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Gergonne and Nagel Points for Simplices in the n-Dimensional Space

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Abstract. Properties of triangles related to so called Gergonne and Nagel points are know in elementary geometry. In this paper we present a discussion on some extensions of these theorems. First, we refer to a relation between a tetrahedron and a sphere inscribed into this tetrahedron in the 3-dimensional space. Next, we generalize the obtained results to simplices in n-dimensional geometry. The problem concerning tetrahedra occurs to be no longer as easy to solve as it is for triangles. It has been shown that there are both tetrahedra, which have Gergonne and Nagel points, and tetrahedra with no such a point. We give conditions necessary and sufficient for a simplex to satisfy the Gergonne and Nagel property.

Key Words: 3-dimensional geometry, n-dimensional geometry, polar transformation, Gergonne point, Nagel point

MSC 2000: 51M04

1. Introduction

The theorems on so called Gergonne and Nagel points in a triangle are well known from literature. These theorems are as follows.

If for a triangle ABC circumscribed to a circle o the sides BC, AC, and AB touch the circle o at points A', B', and C', respectively, then the lines AA', BB', CC' pass through a common point G, which is called Gergonne point.

In other words, the three segments joining the vertices of a triangle with the points of tangency with the incircle intersect at a single point which is called Gergonne point. In a given triangle ABC we can also consider external bisectors of the three angles between sides of the triangle, each two of them meeting the appropriate internal bisector to produce three

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additional excenters, i.a. centers of the three escribed circles, i.e. *excircles.* The following theorem is true:
If for a triangle ABC the points K, L and M are the points of contact between the sides BC, AC, and AB and the three excircles, respectively, then the segments AK, BL, and CM meet at one point, which is called Nagel point.

Shortly: The three segments connecting the vertices of a triangle with the corresponding tangency points of the excircles pass through one point, the Nagel point. The question arises if similar points exist for a tetrahedron. The problem is not as simple as it is for triangles. It can be proved that there is a class of tetrahedra for which these points exist, and another class for which such points do not exist. What conditions should be fulfilled by a tetrahedron so that Gergonne and Nagel points would exist? In the paper the authors try to answer this question.

2. Gergonne point in the 3-dimensional space

At the very beginning let us consider if a sphere can be inscribed into an arbitrary tetrahedron.

Let $A_1A_2A_3A_4$ be a tetrahedron. At first, we consider the trihedron with vertex A_1, edges $k_{12}(A_1A_2)$, $k_{13}(A_1A_3)$, $k_{14}(A_1A_4)$, and faces $\alpha_{123}(A_1A_2A_3)$, $\alpha_{124}(A_1A_2A_4)$, and $\alpha_{134}(A_1A_3A_4)$. It is easy to see, that three bisecting planes of the dihedral angles with the edges $k_{12}(A_1A_2)$ $k_{13}(A_1A_3)$, and $k_{14}(A_1A_4)$ meet at a single straight line. Actually, bisecting planes δ_{2324} and δ_{2334} of corresponding angles $\angle(\alpha_{123}, \alpha_{124})$, and $\angle(\alpha_{123}, \alpha_{134})$ meet in a line k_{1213}. The points on line k_{1213} are equidistant from the faces α_{123}, α_{124} and α_{134}. Thus the line k_{1213} lies in a bisecting plane of the angle $\angle(\alpha_{124}, \alpha_{134})$. Let us take an arbitrary point $O' \in k_{1213}$. Orthogonal projections of O' onto the planes α_{123}, α_{124}, and α_{134} will be denoted by O_{23}, O_{24}, and O_{34}, respectively. The segments $O'O_{23} = O'O_{24} = O'O_{34}$ have equal length. Let us consider the sphere $S(O', O'O_{23})$, which is tangent to the planes α_{123}, α_{124}, and α_{134}. Subsequently, let us consider the plane α'_{234}, which is tangent to the given sphere S and parallel to the plane $\alpha_{234}(A_2A_3A_4)$. Let A'_2, A'_3 and A'_4 be points in which the plane α'_{234} intersects the edges A_1A_2, A_1A_2, and A_1A_2, respectively. The tetrahedron $A_1A'_2A'_1A'_2$ is circumscribed to the sphere $S(O', O'O_{23})$. Let C'_1 be the common point of the line A_1O' and the plane α'_{234}, and C_1 be the common point of the line A_1O' and the plane α_{234}.

