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Ceva's and Menelaus' Theorems for the n-Dimensional Space

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Abstract. This article presents generalizations of the theorems of Ceva and Menelaus for n-dimensional Euclidean space.

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1. Introduction

There are many generalizations of well known Ceva's and Menelaus' theorems. In particular, Witczyński (see [2] and [3]) considered a tetrahedron $A_1A_2A_3A_4$ and six points $B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}$ on its edges ($B_{ij} \in A_iA_j$ for $i, j = 1, \ldots, 4, i < j$). Then he gave two propositions:

P1. A necessary and sufficient condition for six planes, each of them determined by an edge and the point B_{ij} on the opposite edge, to have a common point.

P2. A necessary and sufficient condition for the points B_{ij} to be coplanar.
Additional examples of similar theorems (which concerns the products of the ratios of the respective lengths) can be found in [1] and [4].

In this paper we refer to n-dimensional Euclidean space, denoted by $\mathbb{E}^n$, and an n-simplex Θ ($n > 3$). Let $A_1, \ldots, A_{n+1}$ be the vertices of Θ, and B_{ij} be the points lying on 1-dimensional edges A_iA_j of Θ, different from A_i, A_j ($i = 1, \ldots, n$, $j = i+1, \ldots, n+1$). For $k = 1, \ldots, n+1$ the symbol Θ_k denotes the hyperplane $A_1 \ldots A_{k-1}A_{k+1} \ldots A_{n+1}$ and, at the same time, the corresponding $(n-1)$-dimensional face of Θ.

2. A generalization of Ceva's Theorem

Theorem 1: *The hyperplanes $A_{i_1} \ldots A_{i_{n-1}}B_{km}$ for $i_1 < i_2 < \ldots < i_{n-1}$, $i_j, k, m \in \{1, \ldots, n+1\}$, and $k < m$, $i_j \neq k, m$ (there exist C_2^{n+1} such subspaces) have a common*

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point if and only if the following $\frac{(n+1)!}{3!}$ *equalities are fulfilled:*

$$\frac{A_iB_{ij}}{B_{ij}A_j} \cdot \frac{A_jB_{jk}}{B_{jk}A_k} \cdot \frac{A_kB_{ik}}{B_{ik}A_i} = 1 \tag{1}$$

for $i = 1, \dots, n-1,\ j = 2, \dots, n,\ \ k = 3, \dots, n+1,$ *and* $i < j < k.$ [1]

Proof: The proof is by induction on the dimension n. Theorem 1 is clearly true for $n = 2$ and $n = 3$.

Necessity:

The induction hypothesis implies that if the $(n-2)$-dimensional subspaces $A_{i_1} \dots A_{i_{n-2}}B_{km}$ $(i_1 < \dots < i_{n-2};\ \ i_j, k, m = 1, \dots, n;\ \ k < m;\ \ i_j \neq k, m)$ in $\mathbb{E}^{n-1}$ have a common point, then $\frac{n!}{3!}$ conditions analogous to (1) hold. Let S be a common point of the hyperplanes $A_{i_1} \dots A_{i_{n-1}}B_{km}$; hence the hyperplanes $A_1A_{i_1} \dots A_{i_{n-2}}B_{km}$ $(i_1 < \dots < i_{n-2};$ $i_j, k, m = 2, \dots, n+1;\ k < m;\ i_j \neq k, m)$ have a common line A_1S.

Let $A_1S \cap \Theta_1 = S_1$ and $A_1A_{i_1} \dots A_{i_{n-2}}B_{km} \cap \Theta_1 = A_{i_1} \dots A_{i_{n-2}}B_{km}$. These $(n-2)$-dimensional subspaces $A_{i_1} \dots A_{i_{n-2}}B_{km}$ contain a point S_1. So by the assumption a subset of the conditions (1), where $i, j, k = 2, \dots, n+1,\ \ i < j < k$, is fulfilled. Conducting the same observations for the faces $\Theta_2, \dots, \Theta_{n+1}$ of the n-simplex Θ, we get the necessary part of our thesis.

