Journal for Geometry and Graphics
Volume 5 (2001), No. 1, 13–22.

The Harmonic Analysis of Polygons and Napoleon's Theorem

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Abstract. Plane closed polygons are harmonically analysed, i.e., they are expressed in the form of the sum of fundamental k-regular polygons. From this point of view Napoleon's theorem and its generalization, the so-called theorem of PETR, are studied. By means of PETR's theorem the fundamental polygons of an arbitrary polygon have been found geometrically.

Key Words: finite Fourier series, polygon transformation

MSC 2000: 51M20

1. Harmonic analysis of polygons

A plane n-gon Π is an ordered n-tuple of points $A_0, A_1, \ldots, A_{n-1}$, which can be represented by complex numbers in the complex plane. The points $A_0, A_1, \ldots, A_{n-1}$ are called vertices of the n-gon Π. We shall write $\Pi = (A_0, A_1, \ldots, A_{n-1})$. Denote the n-roots of unity by

$$\omega_\nu^j = e^{i\nu j \frac{2\pi}{n}} = \cos \nu j \tfrac{2\pi}{n} + i \sin \nu j \tfrac{2\pi}{n}\,,$$

where $\nu, j = 0, 1, \ldots, n-1$. Then the system of linear equations

$$A_\nu = \sum_{k=0}^{n-1} \vartheta_k \omega_\nu^k, \qquad \nu = 0, 1, \ldots, n-1 \tag{1}$$

with unknown complex numbers ϑ_k has a unique solution. The determinant of the system (1) is a determinant of the *Fourier matrix* $\Omega = (\omega_j^k)_{k,j=0}^{n-1}$, which is not vanishing (Vandermonde). Multiplying each equation from (1) by ω_ν^{-j} and adding these equations for $\nu = 0, 1, \ldots, n-1$ we get

$$\sum_{\nu=0}^{n-1} A_\nu \omega_\nu^{-j} = \sum_{k=0}^{n-1} \sum_{\nu=0}^{n-1} \vartheta_k \omega_\nu^{k-j} = \sum_{k=0}^{n-1} \vartheta_k \sum_{\nu=0}^{n-1} \omega_{k-j}^\nu = n\vartheta_j\,.$$

ISSN 1433-8157/$ 2.50

The *Fourier coefficients* $\vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}$ of the polygon Π are then

$$\vartheta_j = \frac{1}{n}\sum_{\nu=0}^{n-1} A_\nu \omega_\nu^{-j}.$$

We can see that $\vartheta_0 = \frac{1}{n}\sum_{\nu=0}^{n-1} A_\nu$, i.e., the coefficient ϑ_0 coincides with the centroid of the n-gon Π. If we place the origin of the coordinate system into the centroid of the n-gon Π, we get $\vartheta_0 = 0$. The system of linear equations (1) may be written in the form

$$\Pi = \vartheta_0\Pi_0 + \vartheta_1\Pi_1 + \ldots + \vartheta_{n-1}\Pi_{n-1} \quad \text{or} \quad \Pi = \Phi_0 + \Phi_1 + \ldots + \Phi_{n-1} \tag{2}$$

with $\Phi_k := \vartheta_k\Pi_k$.

The right-hand side of (2) is expressed as a linear combination (with complex Fourier coefficients) of basic k-regular polygons Π_j, where $\Pi_j = (\omega_j^0, \omega_j^1, \ldots, \omega_j^{n-1})$. From (2) it is seen that every closed plane n-gon Π has been uniquely represented as a sum of *fundamental regular n-gons* $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$, i.e., it has been analysed harmonically, see Schoenberg [7].

The expansion (2) may also be expressed in the following form:

$$\Pi = \vartheta_0\Pi_0 + (\vartheta_1\Pi_1 + \vartheta_{n-1}\Pi_{n-1}) + (\vartheta_2\Pi_2 + \vartheta_{n-2}\Pi_{n-2}) + \ldots. \tag{3}$$

It is easily seen that the sum $\vartheta_k\Pi_k + \vartheta_{n-k}\Pi_{n-k}$ is an affine regular n-gon, i.e., the affine image of a k-regular n-gon. From (3) we get the fact, that every n-gon Π can be uniquely expressed as a sum of *fundamental affine regular n-gons* $\vartheta_k\Pi_k + \vartheta_{n-k}\Pi_{n-k}$. It is a question *how to find* the fundamental n-gons of an arbitrary plane n-gon geometrically? For the sake of it, we shall concern with Napoleon's theorem and its generalization, the theorem of Petr, see Naas and Schmid [4]. These theorems are based on polygon to polygon transformation.

