Journal for Geometry and Graphics
Volume 5 (2001), No. 2, 121–132.

The Rigidity Rate of Positions of Stewart-Gough Platforms

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Abstract. We consider a 6-legged Stewart-Gough platform. The following investigations of such platforms will always be carried out at an arbitrary given position. If the leg lengths are kept constant, the platform in general will be rigid within the Euclidean displacement group, whereas viewed within the Euclidean similarity group it will yet be movable. There exists an infinitesimal transformation of this motion. Its deviation from the Euclidean displacement group is used to define the *'rigidity rate'* of the platform at this position. In order to obtain some geometric invariant measurement Lie group methods are applied. An example eventually demonstrates the efficiency of the presented method.

Key Words: Parallel manipulator, Stewart-Gough platform, rigidity rate, equiform motions, infinitesimal motion

MSC 2000: 53A17

1. Introduction

It is a common task to evaluate the robustness of a platform in a given position by a *'performance index'* (see J. Angeles [1], p. 174)[1]. Several papers were dealing with the definition of some sort of *'quality mark'* for positions of parallel manipulators with prismatic legs (see [6, 7, 8] and the references in [1]).

The oriented volume of a framework which admits self-motions remains unchanged (see I. Sabitov [9]). Therefore the infinitesimal change of this volume can be used to rate the rigidity (see I. Sabitov [9]).

Some authors ([6, 7, 8]) use the Jacobian J of the legs at the given position and the value

$$\frac{\sqrt{det(JJ^t)}}{\sqrt{det(J_m J_m^t)}}$$

[1] It is important to assess whether the manipulator is going to approach a critical position and to to be able to compare different states of the robot.

ISSN 1433-8157/$ 2.50

in order to define such a rate. The Jacobian J_m belongs to a platform position where $J_m J_m^t$ is maximal. The determination of this maximum is a nonlinear task.

Another contribution to this topic is due to Pottmann/Peternell/Ravani [10]: They determine the linear line complex which best fits the legs of the platform at the given position. A least square procedure is applied in order to find the line complex.

For 6-legged Stewart-Gough platforms geometric strategies lead to a new approach which will be presented here. A 6-legged Stewart-Gough platform at any position permits a self-motion within the group of Euclidean similarities (i.e., equiform transformations). The corresponding instantaneous motion in general does not belong to a Euclidean displacement. If it did, we would have a shaky position. Thus its deviation from the Euclidean displacement group can be used as a measure of the rigidity of the regarded position.

The Euclidean similarities form a Lie group. The instantaneous motion from above is represented by a tangent vector of this differentiable manifold. Established ideas of differential geometry enable us to measure its deviation from the Euclidean displacement group. It is essential to do it in an *'invariant way'*: Changing the coordinate system or changing the scale must not affect the outcome (i.e., it has to be invariant with respect to similarities in 3-space). This is exactly what we are aiming at.

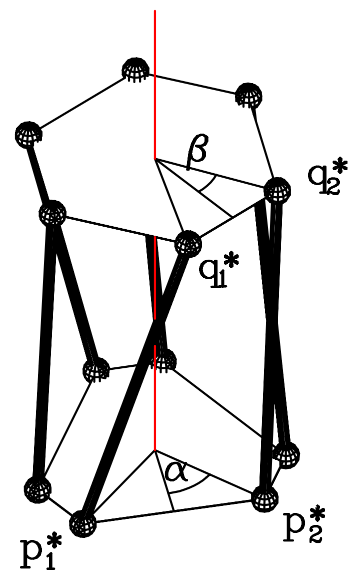

Figure 1: A Stewart-Gough platform

We will define a *'rigidity rate'* of a platform position which meets all these demands. Its computation is straightforward, as it requires the mere solution of a linear system of equations.

