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Packing Congruent Bricks into a Cube

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Abstract. L. Lovász raised the problem in [1] whether 27 congruent bricks of edge lengths a, b, c ($0 < a < b < c,\ a+b+c = s$) can be packed into a cube of edge length s without overlaps so that the arrangement is *universal,* in other words, it should be independent from the choice of a, b and c. If that were possible, we could obtain a geometric proof of the inequality $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$ between the arithmetic and geometric means of three positive numbers. (This would be an analogous method to the well-known proof of the inequality $\frac{1}{2}(a+b) \geq \sqrt{ab}$, $(a, b > 0)$, concerning the packing or four rectangles of edge lengths a, b into a square of edge length $a+b$.)

Hence, fundamentally, this is a special packing problem: some bricks having fixed volume must be put into a container of given volume. From the combinatorial point of view, similar container problems were investigated by D. Jennings in [2, 3].

The first author has found a possible universal arrangement, and someone else has found an additional one which has proved to be different under the symmetries of the cube. In the paper we introduce an algorithm for finding all the different universal arrangements. As a result we obtain 21 possibilities (listed in Section 4) by the corresponding computer program. Our method seems to be suitable for solving the analogous problem in higher dimensions.

Key Words: packing problem, classification problem

MSC 2000: 52C17

1. Introductory definitions

The problem of this paper is whether 27 congruent bricks of edge lengths $0 < a < b < c$ can be packed into a cube of edge length $s = a + b + c$ without overlapping in such a way that the arrangement should be independent from the choice of the lengths. Moreover, if such arrangements exist, they should be listed.

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Because of the inequality between the arithmetic and geometric means, the sum of the volume of the bricks is smaller then the volume of the containing cube: $27abc < (a+b+c)^3$. So we have a chance to find such arrangements.

We can realize that 27 congruent bricks whose edge lengths are significantly different can be put into the cube easily with many freedom. For example, if $a, b < \frac{1}{9}s$, then all bricks can be placed with the smallest face down like columns on the "horizontal" face of the containing cube, and additionally they can be tilted a bit. If, however, the differences between the edge lengths are small enough, then the problem becomes much harder, and the number of the possible arrangements decreases. In this case, as we will see, the edges of the bricks must be parallel to the corresponding edges of the containing cube, and the arrangement has a 3×3×3 structure similarly to the trivial dissection of the cube into 27 small cubes of edge length $\frac{1}{3}s$. The universal (edge length independent) arrangements can be derived exactly from this case.

Considering the 3×3×3 structure, for the simplicity of our formulas, we choose the small cubes of the trivial dissection as unit cubes, consequently, the edge length of the containing cube is $s = 3$. Moreover, in the future we assume that the symbols a, b, c with indices, primes and stars also mean edge lengths satisfying the original properties (e.g. $0 < a' < b' < c'$ and $a' + b' + c' = s = 3$).

We introduce the notation $B(x, y, z)$ and $C(x)$ for a brick of edge lengths x, y, z and a cube of edge length x, respectively. The containing cube ($C(3)$) will be denoted by $\overline{C}$. Finally, $\mathcal{A} = \{B_i(a, b, c),\ O_i\}$ will denote an arrangement of the 27 bricks of edge lengths a, b, c and centres O_i $(i = 1, 2, \ldots, 27)$.

Definition 1 We say that two bricks $B_1(a_1, b_1, c_1)$ and $B_2(a_2, b_2, c_2)$ are *parallel* ($B_1 \parallel B_2$) if the edges of lengths a_1 and a_2, b_1 and b_2, c_1 and c_2 are parallel, respectively.

Remark 1 In the description of the relative situation of a unit cube $C(1)$ or $\overline{C}$ and a brick $B(a, b, c)$ we also use the terminology of *parallelism*. In this case we require that each edge of the brick should be parallel to an edge of the cube.

Definition 2 Let two parallel bricks $B_1(a_1, b_1, c_1)$ and $B_2(a_2, b_2, c_2)$ be given. Then, the parallel brick $T(|a_1 - a_2|,\ |b_1 - b_2|,\ |c_1 - c_2|)$ $(T \parallel B_1 \parallel B_2)$, having common centre with B_1, is called the *brick of admissible translations* of B_1 and B_2.

Definition 3 Two arrangements $\mathcal{A}^1 = \{B_i^1(a_1, b_1, c_1),\ O_i^1\}$ and $\mathcal{A}^2 = \{B_i^2(a_2, b_2, c_2),\ O_i^2\}$ are said to be *compatible* if $B_i^1 \parallel B_i^2$ and O_i^2 belongs to the closed brick T_i of admissible translations of B_i^1 and B_i^2 for each i.

Remark 2 It is easy to see that the *compatibility* is a reflexive and symmetric relation between two arrangements.

Definition 4 We say that the arrangement $\mathcal{A} = \{B_i(a, b, c),\ O_i\}$ is *universal* if there exists a compatible arrangement $\mathcal{A}' = \{B_i'(a', b', c'),\ O_i'\}$ for each choice of edge lengths a', b', c' satisfying the original properties.

2. Universal arrangements

In this section we assume that there exist universal arrangements and examine their structure. For the time being, we do not know even that the bricks are parallel to the containing cube $\overline{C}$ in these arrangements. In Theorem 1, which will be prepared by Lemma 1, we will prove this parallelism and the $3 \times 3 \times 3$ structure. Then, on the basis of Lemma 2, we will prove in Theorem 2 that in these arrangements there are only finitely many possible positions of a brick in $\overline{C}$.