2. Polygon transformation

Let $\Pi = (A_0, A_1, \ldots, A_{n-1})$ be an arbitrary plane n-gon. Let us suppose that P is a such polygon to polygon transformation that assigns to the n-gon Π an n-gon $\Pi' = (A'_0, A'_1, \ldots, A'_{n-1})$, whose vertices are a linear combination of two consecutive vertices of the n-gon Π, i.e.,

$$A'_j = aA_j + bA_{j+1}, \qquad j = 0, 1, \ldots, n-1, \tag{4}$$

where coefficients a, b are complex. We shall concern with a special type of (4) whose coefficients a, b fulfil $a + b = 1$. This condition assures, that all the triangles with vertices A'_k erected on sides A_kA_{k+1} of the original triangle, are similar.

If isocseles triangles with the angle $j\frac{2\pi}{n}$ are constructed on sides of an arbitrary n-gon Π, we get the relation

$$(A_k - A'_k)\omega_j = A_{k+1} - A'_k. \tag{5}$$

From (5) we obtain that for a, b in (4)

$$a = \frac{-\omega_j}{1-\omega_j} \quad \text{and} \quad b = \frac{1}{1-\omega_j}$$

holds (Fig. 1).

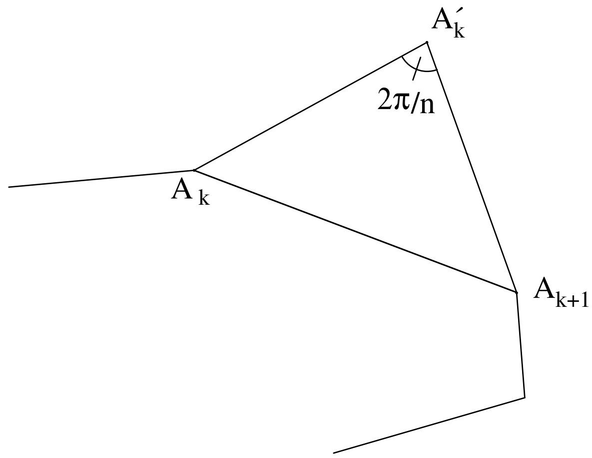

Figure 1: PETR's construction

If we denote a transformation, which maps a j-regular polygon Π_j into the origin, by P_j instead of P, we may write the image of Π in the form of the expansion

$$P_j(\Pi) = \vartheta_0 P_j(\Pi_0) + \vartheta_1 P_j(\Pi_1) + \ldots + \vartheta_{n-1} P_j(\Pi_{n-1}), \tag{6}$$

where

$$P_j(\Pi_\nu) = \frac{\omega_\nu - \omega_j}{1 - \omega_j}\,\Pi_\nu\,, \qquad j = 1, 2, \ldots, n-1. \tag{7}$$

From (6) and (7) we see that $P_j(\Pi_0) = \Pi_0$ and $P_j(\Pi_j) = 0$. Let P_k be another polygon transformation with $k \neq j$. The Fourier expansion of the image of $P_j(\Pi)$ in the transformation P_k is

$$P_k P_j(\Pi) = \sum_{\nu=0}^{n-1} \vartheta_\nu \,\frac{\omega_\nu - \omega_j}{1 - \omega_j}\,\frac{\omega_\nu - \omega_k}{1 - \omega_k}\,\Pi_\nu\,. \tag{8}$$

From (8) it is seen that two of the regular polygons Π_j, Π_k of the Fourier expansion of the polygon $P_k P_j(\Pi)$ vanished. Continuing this process, after $n-1$ steps of using the polygon transformation P_ν for all values $\nu = 1, 2, \ldots, n-1$, we arrive at the point $\vartheta_0 \Pi_0$ — the common centroid of all polygons, which occur in this process. We have proved the theorem that was established by the Czech mathematician K. PETR in 1905 [6].

