Ceva’s and Menelaus’ Theorems for the n-Dimensional Space

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Abstract. This article presents generalizations of the theorems of Ceva and Menelaus for n-dimensional Euclidean space.

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1. Introduction

There are many generalizations of well known Ceva’s and Menelaus’ theorems. In particular, Witczyński (see [2] and [3]) considered a tetrahedron $A_1A_2A_3A_4$ and six points $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ on its edges ($B_{ij} \in A_iA_j$ for $i, j = 1, \ldots, 4, i < j$). Then he gave two propositions:

P1. A necessary and sufficient condition for six planes, each of them determined by an edge and the point $B_{ij}$ on the opposite edge, to have a common point.

P2. A necessary and sufficient condition for the points $B_{ij}$ to be coplanar.

Additional examples of similar theorems (which concerns the products of the ratios of the respective lengths) can be found in [1] and [4].

In this paper we refer to n-dimensional Euclidean space, denoted by $\mathbb{E}^n$, and an n-simplex $\Theta$ ($n > 3$). Let $A_1, \ldots, A_{n+1}$ be the vertices of $\Theta$, and $B_{ij}$ be the points lying on 1-dimensional edges $A_iA_j$ of $\Theta$, different from $A_i, A_j (i = 1, \ldots, n, j = i+1, \ldots, n+1)$. For $k = 1, \ldots, n+1$ the symbol $\Theta_k$ denotes the hyperplane $A_1 \ldots A_{k-1}A_{k+1} \ldots A_{n+1}$ and, at the same time, the corresponding $(n-1)$-dimensional face of $\Theta$.

2. A generalization of Ceva’s Theorem

Theorem 1: The hyperplanes $A_{i_1} \ldots A_{i_{n-1}}B_{km}$ for $i_1 < i_2 < \ldots < i_{n-1}$, $i_j, k, m \in \{1, \ldots, n+1\}$, and $k < m$, $i_j \neq k, m$ (there exist $C_{n+1}^2$ such subspaces) have a common
point if and only if the following \( \frac{(n+1)!}{3!} \) equalities are fulfilled:

\[
\frac{A_iB_{ij}}{B_{ij}A_j} \cdot \frac{A_jB_{jk}}{B_{jk}A_k} \cdot \frac{A_kB_{ik}}{B_{ik}A_i} = 1
\]

for \( i = 1, \ldots, n-1, \ j = 2, \ldots, n, \ k = 3, \ldots, n+1, \) and \( i < j < k. \)

Proof: The proof is by induction on the dimension \( n. \) Theorem 1 is clearly true for \( n = 2 \) and \( n = 3. \)

Necessity: The induction hypothesis implies that if the \((n-2)\)-dimensional subspaces \( A_{i_1} \ldots A_{i_{n-2}}B_{km} \) \( (i_1 < \ldots < i_{n-2}; \ i_j, k, m = 1, \ldots, n; \ k < m; \ i_j \neq k, m) \) in \( \mathbb{E}^{n-1} \) have a common point, then \( \frac{n!}{3!} \) conditions analogous to (1) hold. Let \( S \) be a common point of the hyperplanes \( A_{i_1} \ldots A_{i_{n-1}}B_{km}; \) hence the hyperplanes \( A_1A_{i_1} \ldots A_{i_{n-2}}B_{km} \) \( (i_1 < \ldots < i_{n-2}; \ i_j, k, m = 2, \ldots, n+1; \ k < m; \ i_j \neq k, m) \) have a common line \( A_1S. \)

Let \( A_1S \cap \Theta_1 = S_1 \) and \( A_1A_{i_1} \ldots A_{i_{n-2}}B_{km} \cap \Theta_1 = A_{i_1} \ldots A_{i_{n-2}}B_{km}. \) These \((n-2)\)-dimensional subspaces \( A_{i_1} \ldots A_{i_{n-2}}B_{km} \) contain a point \( S_1. \) So by the assumption a subset of the conditions (1), where \( i, j, k = 2, \ldots, n+1, \ i < j < k, \) is fulfilled. Conducting the same observations for the faces \( \Theta_2, \ldots, \Theta_{n+1} \) of the \( n \)-simplex \( \Theta, \) we get the necessary part of our thesis.