We will start with some basic notions on the equiform self-motion of a Stewart-Gough platform (Section 2 and 3). Sections 4 and 5 are devoted to the corresponding Lie algebra tools. In Section 6 we will study bilinear forms on which our measurement will be based (Section 7). Section 8 will give the definition of the rigidity rate of a Stewart-Gough platform position; an example and some conclusions will round off the paper (Sections 9 and 10).

2. Getting started

The transformation group considered is the 6-parameter group G_6 of the Euclidean displacements in 3-space. We start with a Stewart-Gough platform, containing six basis points (3-

vectors) in the fixed space Σ^*

$$\mathbf{p}_i^*, \quad i = 1, 2, \ldots, 6, \tag{1}$$

and six linked points $\mathbf{q}_i$ in the moving space Σ (moving platform). Its positions in the fixed space are determined by a 4×4-matrix $\mathcal{B} \in G_6$. $\mathcal{B}$ is built up by an orthogonal 3×3-block $\mathbf{B} = (b_{ij})$ and a translation vector $\mathbf{b}^*$ in the following way:

$$\mathcal{B} = \begin{pmatrix} 1 & \mathbf{o}^t \\ \mathbf{b}^* & \mathbf{B} \end{pmatrix}. \tag{2}$$

As we use homogeneous vector columns for the description of points, the position of the six regarded points is being described by

$$\begin{pmatrix} 1 \\ \mathbf{q}_i^* \end{pmatrix} = \mathcal{B} \begin{pmatrix} 1 \\ \mathbf{q}_i \end{pmatrix}, \quad i = 1, 2, \ldots, 6. \tag{3}$$

3. An equiform motion assigned to a given position of the platform

We are using the input data of above. The six rods have the lengths

$$d(\mathbf{q}_i^*, \mathbf{p}_i^*) = d_i = const., \quad i = 1, 2, \ldots, 6. \tag{4}$$

If they are kept constant, the platform, in general, keeps its position in 3-space.

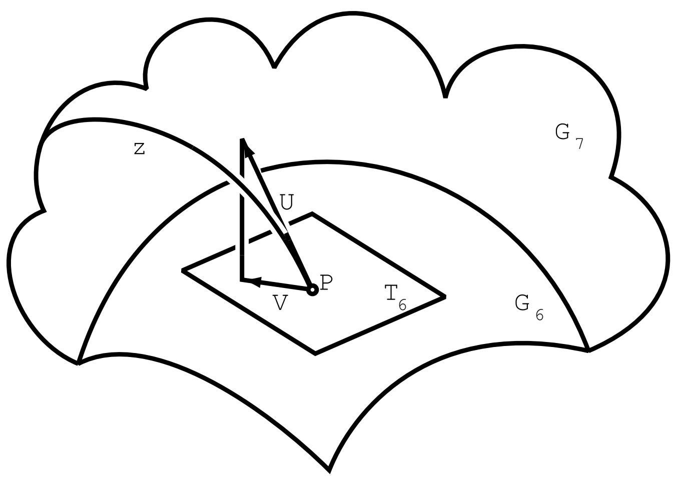

Figure 2: The tangent vector of ζ and the regarded subspace

Let us now watch the 7-parameter group G_7 of equiform displacements in 3-space (*'Euclidean similarities'*). Equiform one-parameter motions ζ (see Bottema/Roth [2], pp. 455–480) are described by matrices $\mathcal{A}$

$$\zeta = \zeta(t) \ \ldots \ \mathcal{A} = \begin{pmatrix} 1 & \mathbf{o}^t \\ \mathbf{a}^*(t) & \alpha(t)\mathbf{A}(t) \end{pmatrix}, \quad t \in I \subset \mathbb{R}, \tag{5}$$

where again the block $\mathbf{A} = (a_{ij}(t))$ is orthogonal for every t. $\mathbf{A} = (a_{ij}(t))$ cares for the spherical, $\mathbf{a}^*(t)$ for the translational part of ζ, and $\alpha(t)$ indicates the scale factor of the equiform transformation ζ. If $\alpha(t) = 1 \ \forall t \in I$, ζ is even contained in the subgroup $G_6 \subset G_7$ of the Euclidean transformations (see (2)).