PETR's Theorem: *On the sides of any closed plane n-gon Π construct isosceles triangles with vertex angle $j_1\frac{2\pi}{n}$, where $j_1 \in \{1, 2, \ldots, n-1\}$. The resulting vertices form a new polygon Π_{j_1}. On the sides of this polygon construct isosceles triangles with vertex angle $j_2\frac{2\pi}{n}$, where $j_2 \in \{1, 2, \ldots, n-1\} \setminus \{j_1\}$. We get a polygon Π_{j_1,j_2}. Continue this construction for all values $1, 2, \ldots, n-1$. Then the final polygon $\Pi_{j_1,j_2,\ldots,j_{n-1}}$ is a point — the common centroid of all polygons $\Pi, \Pi_{j_1}, \Pi_{j_1,j_2}, \ldots, \Pi_{j_1,j_2,\ldots,j_{n-1}}$ and the polygon $\Pi_{j_1,j_2,\ldots,j_{n-2}}$ is j_{n-1}-regular.*

PETR's theorem has been rediscovered several times. This theorem has often been attributed to J. DOUGLAS [2] and B.H. NEUMANN [5], but it seems the priority belongs to K. PETR. We shall call a polygon to polygon transformation, described in PETR's theorem, the PETR*'s construction P*.

Now we will investigate polygons, which we obtain from the original n-gon Π using successively $n-2$ polygon transformations P_ν for different values $\nu \in \{1, 2, \ldots, n-1\}$. Denote such an n-gon by $P^j(\Pi)$, where the upper index j denotes the value, which has been omitted in this process. According to (8) we have

$$P^j(\Pi) = \vartheta_0\Pi_0 + \vartheta_j \prod_{k=1,k\neq j}^{n-1} \frac{\omega_j - \omega_k}{1-\omega_k}\,\Pi_j\,, \tag{9}$$

and after a short calculation we get

$$P^j(\Pi) = \vartheta_0\Pi_0 - \vartheta_j\omega^{-j}\Pi_j\,. \tag{10}$$

From (8) and (9) we see that the n-gon $P^j(\Pi)$ doesn't depend on the order of polygon transformations P_ν and therefore to a given n-gon Π there exist the only n-gon $P^j(\Pi)$ for every $j = 1, 2, \ldots, n-1$. In this way we can assign to an arbitrary plane n-gon Π $n-1$ firmly determined n-gons $P^j(\Pi)$. If we place the origin of the coordinate system into the centroid of Π, then $\vartheta_0 = 0$ and instead of (10) we may write $P^j(\Pi) = -\vartheta_j\omega^{-j}\Pi_j$. The polygon $P^j(\Pi)$ is a j-regular n-gon, which differs from the fundamental n-gon $\vartheta_j\Pi_j$ in the Fourier expansion (2) of the n-gon Π only by the multiply $-\omega^{-j}$. This means that $P^j(\Pi)$ is a centrally symmetric image of $\omega^{-j}\vartheta_j\Pi_j$, which has the same vertices as $\vartheta_j\Pi_j$. With respect to this fact, we can state:

Theorem: *Let Π be an arbitrary plane closed n-gon, whose centroid is placed at the origin of the Cartesian coordinate system. By the construction described above, we can assign to an arbitrary plane n-gon Π $n-1$ firmly determined regular n-gons $P^j(\Pi)$ for every $j = 1, 2, \ldots, n-1$. A centrally symmetric image of $P^j(\Pi)$ shifted by ω^j gives the fundamental n-gon Φ_j and Π can be expressed as the sum of n-gons Φ_j, $j = 1, 2, \ldots, n-1$.*

2.1. The case $n = 3$

As an application of the above theory we will mention *Napoleon's theorem:*

If equilateral triangles are erected externally (or internally) on the sides of an arbitrary triangle, their centers form an equilateral triangle.

This well known theorem has been attributed to Napoleon Bonaparte, although there is some doubt whether its proof is due to him[1]. In the recent survey by Martini [3], more than hundred references to this theorem and its generalizations are collected.

According to this theorem we can assign to an arbitrary triangle ABC two regular triangles $A'B'C'$ and $A''B''C''$, which are called outer and inner Napoleon triangles (Fig. 2). Our theory says that images of these Napoleon triangles in a point reflection with center T are fundamental triangles Φ_1, Φ_2 of the original triangle. In Fig. 3 it is shown, that the original triangle $\Pi = (A, B, C)$ is a sum of its two fundamental triangles $\Phi_1 = (A_1, B_1, C_1)$ and $\Phi_2 = (A_2, B_2, C_2)$, i.e., $\Pi = \Phi_1 + \Phi_2$. For a better view a decomposition of a triangle is shown in Fig. 4.

