3^2+r_4^2) & 2(r_2r_3 - r_1r_4) & 2(r_1r_3 + r_2r_4) \\ 0 & 2(r_2r_3 + r_1r_4) & 1-2(r_2^2+r_4^2) & 2(r_3r_4 - r_1r_2) \\ 0 & 2(r_2r_4 - r_1r_3) & 2(r_1r_2 + r_3r_4) & 1-2(r_2^2+r_3^2) \end{pmatrix} \in O^+(3).$$

It is easy to show, that for $r_1 = q_1 = 0$ and all $\lambda \in \mathbb{R}$ holds

$$\mathbf{r}\,(\lambda\mathbf{r})\,\overline{\mathbf{r}} = \lambda\mathbf{r}, \quad \operatorname{Re}(\mathbf{r}\,\mathbf{q}\,\overline{\mathbf{r}}) = 0, \quad \mathbf{r}\cdot\mathbf{q} = 0 \implies \mathbf{r}\cdot(\mathbf{r}\,\mathbf{q}\,\overline{\mathbf{r}}) = 0\,.$$

From this follows that the matrix $\mathrm{M}^{\triangleright}$ which is the matrix of a revolution (about the plane determined by the real axis and $\operatorname{Im}\mathbf{r}$) acts only on $\operatorname{Im}\mathbb{H}$. The matrix $(-\mathrm{M}^{\triangleright})$ operates in $\operatorname{Im}\mathbb{H}$ as the symmetry with respect to the plane passing through the origin and perpendicular to $\operatorname{Im}\mathbf{r}$ (proof in [1]).

All unit quaternions form a hypersphere $S^3 \subset \mathbf{R}^4$, and any point $\mathbf{r}$ located on this hypersphere corresponds to an orthogonal matrix $\mathrm{R}^{\triangleright}$ with $\det \mathrm{R}^{\triangleright} = 1$. The quaternion $\mathbf{r}$ will be related to the oriented revolution in $\operatorname{Im}\mathbb{H}$ about the axis determined by $\operatorname{Im}\mathbf{r}$ according to $\mathbf{q} \mapsto \mathbf{r}\,\mathbf{q}\,\overline{\mathbf{r}}$. Instead of the unit quaternion $\mathbf{r}$ it is possible to choose a non-zero quaternion $\mathbf{d}$. Any such revolution is regarded as the element of the group $O^+(3)$. The angle θ of revolution, $0 \le \theta \le \pi$, can be expressed as (see [1] or [3])

$$\theta = 2\,\arctan\left(\frac{\|\operatorname{Im}\mathbf{r}\|}{|\operatorname{Re}\mathbf{r}|}\right) = 2\,\arctan\left(\frac{\|\operatorname{Im}\mathbf{d}\|}{|\operatorname{Re}\mathbf{d}|}\right).$$

$\operatorname{Re}\mathbf{r} = r_1 = 0$ implies $\theta = \pi$, and we have an axial symmetry in $\operatorname{Im}\mathbb{H}$ with respect to the line $\lambda\mathbf{r}$.

To determine the angle θ in the easier way, we can express, while choosing the unit quaternion as $\mathbf{c} \in \operatorname{Im}\mathbb{H}$, the unit quaternion $\mathbf{r}$ of the revolution $\mathbf{q} \mapsto \mathbf{r}\,\mathbf{q}\,\overline{\mathbf{r}}$ in the form presented in [3]:

$$\mathbf{r} = \cos\frac{\theta}{2} + \mathbf{c}\,\sin\frac{\theta}{2}\,.$$

The group $O^+(3)$ has three free parameters, the same as its model $S^3 \subset \mathbb{H}$.