By applying the homothety $J_{A_1}^{\frac{A_1C_1}{A_1C'_1}}$ we can write $J_{A_1}^{\frac{A_1C_1}{A_1C'_1}}(A_1A'_2A'_3A'_4) = A_1A_2A_3A_4$. Thus the sphere $J_{A_1}^{\frac{A_1C_1}{A_1C'_1}}(S(O', O'O_{23}))$ is circumscribed to the tetrahedron $A_1A_2A_3A_4$.

Statement 1 *Into each tetrahedron exactly one sphere can be inscribed.*

Let us consider a tetrahedron $A_1A_2A_3A_4$ and a sphere S inscribed into this tetrahedron. Let B_i be a point in which the face $A_jA_kA_l$ touches the sphere S (for $\neq (i, j, k, l)$, where $i, j, k, l = 1, 2, 3, 4$). A tetrahedron will be called a *Gergonne tetrahedron* if the lines A_iB_i $(i = 1, 2, 3, 4)$ meet at one common point. Regular triangular pyramids (in particular regular tetrahedra) are Gergonne tetrahedra.

Can we state that each triangular pyramid is a Gergonne tetrahedron? We shall prove in the following that the answer to this question is negative.

Let us construct a tetrahedron, which is not a Gergonne tetrahedron. Let a sphere S and an arbitrary line a, which does not meet the sphere, be given. We get two planes α_1, α_3 passing through the line a and tangent to the sphere S in points B_1 and B_3. Now, let us consider another line b skew with respect to line B_1B_3, but also not meeting the sphere S, and let α_2, α_4 be two planes passing through line b and tangent to the sphere S at points B_2 and B_4, respectively. The planes α_1, α_2, α_3, α_4 determine a tetrahedron circumscribed to the sphere S. Let us denote the vertices lying on the line b by A_1 and A_3 and the vertices lying on the line a by A_2, A_4. Consider the lines A_1B_1 and A_3B_3. From construction it is clear that these lines are non-coplanar. If contrary, the lines A_1A_3 and B_1B_3 could not be skew as assumed at the beginning of the construction. Thus the lines A_1B_1 and A_3B_3 do not meet at one point. The pyramid we have constructed is not a Gergonne tetrahedron. Fig. 1 shows two overlapping views (top and front views) of such a pyramid. For convenience of the presentation the edges A_1A_2 and A_3A_4 have been assumed as mutually perpendicular. This pyramid has two symmetry planes and is not regular.

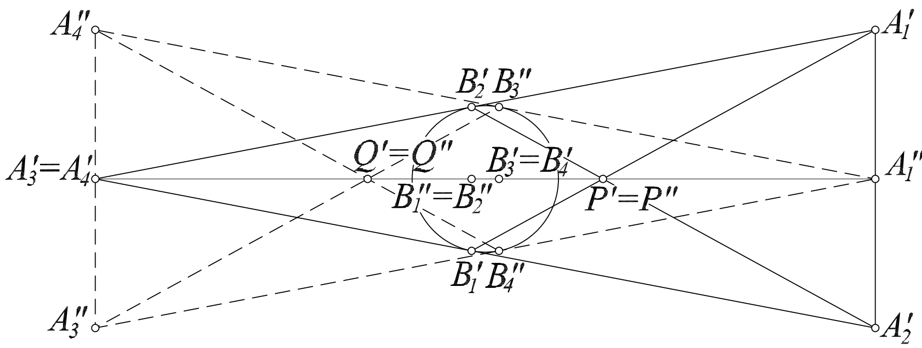

Figure 1: Two overlapping views (top (") and front (') view) of a pyramid $A_1A_2A_3A_4$, which does not have a Gergonne point ($P \neq Q$)

The lines (A_1B_1, A_2B_2) and (A_3B_3, A_4B_4) intersect in pairs respectively at points P and Q and do not meet at one point. So we need to determine necessery and sufficient conditions to be fulfilled so that a pyramid is a Gergonne tetrahedron. First we formulate a simple lemma (cf. [2], p. 1073, [3]). Let us consider m straight lines $p_1, p_2, \ldots, p_m$ $(3 \leq m)$ in the n-dimensional projective space $\mathbb{P}^n$ (for $3 \leq m$). We have

Lemma 1 *Straight lines $p_1, p_2, \ldots, p_m$ $(3 \leq m)$ intersect each other if and only if they are coplanar or if they meet at one point (belong to one bundle).*

Proof: Let us assume that the lines $p_1, p_2, \ldots, p_m$ $(3 \leq m)$ intersect each other. Consider two cases:

(1) all lines pass through the same point – and this is the end of the proof.