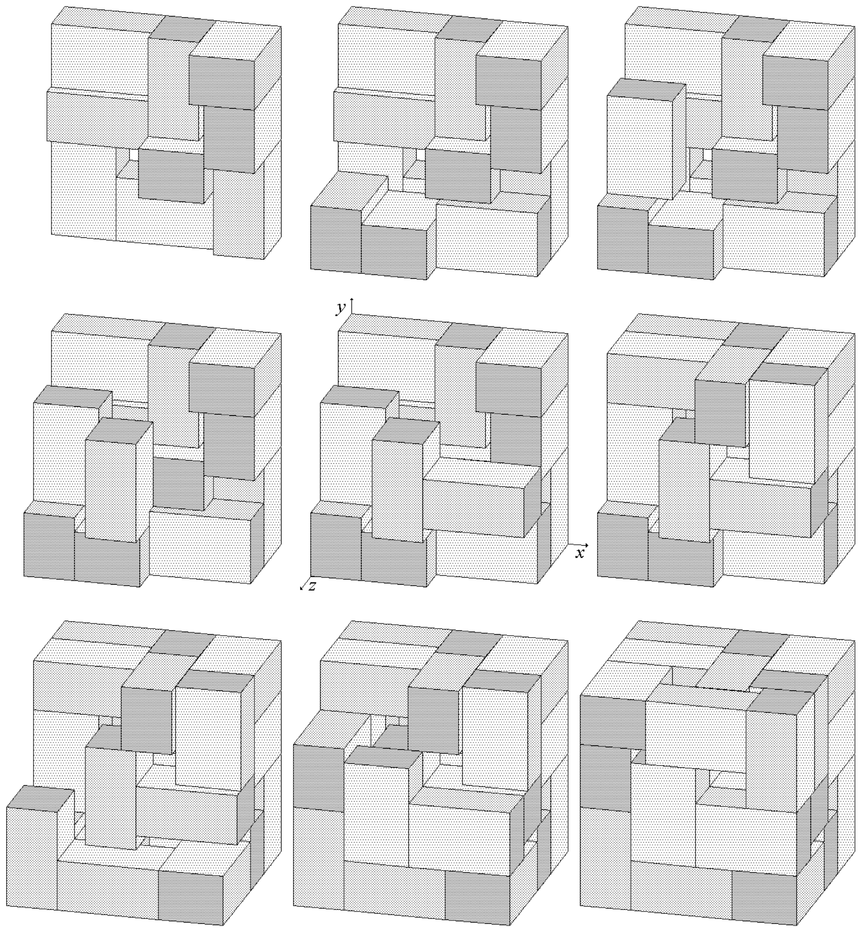

Figure 1: Steps of packing the universal arrangement no. 1

In order to demonstrate these statements, in Fig. 1 we display some steps of packing according to a universal arrangement. The faces of edge lengths $a \times b$, $a \times c$, $b \times c$ are denoted by dark, medium and light shades of gray, respectively. In Section 4 this arrangement will occur with number 1 in the list of the 21 different universal arrangements, describing with coordinates. The direction of coordinate axes are displayed in the central small figure.

Lemma 1 *Let* $\mathcal{A} = \{B_i(a, b, c),\ O_i\}$ *be a universal arrangement and* $\mathcal{A}^\star = \{C_i^\star(1),\ O_i^\star\}$ *be the trivial dissection of the cube into 27 unit cubes* $C_i^\star(1)$*. Then there exists an infinite sequence of arrangements* $\mathcal{A}^k = \{B_i^k(a_k, b_k, c_k),\ O_i^k\}$ *from* $\mathcal{A}^0 \equiv \mathcal{A}$ *tending to* $\mathcal{A}^\star$ *so that* $\mathcal{A}_k$ *is compatible with* $\mathcal{A}$ *for each* k $(k = 0, 1, 2, \ldots)$*.*

Proof: We construct the claimed sequence of arrangements. First we define the edge lengths of B_i^k in $\mathcal{A}^k$ in the following way:

$$a_k = 1 - \frac{1-a}{2^k}, \qquad b_k = 1 - \frac{1-b}{2^k}, \qquad c_k = 1 - \frac{1-c}{2^k}$$

(preserving the original properties: $a_k + b_k + c_k = 3$ and $0 < a_k < b_k < c_k$). Thus the differences between the edge lengths of B_i^k and B_i are

$$|a_k - a| = \frac{2^k-1}{2^k}|1-a|, \qquad |b_k - b| = \frac{2^k-1}{2^k}|1-b|, \qquad |c_k - c| = \frac{2^k-1}{2^k}|1-c|.$$

We can see that these differences are less than $|1-a|$, $|1-b|$, $|1-c|$, respectively.

Since $\mathcal{A}$ is universal, a compatible arrangement $\mathcal{A}^k = \{B_i^k(a_k, b_k, c_k),\ O_i^k\}$ must exist for each k (by Definition 3), so that the brick centre O_i^k belongs to the closed brick $T_i(|1-a|,\ |1-b|,\ |1-c|)$ of centre O_i. Thus, there is a subsequence of arrangements $\mathcal{A}^{k_\ell}$ in which the corresponding centres $O_i^{k_\ell}$ have a limit point O_i' for each i, simultaneously.

Since the edge lengths of $B_i^{k_\ell}$ tend to 1, the limit brick B_i' is a unit cube with centre O_i' for each i. As $\mathcal{A}^{k_\ell}$ is an arrangement in $\overline{C}$ for each k_ℓ, the limit system $\mathcal{A}' = \{B_i'(1,1,1),\ O_i'\}$ must also be an arrangement. However, 27 unit cubes can be packed into $\overline{C}$ only in the trivial manner, which means that $\mathcal{A}'$ is identical with the trivial dissection $\mathcal{A}^\star$, moreover $B_i' \equiv C_i^\star$ and $O_i' \equiv O_i^\star$ for each i. $\square$

Theorem 1 *Let $\mathcal{A} = \{B_i(a,b,c),\ O_i\}$ be a universal arrangement. Then the bricks B_i of $\mathcal{A}$ are parallel to the containing cube $\overline{C}$* (see Remark 1). *Furthermore, if $|1-a|$, $|1-b|$, $|1-c|$ are sufficiently small, then the bricks have a $3{\times}3{\times}3$ structure in $\overline{C}$. In other words, they can be partitioned into 3 layers containing $3{\times}3$ bricks one by one, or into $3{\times}3$ columns containing 3 bricks one by one.*

Proof: By Lemma 1, there exists a sequence of arrangement $\mathcal{A}^k = \{B_i^k(a_k, b_k, c_k),\ O_i^k\}$ which tends to the trivial dissection $\mathcal{A}^\star = \{C_i^\star(1),\ O_i^\star\}$ and contains only elements being compatible with $\mathcal{A}$. Thus, the limit arrangement $\mathcal{A}^\star$ is also compatible with $\mathcal{A}$ (Definition 3, Remark 2). In this way $B_i \parallel C_i^\star \parallel \overline{C}$, which means that B_i is parallel to $\overline{C}$ for each i, as we stated. Moreover, the centre O_i belongs to the closed brick of admissible translation $T_i(|1-a|,\ |1-b|,\ |1-c|)$ of centre $O_i^\star$ for each i. These bricks are disjoint if $\max(|1-a|,\ |1-b|,\ |1-c|) < \frac{1}{2}$. In this case $O_j \in T_i$ if and only if $j = i$. Thus, we get a one-to-one correspondence between the sets of centres $\{O_i^\star\}$ and $\{O_i\}$, and similarly, between the cubes $C_i^\star$ and the bricks B_i. Therefore, $\mathcal{A}$ gets the $3{\times}3{\times}3$ structure of $\mathcal{A}^\star$. $\square$

Lemma 2 *Let $\mathcal{A} = \{B_i(a,b,c),\ O_i\}$ be a universal arrangement. Then there is no brick in $\mathcal{A}$ having a free face. In other words, each face of a brick B_i has a common rectangular surface with a face of another brick B_j or that of the containing cube $\overline{C}$ for each i.*

Proof: Considering the equation $a+b+c = 3$, we can determine the edge lengths of the bricks by two independent parameters μ and ν as follows:

$$a := 1 - \mu, \qquad c := 1 + \nu, \qquad b := 1 + \mu - \nu.$$

The inequalities $0 < a < b < c$ yield the permissible values of μ and ν (Fig. 2a):

$$0 < \mu < 1, \qquad 0 < \nu < 2, \qquad \frac{1}{2}\mu < \nu < 2\mu.$$

Let δ denote the difference between the volume of $\overline{C}$ and the summarized volume of the 27 bricks. Then

$$\frac{1}{27}\delta(\mu,\nu) = 1 - abc = \mu^2\nu - \mu\nu^2 + \mu^2 + \nu^2 - \mu\nu.$$

In contrast with the statement, we suppose that a face f of a brick B_i of $\mathcal{A}$ is free. Since $\mathcal{A}$ is universal, there exists a compatible arrangement $\mathcal{A}'$ with $\mathcal{A}$ for each pair (μ,ν), satisfying the same incidence properties. In this way the corresponding face f' in $\mathcal{A}'$ is also free.

