Projection from 4D to 3D

Svatopluk Zachariáš\(^1\), Daniela Velichová\(^2\)

\(^1\)Faculty of Applied Sciences, West Bohemian University
Univerzitní 22, CZ 306 14 Plzeň, Czech Republic

\(^2\)Dept. of Mathematics, Mechanical Eng’g Faculty, Slovak Technical University
Nám. slobody 17, SK 812 31 Bratislava, Slovak Republic
email: velichov@sjf.stuba.sk

Abstract. The aim of this paper is to give a survey on analytic representations of central and orthographic projections from \(\mathbb{R}^4\) to \(\mathbb{R}^3\) or \(\mathbb{R}^2\). There are discussed various aspects of these projections, whereby some special relations were revealed, e.g., the fact that homogeneous coordinates or barycentric coordinates in \(\mathbb{R}^3\) can be obtained by applying particular projections on a point with given cartesian coordinates in \(\mathbb{R}^4\). We would also like to demonstrate that by projecting curves or 2-surfaces of \(\mathbb{R}^4\) interesting shapes in \(\mathbb{R}^3\) and \(\mathbb{R}^2\) can be obtained.

Key Words: geometry in 4D, projections, quaternions

MSC 1994: 51N20, 51N05.

1. Introduction

Geometric objects in the space \(\mathbb{R}^4\) can be projected first into the space \(\mathbb{R}^3\) and then into the plane \(\mathbb{R}^2\). We prefer orthographic projections against other parallel projections, as they are an approximation of the central projection with large distance. The advantage of the orthographic projection is a rather good realism in visualization of unknown geometric objects. The basic aspect of a realistic view of smooth surfaces after a projection \(\mathbb{R}^3 \to \mathbb{R}^2\) is to find the outline curve; an algorithm is described in [10].

In the projection \(\mathcal{P} : \mathbb{R}^4 \to \mathbb{R}^3\) those 3-dimensional objects are visible, that in the case of the central projection close to the centre of projection (we restrict our consideration on the projection of only one of the open semi-spaces determined by the hyperplane parallel to the projection plane and incident to the centre of projection). In the case of a parallel projection visible objects are in larger distance from the 3-dimensional projection plane, if these distances are oriented opposite to the rays of sight. Visibility defined in this way will be denoted by \(\mathcal{W}^3\).

The visibility \(\mathcal{W}^4\) in the projection \(\mathcal{P}\) is different from the visibility \(\mathcal{W}^3\) in the space \(\mathbb{R}^3\) that is applied on any projection \(\mathbb{R}^3 \to \mathbb{R}^2\). E.g., in the projection of a simplex \(\mathcal{S}^4 \subset \mathbb{R}^4\)
we suppose that the centre of projection is an exterior point (for parallel projections this condition is always satisfied). A simplex $S^4$ with vertices $A_1, \ldots, A_5$ is projected onto the convex hull of the five image points $A^*_1, \ldots, A^*_5$. If one of these points $A^*_i$ is an interior point of the tetrahedron formed by the remaining four image points $A^*_k$, then the four edges passing through $A^*_i$ are non-visible in the visibility $W^3$, but these points are visible in the visibility $W^4$. If any point $A^*_i$, $i = 1, \ldots, 5$ is an exterior point of the tetrahedron determined by the remaining four points, then one and only one edge is non-visible in both visibilities $W^4$ and $W^3$. If one point $A^*_i$ is located on one edge of the tetrahedron of the remaining $A^*_k$, then all edges are visible in the visibility $W^3$.

Generally, there is no chance to define a visibility when any $R^4$-object is projected into $R^2$. The reason is that the "rays of sight" are planes, and for any two points in a plane one cannot define that one point "hides" the other. Such a remark can be found also in [6].

Visibility $W^4$ is suitable for enlightening the space $R^4$. In the figures included in the paper the visibility $W^3$ was applied, as it is easier to realize in the projection plane $R^2$.

2. Central projections, modelling curves and surfaces

2.1. Central projections

Let $V$ be a curve or a 2-surface in the space $R^{n+1}$, $n > 1$. There is a central projection of $R^{n+1}$ from the origin $O$ of the coordinate system onto any hyperplane $R$. Under this projection any point $B = (x_1, \ldots, x_{n+1})$ of the figure $V \setminus \{O\}$ can be connected with $O$ by the line $b = OB$ intersecting the hyperplane $R$ in the image $(x^*_1, \ldots, x^*_n)$ of $B$.