(2) there exist three lines, which do not pass through the same single point.

In the latter case without loss of generality we can assume that these are the lines p_1, p_2, p_3. These lines form a triangle and determine a plane $\pi(p_1, p_2, p_3)$. An arbitrary line p_i $(4 \leq i)$, intersecting each one of the lines p_1, p_2, p_3 has to belong to the plane π. For if p_i did not belong to the plane π, the line p_i would intersect the plane π at a unique point, and this point should belong to p_1, p_2, and p_3, contrary to the assumption of the case (2). △

The proof in opposite direction is trivial. △ ◇

Let $(A_i)_{i=1,2,3,4}$ be a tetrahedron and let $(B_i)_{i=1,2,3,4}$ be points, in which the sphere inscribed into this tetrahedron touches its faces, such that B_i lies on the face opposite to A_i for $i = 1, 2, 3, 4$. The following is true

Theorem 1 *The triangular pyramid $(A_i)_{i=1,2,3,4}$ is a Gergonne tetrahedron if and only if the points A_i, A_j, B_i, B_j are coplanar for all possible i, j with $i \neq j$.*

Proof, "$\Rightarrow$": Let us assume that the triangular pyramid $(A_i)_{i=1,2,3,4}$ is a Gergonne tetrahedron. Then the lines A_iB_i and A_jB_j (for $i \neq j$ and $i, j = 1, 2, 3, 4$) meet at one point. It means that the points A_i, A_j, B_i, B_j are coplanar. $\triangle$

"$\Leftarrow$": Let the points A_i, A_j, B_i, and B_j be coplanar for all possible i, j with $i \neq j$. Thus the lines A_iB_i (for $i, j = 1, 2, 3, 4$) intersect each other. The points $A_1, \dots, A_4, B_1, \dots, B_4$ can not lie in one plane, as the tetrahedron $(A_i)_{i=1,2,3,4}$ (and also $(B_i)_{i=1,2,3,4}$) is not a planar figure. Consequently, the lines A_iB_i for $i = 1, 2, 3, 4$ are not contained in a plane. Applying Lemma 1 we see that the lines A_iB_i for $i, j = 1, 2, 3, 4$ belong to a bundle, so the triangular pyramid $(A_i)_{i=1,2,3,4}$ is a Gergonne tetrahedron. $\triangle$ $\diamondsuit$

Corollary 1 *Each regular triangular pyramid is a Gergonne tetrahedron.*

Proof: Let a pyramid $A_1A_2A_3A_4$ be a regular triangular pyramid with the equilateral triangle $A_1A_2A_3$ as the basis. Let a sphere inscribed into this pyramid be tangent to its faces $A_1A_2A_4$, $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_3$ at points B_3, B_1, B_2, B_4, respectively. The segments B_3B_1, B_1B_2, and B_2B_3 are parallel, respectively, to the edges A_1A_3, A_2A_1, A_3A_2, and hence the followig pairs of segments are coplanar: B_3B_1, A_1A_3; B_1B_2, A_2A_1; B_2B_3, A_3A_2. Similarly, the segments B_4B_3, B_4B_1, B_4B_2 span planes respectively with parallel segments A_3A_4, A_1A_4, A_2A_4. These planes are the planes of symmetry for a tetrahedron. Theorem 1 holds in this case. $\diamondsuit$

The question arises if there are other Gergonne tetrahedra? The answer is negative. We will refer to Fig. 2 and Fig. 3 in the following discussion. In Fig. 2 the idea for creating Fig. 3 and Fig. 4 has been presented. Let A_1, k_{12}, k_{13} denote the vertex and two edges of the triangular pyramid. Let O', B_2', B_3' denote the orthogonal projection of the center O of the sphere and of two tangency points B_2, B_3 onto the plane (face) $A_1A_2A_3$, and let B_2^0, O_{II}^0; B_3^0, O_{III}^0 be two distinct revolved sections of points B_2, O; B_3, O onto the plane (face) $A_1A_2A_3$.