Let us consider the rays intersecting f perpendicularly. These rays form a prism of height 3 in the cube. Let $\delta^\star$ denote the summarized empty volume in the interior of the prism. We will see that $\delta^\star > \delta$ for certain values of μ and ν, and this fact will be a contradiction.

First of all, we determine the minimal total length $\lambda(\mu,\nu)$ of free segments along a ray intersecting f perpendicularly. Because of the 3×3×3 structure of the bricks (see Theorem 1) a ray can go through at most 3 bricks. In this way

$$\lambda \in \{3-p,\ 3-p-q,\ 3-p-q-r \mid p,q,r \in \{a,b,c\}\} \cap \mathbb{R}^+.$$

Looking for the minimal positive λ, we have to solve a system of linear inequalities. The result is a partition of the parameter domain (μ,ν). In Fig. 2a the shading of sub-domains refers to the minimal positive expression of $\lambda(\mu,\nu)$.

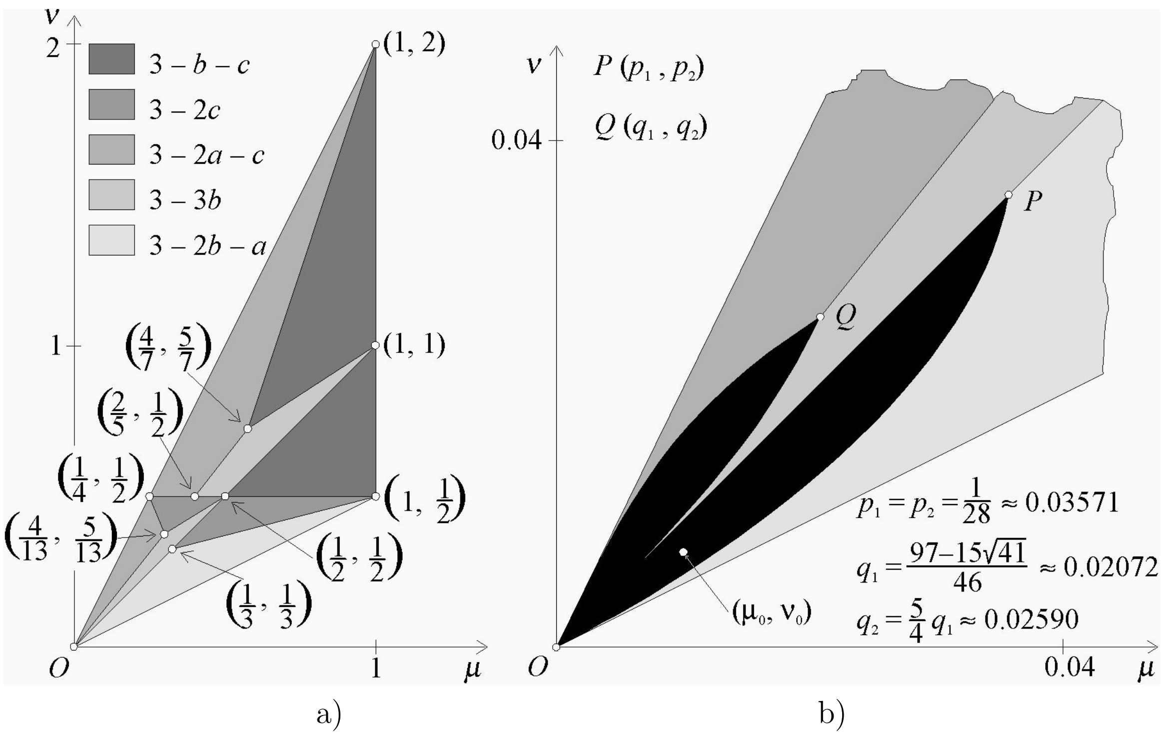

Figure 2: Partition of the parameter domain for (μ,ν)

The area of the face f is at least $a(\mu)b(\mu,\nu) = 1-\nu-\mu^2+\mu\nu$, so we obtain that

$$\delta^\star \geq (1-\nu-\mu^2+\mu\nu)\cdot\lambda(\mu,\nu).$$

It is enough to show that there is a pair (μ,ν) for which $\delta^\star > \delta(\mu,\nu)$. Let $\mu_0 = 0.01$ and $\nu_0 = 0.0075$ (Fig. 2b). The pair (μ_0,ν_0) belongs to the sub-domain in which $\lambda(\mu,\nu) = 3-2b-a = -\mu+2\nu$, and so

$$\delta^\star - \delta \geq -\mu+2\nu-27\mu^2-29\nu^2+28\mu\nu+\mu^3-30\mu^2\nu+29\mu\nu^2.$$

Substituting μ_0 and ν_0, we get exactly 0.027 635 625 on the right side, which means that $\delta^\star > \delta(\mu_0,\nu_0)$.

The universal arrangement $\mathcal{A}$ should also be realized with bricks of edge lengths $a = 1-\mu_0 = 0.99$, $b = 1+\mu_0-\nu_0 = 1.0025$, $c = 1+\nu_0 = 1.0075$. However, assuming the existence of a free face f in $\mathcal{A}$, we obtain that there is a larger empty space in the prism with cross-section f than in the whole $\overline{C}$. This is a contradiction. □

Remark 3 By detailed elementary computations, the inequality $a(\mu)b(\mu,\nu)\lambda(\mu,\nu) > \delta(\mu,\nu)$ can be solved. In Fig. 2b the domains coloured black contain the points whose coordinates (μ,ν) satisfy the inequality. The edges of partition support the curves at the origin: the equations of tangent lines are $\mu-2\nu=0$, $\mu-\nu=0$, $2\mu-\nu=0$. Moreover, the line segment OP belongs to the domain.

