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Generation and Recovery of Highway Lanes

Hesham Abdelmoez Mohamed

Civil Engineering Department, Mataria Faculty of Engineering
Helwan University, P.O. Box 11718, Masaken Elhelmia, Egypt
email: habdelmoez@yahoo.com

Abstract. Highway lanes of planar shapes can be defined by specifying an arc or a straight line called the axis and a geometrical figure such as a disk or a line segment called the generator that wipes the internal boundary of the lane by moving along the axis, possibly changing size as it moves. Medial axis transformations of this type have been considered by Blum, Schwarts, Sharir and others. This research work considers such transformations for both the generation and the recovery processes. For a given highway lane generated in this way, we determine the medial axis and the generation rule that gave rise to it.

Key Words: Computer Graphics, medial axis, swept areas, highway lanes

MSC 2000: 68U05, 51M05

1. Introduction

There are many authors that have considered methods for describing *planar shapes* in terms of an arc called the axis about which the shape is locally symmetric [1,2,3,4]. This paper considers the highway lanes from the standpoints of both *generation* (of the lane, given the axis) and *recovery* (of the axis, given the lane).

Generally, we are given the *center line* (or the *axis*) of the lane and a geometric figure – called the *generator* – such as a disk or a line segment that wipes the area of the lane by moving along the axis, possibly changing size as it moves. More precisely, we assume that the generator contains a unique reference point, e.g., the center of the disk or the midpoint of the line segment. At each point O of the axis we place a copy of the generator so that its reference point coincides with O. The union of all these copies, which may be of different sizes, is the generated *highway lane*.

An early use of this approach to define planar shapes, due to Blum [1], used a disk as generator. He was more interested in description (i.e., recovery) than in generation. Blum's method was developed to describe arbitrary shapes that are not necessarily like highway lanes, using axes or skeletons that are not necessarily simple arcs. We will call Blum's case *lane of type 1.* But in this paper we will define the lane of type 1 as a highway lane only in the case where its axis is a simple arc or a straight line segment.

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There are two more cases to consider. The *lane of type 2* defines a class of highway lane that uses a line segment as generator and requires it to make a fixed angle with the axis. Lane of type 2 is more flexible than that of type 1 from a generative point of view, but as we will see, it does not allow unique recovery.

The third lane shape – *lane of type 3* – depends on local symmetry. Here, the generator is also a line segment, but it is required to make equal angles with the sides of the lane, rather than making a fixed angle with the axis.

We will refer to these types and methods for defining highway lanes as *axial representation*, since they all involve an axis that is a planar arc. Our interest in this paper is in the use of these methods to define highway lane, and we will consider various ways of restricting them so that they do indeed tend to define such lanes. We will discuss these types of lanes from the generation as well as the recovery point of view.

2. General considerations

Before considering specific types of axial representations, we make a few general observations about them, and introduce some general remarks and notations. Fig. 1 shows a piece of an *axis* s and an initial position G_O of the *generator*. We will usually assume that s is a simple, rectifiable arc with a tangent at every point, and that the generator G is a simply connected set. The reference point of the generator O will be called its *center*, and the generator whose center is at position O on s will be denoted by G_O. Then the *lane* ℓ is defined as the union of the generators G_O for all $O \in s$.

Since s and G are connected, ℓ is connected, too. One can get from any point P of any G_1 to any point Q of any G_2 by moving within G_1 from P to the center $O_1 \in s$, then along s to the center $O_2 \in s$, then within G_2 from O_2 to Q.

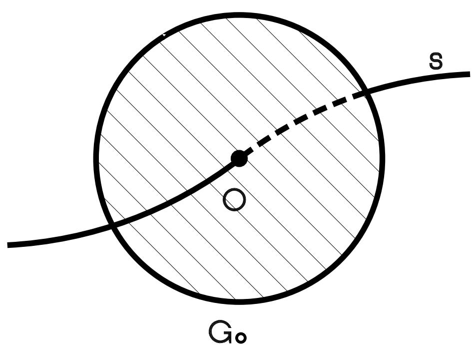

Figure 1: The generation process: G sweeps out the lane ℓ while O moves along s

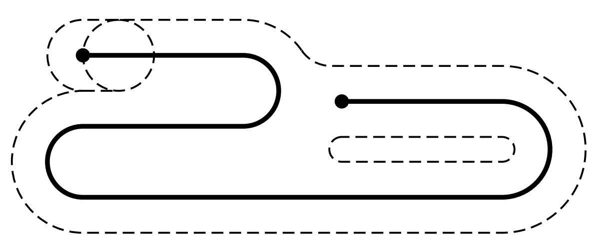

Figure 2: If ℓ is allowed to intersect itself we get shapes which do not look like highway lanes

The generators G are supposed as geometrically similar figures. They all have the same shape, and differ only in size. We will measure the size of G_O by its semidiameter (or radius) r_O. We will usually assume that r_O, as well as the orientation of G_O, vary continuously and differentiably as O moves along s.

Since we want ℓ to form a highway lane, it is reasonable to require that as G moves along s, it should not intersect any G centered at other parts of s. If such intersections were allowed, we could get shapes that were not like highway lanes, as illustrated in Fig. 2.

A closely related requirement, based on the demand that ℓ looks like a highway lane, is that no G should contain another one. By requiring this, we assure that the axis s influences the shape of ℓ. Fig. 3 shows an example of what could happen if we did not impose this restriction. We will impose the stronger requirement that each G is *maximal*, in other words, no G is strictly contained in any G-shaped region which is subset of ℓ. It follows that every G contains at least two border points of ℓ. Otherwise we could expand it slightly to obtain a larger G-shaped region still contained in ℓ, contradicting the maximality of G. Conversely, note that every border point of ℓ belongs to some G since ℓ is the union of all Gs.

Let O' and O'' be the endpoints of s. Those parts of the border of ℓ that are in $G_{O'}$ or $G_{O''}$, but not in any other G, will be called the *ends* of ℓ. The rest of the border of ℓ splits into two components, the *sides* of ℓ. These concepts are illustrated in Fig. 4. Since s is smooth (i.e., differentiable), and since the size and orientation of G_O vary smoothly as O moves along s, it is clear to see that the border of ℓ must also be smooth. We will denote the border of ℓ by b_ℓ.