In this 7-parameter group G_7 six constant leg lengths (4) fall short of keeping the regarded position constant. They rather define an equiform motion $\mathbf{x}^*(t,\mathbf{x}) = \mathcal{A}(t)\,\mathbf{x}$, generally depending on one parameter t.

The motion defines a curve ζ on the manifold G_7. At the given position the infinitesimal transformation determines the tangent vector of ζ at the corresponding point Z. The tangent vectors belonging to the inifinitesimal transformations of Euclidean motions form a 6-dimensional subspace of the tangent space to G_7 at Z. Our approach will use some appropriately defined angle between this subspace and the tangent vector to ζ (Fig. 2).

In fact this angle is an indicator for the rigidity of the position. A zero angle characterizes a shaky position. The angle is closely related to the scaling rate with respect to the Euclidean displacement group. Positions with a larger angle are stiffer compared to those with smaller values. Orthogonality (angle equals $\pi/2$) denotes the stiffest possible situation.

In order to guarantee the invariance of the evaluation we will use some notions from the theory of Lie groups.

4. The Lie group G_7 and its tangent space

The manifold G_7 of all equiform transformations of the 3-space is a 7-dimensional Lie group (5) whose subset $G_6 \ \dots\ [\alpha = 1]$ is the 6-dimensional Lie-subgroup of the Euclidean displacements. A 1-parameter motion (5) represents a curve $\zeta = \zeta(t) \subset G_7$ on the Lie group. Each differentiable curve through a given point P (w.l.o.g. say $P = id_{G_7}$ at $t = 0$, $\alpha(0) = 1, \mathbf{A}(0) = (a_{ij}(0)) = \mathbf{I}_3, \mathbf{a}^*(0) = \mathbf{o}$) defines a tangent vector[2].

Because of $\mathbf{A}(t)\cdot\mathbf{A}^t(t) = \mathbf{I}_3$ we get by differentiation $\dot{\mathbf{A}}(t)\cdot\mathbf{A}^t(t) + (\dot{\mathbf{A}}(t)\cdot\mathbf{A}^t(t))^t = \mathbf{O}_3$. So the matrix $\dot{\mathbf{A}} = (\dot{a}_{ij})$ is skew symmetric

$$\dot{\mathbf{A}} = (\dot{a}_{ij}) = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}. \tag{6}$$

If we name the 3-vector $\begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix}^t =: \mathbf{m}$, we have for any other 3-vector $\mathbf{x}$:

$$\dot{\mathbf{A}}\mathbf{x} = \mathbf{m}\times\mathbf{x} \tag{7}$$

Therefore, the vector $\mathbf{m}$ is being referred to as the *'rotation axis vector asigned to $\dot{\mathbf{A}}$'*.

At $P = id_{G_7}$ the tangent vector determines a mapping X, which is called the *'infinitesimal transformation'* at $t = 0$. It assigns to any point $\mathbf{z}^*$ in the fixed space Σ^* the instantaneous tangent vector $\dot{\mathbf{z}}^*$ of its path. It reads as

$$X:\quad \mathbf{z}^*(0,\mathbf{z}) \longrightarrow \begin{pmatrix} 0 \\ \dot{\mathbf{z}}^* \end{pmatrix} = \mathcal{M}\begin{pmatrix} 1 \\ \mathbf{z}^* \end{pmatrix} \tag{8}$$

with

$$\mathcal{M} = \begin{pmatrix} 0 & \mathbf{o}^t \\ \dot{\mathbf{a}}^* & \dot{\alpha}\mathbf{A} + \dot{\mathbf{A}} \end{pmatrix} \tag{9}$$

where $\dot{\mathbf{A}} = (\dot{a}_{ij})$ is the skew-symmetric matrix (6) and $\dot{\mathbf{a}}^*$ is the vector $\dot{\mathbf{a}}^* = (\dot{a}_1^*, \dot{a}_2^*, \dot{a}_3^*)^t$ (all elements are regarded at $t = 0$). All such tangent vectors form a 7-dimensional vector space

[2]Let I_3 and O_3 denote the 3×3-unit and the 3×3-zero matrix, respectively.