Theorem 2 *Let the origin of a cartesian coordinate system be fixed at a vertex of the containing cube $\overline{C}$, and let the positive half-axes be determined by the edges joining this vertex. If $\mathcal{A} = \{B_i(a,b,c),\ O_i\}$ is a universal arrangement, then the coordinates of vertices of the bricks B_i belong to the set $\mathcal{S} = \{0,a,b,c,\ 3-c,\ 3-b,\ 3-a,\ 3\}$ for each i.*

Proof: By Theorem 1, the bricks of $\mathcal{A}$ are parallel to $\overline{C}$, therefore, the faces of the bricks are perpendicular to the corresponding coordinate axes. Choosing an arbitrary brick B of $\mathcal{A}$, we consider its opposite faces f and g normal to the axis x. Let x_f and x_g denote the first coordinates of points of these faces, respectively. We can suppose that $x_f < x_g$. Then the difference $x_g - x_f$ belongs to $\{a,b,c\}$. By Lemma 2, nor f neither g can be free. The following cases must be distinguished:

1. f joins $\overline{C}$ in the plane $x=0$. Then $x_f = 0$ and $x_g \in \{a,b,c\}$.
2. g joins $\overline{C}$ in the plane $x=3$. Then $x_g = 3$ and $x_f \in \{3-c,\ 3-b,\ 3-a\}$.
3. Neither f nor g join $\overline{C}$. Then there are two bricks B^f and B^g of $\mathcal{A}$ joining f and g, respectively. Let us consider the opposite faces f' and g' of B^f and B^g, respectively. By Theorem 1, f' and g' cannot join another brick (in the $3\times3\times3$ structure each column contains exactly 3 bricks). In this way, by Lemma 2, f' and g' must join $\overline{C}$ along its opposite faces, so $x_f \in \{a,b,c\}$ and $x_g \in \{3-c,\ 3-b,\ 3-a\}$.

Collecting the possible first coordinates above, we get $\mathcal{S}$. Similarly, we obtain that the second and third coordinates also belong to $\mathcal{S}$. □

3. Data structures and procedures of the algorithm

Data 1 *Coordinates.* We will use the following notations: $u = 3-a = b+c$, $v = 3-b = a+c$, $w = 3-c = a+b$. By Theorem 2, the coordinates of vertices belong to $\mathcal{S} = \{0,a,b,c,w,v,u,3\}$. *In our algorithm we will use only these symbols for the coordinates and consider $\mathcal{S}$ as a set of symbols.* Assuming that the differences between a, b and c are sufficiently small and so $c < w$ ($c < \frac{3}{2}$), the list $0,\ a,\ b,\ c,\ w,\ v,\ u,\ 3$ is a strictly monotonic sequence. Thus we will be able to compare two coordinates determining their positions in the list. Similarly, we can assume that $w < c$. Then our strictly monotonic sequence is $0,\ a,\ b,\ w,\ c,\ v,\ u,\ 3$. However, as we will see in Procedure 4, this ordering leads not only to universal arrangements.

The addition (and subtraction) of symbolic coordinates can be determined by an 8×8 table (or matrix) $\mathcal{M}$ indexing on $\mathcal{S}$. Considering the symbols $p,\ q \in \mathcal{S}$, the element m_{pq} of $\mathcal{M}$ contains the sum $p+q$ if it belongs to $\mathcal{S}$, or the symbol "$*$" if $p+q \notin \mathcal{S}$. It is easy to see that $\mathcal{M}$ is independent from the relation between c and w.

Data 2 *Bricks.* By Theorem 1, a brick B can be represented uniformly by its nearest and farthest (so-called *canonic*) vertices, V_1 and V_2, from the origin. Moreover, by Theorem 2, the coordinates of V_1 and V_2 belong to $\mathcal{S}$. In this way, a brick B will be represented by two triples of symbols, and we will use the notation $B(V_1, V_2)$. Finally, we mention that the coordinates of the canonic diagonal vector $\overrightarrow{V_1V_2}$ form a permutation of a, b, c.

Procedure 1 *Intersection of two bricks.* It is easy to verify that two closed intervals $[e_1, e_2]$ and $[f_1, f_2]$ have common inner points if and only if $e_1 < f_2$ and $f_1 < e_2$. From this fact we get the simple condition for deciding whether two bricks $B'(V_1'(x_1', y_1', z_1'),\ V_2'(x_2', y_2', z_2'))$ and $B''(V_1''(x_1'', y_1'', z_1''),\ V_2''(x_2'', y_2'', z_2''))$ have common inner points. Indeed, those exist if and only if the inequalities $x_1' < x_2''$, $x_1'' < x_2'$, $y_1' < y_2''$, $y_1'' < y_2'$, $z_1' < z_2''$, $z_1'' < z_2'$ hold simultaneously. Examining these conditions our procedure decides whether two bricks intersect each other. However, as we can see, the result may depend on the relation between c and w.

Procedure 2 *Possible bricks.* We shall collect all the possible bricks (data 2) into a list $\mathcal{B}$. In the beginning let the list be empty: $\mathcal{B} = \emptyset$. We enumerate all the $8^3 = 512$ possible positions of vertices as points, choosing the three coordinates independently from $\mathcal{S}$. Considering the current point $P_0(x_0, y_0, z_0)$ in each step, we translate it by the possible diagonal vectors (a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b) and (c, b, a) on the basis of table $\mathcal{M}$ (data 1). So we get the points $P_i(x_i, y_i, z_i)$ of symbolic coordinates $(i = 1, 2, \ldots, 6)$. P_0 and P_i determine a possible brick if and only if x_i, y_i, $z_i \in \mathcal{S}$, namely, each of them is different from "$*$". Then we add the brick $B(V_1 \equiv P_0,\ V_2 \equiv P_i)$ to our list: $\mathcal{B} := \mathcal{B} \cup \{B\}$. In this way, finally, we obtain $\mathcal{B}$ containing exactly 288 possible bricks.

According to the $3\times3\times3$ structure of universal arrangements, we classify the possible bricks into 27 disjoint *position classes,* associating each brick of $\mathcal{B}$ with a unit cube of the trivial dissection $\mathcal{A}^\star$. By Theorem 1 and 2, the 27 bricks of a universal arrangement will belong to different classes.

Definition 5 Let us consider the trivial arrangement $\mathcal{A}^\star$ of 27 unit cubes and the system of their centres

$$O_{pqr}(\frac{2p-1}{2},\quad \frac{2q-1}{2},\quad \frac{2r-1}{2}),$$

where $p, q, r \in \{1, 2, 3\}$. Assuming that $c < w$ and so $0 < \frac{1}{2} < a < b < c < \frac{3}{2} < w < v < u < \frac{5}{2} < 3$, we can define the sets $\mathcal{C}_{pqr} \subset \mathcal{B}$ by the equation $\mathcal{C}_{pqr} = \{B \,|\, B \in \mathcal{B},\ O_{pqr} \in B\}$. Then a possible brick can contain exactly one centre O_{pqr}, which means that the sets $\mathcal{C}_{pqr}$ are disjoint. Thus, we get a classification of $\mathcal{B}$ (see also the proof of Theorem 1). The sets $\mathcal{C}_{pqr}$ are said to be *position classes.*

Procedure 3 *Classification.* Taking the possible bricks from $\mathcal{B}$, we form the position classes $\mathcal{C}_{pqr}$ as lists according to Definition 5. In this way we obtain the class $\mathcal{C}_{222}$, associated to the central unit cube, containing exactly 48 bricks. The other $8 + 12 + 6$ classes, associated to "vertex-", "edge-" or "face-fitting" unit cubes, contain exactly 6, 12 or 24 bricks, respectively.