When the equation of the hyperplane $R$ is in the form $x_{n+1} = 1$, we get the same relation as between homogeneous coordinates $(x_1, \ldots, x_{n+1})$ and cartesian coordinates

$$\left( \frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}} \right), \quad x_{n+1} \neq 0$$

doing points of the projective extension $P^n$ of the Euclidean space $R^n$.

When the equation of the hyperplane $R$ is in the form

$$x_1 + x_2 + \ldots + x_n + x_{n+1} = 1,$$

we speak about barycentric coordinates in $R^n$

$$x^b_i := \frac{x_i}{x_1 + \ldots + x_{n+1}}, \quad i = 1, \ldots, n,$$

and we even have to assume that no point of the figure $V$ is located in the hyperplane $x_1 + \ldots + x_{n+1} = 0$ parallel to $R$. In this situation we do not speak of the projective space $P^n$, but of barycentric coordinates in $R^n$.

2.2. Modelling curves and surfaces

Any parabola in $R^{n+1}$, $n > 1$, can be easily determined by parametric equations using quadratic polynomials like

$$x_i(t) := a_{i,2} t^2 + a_{i,1} t + a_{i,0}, \quad -\infty < t < \infty.$$
It is clear that generally a conic section in the projective extension of $\mathbb{R}^n$ can be determined by the ratio of quadratic polynomials $x_i(t)$ in the form

$$x_i^*(t) = \frac{x_i(t)}{x_{n+1}(t)}, \quad i = 1, \ldots, n.$$ 

In addition to the presented polynomial representation of a parabola, any ellipse in $\mathbb{R}^{n+1}$ can be represented by the equations

$$x_i(t) = a_i \cos t + b_i \sin t + c_i, \quad 0 \leq t < 2\pi, \quad a_i, b_i, c_i \in \mathbb{R} \quad \text{for} \quad i = 1, \ldots, n+1,$$

any hyperbola by

$$x_i(t) = \pm a_i \cosh t + b_i \sinh t + c_i, \quad -\infty < t < \infty, \quad a_i, b_i, c_i \in \mathbb{R} \quad \text{for} \quad i = 1, \ldots, n+1$$
or an ellipse in the exponential form

$$x_i(t) = a_i \exp(it) + \pi_i \exp(-it) + c_i, \quad -\infty < t < \infty, \quad a_i, \pi_i, c_i \in \mathbb{C} \quad \text{for} \quad i = 1, \ldots, n+1,$$
or a hyperbola in the exponential form

$$x_i(t) = \pm a_i \exp t + b_i \exp(-t) + c_i, \quad -\infty < t < \infty, \quad a_i, b_i, c_i \in \mathbb{R} \quad \text{for} \quad i = 1, \ldots, n+1.$$

Generally, a conic section in $\mathbb{R}^{n+1}$ can be represented by a linear vector combination of different basic functions

$$\{1, t, t^2\}, \quad \{1, \sin t, \cos t\}, \quad \{1, \exp t, \exp(-t)\},$$

and so on. In the space $\mathbb{R}^n$ we get the corresponding "rational" functions.

Quite a wide variety of curves in $\mathbb{R}^3$ that are useful for technical applications can be determined with the basis $\{1, t, t^2, t^3\}$. These are curves generated from cubic curves in $\mathbb{R}^4$, while the vector coefficients can be four linearly independent vectors in $\mathbb{R}^4$.

Any affine transformation $\mathbb{R}^n \to \mathbb{R}^n$ or parallel projection $\mathbb{R}^n \to \mathbb{R}^{n-1}$ transforms the control polygon (or net) to the control polygon (or net). An affine transformation $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ or the central projection from the origin $O, \mathbb{R}^{n+1} \to \mathbb{P}^n$ (that is the extension of $\mathbb{R}^n$ by the hyperplane $x_{n+1} = 0$), transforms the control polygon $\{Q_j | j = 1, \ldots, k\} \subset \mathbb{R}^{n+1}$ onto the polygon $\{Q_j^* | j = 1, \ldots, k\} \subset \mathbb{P}^n$. If all vertices $Q_j^*$ of the polygon are real points of the projective extension of the space $\mathbb{R}^n$, then the function coefficient at the vertex $Q_j^*$ will be of the form

$$f_j^* = \frac{f_j}{\sum_j f_j Q_{j,n+1}},$$

provided the function coefficients $f_j, j = 1, \ldots, k$, are linear combinations of polynomial functions in the basis $\{1, t, t^2, t^3\}$, and the coordinates of the control point $Q_j$ are denoted as

$$Q_j = (Q_{j,1}, \ldots, Q_{j,n+1}).$$

The situation is a bit more complicated at the transition to barycentric coordinates: The representation of the point $B \in \mathbb{R}^{n+1}$ in terms of the control polygon $\{Q_j | j = 1, \ldots, k\}$