T_7 called the *'tangent space of G_7 at $P = id_{G_7}$'*. T_7 has the structure of the Lie algebra of G_7 (see KARGER/NOVAK [5]). The seven components of its vectors are

$$U := (\dot{\alpha}, \dot{a}_1^*, \dot{a}_2^*, \dot{a}_3^*, m_1, m_3, m_3) =: (u_0, \mathbf{u}^t, \hat{\mathbf{u}}^t). \tag{10}$$

Any instantaneous equiform motion defines such a vector U and vice versa. Changing the instantaneous velocity results in a multiplication of U by a scalar. $\dot{\alpha} = u_0 = 0$ characterizes the 6-dimensional subspace T_6, the *'Lie algebra of the Euclidean displacement group G_6 in $P = id_{G_6}$'*.

5. The adjoint group $Ad(G_7)$

We are interested in geometric results which are invariant with respect to changes of coordinates and scaling. The action of these changes on the Lie algebra vectors U is described by the so-called adjoint group $Ad(G_7)$ of G_7 (see KARGER/NOVAK [5]).

For any similarity

$$\mathcal{C} = \begin{pmatrix} 1 & \mathbf{o}^t \\ \mathbf{c}^* & \gamma\,\mathbf{C} \end{pmatrix} \quad \in G_7 \ (\gamma \in \mathbb{R} - \{0\}) \tag{11}$$

– the block $\mathbf{C}$ being orthogonal – the inverse of the matrix $\mathcal{C}$ is

$$\mathcal{C}^{-1} = \begin{pmatrix} 1 & \mathbf{o}^t \\ -1/\gamma\ \mathbf{C}^t\ \mathbf{c}^* & 1/\gamma\ \mathbf{C}^t \end{pmatrix}. \tag{12}$$

The action of the similarities (11) and (12) changes the description of the one-parametric motion $G_7 \ni \zeta(t) \ \ldots\ \mathcal{A}(t)$ with $\mathcal{A}(0) = id_{G_7}$ to

$$\bar{\mathcal{A}}(t) \longrightarrow \mathcal{C}\ \mathcal{A}(t)\ \mathcal{C}^{-1}. \tag{13}$$

It is called the *'conjugate motion to $\mathcal{A}(t)$ with respect to $\mathcal{C}$'*. Switching over to the conjugate motion also affects the tangent vector at the point id_{G_7} (see KARGER/NOVAK [5], p. 87). The transformation in the tangent space T_7 caused by this change is called the *'adjoint mapping'* $Ad(G_7)$. It is a linear mapping, which transforms the tangent vector U belonging to $\dot{\mathcal{A}}(0)$ into a tangent vector V belonging to $\mathcal{C}\dot{\mathcal{A}}(0)\mathcal{C}^{-1}$, both in T_7:

$$Ad(G_7) : U \mapsto V \text{ with } V = \mathcal{T}U. \tag{14}$$

As the input U we take the matrix $\dot{\mathcal{A}} = \begin{pmatrix} 0 & \mathbf{o}^t \\ \mathbf{u} & u_0\ \mathbf{I}_3 + \dot{\mathbf{A}} \end{pmatrix}$, where the skew-symmetric block $\dot{\mathbf{A}}$ belongs to the rotation axis vector $\hat{\mathbf{u}}$, getting:

$$\mathcal{C} \cdot \dot{\mathcal{A}} \cdot \mathcal{C}^{-1} = \begin{pmatrix} 0 & \mathbf{o}^t \\ \gamma\ \mathbf{C}\ \mathbf{u} - u_0\ \mathbf{c}^* - \mathbf{C}\dot{\mathbf{A}}\mathbf{C}^t\mathbf{c}^* & u_0 I_3 + \mathbf{C}\dot{\mathbf{A}}\mathbf{C}^t \end{pmatrix}. \tag{15}$$