Definition 6 Let Σ denote the symmetry group of the containing cube $\overline{C}$. We say that two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *essentially different* if there is not any symmetry of $\overline{C}$ (in Σ) mapping the bricks of $\mathcal{A}_1$ onto the bricks of $\mathcal{A}_2$.

Remark 4 Let X be a point whose coordinates belong to $\mathcal{S}$. It is easy to see that the coordinates of the image point X^σ also belong to $\mathcal{S}$ for each symmetry $\sigma \in \Sigma$. Similarly, if

$B \in \mathcal{B}$ is a possible brick, then its image B^σ also belongs to $\mathcal{B}$. Indeed, the vertex coordinates of B^σ belong to $\mathcal{S}$, moreover, since B is parallel to $\overline{C}$, B^σ is also parallel to $\overline{C}$. Finally, B^σ is congruent with B, so the edge lengths of B^σ is a, b and c, too.

Remark 5 Now we consider a brick $B_1 \in \mathcal{C}_{222} \subset \mathcal{B}$. For example (according to our list of results in Section 4) we choose the brick $B_1(V_1(b,a,c),\ V_2(w,v,u))$, whose distinguished diagonal vector is (a,c,b). Since B_1 contains the centre centre of the central unit cube and $\overline{C}$ if $c < w$, the images B_1^σ also contain that point, so they also belong to the position class $\mathcal{C}_{222}$. Moreover, an image brick B_1^σ is identical with B_1 if and only if σ is the identity element of Σ (assuming that $\frac{1}{2}(a+c) \neq b$, of course). Taking the result of Procedure 3 into consideration, $\mathcal{C}_{222}$ is just the set of the 48 images of B_1 under Σ. It is an important consequence of this fact, that *two different arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ containing the same "central" brick from $\mathcal{C}_{222}$ are essentially different.*

Procedure 4 *Searching for universal arrangements.* By Remark 5, we consider a list $\widetilde{\mathcal{A}}$ of bricks containing only $B_1 \in \mathcal{C}_{222}$ in the beginning. Choosing disjoint bricks from the other position classes $\mathcal{C}_{ijk}$ (exactly one brick from each class), we will complete $\widetilde{\mathcal{A}}$ to get arrangements. The basis of our procedure is the so-called *back-track algorithm* (see e.g. [4]).

First of all we fix an order of position-classes where $\mathcal{C}_1 = \mathcal{C}_{222}$. For example $\mathcal{C}_2 = \mathcal{C}_{111}$, $\mathcal{C}_3 = \mathcal{C}_{211}$, …, $\mathcal{C}_{27} = \mathcal{C}_{333}$, (making distinction between one and three subscript form of $\mathcal{C}$). $B_2, B_3, \ldots, B_{27}$ will denote the chosen elements of these classes in $\widetilde{\mathcal{A}}$, respectively. We introduce a list of variables $n_1, n_2, \ldots, n_{27}$. During the procedure, n_i shows that we just examine whether the n_i-th element of $\mathcal{C}_i$ fits to the bricks $B_1, B_2, \ldots, B_{i-1}$ of $\widetilde{\mathcal{A}}$. In the beginning $n_1 = 1$, and $n_i = 0$ for $i = 2, 3, \ldots, 27$. The variable i points the class $\mathcal{C}_i$ from which we just try to choose an appropriate brick B_i into $\widetilde{\mathcal{A}}$. In the beginning $i = 2$.

1. We examine the value of i:
   – If $i = 1$, then we cannot create additional essentially different arrangements, so *the procedure comes to an end.*
   – If $i = 28$ then $\widetilde{\mathcal{A}}$ contains exactly one brick from each class $\mathcal{C}_i$ determining an arrangement $\mathcal{A} = \widetilde{\mathcal{A}}$, so we print the symbolic coordinates (see data 1) of $B_1, B_2, \ldots, B_{27}$ as a result. In order to find additional arrangements, we delete the last brick B_{27} from $\widetilde{\mathcal{A}}$, modify i to 27, and go to 2.
   – Otherwise we go to 2.
2. We increment n_i and examine whether $n_i > |\mathcal{C}_i|$.
   – If so, then we cannot choose an appropriate new brick B_i from $\mathcal{C}_i$. Thus we modify n_i to 0, delete the brick B_{i-1} from $\widetilde{\mathcal{A}}$, decrease i, and go to 1.
   – Otherwise we go to 3.
3. Applying Procedure 1, we examine whether the n_i-th brick B of $\mathcal{C}_i$ intersects the existing bricks $B_1, B_2, \ldots, B_{i-1}$ of $\widetilde{\mathcal{A}}$.
   – If so, then B cannot be added to A, so we go to 2.
   – Otherwise we add B to $\widetilde{\mathcal{A}}$ as B_i, increment i, and go to 1.

We can see that Step 3 depends on the relation between c and w. In this way we obtain two sets $\Omega_{c<w}$ and $\Omega_{w<c}$ containing 21 and 144 essentially different arrangements, respectively. Considering the symbolic coordinates of bricks, we can compare the arrangements of $\Omega_{c<w}$ and $\Omega_{w<c}$, and we get the result: $\Omega_{c<w} \subset \Omega_{w<c}$. Moreover, if $\mathcal{A}$ is an arbitrary arrangement of $\Omega_{w<c} \setminus \Omega_{c<w}$ and we realize $\mathcal{A}$ with bricks of edge length condition $c < a + b = w$, then we can always find intersecting bricks in $\mathcal{A}$. Similarly, examining an arbitrary arrangement of

$\Omega_{c<w}$, we find that the bricks are disjoint independently from the relation between c and w. Thus, $\Omega_{c<w}$ is the complete set of essentially different universal arrangements.

4. Summary. The list of the universal arrangements

Summarizing Theorem: *There are exactly* 21 *essentially different universal arrangements of* 27 *congruent bricks of edge lengths a, b and c in a cube of edge length $a+b+c$.*
These arrangements are enumerated below, describing the coordinates of canonic vertices of their bricks, excepting the common central brick $B_1(V_1(b,a,c),\ V_2(w,v,u))$ (briefly $(b,a,c;\ w,v,u)$).

1: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,b,a)$, $(u,0,0;\ 3,c,b)$, $(0,c,0;\ c,v,b)$, $(c,b,0;\ u,w,c)$, $(u,c,0;\ 3,u,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ v,3,b)$, $(v,u,0;\ 3,3,c)$, $(0,0,a;\ a,b,v)$, $(a,0,a;\ w,a,v)$, $(w,0,b;\ 3,b,w)$, $(0,b,b;\ b,u,w)$, $(w,b,c;\ 3,w,u)$, $(0,u,a;\ c,3,w)$, $(c,v,b;\ v,3,u)$, $(v,w,c;\ 3,3,v)$, $(0,0,v;\ a,c,3)$, $(a,0,v;\ v,a,3)$, $(v,0,w;\ 3,a,3)$, $(0,c,w;\ a,u,3)$, $(a,a,u;\ w,v,3)$, $(w,a,u;\ 3,w,3)$, $(0,u,w;\ b,3,3)$, $(b,v,u;\ u,3,3)$, $(u,w,v;\ 3,3,3)$ (see also Fig. 1).