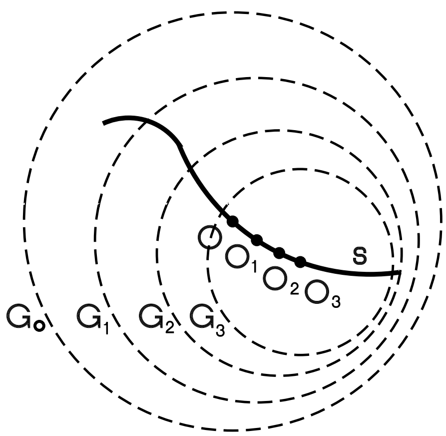

Figure 3: If any G contains another one. Here G_O is large, and the Gs get rapidly smaller as we move away from O, so that G_O is all of ℓ. The shape of ℓ is not influenced by that of s.

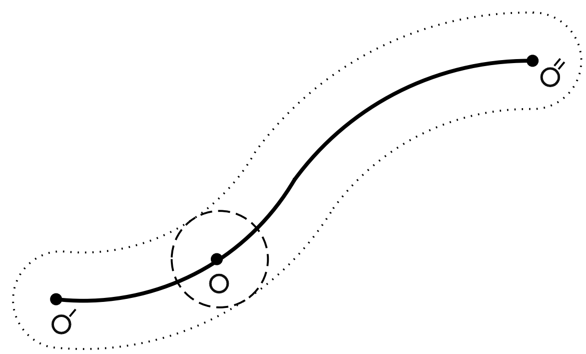

Figure 4: The *sides* and the *ends* of ℓ.

We assume that G is symmetric about its center O in the examples coming later. G will be a disk and O its center, or G will be a line segment and O its midpoint. The symmetry of G tends to make ℓ locally symmetric, but it does not imply any type of global symmetry, since the axis may be curved, and the orientation of G relative to the axis may vary.

3. Lanes of type 1

Lemma 1: *Let the lane ℓ be simply connected and let its borderline b_ℓ be smooth. Then any maximal disk D contained in ℓ is tangent to b_ℓ.*

Proof: D must intersect b_ℓ somewhere, say at P. If D is not tangent to b_ℓ at P, it would cross b_ℓ and so contain points not in ℓ — contradiction. $\square$

Lemma 1 holds not only for disks, but for any class of shapes that have smooth borders.

Lemma 2: *If ℓ is a lane of type 1, every maximal disk D contained in ℓ is one of the Gs and in particular has its center on s. Thus the set of maximal disks equals the set of its generators.*

Proof: Let D be tangent to b_ℓ at P. As P must be contained in some G_O and G_O is maximal, due to Lemma 1 G_O is tangent to b_ℓ at P, too. Thus D and G_O are both tangent to b_ℓ at P, and since both are maximal disks, they must be identical. □

Lemma 3: *If ℓ is a lane of type 1, then the axis and the set of generators of ℓ are uniquely determined.*

Proof: For every $P \in b_\ell$ we can construct the set of disks tangent to b_ℓ and contained in ℓ. Let D_P be the largest of these disks, so that D_P is a maximal disk. According to Lemma 1, the set $\{D_P \mid P \in b_\ell\}$ contains all maximal disks. By Lemma 2, this is the set of generators G, and the axis is the locus of their centers.[1] □

We thus see that lanes of type 1 are very well behaved with respect to their recoverability. However, they are more limited or harder to deal with in other respects. One limitation is that a thick lane of type 1 cannot have points of high positive curvature (=convex) on its border, provided the border b_ℓ is oriented such that the interior of ℓ lies left hand. For example, the shape shown in Fig. 5a cannot be a lane of type 1; the set of centers of its maximal disks is not a simple arc. On the other hand, points of high negative curvature are admitted, as Fig. 5b shows.

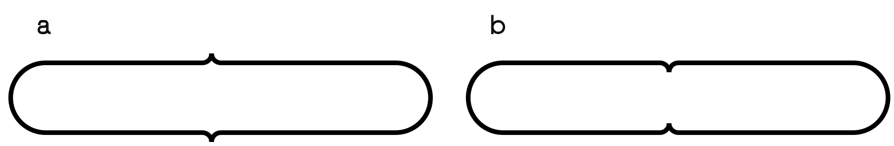

Figure 5: A thick lane of type 1 can have points of high negative curvature on its border (b), but not points of high positive curvature (a).

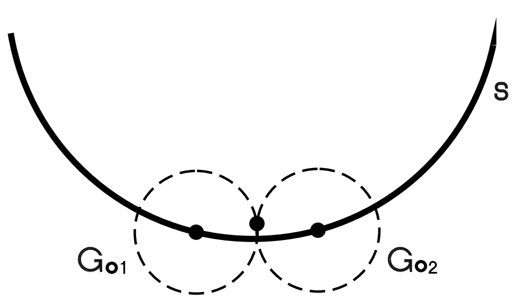

Figure 6: When the axis is curved, then two touching disks need not touch on the axis.

There are several ways to exclude self-intersection for lanes of type 1. One approach is to require that the sides do not intersect themselves or each other. To define this concept more precisely, note that by the proof of Lemma 3, there is a one-to-one correspondence between the points P on one side of ℓ and the points O on the axis. Hence, if the side intersects itself, two different centers O will correspond to the same P.

[1] A descriptive geometry approach to the construction of the axis is presented in [1] and [6].

From the purely generative point of view, it would be more appropriate to define non-self-intersection in terms of the generators themselves. But it is not easy to do this. For lanes generated by straight line segments, we could simply require that no two generators intersect, but we cannot require this in the case of disk generators, since disks having sufficiently close centers on the axis must intersect as long as their radii are bounded away from zero.

Another possibility would be to require that whenever two generators intersect, their intersection contains a part of the axis. But this does not work either in the case of disk generators since (cf. Fig. 6) at a curved axis two touching generators need not touch at a point of the axis.

A better definition seems to be the following: Let s be the set of centers of all the generators that intersect any given generator G. Then s is an arc. This excludes cases like those displayed in Figs. 7 and 8. The case of Fig. 7 is forbidden in any case because not all generators are maximal disks. But Fig. 8 cannot be excluded by non-maximality. Note that Fig. 7 also shows that thick lanes of type 1 are limited in the rate at which they can turn. A concave side of a lane of type 1 can turn rapidly, as we see in Fig. 5b. An even simpler example is that of a thick annulus with a very small inner radius.