$$B = \sum_{j=1}^k f_j Q_j \quad \text{will be replaced by} \quad B_i^* = \frac{\sum_j f_j Q_{j,i}}{\sum_{j,m} f_j Q_{j,m}}, \quad i = 1, \ldots, n+1.$$
Let the control polygon \( \{Q_1, \ldots, Q_k\} \) be a simplex in \( \mathbb{R}^{n+1} \), i.e., \( k = n+1 \), then the vertex \( Q_1 \) can be associated to the \((n+1)\)-tuple \((1,0,\ldots,0)\), and vertex \( Q_{n+1} \) to the \((n+1)\)-tuple \((0,\ldots,0,1)\). For \( k > n + 1 \) the vertices \( Q_i \in \mathbb{R}^{n+1}, i = 1, \ldots, k \), can be regarded as the parallel views of the vertices of any simplex in \( \mathbb{R}^{k-1} \).

The practical advantage of the determination of barycentric coordinates for the control polygons is that all barycentric coordinates of any point \( B \) are positive numbers, if and only if the point is located inside the simplex. In connection to the parallel projection of the simplex, the following statement is valid: If all coefficients determining the point \( B \) with respect to the control polygon are positive and the sum of them equals 1, then \( B \) is located inside the convex hull of the control polygon.

2-surfaces in \( \mathbb{R}^4 \) can sometimes be modelled as the graph of any complex function \( f(z) \) in one complex variable \( z = x + iy \). This gives for \( f(z) = u(x, y) + iv(x, y) \)

\[
x_1 = \text{Re} z, \quad x_2 = \text{Im} z, \quad x_3 = u = \text{Re} f(z), \quad x_4 = v = \text{Im} f(z).
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Central views of 2-surfaces defined by complex functions}
\end{figure}

In Fig. 1 central views of graphs of the functions \( f(z) = 4 + i + z^2 \) (on the left) and \( f(z) = 4 + z^2 \) (on the right) under the central projection from the origin \((0,0,0,0)\) onto the hyperplane \( x_4 = 1 \) are displayed.

2-surfaces in \( \mathbb{R}^4 \) can also be determined by basic functions

\[
\{1, \sin u, \cos u, \sin v, \cos v\}, \quad 0 \leq u, v \leq 2\pi.
\]

When \( x_4(u, v) \) is sufficiently far from zero, we receive in \( \mathbb{R}^3 \) a closed torus-like surface. Some examples are shown in the Figures 2 and 3. The orthographic view of the surface defined by the parametric equations

\[
w = 8 + \cos u + 2\sin u + \cos v + \sin v
\]
Figure 2: Central views of torus-like surfaces

Figure 3: Central views of 2-surfaces defined by linear rational functions

\[ x = \frac{1}{w}, \quad y = \frac{(2 + \sin u)}{w}, \quad z = \frac{0.5 \cos v}{w}, \quad 0 \leq u, v < 2\pi \]

is presented in Fig. 4. The outline of the orthographic view is shown on the left, the net of isoparametric curves is displayed on the right.

In the space \( \mathbb{R}^4 \) with the coordinates \( x_1, x_2, x_3, x_4 \) two tori can share no more than two meridian circles. Let one of them be located in the hyperplane \( x_4 = 0 \). Both tori in \( \mathbb{R}^4 \) can be projected from the point \( (0, 0, 0, 0) \) to the hyperplane \( x_4 = 0 \) (see Fig. 5).
3. Spherical coordinates and orthographic projections

In $\mathbb{R}^4$ the norm of a vector is the $l^2$-norm

$$\|(x, y, z, w)\| := \sqrt{x^2 + y^2 + z^2 + w^2}.$$  

The hypersphere $S^3$ with the implicit equation $x^2 + y^2 + z^2 + w^2 = r^2$ can be parametrized in many ways, from which that one will be chosen that is the extension of the geographic spherical coordinates from $\mathbb{R}^3$ to $\mathbb{R}^4$:

$$
\begin{align*}
x(r, t, u) & = r \cos t \cos u & x(r, t, u, v) & = r \cos t \cos u \cos v \\
y(r, t, u) & = r \sin t \cos u & y(r, t, u, v) & = r \sin t \cos u \cos v \\
z(r, u) & = r \sin u & z(r, u, v) & = r \sin u \cos v \\
w(r, v) & = r \sin v & w(r, v) & = r \sin v
\end{align*}
$$
where \( r = 1 \) is the radius, \( t \) the longitude obeying \( 0 \leq t < 2\pi \); \( u \) is the classical latitude with \(-\pi/2 \leq u \leq \pi/2\), \( v \) the additional new latitude with \(-\pi/2 \leq v \leq \pi/2\). The parametrization of the sphere \( S^2 \in \mathbb{R}^2 \) has a singular subsphere \( S^0 \), i.e., the two poles \( v = \pm\pi/2 \). The parametrization of the hypersphere \( S^3 \) has a singular subsphere \( S^1 \), i.e., the circle \( \kappa: x = y = 0, \ z^2 + w^2 = 1 \). Excluding this singularity of our parametrization, we receive:

\[
v = \arcsin w, \quad u = \arcsin \frac{z}{\sqrt{1 - w^2}}, \quad t = \arg(x + iy).
\]

A spherical motion \( \mathcal{O} \) in \( \mathbb{R}^4 \) keeping invariant the origin \( O = (0,0,0,0) \) is represented by an orthogonal matrix \( Q \) with \( \det Q = 1 \). This matrix \( Q \) is an element of the group \( O^+(4) \) (see [1]). Similarly, the group of revolutions in \( \mathbb{R}^3 \) is represented by the group of orthogonal matrices of degree 3 with determinant 1 and denoted by \( O^+(3) \).

An orthogonal matrix \( Q \) can be obtained from the Jacobian matrix of the transformation \((r, t, u, v) \mapsto (x, y, z, w) \) in \( \mathbb{R}^4 \), i.e., from the partial derivatives of the vector

\[
[r \cos t \cos u \cos v, \ r \sin t \cos u \cos v, \ r \sin u \cos v, \ r \sin v]
\]

by normalizing:

\[
Q = \begin{pmatrix}
\cos t \cos u \cos v & \sin t \cos u \cos v & \sin u \cos v & \sin v \\
-\sin t & \cos t & 0 & 0 \\
-\cos t \sin u & -\sin t \sin u & \cos u & 0 \\
-\cos t \cos u \sin v & -\sin t \cos u \sin v & -\sin u \sin v & \cos v
\end{pmatrix}.
\]

It is easy to prove, that \( Q^T = Q^{-1} \) and \( Q \in O^+(4) \).

At parallel projections from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) usually the image of the last coordinate axis is specified as a vertical line in the projection plane. This can be assured by excluding the row with the partial derivatives with respect to \( r \) in the matrix of the projection. After erasing the first row in \( Q \) the matrix

\[
P = \begin{pmatrix}
-\sin t & \cos t & 0 & 0 \\
-\cos t \sin u & -\sin t \sin u & \cos u & 0 \\
-\cos t \cos u \sin v & -\sin t \cos u \sin v & -\sin u \sin v & \cos v
\end{pmatrix}
\]

of an orthographic projection \( \mathbb{R}^4 \to \mathbb{R}^3 \) onto the hyperplane passing through the origin \( O \) is obtained.

The rows of the matrix \( P \) are basic vectors of the tangent space of the hypersphere \( S^3 \) at an arbitrary point determined by parameters \((t, u, v) \). The image hyperplane \( \mathcal{R} \) is in general position, parallel to this tangent hyperplane of \( S^3 \). Coordinates of the orthographic views will be denoted by upper asterixes,

\[
[x^*, y^*, z^*] = P \cdot [x, y, z, w]^T.
\]

The sum of squared norms of the columns in the matrix \( Q \) equals 4. The columns in the matrix \( P \) determine the orthographic views of four unit vectors along the coordinate axes \(+x, +y, +z, +w\). The sum of squared norms of the columns of \( P \) is 3. This is a well-known property when the unit points of a cartesian frame are mapped under an orthogonal projection; this result can be found in [8, 5, 7]. Using the matrix \( Q \) instead of \( P \), the first additional coordinate will determine the oriented distance of the projected point in \( \mathbb{R}^4 \) to the image hyperplane \( \mathcal{R} \) passing through \( O \).
Omitting the visibility, the parametrized hypersphere $S^3$ is projected orthogonally to the ball in $\mathbb{R}^3$ with two-parametric nets of space curves. Iso-parametric $t$-curves on $S^3$ are circles or points (for $u = \pm \pi/2$). These points are located in the plane $(z, w)$ on the unit circle $\kappa$ with the centre in the origin. Iso-parametric $u$-curves are semi-circles or the points $S$, $J$ (for $v = \pm \pi/2$), the North and South Pole located on the axis $w$. The planes of the arcs are perpendicular to the axis $w$. Iso-parametric $v$-curves are semi-circles with the end points in $S$ and $J$.