A generalization of Napoleon's theorem is the following

Theorem: *If similar triangles are constructed on the sides of an arbitrary triangle ABC, their vertices form a new triangle whose fundamental triangles arise from fundamental triangles of ABC in a similar way* (Fig. 5).

For isosceles triangles with the angle of 120°, we get the subcase of Napoleon's theorem.

[1] see J. Fischer: *Napoleon und die Naturwissenschaften*, Franz Steiner Verlag Wiesbaden, Stuttgart 1988

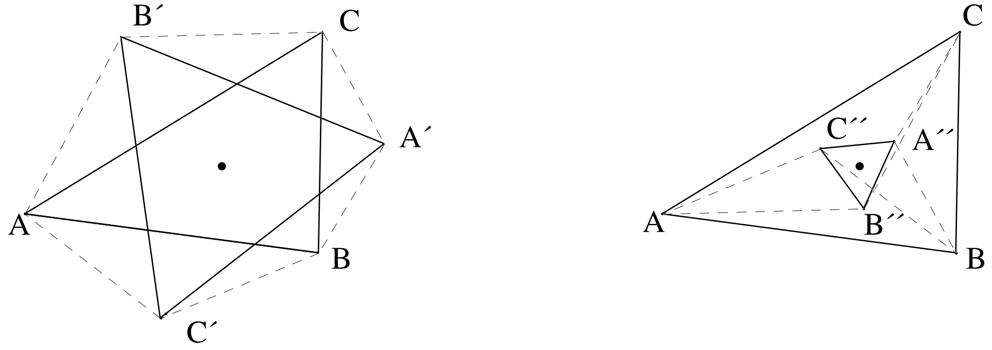

Figure 2: Outer Napoleon triangle Inner Napoleon triangle

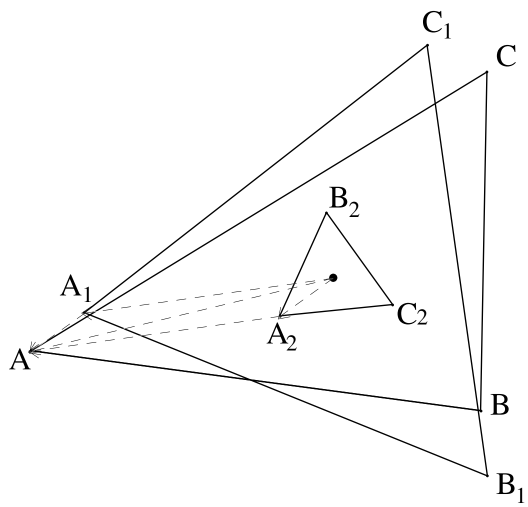

Figure 3: The triangle ABC is a sum of two fundamental triangles $A_1B_1C_1$ and $A_2B_2C_2$.

2.2. The case $n = 4$

Similarly as in the case of triangles we can assign to an arbitrary quadrangle $ABCD$ three fundamental regular quadrangles (squares), one of which is a segment (Fig. 6).

A decomposition of a quadrangle into three regular quadrangles is shown in Fig. 7.

In Fig. 8 isosceles right angled triangles are erected on the sides of a quadrangle $ABCD$. We obtain a new quadrangle which is the sum of $A_1'B_1'C_1'D_1'$ (square) and $A_2'B_2'C_2'D_2'$ (segment) whereas the third fundamental quadrangle vanished. From this decomposition it is easy to