2: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,b,a)$, $(u,0,0;\ 3,c,b)$, $(0,c,0;\ c,u,a)$, $(c,b,0;\ u,w,c)$, $(u,c,0;\ 3,u,c)$, $(0,u,0;\ c,3,b)$, $(c,w,0;\ v,3,b)$, $(v,u,0;\ 3,3,c)$, $(0,0,a;\ a,c,w)$, $(a,0,a;\ w,a,v)$, $(w,0,b;\ 3,b,w)$, $(0,c,a;\ b,v,v)$, $(w,b,c;\ 3,w,u)$, $(0,v,b;\ c,3,w)$, $(c,v,b;\ v,3,u)$, $(v,w,c;\ 3,3,v)$, $(0,0,w;\ a,b,3)$, $(a,0,v;\ v,a,3)$, $(v,0,w;\ 3,a,3)$, $(0,b,v;\ a,u,3)$, $(a,a,u;\ w,v,3)$, $(w,a,u;\ 3,w,3)$, $(0,u,w;\ b,3,3)$, $(b,v,u;\ u,3,3)$, $(u,w,v;\ 3,3,3)$.

3: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,b,a)$, $(u,0,0;\ 3,c,b)$, $(0,c,0;\ c,v,b)$, $(c,b,0;\ u,w,c)$, $(u,c,0;\ 3,u,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ v,3,b)$, $(v,u,0;\ 3,3,c)$, $(0,0,a;\ a,b,v)$, $(a,0,a;\ v,a,w)$, $(v,0,b;\ 3,a,u)$, $(0,b,b;\ b,u,w)$, $(w,a,c;\ 3,w,v)$, $(0,u,a;\ c,3,w)$, $(c,v,b;\ v,3,u)$, $(v,w,c;\ 3,3,v)$, $(0,0,v;\ a,c,3)$, $(a,0,w;\ w,a,3)$, $(w,0,u;\ 3,b,3)$, $(0,c,w;\ a,u,3)$, $(a,a,u;\ w,v,3)$, $(w,b,v;\ 3,w,3)$, $(0,u,w;\ b,3,3)$, $(b,v,u;\ u,3,3)$, $(u,w,v;\ 3,3,3)$.

4: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,b,a)$, $(u,0,0;\ 3,c,b)$, $(0,c,0;\ c,u,a)$, $(c,b,0;\ u,w,c)$, $(u,c,0;\ 3,u,c)$, $(0,u,0;\ c,3,b)$, $(c,w,0;\ v,3,b)$, $(v,u,0;\ 3,3,c)$, $(0,0,a;\ a,c,w)$, $(a,0,a;\ v,a,w)$, $(v,0,b;\ 3,a,u)$, $(0,c,a;\ b,v,v)$, $(w,a,c;\ 3,w,v)$, $(0,v,b;\ c,3,w)$, $(c,v,b;\ v,3,u)$, $(v,w,c;\ 3,3,v)$, $(0,0,w;\ a,b,3)$, $(a,0,w;\ w,a,3)$, $(w,0,u;\ 3,b,3)$, $(0,b,v;\ a,u,3)$, $(a,a,u;\ w,v,3)$, $(w,b,v;\ 3,w,3)$, $(0,u,w;\ b,3,3)$, $(b,v,u;\ u,3,3)$, $(u,w,v;\ 3,3,3)$.

5: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,a,b)$, $(u,0,0;\ 3,b,c)$, $(0,c,0;\ c,v,b)$, $(c,a,0;\ v,w,c)$, $(v,b,0;\ 3,w,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ v,3,b)$, $(v,w,0;\ 3,3,a)$, $(0,0,a;\ a,b,v)$, $(a,0,b;\ w,a,u)$, $(w,0,c;\ 3,a,u)$, $(0,b,b;\ b,u,w)$, $(w,a,c;\ 3,w,v)$, $(0,u,a;\ b,3,v)$, $(b,v,b;\ u,3,w)$, $(u,w,a;\ 3,3,w)$, $(0,0,v;\ a,c,3)$, $(a,0,u;\ v,b,3)$, $(v,0,u;\ 3,c,3)$, $(0,c,w;\ a,u,3)$, $(a,b,u;\ w,u,3)$, $(w,c,v;\ 3,v,3)$, $(0,u,v;\ c,3,3)$, $(c,u,w;\ u,3,3)$, $(u,v,w;\ 3,3,3)$.

6: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,a,b)$, $(u,0,0;\ 3,b,c)$, $(0,c,0;\ c,v,b)$, $(c,a,0;\ v,w,c)$, $(v,b,0;\ 3,w,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ v,3,b)$, $(v,w,0;\ 3,3,a)$, $(0,0,a;\ a,b,v)$, $(a,0,b;\ w,a,u)$, $(w,0,c;\ 3,b,v)$, $(0,b,b;\ b,u,w)$, $(w,b,c;\ 3,w,u)$, $(0,u,a;\ b,3,v)$, $(b,v,b;\ u,3,w)$, $(u,w,a;\ 3,3,w)$, $(0,0,v;\ a,c,3)$, $(a,0,u;\ w,c,3)$, $(w,0,v;\ 3,a,3)$, $(0,c,w;\ a,u,3)$, $(a,c,u;\ v,u,3)$, $(v,a,u;\ 3,v,3)$, $(0,u,v;\ c,3,3)$, $(c,u,w;\ u,3,3)$, $(u,v,w;\ 3,3,3)$.

7: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,a,b)$, $(u,0,0;\ 3,b,c)$, $(0,c,0;\ c,v,b)$, $(c,a,0;\ v,w,c)$, $(v,b,0;\ 3,w,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ u,3,a)$, $(u,w,0;\ 3,3,b)$, $(0,0,a;\ a,b,v)$, $(a,0,b;\ w,a,u)$, $(w,0,c;\ 3,a,u)$, $(0,b,b;\ b,u,w)$, $(w,a,c;\ 3,w,v)$, $(0,u,a;\ c,3,w)$, $(c,v,a;\ v,3,v)$, $(v,w,b;\ 3,3,w)$, $(0,0,v;\ a,c,3)$, $(a,0,u;\ w,c,3)$, $(w,0,u;\ 3,b,3)$, $(0,c,w;\ b,v,3)$, $(b,c,u;\ u,u,3)$, $(u,b,v;\ 3,u,3)$, $(0,v,w;\ a,3,3)$, $(a,u,v;\ v,3,3)$, $(v,u,w;\ 3,3,3)$.