The 2-surface $v = \text{const.}$, a sphere $S^2$, is projected to the ellipsoid, that can be also contracted to the point $S^*$ or $J^*$. The 2-surface $u = \text{const.}$ is projected to the “garlic pod” (closed surface) with point $S^*$, $J^*$ that can appear in the form of an ellipsoid ($u = 0$) transformed contractively up to a semi-ellipse ($u = \pm \pi/2$). The 2-surface $t = \text{const.}$, a semi-sphere, is projected to a semi-ellipsoid.

![Figure 6: Orthographic views of parametric surfaces on the hypersphere $S^3$](image)

In Fig. 6 orthographic views of 2-surfaces $\subset S^3$ defined by the constant parameters $v = 0$ and $v = 1.2$, $t = 0$ and $u = 0, u = 0.7854, u = 1.2$ are displayed. Orthographic views of a 2-surface patch determined by parametric equations in the form

$$x = \cos u \cos v, \quad y = \sin u \cos v, \quad z = \sin v, \quad w = v, \quad -\pi/2 \leq u, v \leq \pi/2,$$

are presented in Fig. 7. Different orthographic projections were derived from the spherical motion determined by omitting one row in the randomly defined matrix $Q \in O^+(4)$,

$$Q = \begin{pmatrix}
-0.699441 & -0.232487 & -0.657456 & 0.156472 \\
-0.628437 & 0.029543 & 0.520789 & -0.577038 \\
-0.317978 & 0.707540 & 0.228128 & 0.588417 \\
-0.121403 & -0.666681 & 0.494458 & 0.544343 \\
\end{pmatrix}. $$
The views in Fig. 7 from left to right are obtained from $Q$ by cancelling the first, the second, the third, or finally the last row, respectively.

Any helix in $\mathbb{R}^3$ is a non-self-intersecting curve, but its orthographic view to the plane is a prolonged cycloidal curve which can be self-intersecting. Similarly, a 2-surface in $\mathbb{R}^4$ which is not self-intersecting can have a self-intersecting orthogonal view in $\mathbb{R}^3$.

Let us inscribe into the hypersphere $S^3$ a regular cross-polytope with vertices $(\pm 1, 0, 0, 0), \ldots, (0, 0, 0, \pm 1)$, the dual to the hypercube. There are four triples of axes, and any triple generates a regular octahedron that is inscribed into a sphere $S_i^2 \subset S^3$, $i = 1, \ldots, 4$. In this way we obtain on the hypersphere $S^3$ four congruent concentric spheres $S_i^2$, $i = 1, \ldots, 4$, any two of them intersect orthogonally in a concentric circle with the same radius. The spheres $S_i^2$ intersect the hypersphere $S^3$ in 16 curve-like tetrahedra, precisely in 16 homeomorphic images of a tetrahedron that compose the hypersphere $S^3$. Four "principal" spheres $S_i^2$ are projected by the orthographic projection $P : \mathbb{R}^4 \to \mathbb{R}^3$ onto four ellipsoids that are inscribed into the "outline" (i.e. a sphere $S^2$) in $\mathbb{R}^3$, and then projected by a perspective, as it can be seen in Fig. 8. Conjugate diameters of these ellipsoids are three lines from four diagonals of the regular cross-polytope.

The central projection from the origin $O$ onto the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ results in the determination of well-known barycentric coordinates in $\mathbb{R}^3$. This hyperplane with the normal vector $\mathbf{n} = (1, 1, 1, 1)$ can be regarded as the image plane for an orthogonal axonometry. The unit normal vector is

$$\mathbf{n}_1 = (1/2, 1/2, 1/2), \text{ while } \sin v = \frac{1}{2}, \cos v = \sqrt{\frac{3}{2}}.$$ 

The unit vector of the orthographic view of $\mathbf{n}_1$ in the plane $(x_1, x_2, x_3)$ is

$$\mathbf{n}_{11} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), \text{ and then } \sin u = 1/\sqrt{3}, \cos u = 2/\sqrt{3}.$$ 

The unit vector of the orthographic view of $\mathbf{n}_{11}$ in the plane $(x_1, x_2)$ is

$$\mathbf{n}_{111} = (1/\sqrt{2}, 1/\sqrt{2}), \text{ and then } \sin t = 1/\sqrt{2}, \cos t = \sqrt{3}/2.$$
Figure 8: Four concentric spheres pairwise intersecting in circles

The matrix of the orthogonal isometry $\mathbb{R}^4 \to \mathbb{R}^3$ is therefore in the form

$$ P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2/3} & 0 \\ -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & \sqrt{3}/2 \end{pmatrix}. $$

Isometric views of the $n$-space have also been studied in [4].