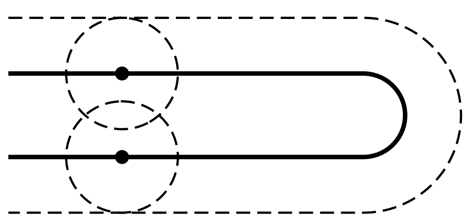

Figure 7: This example violates the condition of non-self-intersection and that of non-maximality, too.

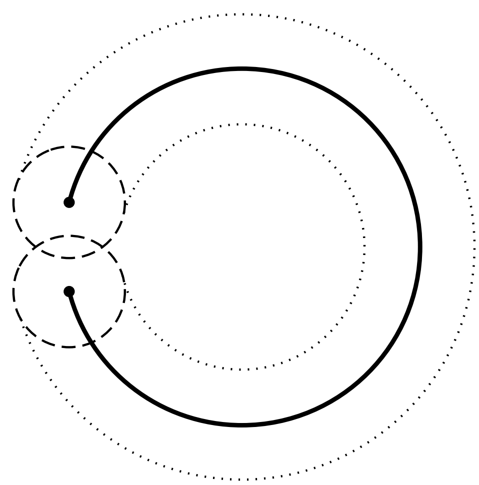

Figure 8: This example violates non-self-intersection

Another approach for defining non-self-intersection would be to regard the generator not as a disk, but as a pair of arcs terminating at the points where the disk is tangent to the sides, as in the proof of Lemma 3, and to require that no two of these arc pairs intersect. It is not easy to define a generation process purely in terms of these arc pairs. The arc-pair approach is more interesting from the recovery point of view: If P and P' are corresponding points on the two sides, the normals at P and P' will intersect each other at point O equidistant to P and P'. This will be the first time that any of these normals meets a normal that has not itself met any other normal previously. In this case, O is the axis point corresponding to P and P'.

4. Lanes generated by line segments

Suppose next that the generator is a line segment with its midpoint on the axis s. In general, we can allow both the length and the direction of the segment (relative to the axis) to vary as it moves along the axis. We will call a shape ℓ generated in this way an *L-lane*. Note that by maximality, the endpoints of any generator must both lie on the border of ℓ. In fact, the sides of ℓ are just the loci of the two endpoints, while the ends of ℓ are just the generators at the two ends of the axis.

It is trivial to formulate the non-self-intersection condition for L-lanes: We simply demand that no two generators intersect. Thick L-lanes are also not strongly limited in their ability to turn, due to the fact that the direction of the generators relative to the axis is allowed to vary (see Fig. 9).

L-lanes can also have points of high positive or negative curvature on their borders. Thus they are also more flexible in this respect than lanes of type 1. An L-lane can have arbitrary long edges (protuberances) on its border (Figs. 10a,b), provided every point on the edge is visible from the axis, in contradiction to (Figs. 10c,d). Since the slope of the generator is supposed variable, an L-lane can even have edges with overhangs (Fig. 10b). On the other hand, some combinations of edges may be impossible even if they are all visible from the axis, as the generators would have to cross one another in order to generate the lane (Fig. 10e). We learn from Fig. 10 that the class of L-lanes is somehow too large. It contains shapes that no longer look like simple lanes. Some of these examples wil be out of discussion by the restrictions that we will impose in the next two sections.

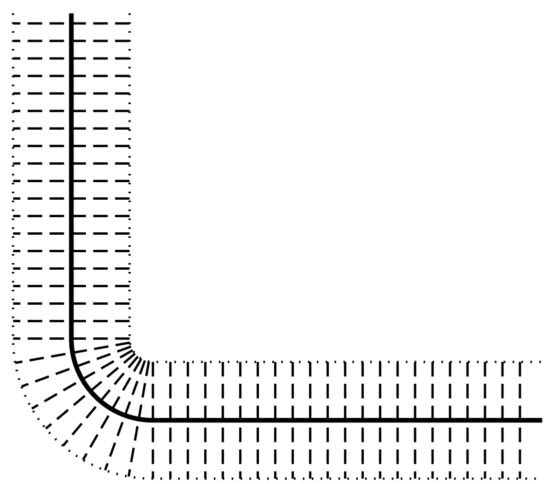

Figure 9: Thick L-lanes can make sharp turns

The problem with the most basic example Fig. 10a is that when we regard a shape as being axially generated, we prefer to choose the axis such that it has maximal length relatively to the thickness of the shape. For example, a rectangle can be generated as an L-lane (with generators perpendicular to the axis) in two ways, as shown in Figs. 11a,b , but we strongly prefer Fig. 11a because it has greater *elongatedness*. Similarly, we prefer not to regard the edges in Fig. 10 as being generated from the axis of the main part of the shape, even when this can be done legally, because the edges have much greater elongatedness w.r.t. their axis (Fig. 12).

Another serious difficulty with L-lanes is that they are highly *ambiguous*. The same shape