We will first examine a regular pyramid $(A_i)_{i=1,2,3,4}$ for which the face $A_2A_3A_4$ is an equilateral triangle (Fig. 3). Its face $A_1A_2A_3$ lies in an arbitrary plane. A sphere has been inscribed into this pyramid. Let us now move the face $A_2A_3A_4$ around the sphere in such a way that it remains tangent to the sphere. At least one of the following pairs of edges A_2A_3,B_3B_2; A_3A_4,B_4B_3; or A_2A_4,B_4B_2 becomes skew (see Fig. 3: lines B_3B_2 and A_2A_3 are not parallel). In case the face $A_1A_2A_3$ is an isosceles triangle, similar considerations are based on drawings in Fig. 1, while the edge A_2A_3 remains parallel to the line B_3B_2. Each distortion of a regular triangular pyramid contradics assumptions of Theorem 1.

Let us now examine the case of an arbitrary tetrahedron (Fig. 4). We will also assume that its face $A_1A_2A_3$ lies in an arbitrary plane, while the faces $A_1A_2A_4$, $A_1A_3A_4$ make arbitrary angles with the face $A_1A_2A_3$. The line B_2B_3 intersects the plane $A_1A_2A_3$ at a point P_{231}. Again, let us move the face $A_2A_3A_4$ around, so that it remains tangent to the sphere. One of the boundary positions of the edge A_2A_3 is to be parallel to $B_3'B_2'$. If the point A_3 is approaching A_1 then the lines A_2A_3 and B_3B_2 will positively not intersect (Fig. 4). Therefore only such positions are possible in which point A_2 approaches A_1. Continuing our discussion we notice that the only possible positions for B_3 are such in which B_3' belongs to the hatched area in Fig. 4. Then, for an arbitrary position, which does not produce a regular pyramid, the

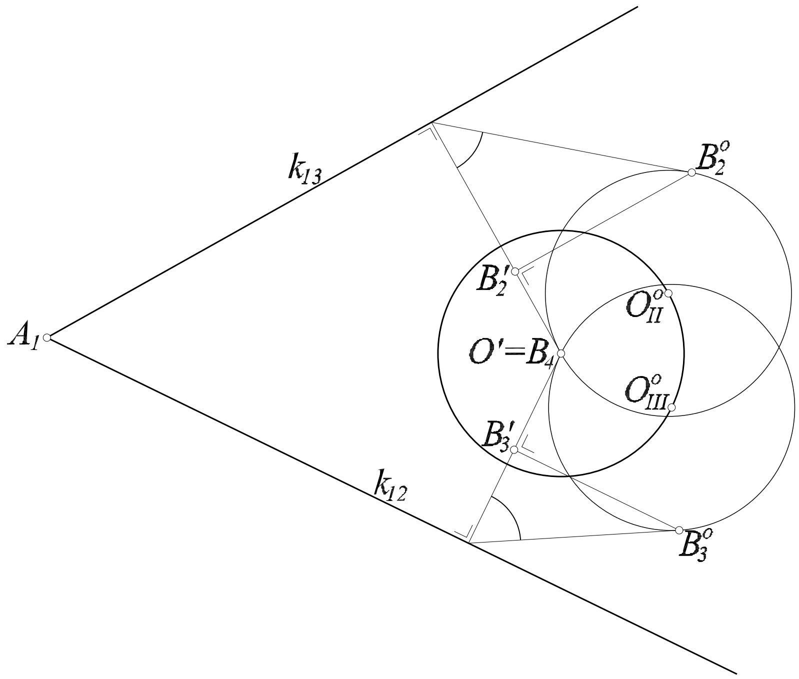

Figure 2: The orthogonal projections (B_2', B_3', O') and revolved sections ($B_2^0, O_{II}^0; B_3^0, O_{III}^0$) of the tangency points B_2, B_3 and the center O of the sphere. The elements without strokes lie in the projection plane ($A_1A_2A_3$)