Another parametrization of the hypersphere $S^3$ of radius $r$,

$$ \overline{x} = r \cos u \cos t, \quad \overline{y} = r \cos u \sin t, \quad \overline{z} = r \sin u \cos v, \quad \overline{w} = r \sin u \sin v $$

for $0 \leq t < 2\pi$, $0 \leq u \leq \pi/2$, $0 \leq v < 2\pi$ produces a matrix

$$ \overline{Q} = \begin{pmatrix} \cos u \cos t & \cos u \sin t & \sin u \cos v & \sin u \sin v \\ -\sin t & \cos t & 0 & 0 \\ -\sin u \cos t & -\sin u \sin t & \cos u \cos v & \cos u \sin v \\ 0 & 0 & -\sin v & \cos v \end{pmatrix}. $$

4. Quaternions

The space $\mathbb{R}^4 = \mathbb{C}^2$ can be regarded as the set $\mathbb{H}$ of *quaternions*. Quaternions were defined by W. Hamilton in the year 1843. They are a generalization of complex numbers $\mathbb{R}^2 = \mathbb{C}$, where the real (scalar) part $\text{Re} \, \mathbb{H}$ remains in $\mathbb{R}$ and a new imaginary part (vector part, pure quaternion) $\text{Im} \, \mathbb{H}$ in $\{\mathbb{R}^3 \setminus \mathcal{O}\}$ with three axes $\{i, j, k\}$ is introduced, i.e. for $a_1, \ldots, a_4 \in \mathbb{R}$

$$ a := a_1 + a_2 i + a_3 j + a_4 k \in \mathbb{H}, \quad \text{Re} \, a = a_1, \quad \text{Im} \, a = a_2 i + a_3 j + a_4 k. $$
The sum of two quaternions $a + b$ can be regarded as the standard vector sum. The multiplication of quaternions is associative and satisfies the distributive laws and the relations
\[ ii = jj = kk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

Hence
\[ ab = (a_1 + a_2 i + a_3 j + a_4 k)(b_1 + b_2 i + b_3 j + b_4 k) = \\
(a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4) + (a_2 b_1 + a_1 b_2 - a_4 b_3 + a_3 b_4) i + \\
+(a_3 b_1 + a_4 b_2 + a_1 b_3 - a_2 b_4) j + (a_4 b_1 - a_3 b_2 + a_2 b_3 + a_1 b_4) k. \]

It follows from these properties that the product of two pure quaternions from $\text{Im} \mathbb{H}$, i.e. $a_1 = b_1 = 0$, can be expressed in the cartesian coordinates as the difference of the vector product $a \times b$ minus the scalar product $a \cdot b$:
\[ (a_2 i + a_3 j + a_4 k)(b_2 i + b_3 j + b_4 k) = \\
= -a_2 b_2 - a_3 b_3 - a_4 b_4 + (a_3 b_4 - a_4 b_3) i - (a_2 b_4 - a_4 b_2) j + (a_2 b_3 - a_3 b_2) k = \\
= (a \times b) - (a \cdot b). \]

If $c \in \text{Im} \mathbb{H}$, then $c^2 < 0$.

In the terms of algebra, the quaternions form an associative non-commutative field. With respect to the FROBENIUS theorem (1877), quaternions form the unique associative non-commutative finitely dimensional algebra with a unit element and without zero divisors. It is interesting to point out that both (scalar and vector) products $a \cdot b$ and $a \times b$ were born historically in the theory of quaternions.

Conjugate quaternions and their norms satisfy the following formulae:
\[ \overline{a} := a_1 - a_2 i - a_3 j - a_4 k = \text{Re} a - \text{Im} a, \quad \overline{a b} = \overline{b a}, \]
\[ ||a|| = \sqrt{a \overline{a}}, \quad a^{-1} = \frac{\overline{a}}{||a||^2} \text{ for } a \neq 0. \]