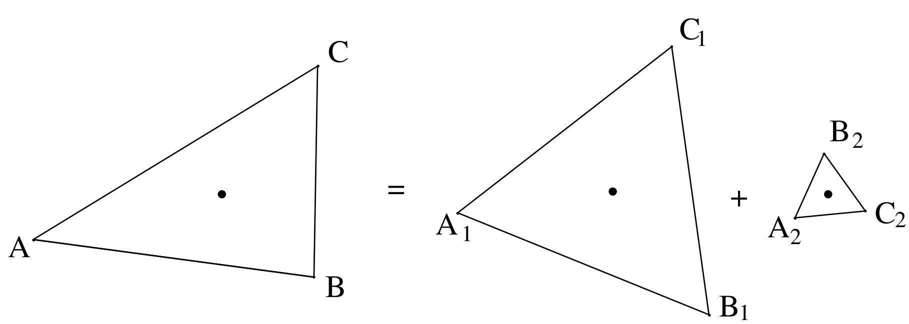

Figure 4: A decomposition of a triangle into two fundamental triangles

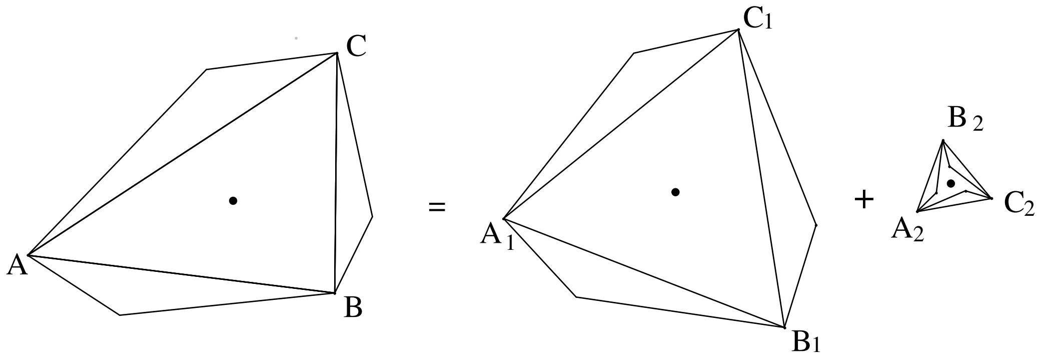

Figure 5: The construction of similar triangles on the sides of an arbitrary triangle is preserved

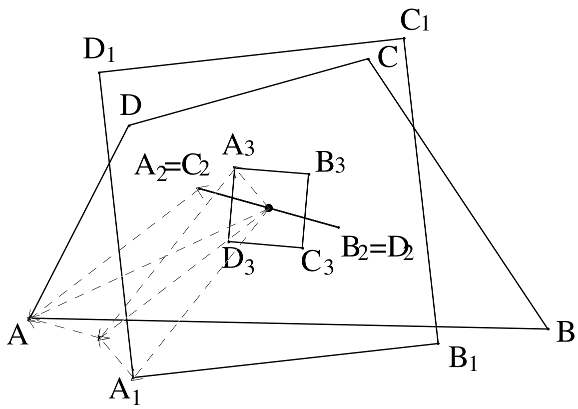

Figure 6: The quadrangle $ABCD$ as the sum of the three squares $A_1B_1C_1D_1, A_2B_2C_2D_2, A_3B_3C_3D_3$.

see that diagonals of the quadrangle $A'B'C'D'$ are perpendicular and have equal length.

If isosceles triangles with vertex angle 180° are erected on the sides of a quadrangle $ABCD$, i.e., $A'B'C'D'$ is formed by centers of sides of $ABCD$, then a quadrangle $A'B'C'D'$ is a sum of two oppositely oriented squares $A_1'B_1'C_1'D_1'$ and $A_3'B_3'C_3'D_3'$, whereas the third $A_2'B_2'C_2'D_2'$ vanished. $A'B'C'D'$ is clearly a parallelogram (Fig. 9).

The outward (or inward) erection of right angled isosceles triangles on the sides of a parallelogram gives the fundamental square as it is shown in Fig. 10, whereas the other square from the decomposition described in Fig. 10 vanishes. This is a special case of a more general theorem, often called *Barlotti theorem* (see Barlotti [1]), but it has already been mentioned by J. Douglas [2]:

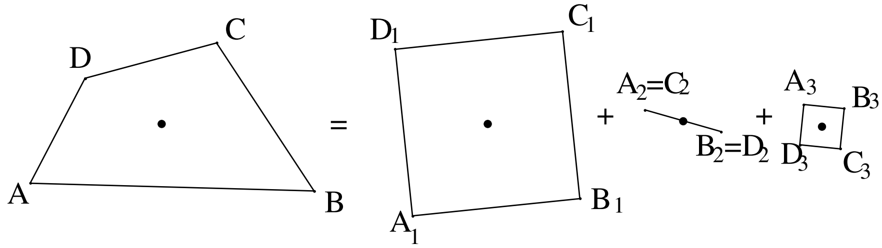

Figure 7: Decomposition of a quadrangle into three fundamental squares

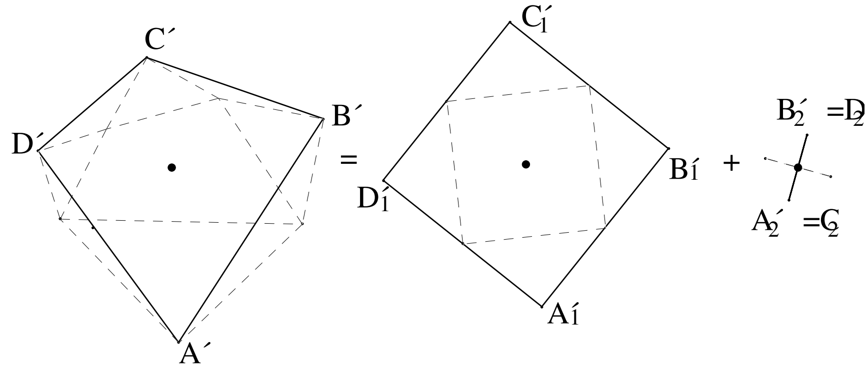

Figure 8: Decomposition of a quadrangle with equal and perpendicular diagonals

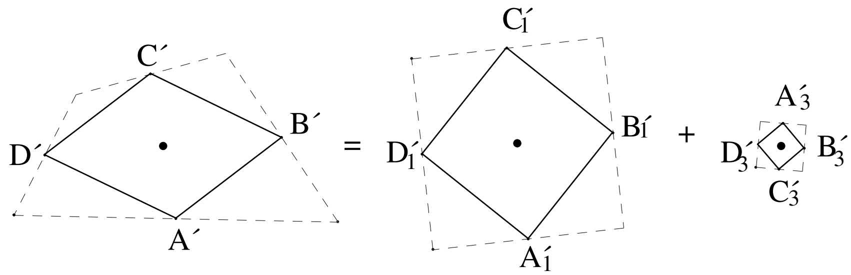

Figure 9: A parallelogram as a sum of two squares

Erecting regular n-gons outwardly (or inwardly) on the sides of any affinely regular n-gon, their centers form a regular n-gon.

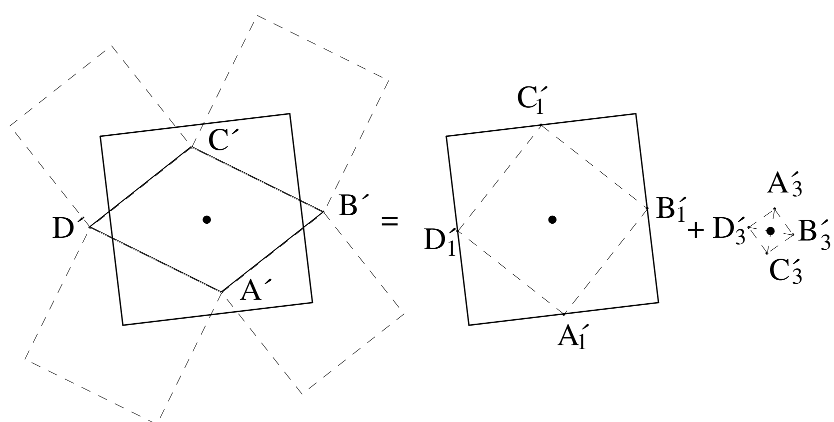

Figure 10: Centers of squares erected on the sides of a parallelogram form a square

From the decomposition of a quadrangle into four fundamental quadrangles as displayed in Fig. 7 another expression of a quadrangle follows. Summing up two squares $A_1B_1C_1D_1$ and $A_3B_3C_3D_3$ we obtain a parallelogram $A'B'C'D'$. The quadrangle $ABCD$ is the sum of a parallelogram $A'B'C'D'$ and a four times calculated segment with endpoints $A_3 = C_3$ and $B_3 = D_3$ (Fig. 11).