Limits for the angle θ in the interval $[0,\pi]$ are not essential, because the complementary interval $[\pi, 2\pi]$ can be achieved by a revolution about the axis determined by the vector $(-\mathbf{c})$ for the angles also in the interval $[0,\pi]$. This topological problem in $\mathbf{R}^3$ will be not a problem in the projective space $\mathbf{P}^3$, because this is homeomorphic to $O^+(3)$. The converse statement is also valid (see in [1]), to any revolution from $O^+(3)$ there can be related a non-zero quaternion generating this revolution in $\operatorname{Im}\mathbb{H}$.

Modelling a spherical motion in $\mathbf{R}^4$ enables us to construct different views of 4-dimensional objects in such a way, that the object is firstly "revolved" and then projected into the 3-dimensional space defined by the first three coordinate axes (similarly as in the projection $\mathbb{R}^3 \to \mathbb{R}^2$).

For an orthographic view it is sufficient to exclude one coordinate. There exists a decomposition of the matrix $\mathrm{Q} \in O^+(4)$ (see Section 1), $\mathrm{Q}^{\triangleright} = \mathrm{R}^{\triangleright\triangleleft}\mathrm{S}$, anyhow the construction

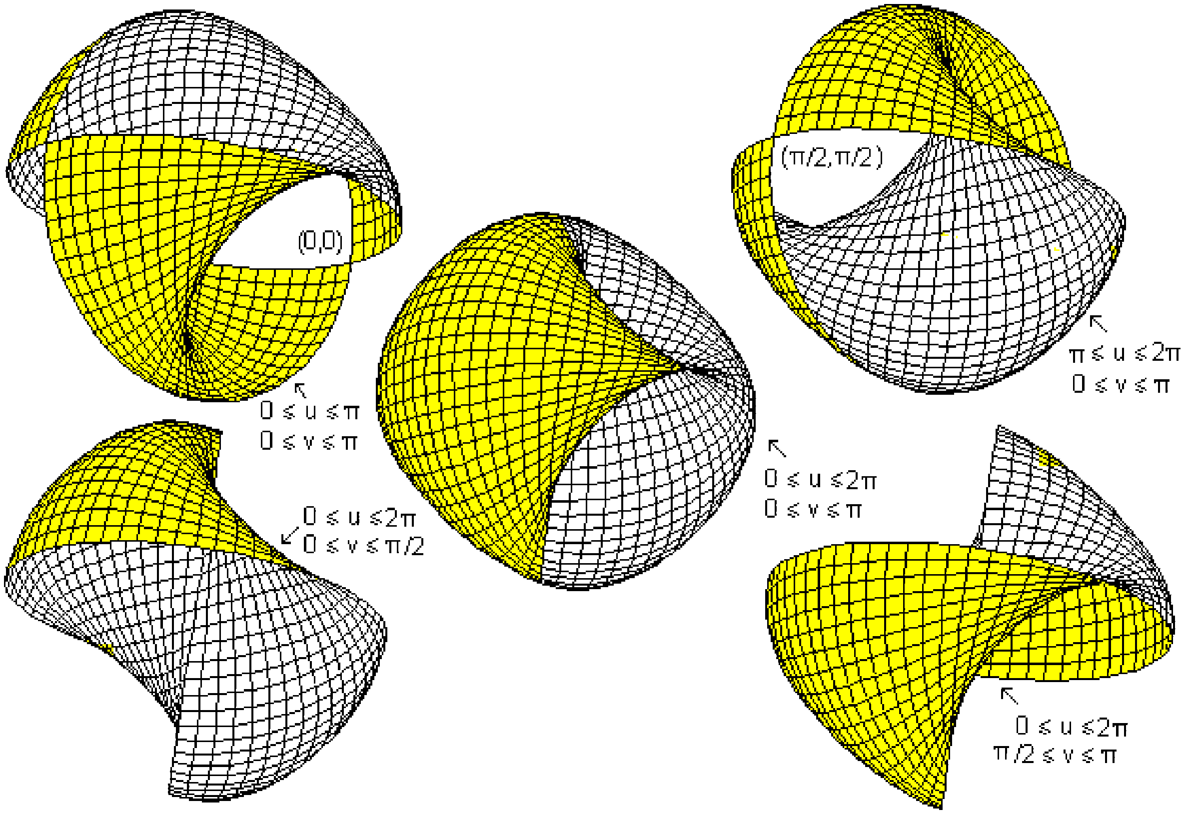

Figure 9: Images of a hypersphere S^3 under projections $\mathbb{R}^4 \to \mathbb{R}^3$