The product $ab$ of quaternions $a$, $b$ can be expressed in several matrix forms:
\[ ab = \begin{pmatrix} i & j & k \end{pmatrix} \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\
                   a_2 & a_1 & -a_4 & a_3 \\
                   a_3 & a_4 & a_1 & -a_2 \\
                   a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \begin{pmatrix} b_1 \\
                   b_2 \\
                   b_3 \\
                   b_4 \end{pmatrix} = \\
= \begin{pmatrix} i & j & k \end{pmatrix} ^AB \begin{pmatrix} b_1 \\
                   b_2 \\
                   b_3 \\
                   b_4 \end{pmatrix} = \\
= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\
                   -b_2 & b_1 & -b_4 & b_3 \\
                   -b_3 & b_4 & b_1 & -b_2 \\
                   -b_4 & -b_3 & b_2 & b_1 \end{pmatrix} \begin{pmatrix} 1 \\
                   i \\
                   j \\
                   k \end{pmatrix} = \\
= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \circ B \begin{pmatrix} 1 \\
                   i \\
                   j \\
                   k \end{pmatrix}. \]
For quaternions from $\text{Im} \mathbb{H}$ the matrices $A^p$ and $^cB$ are antisymmetric. For unit quaternions the matrices $A^p$ and $^cB$ are orthogonal and their determinant is equal 1; they are elements of the group $O^+(4)$.

In analogy to the fact that the complex number $b_1 + b_2i$ corresponds homomorphically to the real matrix

$$
\begin{pmatrix}
  b_1 & b_2 \\
  -b_1 & b_1
\end{pmatrix},
$$

CAYLEY showed 1858 that the quaternion $b_1 + b_2i + b_3j + b_4k$ corresponds to the complex matrix (see [2])

$$
\begin{pmatrix}
  b_1 + b_2i & b_3 + b_4i \\
  -b_3 + b_4i & b_1 - b_2i
\end{pmatrix}.
$$

In this sense the imaginary unit $i \in \mathbb{C}$ corresponds to the unit $j \in \mathbb{H}$.

$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $j \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

In this model the quaternion $b$ clearly refers to the real matrix $^cB$.

J.W. GIBBS is the founder of the vector analysis without quaternions, which is today widely used.

5. Spherical motions and orthographic projections

A spherical motion $O$ fixing the origin $O \in \mathbb{R}^4$ can be expressed by an orthogonal matrix $M$ with $\det M = 1$, which is regarded as an element of the group $O^+(4)$. Compared with the group of revolutions $O^+(3)$ that is simple, the group $O^+(4)$ has a non-trivial normal subgroup (see [1]).

Any pair of unit quaternions $(r, s)$ defines an element $O$ of the group $O^+(4)$ according to

$$
O : \mathbb{H} \rightarrow \mathbb{H}, \ q \mapsto r q s.
$$

The converse statement is also valid (see [1]) which means, that for any $O \in O^+(4)$ there exist two unit quaternions $r$, $s$ generating this spherical motion. The pairs $(r, s) = (1, 1)$ and $(-1, -1)$ correspond to the identity $\text{Id}_{\mathbb{R}^4}$ in $O^+(4)$.

The multiplication in the group $O^+(4)$ can be expressed by means of the quaternion product: Let two elements from $O^+(4)$ correspond to two pairs $(r, s)$ and $(r', s')$ of unit quaternions, respectively: The product of these pairs is the pair $(r s, r' s')$, because the following holds:

$$
q \mapsto rr'q s s' = r(r' q s s') .
$$

Comparing quaternion and matrix expression of the elements of $O^+(4)$ we have

$$
r(q s) = \begin{pmatrix} 1 & i & j & k \end{pmatrix} \mathbb{R}^p <S^T \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 1 & i & j & k \end{pmatrix} M^p \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix},
$$

where

$$
M^p = \mathbb{R}^p <S^T = \mathbb{R}^p <S.
$$
For \( r = s \) we receive the classical revolution in \( \text{Im} \mathbb{H} \):

\[
M^o = R^o q R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - 2(r_3^2 + r_4^2) & 2(r_2 r_3 - r_1 r_4) & 2(r_1 r_3 + r_2 r_4) \\
0 & 2(r_2 r_3 + r_1 r_4) & 1 - 2(r_2^2 + r_4^2) & 2(r_3 r_4 - r_1 r_2) \\
0 & 2(r_2 r_4 - r_1 r_3) & 2(r_1 r_2 + r_3 r_4) & 1 - 2(r_2^2 + r_3^2)
\end{pmatrix} \in O^+(3).
\]

It is easy to show, that for \( r_1 = q_1 = 0 \) and all \( \lambda \in \mathbb{R} \) holds

\[
\lambda r \overline{r} = \lambda r, \quad \text{Re}(r q \overline{r}) = 0, \quad r \cdot q = 0 \implies r \cdot (r q \overline{r}) = 0.
\]

From this follows that the matrix \( M^o \) which is the matrix of a revolution (about the plane determined by the real axis and \( \text{Im} r \)) acts only on \( \text{Im} \mathbb{H} \). The matrix \((-M^o)\) operates in \( \text{Im} \mathbb{H} \) as the symmetry with respect to the plane passing through the origin and perpendicular to \( \text{Im} r \) (proof in [1]).