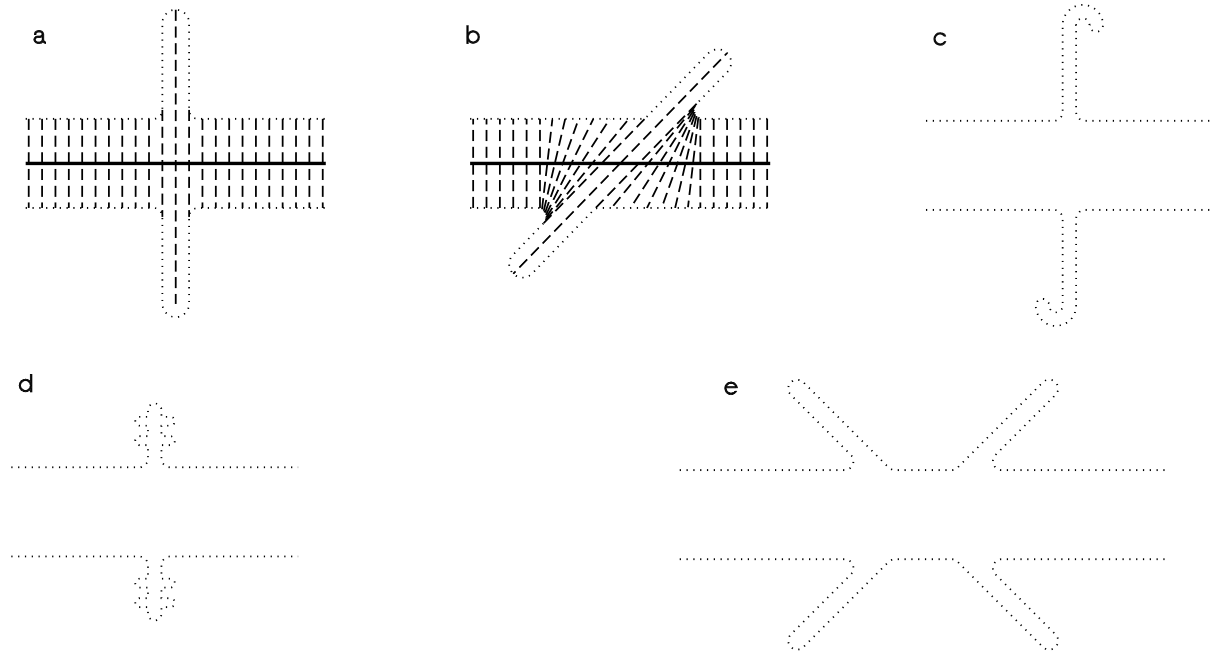

Figure 10: L-lanes can have long edges on their borders (a, b), but not if they look like in c, d, or e.

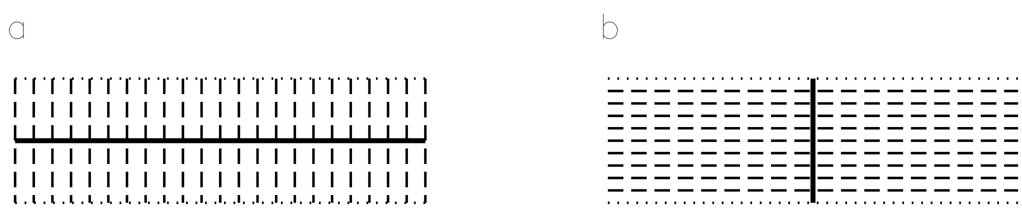

Figure 11: We prefer the generation (a) against (b) because it is more elongated.

can be generated in many different ways, even using the same axis, as illustrated in Fig. 13. Of course, we strongly prefer the generation process in Fig. 13a against that of Fig. 13b because the former is much simpler. In Fig. 13a both the size and the direction of the generator remain

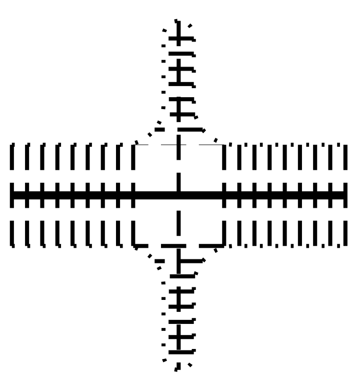

Figure 12: The edges have much greater elongatedness with respect to axis of their own.

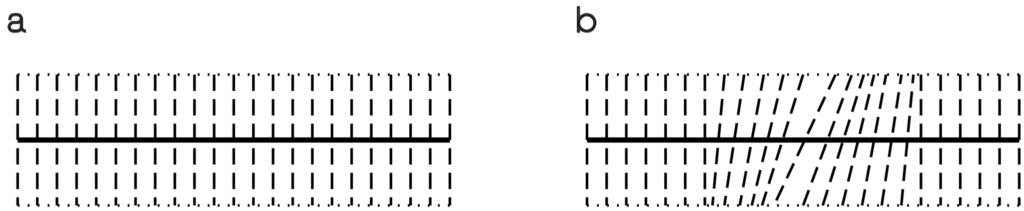

Figure 13: L-lanes are highly ambiguous; a given shape can be generated in many different ways.

constant, while in Fig. 13b they both vary. For an arbitrary shape, however, it would not be easy to formulate a certain measure of simplicity. For example, which would be prefered – constant size and variable slope, or constant slope and variable size? The priority between simplicity and degree of elongatedness is also far from being clear.

Figures 11 and 13 show that L-lanes are not recoverable. For a given highway lane, the axis and the set of generators are both far from being uniquely determined. Even if we had reliable criteria, based on simplicity and elongatedness, for preferring one generation process against another, we would still not have a constructive method for determining the best generation process for a given lane.

We can reduce the ambiguity of L-lanes if we require additional constraints. In the next two sections we consider two such constraints:

(1) requiring the generators to make a fixed angle with the axis, and

(2) requiring them to make equal angles with the sides of the lane.

5. Lanes of type 2

Our first restriction on L-lanes is that the generator is required to make a fixed angle with the axis. We will assume here, for simplicity, that the angle between the generators and the axis is always 90°.

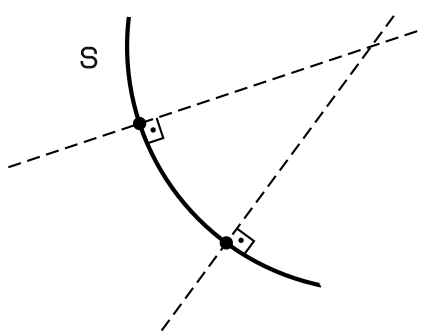

Figure 14: The thickness of a lane of type 2 cannot exceed twice the radius of curvature of its axis.

Fixing the angle has the undesirable consequence of limiting the ability of lanes of type 2 to make sharp turns. In fact, as Fig. 14 shows, the thickness of a lane of type 2 cannot exceed twice the radius of curvature of its axis. Lanes of type 2 still allow some pathological cases, such as that in Fig. 10a.

A shape can be globally ambiguos w.r.t. the type 2-generation, as we saw in Fig. 11. However, the ambiguity in Fig. 11 results from interchanging the roles of the end and the sides. If we specify which are the sides, Fig. 11 becomes unambiguous.