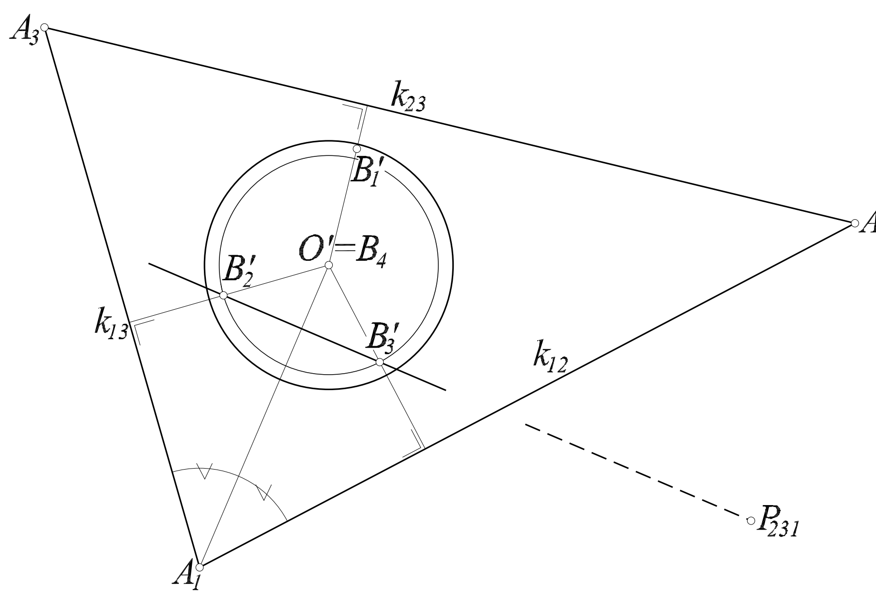

Figure 3: The orthogonal projection of two tetrahedra $(A_i)_{i=1,2,3,4}$, $(B_i)_{i=1,2,3,4}$ and the sphere with center O. The elements without strokes lie in the projection plane ($A_1A_2A_3$)

segments of at least one pair among A_2A_3, B_3B_2; A_1A_3, B_3B_1; A_1A_2, B_2B_1 do not intersect each other. In our discussion we take into consideration the incidence of the point B_3 to the hatched area presented in Fig. 4 and the inclination of pairs of lines A_2A_3, $B_3'B_2'$; A_1A_3, $B_3'B_1'$; A_1A_2, $B_2'B_1'$.

Hence we conclude that the only Gergonne tetrahedra are regular triangular pyramids.