8: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,a,b)$, $(u,0,0;\ 3,b,c)$, $(0,c,0;\ c,v,b)$, $(c,a,0;\ v,w,c)$, $(v,b,0;\ 3,w,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ u,3,a)$, $(u,w,0;\ 3,3,b)$, $(0,0,a;\ a,b,v)$, $(a,0,b;\ w,a,u)$, $(w,0,c;\ 3,a,u)$, $(0,b,b;\ b,u,w)$, $(w,a,c;\ 3,w,v)$, $(0,u,a;\ c,3,w)$, $(c,v,a;\ v,3,v)$, $(v,w,b;\ 3,3,w)$, $(0,0,v;\ a,c,3)$, $(a,0,u;\ v,b,3)$, $(v,0,u;\ 3,c,3)$, $(0,c,w;\ a,u,3)$, $(a,b,u;\ w,u,3)$, $(w,c,v;\ 3,v,3)$, $(0,u,w;\ b,3,3)$, $(b,u,v;\ u,3,3)$, $(u,v,w;\ 3,3,3)$.

9: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,a,b)$, $(u,0,0;\ 3,b,c)$, $(0,c,0;\ c,v,b)$, $(c,a,0;\ v,w,c)$, $(v,b,0;\ 3,w,c)$, $(0,v,0;\ c,3,a)$, $(c,w,0;\ u,3,a)$, $(u,w,0;\ 3,3,b)$, $(0,0,a;\ a,b,v)$, $(a,0,b;\ w,a,u)$, $(w,0,c;\ 3,b,v)$, $(0,b,b;\ b,u,w)$, $(w,b,c;\ 3,w,u)$, $(0,u,a;\ c,3,w)$, $(c,v,a;\ v,3,v)$, $(v,w,b;\ 3,3,w)$, $(0,0,v;\ a,c,3)$, $(a,0,u;\ w,c,3)$, $(w,0,v;\ 3,a,3)$, $(0,c,w;\ a,u,3)$, $(a,c,u;\ v,u,3)$, $(v,a,u;\ 3,v,3)$, $(0,u,w;\ b,3,3)$, $(b,u,v;\ u,3,3)$, $(u,v,w;\ 3,3,3)$.

10: $(0,0,0;\ b,c,a)$, $(b,0,0;\ u,a,b)$, $(u,0,0;\ 3,b,c)$, $(0,c,0;\ c,u,a)$, $(c,a,0;\ v,w,c)$, $(v,b,0;\ 3,w,c)$, $(0,u,0;\ c,3,b)$, $(c,w,0;\ u,3,a)$, $(u,w,0;\ 3,3,b)$, $(0,0,a;\ a,c,w)$, $(a,0,b;\ w,a,u)$, $(w,0,c;\ 3,a,u)$, $(0,c,a;\ b,v,v)$, $(w,a,c;\ 3,w,v)$, $(0,v,b;\ c,3,w)$, $(c,v,a;\ v,3,v)$, $(v,w,b;\ 3,3,w)$, $(0,0,w;\ a,b,3)$, $(a,0,u;\ v,b,3)$, $(v,0,u;\ 3,c,3)$, $(0,b,v;\ a,u,3)$, $(a,b,u;\ w,u,3)$, $(w,c,v;\ 3,v,3)$, $(0,u,w;\ b,3,3)$, $(b,u,v;\ u,3,3)$, $(u,v,w;\ 3,3,3)$.

11: $(0,0,0;\,b,c,a)$, $(b,0,0;\,u,a,b)$, $(u,0,0;\,3,b,c)$, $(0,c,0;\,c,u,a)$, $(c,a,0;\,v,w,c)$, $(v,b,0;\,3,w,c)$, $(0,u,0;\,c,3,b)$, $(c,w,0;\,u,3,a)$, $(u,w,0;\,3,3,b)$, $(0,0,a;\,a,c,w)$, $(a,0,b;\,w,a,u)$, $(w,0,c;\,3,b,v)$, $(0,c,a;\,b,v,v)$, $(w,b,c;\,3,w,u)$, $(0,v,b;\,c,3,w)$, $(c,v,a;\,v,3,v)$, $(v,w,b;\,3,3,w)$, $(0,0,w;\,a,b,3)$, $(a,0,u;\,w,c,3)$, $(w,0,v;\,3,a,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,a,u;\,3,v,3)$, $(0,u,w;\,b,3,3)$, $(b,u,v;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

12: $(0,0,0;\,a,c,b)$, $(a,0,0;\,v,a,b)$, $(v,0,0;\,3,a,c)$, $(0,c,0;\,b,v,c)$, $(b,a,0;\,w,w,c)$, $(w,a,0;\,3,w,a)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,u,3,a)$, $(u,w,0;\,3,3,b)$, $(0,0,b;\,a,b,u)$, $(a,0,b;\,w,a,u)$, $(w,0,c;\,3,b,v)$, $(0,b,c;\,b,u,v)$, $(w,b,a;\,3,w,w)$, $(0,u,a;\,c,3,w)$, $(c,v,a;\,v,3,v)$, $(v,w,b;\,3,3,w)$, $(0,0,u;\,c,b,3)$, $(c,0,u;\,u,c,3)$, $(u,0,v;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,c,w;\,3,v,3)$, $(0,u,w;\,b,3,3)$, $(b,u,v;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

13: $(0,0,0;\,a,c,b)$, $(a,0,0;\,v,a,b)$, $(v,0,0;\,3,a,c)$, $(0,c,0;\,b,v,c)$, $(b,a,0;\,w,w,c)$, $(w,a,0;\,3,w,a)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,v,3,b)$, $(v,w,0;\,3,3,a)$, $(0,0,b;\,a,b,u)$, $(a,0,b;\,w,a,u)$, $(w,0,c;\,3,b,v)$, $(0,b,c;\,b,u,v)$, $(w,b,a;\,3,w,w)$, $(0,u,a;\,b,3,v)$, $(b,v,b;\,u,3,w)$, $(u,w,a;\,3,3,w)$, $(0,0,u;\,c,b,3)$, $(c,0,u;\,u,c,3)$, $(u,0,v;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,c,w;\,3,v,3)$, $(0,u,v;\,c,3,3)$, $(c,u,w;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

14: $(0,0,0;\,b,c,a)$, $(b,0,0;\,w,b,c)$, $(w,0,0;\,3,a,b)$, $(0,c,0;\,c,v,b)$, $(c,b,0;\,u,w,c)$, $(u,a,0;\,3,w,c)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,v,3,b)$, $(v,w,0;\,3,3,a)$, $(0,0,a;\,a,b,v)$, $(a,0,c;\,v,a,u)$, $(v,0,b;\,3,a,u)$, $(0,b,b;\,b,u,w)$, $(w,a,c;\,3,w,v)$, $(0,u,a;\,b,3,v)$, $(b,v,b;\,u,3,w)$, $(u,w,a;\,3,3,w)$, $(0,0,v;\,a,c,3)$, $(a,0,u;\,v,b,3)$, $(v,0,u;\,3,c,3)$, $(0,c,w;\,a,u,3)$, $(a,b,u;\,w,u,3)$, $(w,c,v;\,3,v,3)$, $(0,u,v;\,c,3,3)$, $(c,u,w;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

15: $(0,0,0;\,b,c,a)$, $(b,0,0;\,w,b,c)$, $(w,0,0;\,3,a,b)$, $(0,c,0;\,c,v,b)$, $(c,b,0;\,u,w,c)$, $(u,a,0;\,3,w,c)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,u,3,a)$, $(u,w,0;\,3,3,b)$, $(0,0,a;\,a,b,v)$, $(a,0,c;\,v,a,u)$, $(v,0,b;\,3,a,u)$, $(0,b,b;\,b,u,w)$, $(w,a,c;\,3,w,v)$, $(0,u,a;\,c,3,w)$, $(c,v,a;\,v,3,v)$, $(v,w,b;\,3,3,w)$, $(0,0,v;\,a,c,3)$, $(a,0,u;\,w,c,3)$, $(w,0,u;\,3,b,3)$, $(0,c,w;\,b,v,3)$, $(b,c,u;\,u,u,3)$, $(u,b,v;\,3,u,3)$, $(0,v,w;\,a,3,3)$, $(a,u,v;\,v,3,3)$, $(v,u,w;\,3,3,3)$.