Lemma 4: *If the sides of the lane of type 2 are straight and parallel, its axis and generators are uniquely determined. In fact, the axis is the line parallel to the sides and midway between them.*

Proof: Let G_O be any generator, as illustrated in Fig. 15. Since O is the midpoint of G_O and the sides are parallel, by similar triangles O is midway between the sides. As this is true for any O, the axis must be the line parallel to the sides and midway between them, and the generators must thus be perpendicular to the axis and the sides. □

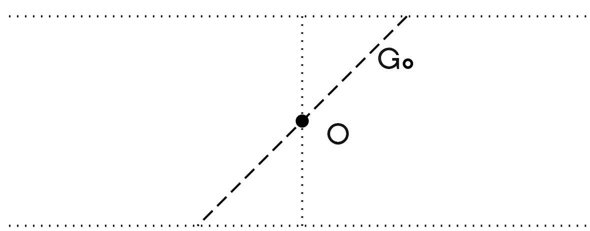

Figure 15: If a lane of type 2 has parallel straight sides, its axis must be parallel to them and midway between them.

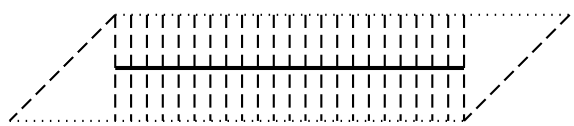

Figure 16: A parallelogram ($\neq$ rectangle) is not a lane of type 2.

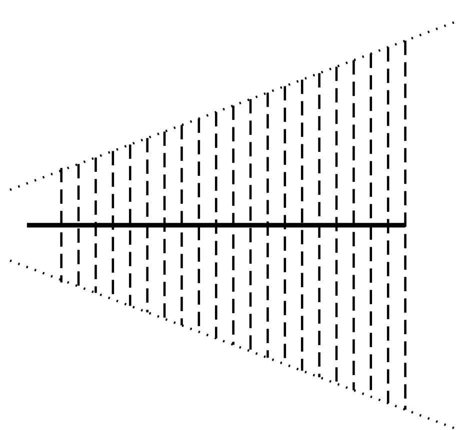

Figure 17: A lane of type 2 with straight sides.

Note that due to Lemma 4 a parallelogram cannot be a lane of type 2 (in our restricted sense) unless it is a rectangle. Since the generator must be perpendicular to the sides, it cannot generate the oblique ends (Fig. 16). To generate oblique parallelograms, we must allow the generator to make an oblique angle with the axis. In the following, we will ignore what happens at the ends of a lane, and consider only the problem of generating the parts of the sides away from the ends.

Let us now consider the case where the sides are straight but not parallel. Evidently, we can generate parts of these sides by taking the axis to be part of the straight line that bisects the angle between the sides, as shown in Fig. 17. However, this is not the only possibility. In fact, as we will next prove:

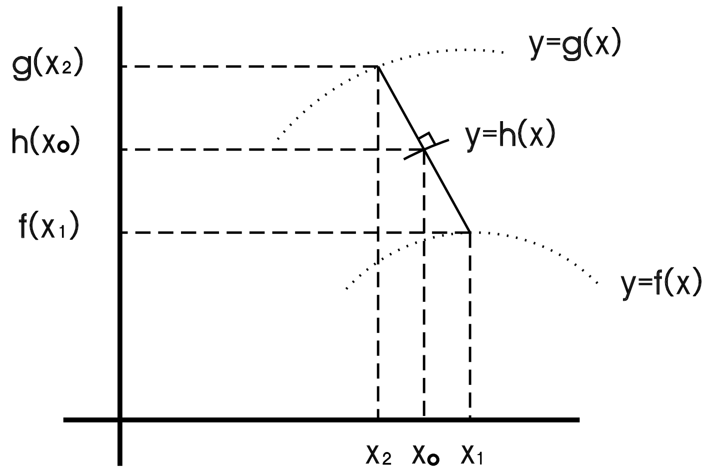

Figure 18: Deriving the differential equation of the axis $y = h(x)$.

Lemma 5: *A lane of type 2 with straight sides need not have a straight axis.*

Proof: Consider first the general case where the sides are arbitrary curves $y = f(x)$ and $y = g(x)$, as shown in Fig. 18. Let the known equation of the axis be $y = h(x)$. Let the generator centered at point $[x_0, h(x_0)]$ of the axis meet the sides at points $[x_1, f(x_1)]$ and $[x_2, g(x_2)]$. Since the midpoint of the generator is on the axis, we must have

$$x_0 = \frac{x_1 + x_2}{2} \quad \text{and} \quad h(x_0) = \frac{f(x_1) + g(x_2)}{2}\,.$$

The slope of the generator at $[x_0, h(x_0)]$ is $-1/h'(x_0)$. Thus its equation is

$$\frac{x_0 - x}{y - h(x_0)} = h'(x_0)\,,$$

so that the intersection points with the sides satisfy

$$f(x_1) = \frac{x_0 - x_1}{h'(x_0)} + h(x_0) \quad \text{and} \quad g(x_2) = \frac{x_0 - x_2}{h'(x_0)} + h(x_0).$$

We can solve these equations for x_1 and x_2 in terms of x_0, $h(x_0)$ and $h'(x_0)$ and substitute the results in the equation $x_0 = \frac{1}{2}(x_1 + x_2)$ to obtain an equation involving only x_0, $h(x_0)$ and $h'(x_0)$, i.e., a first-order differential equation for the unknown function h.

As an example, let the sides be portions of straight lines, say with equations $y = 0$ and $x = 0$. Thus the intersection points satisfy

$$0 = \frac{x_0 - x_1}{h'(x_0)} + h(x_0) \quad \text{and} \quad mx_2 = \frac{x_0 - x_2}{h'(x_0)} + h(x_0).$$

This yields

$$x_1 = x_0 + h(x_0)h'(x_0) \quad \text{and} \quad x_2[mh'(x_0) + 1] = x_0 + h(x_0)h'(x_0).$$