Sufficiency:

We have to assume that our Theorem 1 is true for all dimensions smaller than n. Define the set Γ of C_2^{n+1} hyperplanes according to

$$\Gamma := \{A_{i_1} \dots A_{i_{n-1}}B_{km} \mid i_1 < \dots < i_{n-1},\ i_j, k, m = 1, \dots, n+1,\ k < m,\ i_j \neq k, m\}.$$

Let Γ_j $(j = 1, \dots, n+1)$ be such a subset of Γ which contains only these hyperplanes with the vertex A_j. Without loss of generality it is enough to consider the sets $\Gamma_1, \Gamma_2, \Gamma_3$ only $(\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq \emptyset,\ \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma)$.

We may treat the face Θ_1 of Θ as an $(n-1)$-dimensional simplex with $(n-2)$-dimensional faces. The conditions (1) hold, so from the inductive assumption for $\mathbb{E}^{n-2}$ we know that for $i = 2, \dots, n+1$ there exist points S_i^1 which are the common points of the $(n-3)$-dimensional subspaces $A_{i_1} \dots A_{i_{n-3}}B_{km}$, where $i_1 < \dots < i_{n-3},\ \ k < m,\ \ i_j \neq k \neq m \neq 1, i$. We shall call them the Cevian points. Every Cevian point S_i^1, $i = 2, \dots, n+1$, belongs to the face of Θ_1 which is opposite to the vertex A_i, $i = 2, \dots, n+1$.

Due to the inductive hypothesis for $\mathbb{E}^{n-1}$, the $(n-2)$-dimensional subspaces $A_{i_1} \dots A_{i_{n-2}}B_{km}$ $(i_1 < \dots < i_{n-2};\ k < m;\ i_j \neq k \neq m \neq 1)$ intersect in one point, say S^1. It can be proved that $S^1 = \bigcap_{i=2}^{n+1} A_iS_i^1$. Similarly, taking into account the faces Θ_2 and Θ_3, we get points S_i^2 $(i = 1, 3, \dots, n+1)$, $S^2 = \bigcap_{i=1, i\neq 2}^{n+1} A_iS_i^2$ and points S_j^3 $(j = 1, 2, 4, \dots, n+1)$, $S^3 = \bigcap_{j=1, j\neq 3}^{n+1} A_iS_i^3$.

We ought to show that $A_1S^1 \cap A_2S^2 \cap A_3S^3 \neq \emptyset$. To do this, we note that $S_2^1 = S_1^2$, $S_3^1 = S_1^3$, $S_2^3 = S_3^2$. Thus $S^1 \in A_3S_3^1 = A_3S_1^3$ and $S_3 \in A_1S_1^3$, hence $A_1S^1 \cap A_3S^3 = P_1 \neq \emptyset$. Similarly, $A_1S^1 \cap A_2S^2 = P_2 \neq \emptyset$ and $A_2S^2 \cap A_3S^3 = P_3 \neq \emptyset$. Moreover, $P_1 \in \Delta A_1A_2S_1^2 \cap A_2A_3S_3^2 = A_2S^2$ and $P_1 \in \Delta A_1A_3S_1^3 \cap A_2A_3S_3^2 = A_3S^3$ (where symbols Δ denote 2-dimensional subspaces of $\mathbb{E}^n$). Then the points P_1, P_2, P_3 coincide. $\square$

[1] Compare [1], where similar sets of equations, but for n-gons in the affine plane, are given.

3. A generalization of Menelaus' Theorem

Theorem 2: *The points B_{ij} for $i, j = 1, \dots, n+1$ and $i < j$ (there exist C_2^{n+1} such points) lie on one hyperplane of $\mathbb{E}^n$ if and only if the following conditions hold:*

$$\frac{A_iB_{ij}}{B_{ij}A_j} \cdot \frac{A_jB_{jk}}{B_{jk}A_k} \cdot \frac{A_kB_{ik}}{B_{ik}A_i} = -1 \tag{2}$$

for $i = 1, \dots, n-1$, $j = 2, \dots, n$, $k = 3, \dots, n+1$, $i < j < k$.