All unit quaternions form a hypersphere \( S^3 \subset \mathbb{R}^4 \), and any point \( r \) located on this hypersphere corresponds to an orthogonal matrix \( R^o \) with \( \det R^o = 1 \). The quaternion \( r \) will be related to the oriented revolution in \( \text{Im} \mathbb{H} \) about the axis determined by \( \text{Im} r \) according to \( q \mapsto r q \overline{r} \). Instead of the unit quaternion \( r \) it is possible to choose a non-zero quaternion \( d \). Any such revolution is regarded as the element of the group \( O^+(3) \). The angle \( \theta \) of revolution, \( 0 \leq \theta \leq \pi \), can be expressed as (see [1] or [3])

\[
\theta = 2 \arctan \left( \frac{\|\text{Im} r\|}{|\text{Re} r|} \right) = 2 \arctan \left( \frac{\|\text{Im} d\|}{|\text{Re} d|} \right).
\]

\( \text{Re} r = r_1 = 0 \) implies \( \theta = \pi \), and we have an axial symmetry in \( \text{Im} \mathbb{H} \) with respect to the line \( \lambda r \).

To determine the angle \( \theta \) in the easier way, we can express, while choosing the unit quaternion as \( c \in \text{Im} \mathbb{H} \), the unit quaternion \( r \) of the revolution \( q \mapsto r q \overline{r} \) in the form presented in [3]:

\[
r = \cos \frac{\theta}{2} + c \sin \frac{\theta}{2}.
\]

The group \( O^+(3) \) has three free parameters, the same as its model \( S^3 \subset \mathbb{H} \).

Limits for the angle \( \theta \) in the interval \([0, \pi]\) are not essential, because the complementary interval \([\pi, 2\pi]\) can be achieved by a revolution about the axis determined by the vector \((-c)\) for the angles also in the interval \([0, \pi]\). This topological problem in \( \mathbb{R}^3 \) will be not a problem in the projective space \( \mathbb{P}^3 \), because this is homeomorphic to \( O^+(3) \). The converse statement is also valid (see in [1]), to any revolution from \( O^+(3) \) there can be related a non-zero quaternion generating this revolution in \( \text{Im} \mathbb{H} \).

Modelling a spherical motion in \( \mathbb{R}^4 \) enables us to construct different views of 4-dimensional objects in such a way, that the object is firstly “revolved” and then projected into the 3-dimensional space defined by the first three coordinate axes (similarly as in the projection \( \mathbb{R}^3 \rightarrow \mathbb{R}^2 \)).

For an orthographic view it is sufficient to exclude one coordinate. There exists a decomposition of the matrix \( Q \in O^+(4) \) (see Section 1), \( Q^o = R^o \circ S \), anyhow the construction
Figure 9: Images of a hypersphere $S^3$ under projections $\mathbb{R}^4 \to \mathbb{R}^3$

of matrices $\mathbb{R}^n$, $\mathbb{S}$ is a different problem. On the other hand, with the help of the random generator of quaternions $r, s$ it is easy to create a random matrix in $O^+(4)$ as the product $(R^\circ S)$. Excluding one of the rows the random matrix $\mathcal{P}$ of a projection from $\mathbb{R}^4$ to $\mathbb{R}^3$ can be obtained. This projection includes also the revolution of the orthographic view in $\mathbb{R}^3$, when the orthographic views of the axis $k$ are not necessarily coincident with the $z$-axis.

Several examples are illustrated in Fig. 9, where the spherical motion is determined by the quaternion $a' = ra_s$, while

$$r = \cos u + i \sin u, \quad s = j \cos v + k \sin v, \quad a = 0.5 + 0.5i + 0.5j + 0.5k,$$

and intervals for parameters $u, v$ are given in the figure. The corresponding projection is applied to the hypersphere $S^3$, and its image is then projected by a linear perspective from $\mathbb{R}^3$ to $\mathbb{R}^2$.

The other possible access is to compose revolutions about six pairs of coordinate axes. Six corresponding angles are in the interval $[0, 2\pi]$. The number of free parameters of the pair $(r, s)$ with $|r| = |s| = 1$ is six. The product of these two revolutions needs not be a classical revolution, but it is a spherical motion.

$$
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & \sin \beta & \cos \beta
\end{pmatrix}
= $$
\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & \sin \beta & \cos \beta
\end{pmatrix}
\]

In general the last matrix has not two real eigenvectors that would determine a plane, about which the classical revolution could be performed.

References


Received February 14, 2000; final form June 12, 2000