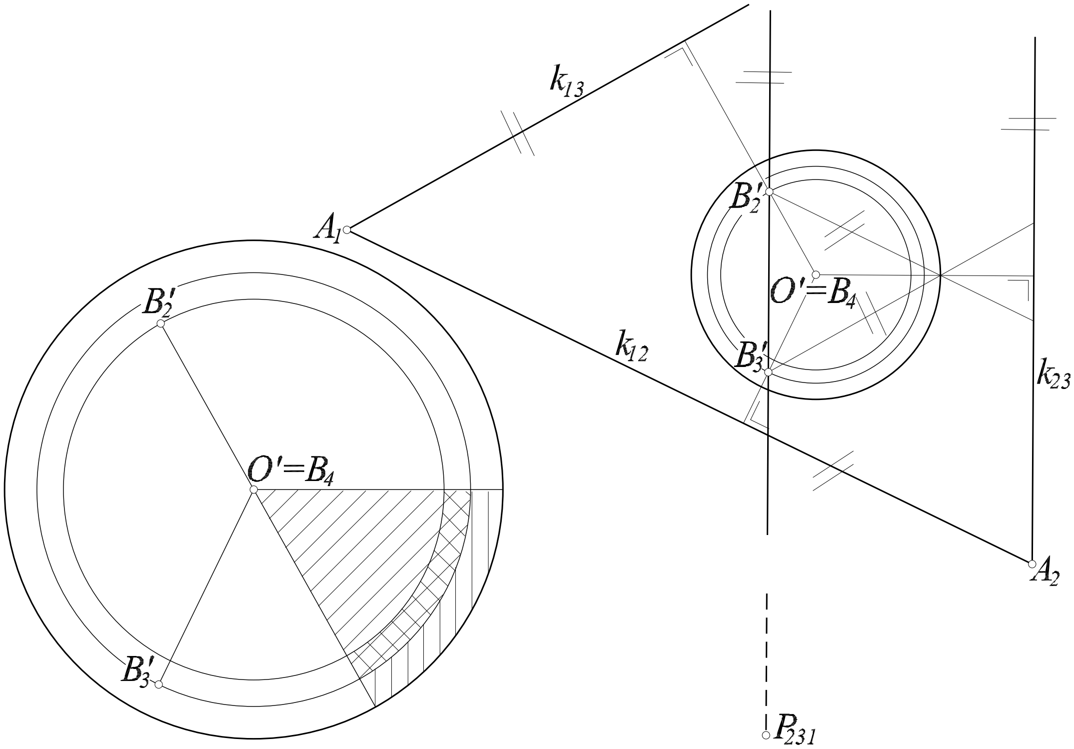

Figure 4: The graphical idea of cases of the examination of Gergonne tetrahedra

3. Polar transformation

Theorem 1 has close connections with a certain polar transformation induced by a sphere (cf. [1], pp. 355–357, [4], p. 334). Let $\mathbb{P}^3$ be the 3-dimensional projective space over the field $\mathbb{R}$ with points at infinity distinguished. Let $\mathrm{Sub}_m\mathbb{P}^3$ be the set of m-dimensional subspaces of the space $\mathbb{P}^3$ ($0 \leq m \leq 3$), and let S be an arbitrary sphere in the space $\mathbb{P}^3$. Let L be the set $\mathrm{Sub}_1\mathbb{P}^3$ (so called a space of lines), we write '$|$' for the incidence relation which may hold between a point and a line, or a point and a plane, or a line and a plane, or between line and another line (the last relation is equivalent to lines intersection). Let us now consider the polar correlation Φ: $\mathrm{Sub}_0\mathbb{P}^3 \longmapsto \mathrm{Sub}_2\mathbb{P}^3$ induced by a sphere. The transformation Φ induces an other transformation Φ^1: $\mathrm{Sub}_1\mathbb{P}^3 \longmapsto \mathrm{Sub}_1\mathbb{P}^3$. The last transformation is an automorphism of the structure $\langle \mathsf{L}, | \rangle$ (cf. [2], p. 1077). The following properties of the maps Φ and Φ^1 become crucial: $\Phi^1(x) \perp x$, and $\Phi(X)|X \iff X \in S$ for each $x \in \mathsf{L}$, $X \in \mathbb{P}^3$. Let the tetrahedron $(A_i)_{i=1,2,3,4}$ be circumscribed to the sphere S and let the correlation Φ be determined by the sphere S, let Φ^1 be defined as above. We have $\Phi^1(A_iA_j) = B_kB_l$ and $\Phi^1(B_iB_j) = A_kA_l$ for $\neq (i,j,k,l)$. If we assume that the lines A_iA_j and B_iB_j are coplanar for all $\neq (i,j)$, then there exist lines p_l such that $p_l|A_iA_j{,}B_iB_j$; $A_iA_k{,}B_iB_k$; $A_jA_k{,}B_jB_k$ (i.e. each triplet of lines (p_l, A_iA_j, B_iB_j), (p_l, A_iA_j, B_iB_j), (p_l, A_iA_j, B_iB_j) belongs to one bundle) for $\neq (i,j,k,l)$ and each triplet of lines is related to planes $\alpha_l(A_iA_jA_k), \beta_l(B_iB_jB_k)$. Based on properties of the transformation Φ^1 we have: $\Phi^1(p_l)|B_kB_l{,}A_kA_l$; $B_jB_l{,}A_jA_l$; $B_iB_l{,}A_lA_l$. Hence $\Phi^1(p_l) = A_lB_l$. Consequently, the lines $(A_iB_i)_{i=1,2,3,4}$ have one common point if and only if lines $(p_i)_{i=1,2,3,4}$ are coplanar. By the above we have

Theorem 2 *The triangular pyramid $(A_i)_{i=1,2,3,4}$ is a Gergonne tetrahedron if and only if the lines $(p_i)_{i=1,2,3,4}$ are coplanar.*