of matrices $\mathrm{R}^{\triangleright}$, ${}^{\triangleleft}\mathrm{S}$ is a different problem. On the other hand, with the help of the random generator of quaternions $\mathbf{r}$, $\mathbf{s}$ it is easy to create a random matrix in $O^+(4)$ as the product $(\mathrm{R}^{\triangleright}{}^{\triangleleft}\mathrm{S})$. Excluding one of the rows the random matrix $\mathcal{P}$ of a projection from $\mathbb{R}^4$ to $\mathbb{R}^3$ can be obtained. This projection includes also the revolution of the orthographic view in $\mathbb{R}^3$, when the orthographic views of the axis $\mathbf{k}$ are not necessarily coincident with the z-axis.

Several examples are illustrated in Fig. 9, where the spherical motion is determined by the quaternion $\mathbf{a}' = \mathbf{r}\,\mathbf{a}\,\bar{\mathbf{s}}$, while

$$\mathbf{r} = \cos u + \mathbf{i} \sin u, \quad \mathbf{s} = \mathbf{j} \cos v + \mathbf{k} \sin v, \quad \mathbf{a} = 0.5 + 0.5\mathbf{i} + 0.5\mathbf{j} + 0.5\mathbf{k}\,,$$

and intervals for parameters u, v are given in the figure. The corresponding projection is applied to the hypersphere S^3, and its image is then projected by a linear perspective from $\mathbb{R}^3$ to $\mathbb{R}^2$.

The other possible access is to compose revolutions about six pairs of coordinate axes. Six corresponding angles are in the interval $[0, 2\pi]$. The number of free parameters of the pair $(\mathbf{r}, \mathbf{s})$ with $|\mathbf{r}| = |\mathbf{s}| = 1$ is six. The product of these two revolutions needs not be a classical revolution, but it is a spherical motion.

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}.$$

In general the last matrix has not two real eigenvectors that would determine a plane, about which the classical revolution could be performed.

References

[1] M. Berger: *Géométrie*. CEDIC and Fernand Nathan, Paris 1977.

[2] B.A. Dubrovin, S.P. Novikov, A.T. Fomenko: *Sovremennaya geometriya, metody i priloženiya*. Nauka, Moskva 1986.

[3] J.C. Hart, G.K. Francis, L.H. Kauffman: *Visualizing Quaternion Rotation*. ACM Transactions on Graphics **13** (3), 256–276 (1994).

[4] G.C. Laghi: *About the Contour Shape of a Hypercube in Isometric Axonometry*. Proceedings of the 3rd ICECGDG, Vienna 1988, vol. 1, 300–304.

[5] H. Stachel: *Mehrdimensionale Axonometrie*. Proceedings Congress of Geometry, Thessaloniki 1987, 159–168.

[6] H. Stachel: *The Right-Angle-Theorem in Four Dimensions*. Journal of Theoretical Graphics and Computing **3**, 4–13 (1990).

[7] H. Stachel: *Parallel Projections in Multidimensional Space*. Proceedings Compugraphics ‘91, Sesimbra (Portugal) 1991, vol. I, 119–128.

[8] W. Wunderlich: *Zur normalen Axonometrie des vierdimensionalen Raumes*. Monatsh. Math. **80**, 231–240 (1975).

[9] S. Zachariáš: *Projection in barycentric coordinates*. Proceedings WSCG 96, Plzeň (CZ), 1996.

[10] S. Zachariáš: *Projekce hladkých ploch*. Proceedings Seminars on Computational Geometry SCG’97, Kočovce (SR), 1997, 189–193.

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