Proof: Again, the proof is inductive. Our Theorem 2 is obviously true for $n = 2$ and 3.

Necessity:

Assume that the points B_{ij}, $i, j = 1, \dots, n+1$, $i < j$ are on one hyperplane, say α. Now we introduce subspaces $\beta_i = \alpha \cap \Theta_i$ ($i = 1, \dots, n+1$) of $\mathbb{E}^n$. Note that $\dim \beta_i = n-2$. First let us exam the subspace β_{n+1}. The points B_{ij}, $i = 1, \dots, n-1$, $j = i+1, \dots, n$, belong to β_{n+1}. Applying the induction hypothesis to these points and the hyperplane (in $\mathbb{E}^{n-1}$) β_{n+1} we get the conditions (2) for $i = 1, \dots, n-2$, $j = 2, \dots, n-1$, $k = 3, \dots, n$, $i < j < k$, respectively. To obtain the next equations (from (2)) it remains to consider the subspaces β_i, $i = 1, \dots, n$, analogously.

Sufficiency:

Now let us assume that our hypothesis is true for every integer smaller than n. Define subspaces (of $\mathbb{E}^n$)

$$\alpha_k := \operatorname{span}\{B_{ij} \mid i = 1, \dots, n,\ j = i+1, \dots, n+1,\ i, j \neq k\}, \quad k = 1, \dots, n+1$$

and take into account the $(n-1)$-simplex Θ_1 in $\mathbb{E}^{n-1}$. Clearly, the points B_{ij}, $i = 2, \dots, n$, $j = i+1, \dots, n+1$, generating the subspace α_1 lie on the 1-dimensional edges of Θ_1. Furthermore, by the inductive hypothesis these points belong to one hyperplane in $\mathbb{E}^{n-1}$ (whose dimension equals $(n-2)$). Conducting in the same way for all α_k, we get that $\dim \alpha_k = n-2$ ($k = 1, \dots, n+1$). Without loss of generality, we may consider only three of the α_k, say $\alpha_1, \alpha_2, \alpha_3$. Observe that all the points B_{ij}, $i = 1, \dots, n$, $j = i+1, \dots, n+1$, belong to $\alpha_1 \cup \alpha_2 \cup \alpha_3$. Applying the inductive assumption for $\mathbb{E}^{n-2}$ and $\mathbb{E}^{n-3}$, we note:

$$\dim(\alpha_1 \cap \alpha_2) = \dim(\alpha_1 \cap \alpha_3) = \dim(\alpha_2 \cap \alpha_3) = n-3 \quad \text{and} \quad \dim(\alpha_1 \cap \alpha_2 \cap \alpha_3) = n-4,$$

respectively. Finally we obtain that each two of the three $(n-2)$-dimensional subspaces of $\mathbb{E}^n$ intersect in an $(n-3)$-dimensional subspace, and the intersection of all three is an $(n-4)$-dimensional subspace. Hence $\alpha_1, \alpha_2, \alpha_3$ are contained in one hyperplane of $\mathbb{E}^n$. Thus all the points B_{ij}, $i = 1, \dots, n$, $j = i+1, \dots, n+1$, lie on this hyperplane. □

References

[1] B. Grünbaum, G.C. Shephard: *Some new transversality properties.* Geom. Dedicata **71**, 179–208 (1998).

[2] K. Witczyński: *Ceva's and Menelaus' theorems for tetrahedra.* Zeszyty Naukowe "Geometry" **21**, 98–107 (1995).

[3] K. Witczyński: *Ceva's and Menelaus' theorems for tetrahedra (II).* Zeszyty Naukowe "Geometry" **29**, 233–235 (1996).

[4] M.S. Klamkin, A. Liu: *Simultaneous generalizations of the theorems of Ceva and Menelaus*. Math. Magazine **65**, 48–53 (1992).

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