Let us now generalize our observations. Let $\mathbb{P}^n$ be the n-dimensional projective space over the field $\mathbb{R}$ with points at infinity distinguished. Let $\mathrm{Sub}_m\mathbb{P}^n$ be the set of m-dimensional subspaces of $\mathbb{P}^n$ $(0 \leq m \leq n)$. Let us consider a simplex $(A_i)_{i=1,2,\dots,n+1}$ and a sphere S inscribed into this simplex. Let B_k be the point in which the hyperplane determined by the vertices $A_1, A_2, \dots, A_{k-1}, A_{k+1}, \dots, A_{n+1}$ is tangent to the sphere S, for $k = 1, 2, \dots, n+1$. The existence of a sphere inscribed into the simplex (circumscribed to an arbitrary simplex) can be proved similarly as it was done previously in 3-dimensional geometry. The simplex $(A_i)_{i=1,2,\dots,n+1}$ will be called the *Gergonne simplex* if the lines A_kB_k $(k = 1, 2, \dots, n+1)$ meet at a common point. Let us consider the polar correlation Φ: $\mathrm{Sub}_0\mathbb{P}^n \longmapsto \mathrm{Sub}_{n-1}\mathbb{P}^n$ induced by the sphere S (cf. [1], pp. 355–357, [4], p. 334). The correlation Φ induces a set of tranformations Φ^m: $\mathrm{Sub}_m\mathbb{P}^n \longmapsto \mathrm{Sub}_{n-m-1}\mathbb{P}^n$ for $0 \leq m \leq n-1$. Let H_i^{n-1} be the hyperplane tangent to the sphere S at point B_i, and determined by the vertices $A_1, A_2, \dots, A_{i-1}, A_{i+1}, \dots, A_{n+1}$ of the simplex $(A_i)_{i=1,2,\dots,n+1}$. Obviously, $\mathrm{H}_i^{n-1} \in \mathrm{Sub}_{n-1}\mathbb{P}^n$ and $A_i \notin \mathrm{H}_i^{n-1}$ for $1 \leq i \leq n+1$. Notice that the lines A_iB_i for $1 \leq i \leq n+1$ do not all belong to one hyperplane and, by Lemma 1, they have a common point if and only if each pair of them intersect each other. Consider an arbitrary k with $1 \leq k \leq n+1$. Every two lines (A_iB_i, A_jB_j), with $i \neq j$, $i, j \neq k$ intersect each other. The lines A_iB_i, A_jB_j have a common point if and only if the lines A_iA_j, B_iB_j have a common point, so there exists a common point P_{ij} of the lines A_iA_j, B_iB_j for all $i \neq j$, $i, j \neq k$. All the points P_{ij} belong to a space $\gamma_k^{n-2} = \mathrm{H}_k^{n-2}$ which is the common part of the hyperplanes $\alpha_k^{n-1} = \mathrm{H}_k^{n-1}(A_1, A_2, \dots, A_{k-1}, A_{k+1}, \dots, A_{n+1})$ and $\beta_k^{n-1} = \mathrm{H}_k^{n-1}(B_1, B_2, \dots, B_{k-1}, B_{k+1}, \dots, B_{n+1})$. But $\Phi(\alpha_k^{n-1}) = B_k$ (point B_k is a tangency point of the hyperplane α_k^{n-1} and the sphere S) and $\Phi(\beta_k^{n-1}) = A_k$ (β_k^{n-1} is a polar hyperplane of the point A_k in reference to the sphere S) for $1 \leq k \leq n+1$. Then it follows that $\Phi^1(\gamma_k^{n-2}) = A_kB_k$ for $1 \leq k \leq n+1$. All the lines A_kB_k with $1 \leq k \leq n+1$ meet in a common point if and only if the subspaces γ_k^{n-2} are incident with a certain hyperplane. We obtain here

Theorem 3 *The simplex $(A_i)_{i=1,2,\dots,n+1}$ is a Gergonne simplex if and only if the spaces $(\gamma_i^{n-2})_{i=1,2,\dots,n+1}$ are incident with a single hyperplane.*

Let us notice that if in a certain simplex $(A_i)_{i=1,2,\dots,n+1}$ the hyperplanes α_k^{n-1}, β_k^{n-1} are parallel for all $1 \leq k \leq n+1$, then the spaces γ_k^{n-2} including only infinite points of the space $\mathbb{P}^n$ are incident with the hyperplane at infinity of the space $\mathbb{P}^n$. Such simplex is a Gergonne simplex. Since it follows that all simplices which have exactly $(n+1)!$ own isometries (*regular simplices*), and simplices which have exactly $n!$ own isometries (*semi-regular simplices*) are Gergonne simplices in $\mathbb{P}^n$. Notice that the above $n!$ own isometries of the semi-regular simplex in $\mathbb{P}^n$ coincide with all isometries of the base of this semi-regular simplex, which is a regular simplex in $\mathbb{P}^{n-1}$.