16: $(0,0,0;\,b,c,a)$, $(b,0,0;\,w,b,c)$, $(w,0,0;\,3,a,b)$, $(0,c,0;\,c,v,b)$, $(c,b,0;\,u,w,c)$, $(u,a,0;\,3,w,c)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,u,3,a)$, $(u,w,0;\,3,3,b)$, $(0,0,a;\,a,b,v)$, $(a,0,c;\,v,a,u)$, $(v,0,b;\,3,a,u)$, $(0,b,b;\,b,u,w)$, $(w,a,c;\,3,w,v)$, $(0,u,a;\,c,3,w)$, $(c,v,a;\,v,3,v)$, $(v,w,b;\,3,3,w)$, $(0,0,v;\,a,c,3)$, $(a,0,u;\,v,b,3)$, $(v,0,u;\,3,c,3)$, $(0,c,w;\,a,u,3)$, $(a,b,u;\,w,u,3)$, $(w,c,v;\,3,v,3)$, $(0,u,w;\,b,3,3)$, $(b,u,v;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

17: $(0,0,0;\,b,c,a)$, $(b,0,0;\,w,b,c)$, $(w,0,0;\,3,a,b)$, $(0,c,0;\,c,u,a)$, $(c,b,0;\,u,w,c)$, $(u,a,0;\,3,w,c)$, $(0,u,0;\,c,3,b)$, $(c,w,0;\,u,3,a)$, $(u,w,0;\,3,3,b)$, $(0,0,a;\,a,c,w)$, $(a,0,c;\,v,a,u)$, $(v,0,b;\,3,a,u)$, $(0,c,a;\,b,v,v)$, $(w,a,c;\,3,w,v)$, $(0,v,b;\,c,3,w)$, $(c,v,a;\,v,3,v)$, $(v,w,b;\,3,3,w)$, $(0,0,w;\,a,b,3)$, $(a,0,u;\,v,b,3)$, $(v,0,u;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,b,u;\,w,u,3)$, $(w,c,v;\,3,v,3)$, $(0,u,w;\,b,3,3)$, $(b,u,v;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

18: $(0,0,0;\,a,c,b)$, $(a,0,0;\,w,a,c)$, $(w,0,0;\,3,b,a)$, $(0,c,0;\,b,v,c)$, $(b,a,0;\,w,w,c)$, $(w,b,0;\,3,w,b)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,u,3,a)$, $(u,w,0;\,3,3,b)$, $(0,0,b;\,a,b,u)$, $(a,0,c;\,v,a,u)$, $(v,0,a;\,3,a,v)$, $(0,b,c;\,b,u,v)$, $(w,a,b;\,3,w,w)$, $(0,u,a;\,c,3,w)$, $(c,v,a;\,v,3,v)$, $(v,w,b;\,3,3,w)$, $(0,0,u;\,c,b,3)$, $(c,0,u;\,u,c,3)$, $(u,0,v;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,c,w;\,3,v,3)$, $(0,u,w;\,b,3,3)$, $(b,u,v;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

19: $(0,0,0;\,a,c,b)$, $(a,0,0;\,w,a,c)$, $(w,0,0;\,3,b,a)$, $(0,c,0;\,b,v,c)$, $(b,a,0;\,w,w,c)$, $(w,b,0;\,3,w,b)$, $(0,v,0;\,c,3,a)$, $(c,w,0;\,v,3,b)$, $(v,w,0;\,3,3,a)$, $(0,0,b;\,a,b,u)$, $(a,0,c;\,v,a,u)$, $(v,0,a;\,3,a,v)$, $(0,b,c;\,b,u,v)$, $(w,a,b;\,3,w,w)$, $(0,u,a;\,b,3,v)$, $(b,v,b;\,u,3,w)$, $(u,w,a;\,3,3,w)$, $(0,0,u;\,c,b,3)$, $(c,0,u;\,u,c,3)$, $(u,0,v;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,c,w;\,3,v,3)$, $(0,u,v;\,c,3,3)$, $(c,u,w;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

20: $(0,0,0;\,c,a,b)$, $(c,0,0;\,v,b,c)$, $(v,0,0;\,3,c,a)$, $(0,a,0;\,a,w,c)$, $(a,b,0;\,w,w,c)$, $(w,c,0;\,3,u,a)$, $(0,w,0;\,b,3,a)$, $(b,w,0;\,w,3,b)$, $(w,u,0;\,3,3,b)$, $(0,0,b;\,b,a,u)$, $(b,0,c;\,u,a,u)$, $(u,0,a;\,3,b,v)$, $(0,a,c;\,b,v,v)$, $(w,b,a;\,3,w,w)$, $(0,v,a;\,a,3,v)$, $(a,v,b;\,v,3,w)$, $(v,w,b;\,3,3,w)$, $(0,0,u;\,c,b,3)$, $(c,0,u;\,u,c,3)$, $(u,0,v;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,c,w;\,3,v,3)$, $(0,u,v;\,c,3,3)$, $(c,u,w;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

21: $(0,0,0;\,a,b,c)$, $(a,0,0;\,v,a,b)$, $(v,0,0;\,3,c,a)$, $(0,b,0;\,b,w,c)$, $(b,a,0;\,w,w,c)$, $(w,c,0;\,3,u,a)$, $(0,w,0;\,b,3,a)$, $(b,w,0;\,w,3,b)$, $(w,u,0;\,3,3,b)$, $(0,0,c;\,c,a,u)$, $(c,0,b;\,u,a,u)$, $(u,0,a;\,3,b,v)$, $(0,a,c;\,b,v,v)$, $(w,b,a;\,3,w,w)$, $(0,v,a;\,a,3,v)$, $(a,v,b;\,v,3,w)$, $(v,w,b;\,3,3,w)$, $(0,0,u;\,c,b,3)$, $(c,0,u;\,u,c,3)$, $(u,0,v;\,3,c,3)$, $(0,b,v;\,a,u,3)$, $(a,c,u;\,v,u,3)$, $(v,c,w;\,3,v,3)$, $(0,u,v;\,c,3,3)$, $(c,u,w;\,u,3,3)$, $(u,v,w;\,3,3,3)$.

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