Sufficiency:

We have to assume that our Theorem 1 is true for all dimensions smaller than \( n. \) Define the set \( \Gamma = C_{2}^{m+1} \) hyperplanes according to

\[
\Gamma := \{ A_{i_1} \ldots A_{i_{n-1}}B_{km} \mid i_1 < \ldots < i_{n-1}, \ i_j, k, m = 1, \ldots, n+1, \ k < m, \ i_j \neq k, m \}. \]

Let \( \Gamma \) \( (j = 1, \ldots, n+1) \) be such a subset of \( \Gamma \) which contains only these hyperplanes with the vertex \( A_j. \) Without loss of generality it is enough to consider the sets \( \Gamma_1, \Gamma_2, \Gamma_3 \) only \( (\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq \emptyset, \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma). \)

We may treat the face \( \Theta_1 \) of \( \Theta \) as an \((n-1)\)-dimensional simplex with \((n-2)\)-dimensional faces. The conditions (1) hold, so from the inductive assumption for \( \mathbb{E}^{n-2} \) we know that for \( i = 2, \ldots, n+1 \) there exist points \( S_1^i \) which are the common points of the \((n-3)\)-dimensional subspaces \( A_{i_1} \ldots A_{i_{n-3}}B_{km}, \) where \( i_1 < \ldots < i_{n-3}, \ k < m, \ i_j \neq k \neq m \neq 1, i. \) We shall call them the Cevian points. Every Cevian point \( i, \ i = 2, \ldots, n+1, \) belongs to the face of \( \Theta_1 \) which is opposite to the vertex \( A_i, \ i = 2, \ldots, n+1. \)

Due to the inductive hypothesis for \( \mathbb{E}^{n-1}, \) the \((n-2)\)-dimensional subspaces \( A_{i_1} \ldots A_{i_{n-2}}B_{km} \) \( (i_1 < \ldots < i_{n-2}; \ k < m; \ i_j \neq k \neq m \neq 1) \) intersect in one point, say \( S_1. \) It can be proved that \( S_1^1 = \bigcap_{i=2}^{n+1} A_iS_1^i. \) Similarly, taking into account the faces \( \Theta_2 \) and \( \Theta_3, \) it can get points \( S_2^i \) \( (i = 1, 3, \ldots, n+1), \) \( S_2^2 = \bigcap_{i=1, i \neq 2}^{n+1} A_iS_2^i \) and points \( S_3^j \) \( (j = 1, 2, 4, \ldots, n+1), \) \( S_3^3 = \bigcap_{i=1, j \neq 3}^{n+1} A_iS_3^i. \)

We ought to show that \( A_1S_1^1 \cap A_2S_2^2 \cap A_3S_3^3 \neq \emptyset. \) To do this, we note that \( S_2^1 = S_2^2, S_1^3 = S_3^3, \) \( S_3^2 = S_2^2. \) Thus \( S_1^1 \in A_3S_3^3 = A_3S_1^3 \) and \( S_3 \in A_1S_1^3, \) hence \( A_1S_1^1 \cap A_3S_3^3 \cap P_1 \neq \emptyset. \) Similarly, \( A_1S_1^1 \cap A_2S_2^2 = P_2 \neq \emptyset \) and \( A_2S_2^2 \cap A_3S_3^3 = P_3 \neq \emptyset. \) Moreover, \( P_1 \in A_1A_2S_2^1 \cap A_2A_3S_3^2 = A_2S_2^2 \) and \( P_1 \in A_1A_2S_1^1 \cap A_2A_3S_3^2 = A_3S_3^3 \) (where symbols \( \Delta \) denote 2-dimensional subspaces of \( \mathbb{E}^n \)). Then the points \( P_1, P_2, P_3 \) coincide.