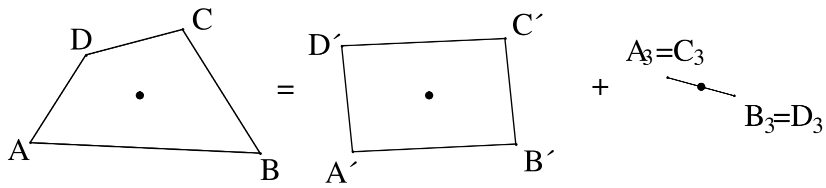

Figure 11: Decomposition of a quadrangle $ABCD$ into a parallelogram and a segment

2.3. The case $n = 5$

A plane pentagon $ABCDE$ could be expressed as a sum of its fundamental pentagons $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ which consist of two convex regular pentagons with the opposite order of vertices and two nonconvex star regular pentagons (Fig. 12). A pentagon $ABCDE$ may also be expessed as a sum of two affine regular pentagons $\Phi_1 + \Phi_4$ and $\Phi_2 + \Phi_3$ (see Schoenberg [7] and Fig. 13).

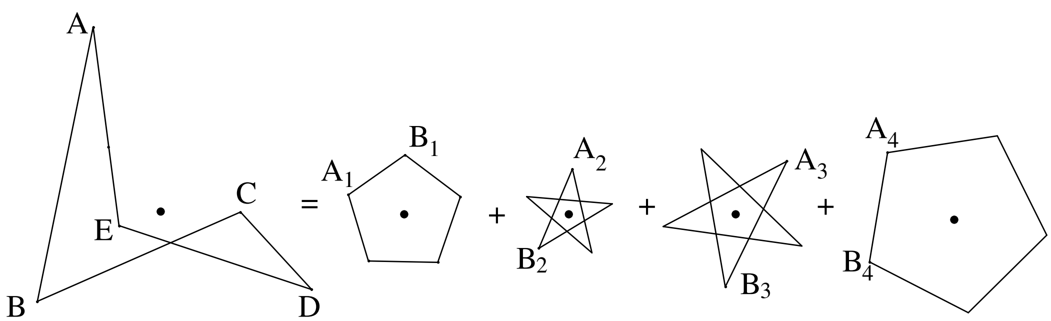

Figure 12: Decomposition of a pentagon into four fundamental pentagons

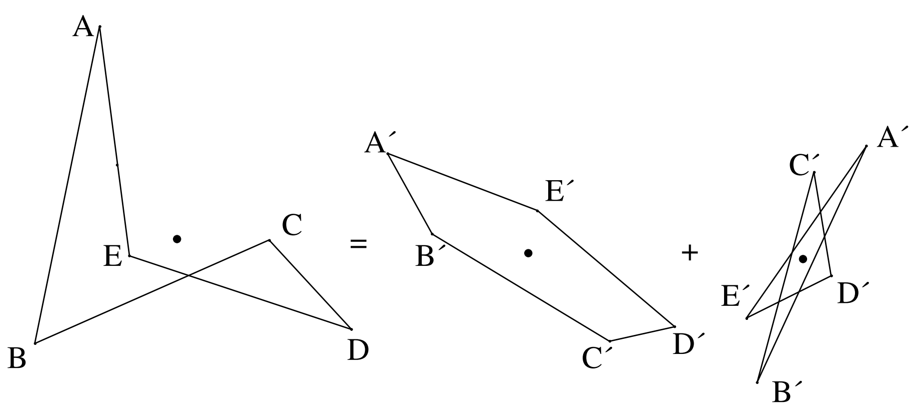

Figure 13: Decomposition of a pentagon into two affine regular pentagons

3. Spatial case

It is possible to transfer our constructions into 3-space:

3.1. The case $n = 4$

Let $ABCD$ be an arbitrary skew and closed quadrangle. It is easy to show that it can be expressed as sum of three squares, two of which lie in the plane of a parallelogram $A'B'C'D'$ which is formed by centers of sides of $ABCD$, the other fundamental square coincides with a segment S_1S_2, whose endpoints are centers of diagonals which join opposite vertices of $ABCD$. In Fig. 14 it is shown that a quadrangle $ABCD$ is the sum of a segment S_1S_2 and a parallelogram, whose centers of sides are vertices of $A'B'C'D'$. Two of the fundamental squares lie in the plane of $A'B'C'D'$ and are formed by the centers of squares erected on the sides of $A'B'C'D'$ outwardly and inwardly.