Thus

$$x_0 = \frac{x_1 + x_2}{2} = \frac{x_0 + h(x_0)h'(x_0)}{2}\left[1 + \frac{1}{1 + mh'(x_0)}\right]$$

or

$$2[1 + mh'(x_0)]x_0 = [x_0 + h(x_0)h'(x_0)][2 + mh'(x_0)]$$

which simplifies to

$$mx_0h'(x_0) = [2 + mh'(x_0)]h(x_0)h'(x_0).$$

Canceling $h'(x_0)$ (clearly h is not a constant, so h' is not identically zero) gives

$$mh(x_0)h'(x_0) + 2h(x_0) - mx_0 = 0$$

so that h satisfies the differential equation

$$myy' + 2y - mx = 0.$$

The general solution to this equation is found as follows: The substitution $y := xw$ gives after canceling by x

$$mw(w + xw') + 2w - m = 0.$$

Thus

$$xww' = \frac{m - 2w - mw^2}{m}$$

or

$$\frac{1}{x} + \frac{ww'}{w^2 + 2\frac{w}{m} - 1} = 0\,.$$

It can be verified that

$$\frac{ww'}{w^2 + 2\frac{w}{m} - 1} = \frac{aw'}{w + c} + \frac{bw'}{w + d}$$

where

$$a = \frac{\sqrt{m^2+1} + 1}{2\sqrt{m^2+1}}, \quad b = \frac{\sqrt{m^2+1} - 1}{2\sqrt{m^2+1}}, \quad c = \frac{1 + \sqrt{m^2+1}}{m}, \quad d = \frac{1 - \sqrt{m^2+1}}{m}\,.$$

Hence

$$\int \frac{dx}{x} + \int \frac{aw'}{w + c}\,dw + \int \frac{bw'}{w + d}\,dw = k$$

or

$$\log x + a\log(w + c) + b\log(w + d) = k$$

or

$$x(w + c)^a(w + d)^b = k'.$$

Since

$$w = \frac{y}{x} \quad \text{and} \quad a + b = 1,$$

this becomes

$$(y + ck)^a (y + dx)^b = k'.$$

If we raise both sides to the power $\dfrac{2\sqrt{m^2+1}}{m}$, we get

$$(y + cx)^c (y + dx)^{-d} = k''.$$

Noting finally that $-d = \frac{1}{c}$, we have

$$(y + cx)^c (y - \frac{x}{c})^{\frac{1}{c}} = k''.$$

The line bisecting the angle between the sides is a special case of this solution. Indeed, the slope of this line is

$$M = \tan\left(\frac{1}{2}\tan^{-1} m\right) = \frac{-1 \pm \sqrt{m^2+1}}{m} = -c \quad \text{or} \quad -d,$$

so that $y = Mx$ is a solution for $k'' = 0$. However, there is also a large family of nonlinear solutions. □

Lemma 6: *Specifying parts of the sides of a lane of type 1 does not uniquely determine the axis.*

Proof: Consider the case where the sides are perpendicular, say $y = 0$ and $x = 0$. This is not a special case of our general formulation, since the second side is not of the form $y = g(x)$. However, we can derive the differential equation for this case by the same method. It turns out to be

$$yy' = x.$$

This can be obtained from our general differential equation

$$myy' + 2y = mx$$

by dividing through m and letting $m \to \infty$. The solution to this equation is simply

$$y^2 = x^2 + C.$$

For $C \neq 0$, this is a family of hyperbolas asymptotic to the line $y = x$, and for $C = 0$ we get the line $y = x$ itself (see Fig. 19).

It can be verified that if we draw any perpendicular to such a hyperbola, the distances along the perpendicular to the two axes are equal. Note, however, that the hyperbola axes do not yield the entire axes as sides. For example the hyperbola $y^2 = x^2 - C$ shown in Fig. 19 cannot generate the interval $[0, 2C]$ of the x-axis. Thus, our straight-sided examples imply the statement. □

In the straight-sided examples, there is only one linear solution. All other solutions have higher degree. This suggests the possibility that in general there might be a unique lowest-degree solution. Unfortunately, this is not so, as we can see from considering the case where one side is a straight line and the other is a parabola, e.g., $y = 0$ and $y = ax^2$.

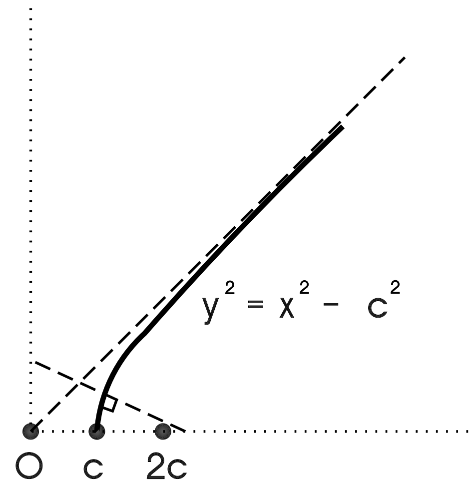

Figure 19: A lane of type 2 with straight sides need not have a straight axis.

Lemma 7: *If one side of a lane of type 2 is a straight line and the other is a parabola, the axis is not algebraic.*

Proof: Here the intersection points satisfy

$$0 = \frac{(x_0 - x_1)}{h'(x_0)} + h(x_0) \quad \text{and} \quad ax_2^2 = \frac{(x_0 - x_2)}{h'(x_0)]} + h(x_0).$$

Thus

$$x_1 = x_0 + h(x_0)h'(x_0) \quad \text{and} \quad x_2 = \frac{-1 \pm \sqrt{1 + 4ah'(x_0)[x_0 + h(x_0)h'(x_0)]}}{2ah'(x_0)}.$$

The substitution $x_0 = \dfrac{x_1 + x_2}{2}$ gives

$$x_0 = h(x_0)h'(x_0) + \frac{-1 \pm \sqrt{1 + 4ah'(x_0)[x_0 + h(x_0)h'(x_0)]}}{2ah'(x_0)}$$

so that h satisfies the differential equation

$$x = yy' + \frac{-1 \pm \sqrt{1 + 4ay'(x + yy')}}{2ay'}.$$

Transposing and squaring gives

$$1 + 4ay'(x + yy') = [1 + 2ay'(x - yy')]^2 = 1 + 4ay'(x - yy') + 4a^2y'^2(x - yy')^2.$$

Thus

$$8ayy'^2 = 4a^2y'^2(x - yy')^2, \quad \text{or} \quad 2y = a(x - yy')^2$$

where we can cancel y'^2 since y is not a constant.