\footnote{Compare [1], where similar sets of equations, but for \( n \)-gons in the affine plane, are given.}
3. A generalization of Menelaus’ Theorem

Theorem 2: The points \( B_{ij} \) for \( i, j = 1, \ldots, n + 1 \) and \( i < j \) (there exist \( C_2^{n+1} \) such points) lie on one hyperplane of \( \mathbb{E}^n \) if and only if the following conditions hold:

\[
\frac{A_i B_{ij}}{B_{ij} A_j} \cdot \frac{A_j B_{jk}}{B_{jk} A_k} \cdot \frac{A_k B_{ik}}{B_{ik} A_i} = -1
\]

(2)

for \( i = 1, \ldots, n - 1, j = 2, \ldots, n, k = 3, \ldots, n + 1, i < j < k \).

Proof: Again, the proof is inductive. Our Theorem 2 is obviously true for \( n = 2 \) and \( 3 \).

Necessity:
Assume that the points \( B_{ij}, i, j = 1, \ldots, n + 1, i < j \) are on one hyperplane, say \( \alpha \). Now we introduce subspaces \( \beta_i = \alpha \cap \Theta_i \) (\( i = 1, \ldots, n + 1 \)) of \( \mathbb{E}^n \). Note that \( \dim \beta_i = n - 2 \). First let us exam the subspace \( \bar{\beta}_{n+1} \). The points \( B_{ij}, i = 1, \ldots, n - 1, j = i + 1, \ldots, n \), belong to \( \bar{\beta}_{n+1} \). Applying the induction hypothesis to these points and the hyperplane (in \( \mathbb{E}^{n-1} \)) \( \bar{\beta}_{n+1} \) we get the conditions (2) for \( i = 1, \ldots, n - 2, j = 2, \ldots, n - 1, k = 3, \ldots, n, i < j < k \), respectively. To obtain the next equations (from (2)) it remains to consider the subspaces \( \beta_i \), \( i = 1, \ldots, n \), analogously.

Sufficiency:
Now let us assume that our hypothesis is true for every integer smaller than \( n \). Define subspaces (of \( \mathbb{E}^n \))

\[
\alpha_k := \text{span} \{ B_{ij} \mid i = 1, \ldots, n, j = i + 1, \ldots, n + 1, i, j \neq k \}, \quad k = 1, \ldots, n + 1
\]

and take into account the \((n-1)\)-simplex \( \Theta_1 \) in \( \mathbb{E}^{n-1} \). Clearly, the points \( B_{ij}, i = 2, \ldots, n, j = i + 1, \ldots, n + 1 \), generating the subspace \( \alpha_1 \) lie on the 1-dimensional edges of \( \Theta_1 \). Furthermore, by the inductive hypothesis these points belong to one hyperplane in \( \mathbb{E}^{n-1} \) (whose dimension equals \((n - 2)\)). Conducting in the same way for all \( \alpha_k \), we get that \( \dim \alpha_k = n - 2 \) \((k = 1, \ldots, n + 1)\). Without loss of generality, we may consider only three of the \( \alpha_k \), say \( \alpha_1, \alpha_2, \alpha_3 \). Observe that all the points \( B_{ij}, i = 1, \ldots, n, j = i + 1, \ldots, n + 1 \), belong to \( \alpha_1 \cup \alpha_2 \cup \alpha_3 \). Applying the inductive assumption for \( \mathbb{E}^{n-2} \) and \( \mathbb{E}^{n-3} \), we note:

\[
\dim(\alpha_1 \cap \alpha_2) = \dim(\alpha_1 \cap \alpha_3) = \dim(\alpha_2 \cap \alpha_3) = n - 3 \quad \text{and} \quad \dim(\alpha_1 \cap \alpha_2 \cap \alpha_3) = n - 4,
\]

respectively. Finally we obtain that each two of the three \((n - 2)\)-dimensional subspaces of \( \mathbb{E}^n \) intersect in an \((n - 3)\)-dimensional subspace, and the intersection of all three is an \((n - 4)\)-dimensional subspace. Hence \( \alpha_1, \alpha_2, \alpha_3 \) are contained in one hyperplane of \( \mathbb{E}^n \). Thus all the points \( B_{ij}, i = 1, \ldots, n, j = i + 1, \ldots, n + 1 \), lie on this hyperplane.

\[\square\]

References


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