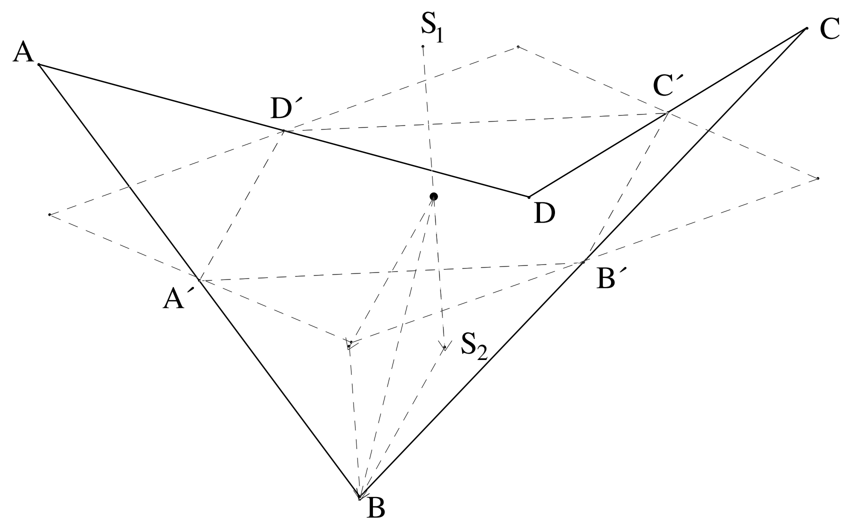

Figure 14: Decomposition of a skew quadrangle into a segment and a parallelogram

3.2. The case $n = 5$

Let $ABCDE$ be an arbitrary skew and closed pentagon. We shall construct pentagons $A'B'C'D'E'$ and $A''B''C''D''E''$ in the following way: Denote by $M_1M_2M_3M_4M_5$ the centers of the sides of $ABCDE$. The vertices A' and A'' lie on the line AM_3 (outwardly and inwardly AM_3) with $A' - M_3 = \frac{1}{\sqrt{5}}(M_3 - A)$ and $A'' - M_3 = -\frac{1}{\sqrt{5}}(M_3 - A)$. Then $A'B'C'D'E'$ and $A''B''C''D''E''$ are plane convex and star affine regular pentagons (see Douglas [2] and Fig. 15). In order to arrive at fundamental pentagons of $ABCDE$, erect in the plane of $A'B'C'D'E'$ on its sides isosceles triangles with vertex angle $\frac{2\pi}{5}$ (outwardly and inwardly). We obtain two regular pentagons. By a homothety with the center at the common centroid of all the pentagons and the ratio $\frac{3-\sqrt{5}}{2}$ we get two of the fundamental pentagons of $ABCDE$. We obtain from $A''B''C''D''E''$ the other two (lying in a different plane) in the same way.

Acknowledgements

This research has been performed within the Grant Project MSM 124100006.

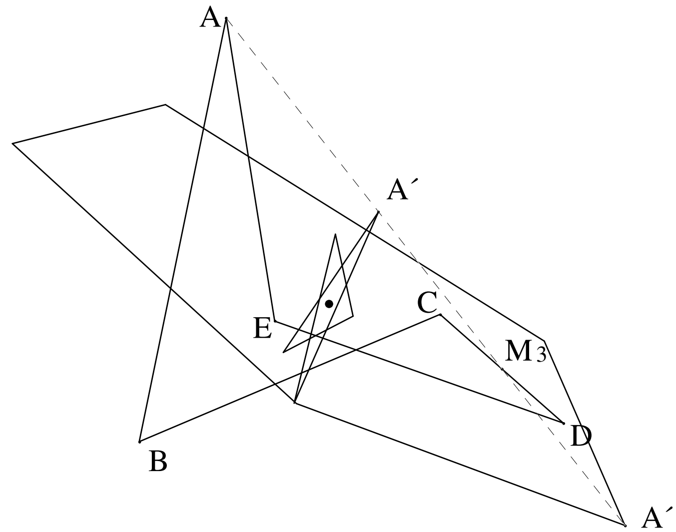

Figure 15: Construction of two plane affine regular pentagons to any pentagon $ABCDE$

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Received August 1, 2000; final form December 20, 2000