It is not difficult to see that this differential equation has no polynomial solution. Note first that it has no linear solution. In fact, if $y = Ax + B$ were a solution, we would have

$$2(Ax + B) = a[x - A(Ax + B)]^2.$$

This must vanish identically in x. Hence the coefficient of each power of x must vanish. Collecting coefficients, we obtain

$$a(1-A^2)^2x^2 - 2A[1+a(1-A^2)B]x + aA^2B^2 - 2B = 0.$$

For the coefficient of x^2 to vanish we must have $A = \pm 1$. But then the coefficient of x does not vanish — contradiction.

Now suppose the equation has an algebraic solution of degree exactly $n > 1$, say $y = Ax^n + \ldots$ (terms of lower degree), where $A \neq 0$. Then the coefficient of $x^{2n(n-1)}$ is naA^2, and since this must vanish, we must have $A = 0$ — contradiction. □

It would be useful to obtain explicit solutions for the axis when the sides are polynomials of low degree, but the differential equation of the axis is extremely complicated when both sides are non-straight, e.g., even when they are both circular arcs.

6. Lanes of type 3

We consider L-lanes satisfying the condition that the generator always makes equal angles with the sides of the lane. Note first that it is not obvious how to generate a lane of type 3 from an arbitrary given axis. Let the equation of the axis be $y = h(x)$, and let the generator centered at point (x_0, y_0) of the axis have half-length r_0 and slope $\tan\theta_0$. Then the endpoints of the generator are at $(x_0 \pm r_0\cos\theta_0, y_0 \pm r_0\sin\theta_0)$, where $y_0 = h(x_0)$. The loci of the sides at the endpoints are the sides of the lane. Thus the slopes at the endpoints are

$$\frac{d(y_0 \pm r_0\sin\theta_0)}{d(x_0 \pm r_0\cos\theta_0)} = \frac{y_0' \pm r_0'\sin\theta_0 \pm r_0\theta_0'\cos\theta_0}{1 + r_0'\cos\theta \mp r_0\theta_0'\sin\theta}.$$

If we denote these slopes by $\tan\theta_1$ and $\tan\theta_2$, respectively, then the equal-angle condition means that we must have $\frac{1}{2}(\theta_1+\theta_2) = \theta_0$. In principle, we can solve this equation to find pairs of functions r_0 and θ_0 that generate lanes of type 3.

Lemma 8: *If a lane of type 3 has straight sides, its axis must be a segment of the angle bisector of the sides, and its generators must be perpendicular to the axis.*

Proof: Let the sides have slopes $\tan\theta_1$ and $\tan\theta_2$. An arbitrary line of slope $\tan\theta$ makes angles $(\theta-\theta_1)$ and $(\theta-\theta_2)$ with the sides. Thus there is only one slope for which the angles are equal, namely $\tan\theta$, where $\theta = \frac{1}{2}(\theta_1+\theta_2)$. Thus all generators must be parallel, and evidently they are perpendicular to the angle bisector. If $\theta_1 = \theta_2$, the angle bisector becomes the line parallel to the two sides and midway between them. □

Lemma 8 shows that if a lane of type 3 has straight sides, its axis and generators are uniquely determined. Lemma 8 also holds if just one side is straight.

Lemma 9: *If a lane of type 3 has just one straight side, its axis and generators are uniquely determined.*

Proof: Let the straight side have slope $\tan\theta_1$, let P be any point on the other side, and let the tangent at P have the slope θ_2, where $\theta_2 \neq \theta_1$. Just as in the proof of Lemma 8, there is a unique line through P that makes equal angles with the straight side and with the tangent at P – namely, the line having the slope $\tan\theta$, where $\theta = \frac{1}{2}(\theta_1 + \theta_2)$. Thus at every P for which $\theta_2 \neq \theta_1$, the generator is uniquely determined. Moreover, at any P for which $\theta_2 = \theta_1$ we must take the generator perpendicular to the two sides in order to assure continuity of its slope. Thus all the generators are uniquely determined, and the axis is the locus of their midpoints. □

For arbitrarily shaped sides s and t, let $P \in s$. If there exists $Q \in t$ such that the normals to s at P and to t at Q are parallel, then the line segment $\overline{PQ}$ is a generator, since it makes equal angles with the normals. And if there exists more than one such Q, there is more than one generator with endpoint P. On the other hand, as Fig. 20 shows, the normals at P and Q need not be parallel for $\overline{PQ}$ being a generator. Thus we cannot say anything simple about uniqueness in the general case.

Thick lanes of type 3 can make sharp turns. In Fig. 21, if t is very tiny, the generator $\overline{PQ}$ will have approximately the same length. Hence the locus of their centers, i.e., the axis, is approximately a circular arc parallel to s and with about half its radius. Thus the lane is more than twice as thick as the radius of curvature of its axis, but it is still able to turn without intersecting itself. This example also shows that there exist lanes of type 3 that are not of type 2.

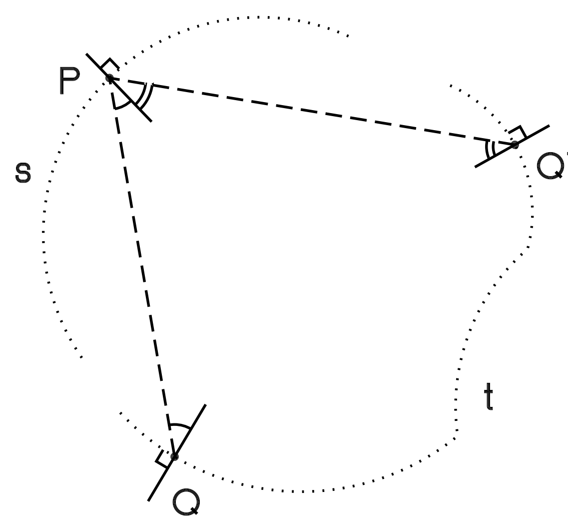

Figure 20: Generators of lanes of type 3 do not require parallel normals.

Finally we show that every lane of type 1 is a lane of type 3. Referring to the last paragraph of Section 3, the triangle OPP' is isosceles. Hence $\overline{PP'}$ makes equal angles with the normals $\overline{OP}$ and $\overline{OP'}$ to the sides of the lane. Thus $\overline{PP'}$ is a generator for a lane of type 3, and the locus of its midpoint is an axis for a lane of type 3. Note that this axis is not necessarily the same as the axis for a lane of type 1, which is the locus of O. If ℓ is a lane of type 1, we can recover its generator as in Section 3, for given pairs (P, P'). Then this determines a set of generators for lanes of type 3 of ℓ, namely the segments $\overline{PP'}$. However, there is no guarantee that this set is unique.