4. About the Nagel point

A sphere S is said to be escribed to a tetrahedron if it is tangent to one of its faces and tangent to three planes which are extensions of the other faces, and which is not inscribed into this tetrahedron. Is it possible to escribe a sphere to each tetrahedron? The answer to this question is positive. Let $(A)_{i=1,2,3,4}$ be a tetrahedron. Into the trihedron with the vertex A_1 we escribe a sphere S' and, in the following, we construct two planes α'_1, α'_2 parallel to the plane $A_2A_3A_4$. One of these planes determines on the edges A_1A_i (for $i = 2, 3, 4$) points

$(A'_i)_{i=2,3,4}$ such that the sphere S' is escribed into the tetrahedron $A_1A'_2A'_3A'_4$. By using the homothety $J_{A_1}^{\frac{A_1A_2}{A_1A'_2}}$ we transform the sphere S' into the sphere S escribed onto the tetrahedron $A_1A_2A_3A_4$.

Let us now take into considerations the tetrahedron $A_1A_2A_3A_4$ and four spheres escribed to this tetrahedron. Let B'_1, B'_2, B'_3, B'_4 be the tangency points of these spheres and the tetrahedron (B'_i lies in the face $A_jA_kA_l$ for $\neq (i, j, k, l)$). The tetrahedron $A_1A_2A_3A_4$ we will called a *Nagel tetrahedron* if and only if the lines $A_iB'_i$ are concurrent for $i = 1, 2, 3, 4$. Each regular triangular pyramid is a Nagel tetrahedron. The proof for the following theorem can be done in much same way as for Theorem 1.

Theorem 4 *The tetrahedron $(A_i)_{i=1,2,3,4}$ is a Nagel tetrahedron if and only if points A_i, A_j, B'_i, B'_j are coplanar for all possible i, j with $i \neq j$.*

One can show an example of tetrahedron which is not a Nagel tetrahedron. In an analogous way we prove that each regular triangular pyramid is a Nagel tetrahedron.

A generalization of the idea of the Nagel point in the n-dimensional space is based on the concepts formulated in paragraph 3. The similar reasoning on the correlation Φ will be applied here. But now a correlation is defined in a slightly different manner:

Let $(A_i)_{i=1,2,\dots,n+1}$ be a simplex and S be an inscribed sphere in the n-dimensional space. Let α_i^{n-1} be the hyperplane tangent to the sphere S and determined by the vertices $A_1, A_2, \dots, A_{i-1}, A_{i+1}, \dots, A_{n+1}$ of the simplex $(A_i)_{i=1,2,\dots,n+1}$, for $1 \leq i \leq n+1$. Let us consider $n+1$ spheres S_i escribed into the simplex $(A_i)_{i=1,2,\dots,n+1}$; each S_i is tangent to the corresponding face α_i^{n-1}, not containing A_i, at a point B'_i. Let ${\beta'}_i^{n-1}$ be the hyperplane determined by the vertices $B'_1, B'_2, \dots, B'_{i-1}, B'_{i+1}, \dots, B'_{n+1}$ of the simplex $(B'_i)_{i=1,2,\dots,n+1}$, for $1 \leq i \leq n+1$. Let us consider a correlation Ψ: $\mathrm{Sub}_0\mathbb{P}^n \longmapsto \mathrm{Sub}_{n-1}\mathbb{P}^n$, which is determined on the points A_i by the condition: $\Psi(A_i) := {\beta'}_i^{n-1}$ for $i = 1, \dots, n+1$. Then $\Psi(\alpha_i^{n-1}) = B'_i$, so $\Psi(B'_i) = \alpha_i^{n-1}$ for all $1 \leq i \leq n+1$. Consequently, $B'_i \in \Psi(B'_i)$. The correlation Ψ induces a set of transformations Ψ_m: $\mathrm{Sub}_m\mathbb{P}^n \longmapsto \mathrm{Sub}_{n-m-1}\mathbb{P}^n$ for $0 \leq m \leq n-1$. If we write ${\gamma'}_k^{n-2}$ for the common part of the hyperplanes α_k^{n-1} and ${\beta'}_k^{n-1}$, we obtain results similar to those established in Theorem 3. All lines $(A_iB'_i)$ with $1 \leq i \leq n+1$ go through a common point if and only if the spaces ${\gamma'}_k^{n-2}$ are included in a certain hyperplane. Thus we obtain

Theorem 5 *The simplex $(A_i)_{i=1,2,\dots,n+1}$ is a Nagel simplex if and only if all the spaces $({\gamma'}_i^{n-2})_{i=1,2,\dots,n+1}$ are contained in one single hyperplane.*

Is a Gergonne simplex also a Nagel simplex? Is it a right simplex? These questions remain to be answered.

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