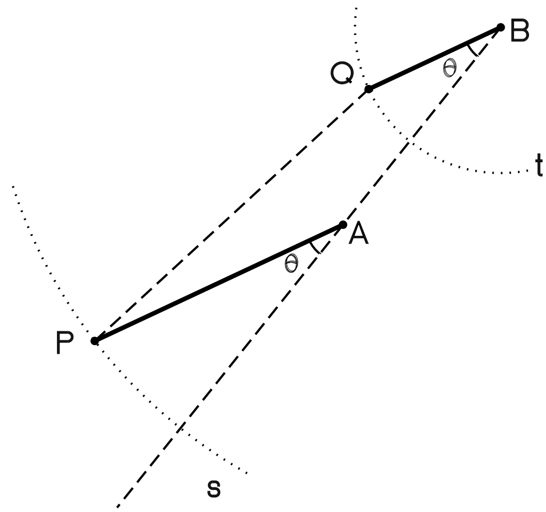

Figure 21: Thick lanes of type 3 can make sharp turns.

7. Special cases

In last two sections we have ignored what happens at the ends of a lane. By definition, the ends of a lane of type 1 must be circular arcs, while those of a lane of types 2 and 3 must be straight. In the case of L-lanes (i.e., lanes having line segment generators) we can shape the ends by attaching to each end what we shall call a *generalized sector*. This is simply a pencil of ray segments coming from each endpoint of the segment over a 180° sector (in the half-plane bounded by the last generator of the lane and not containing the adjacent generators). The lengths of the ray segments can vary in any desired way. In particular, if we make the segments all equal, the generalized sector becomes a semidisk, so that the ends of the lane are bounded. Another way of shaping the ends of an L-lane is to attach to each end a *generalized wedge*. This is a pencil of ray segments coming from the endpoint of one of the sides, and covering an angular sector bounded on one side by the last generator of the lane. Generalized wedges would be a natural way of completing lanes such as that shown in Fig. 16. Note that in both generalized sectors and generalized wedges, the generating ray segments all have one endpoint in common, rather than being disjoint as in the lane case.

If we do not ignore what happens at the ends, our three special classes of lanes types 1, 2, 3 are all incomparable. Lanes of type 1 have rounded ends, while lanes of type 2 and 3 have flat ends. There exist lanes of type 3 that are not of type 2, as we saw at the end of Section 6. Conversely, a lane of type 2 such as that in Fig. 19 is not a lane of type 3, the method of the lane of type 3 can generate straight-sided lanes, but the sides must be symmetric around their angle bisector.

Even if we ignore the ends, there are many types of lanes of type 2 or 3 that are not of type 1, e.g. Fig. 10a. And there are lanes of type 3 that are not of type 2 or 3, as we saw at the end of Section 6.

The remaining questions are: Ignoring the ends, is every lane of type 1 a lane of type 2 ? Is every lane of type 2 a lane of type 3 ? We will not settle these questions here in general, but we will settle them in the case where the axis is straight.

Lemma 10: *If the axis is straight, and we ignore the ends, every lane of type 1 is a lane of type 2, too.*

Proof: Let ℓ be a lane of type 1 whose axis lies along the x-axis, and let O be any point on the axis. Since ℓ is a union of disks centered on the x-axis, it is clear that the vertical line through O can only meet ℓ in a single connected segment. Let P and Q be the border points of ℓ directly above and below O. Any disk centered on the x-axis that contains P also contains Q, and vice versa. Hence the unique generator of ℓ that touches the boundaries b_ℓ at P also touches it at Q, so that P and Q are equidistant from the x-axis. At each point O of the axis, $\overline{PQ}$ is a generator for a lane of type 2, since it is perpendicular to the axis and O is its midpoint. The lane of type 2 generated by this set of generators is evidently ℓ (except at the ends). □

Lemma 11: *If the axis is straight, every lane of type 2 is a lane of type 3.*

Proof: Let ℓ be a lane of type 2 with a straight axis, say, located on the x-axis. Let $G(x)$ be the generator of ℓ at x so that $G(x)$ is a vertical line segment with x as its midpoint, and let $r(x)$ be the half-length of $G(x)$. The slopes of the sides of ℓ at the endpoints of $G(x)$ are evidently $\pm r'(x)$. Hence the sides make equal angles with $G(x)$, so that ℓ is a lane of type 3. □

Even if the axis is straight, a lane of type 3 need not be a lane of type 2 (see Fig. 10b). Similarly, a lane of type 2 need not be a lane of type 1 (Fig. 10a). Thus for straight axes, ignoring the ends, the three classes of lanes are strictly nested; that is

$$\text{type 1} \subset_{\neq} \text{type 2} \subset_{\neq} \text{type 3}\,.$$

8. Conclusion

We have discussed various ways of defining generalized highway lanes. In particular, we have considered three specific models, due to lanes of types 1, 2 and 3, respectively. The model of type 1 seems to have the least generative capacity. But it has unique recoverability. Lanes of type 1 are limited in their flexibility (e.g., in terms of turn radius), but they are also limited in their ability to generate not lane-like shapes. The model of type 3 has somehow more generative capacity than that of type 2, and its recoverability properties also seem to be better. Its main disadvantage is that the generation is not a straightforward process. It is not easy to specify how to define the set of generators so as to assure that they satisfy the equal angle condition of lanes of type 3.

It would be of interest to generalize the results of this research work to three dimensions by defining and comparing various classes of generalized cylinders (or cones). In 3D the axis is a space curve (or rather arc). The 3D analog of type-1-lanes uses a sphere as generator, while the analoges of the 2D schemes based on line segment generators use a planar figure (such as a disk) as generators. Here we can consider lanes of type-2-restriction in which the disk is required to remain at a fixed angle to the axis. It is less obvious how to define a 3D analog of the equal angle restriction of lanes of type 3. Note that if we use a line segment as generator in 3D, and allow its length and spatial orientation to vary, we obtain generalized space lanes rather than cylinders or cones.

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