The matrix $\mathbf{C}\dot{\mathbf{A}}\mathbf{C}^t$ is again skew-symmetric and belongs to the vector $C\hat{\mathbf{u}}$. Conversely the vector $\mathbf{c}^* = (c_1^*, c_2^*, c_3^*)^t$ (see (11)) gives rise to the skew-symmetric matrix

$$\mathbf{C}^* := \begin{pmatrix} 0 & -c_3^* & c_2^* \\ c_3^* & 0 & -c_1^* \\ -c_2^* & c_1^* & 0 \end{pmatrix}$$

which helps us to display $\mathcal{T}$ (14) in an elegant way:

$$\mathcal{T} = \begin{pmatrix} 1 & \mathbf{o}^t & \mathbf{o}^t \\ -\mathbf{c}^* & \gamma\mathbf{C} & -\mathbf{C}^t\mathbf{C}^* \\ \mathbf{o} & \mathbf{O} & \mathbf{C} \end{pmatrix}. \tag{16}$$

$\mathcal{T}$ is a regular 7×7-matrix with the determinant $det\mathcal{T} = \gamma^3$. All these linear transformations (14) of the 7-dimensional tangent space T_7 form the 7-parameter adjoint group $Ad(G_7)$.

6. Invariant bilinear forms with respect to the adjoint group

There are well-tried ways to define non-Euclidean Cayley-Klein geometries, if a vector space and a linear transformation group are given (see O. Giering [4]). A lot of the corresponding geometric invariants can be computed by the use of symmetric bilinear forms. The method is quite convenient if it is based on positively semi-definite bilinear forms. This is why we now look for such invariant bilinear forms in the Lie algebra T_7 with respect to the adjoint group[3] $Ad(G_7)$ (16):

Any symmetric bilinear form $F(U, V)$ for $U, V \in T_7$ is uniquely determined by a symmetric $7\times 7-$ matrix $\mathbf{Q}$ (see W. Greub [4]). Its invariance with respect to the adjoint group $Ad(G_7)$ is characterized by

$$\mathbf{Q} = \mathcal{T}^t \, \mathbf{Q} \, \mathcal{T} \tag{17}$$

for all matrices $\mathcal{T}$ (16). We drop the calculation and end up with

Theorem 1: *All symmetric bilinear forms $F(U, V) = U^t \, \mathbf{Q} \, V$ in the tangent space T_7 which meet the condition (17) are described by*

$$\mathbf{Q} = \begin{pmatrix} \lambda & \mathbf{o}^t & \mathbf{o}^t \\ \mathbf{o} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{o} & \mathbf{0}_3 & \kappa\mathbf{I}_3 \end{pmatrix} \tag{18}$$

with any values $\lambda, \kappa \in \mathbb{R}$. All positively semi-definite invariant bilinear forms are gained by $\lambda, \kappa \geq 0$.

We put $U := (u_0, \mathbf{u}^t, \hat{\mathbf{u}}^t)$ and $V := (v_0, \mathbf{v}^t, \hat{\mathbf{v}}^t)$. According to (18) we get the invariant values

$$F(U, V) = \lambda \, u_0 v_0 \; + \kappa \, \hat{\mathbf{u}}^t \hat{\mathbf{v}} \tag{19}$$

with $\lambda, \kappa \in \mathbb{R}$.

Remarks:

1) The definition of measures in the sense of non-Euclidean geometry uses invariant symmetric bilinear forms (see O. Giering [3]). In order to avoid the handling of many different cases it is better to make use of a quasi-elliptic structure, i.e., to use positively semi-definite bilinear forms.
2) Of course, the Killing bilinear form is contained in the set (19). A lengthy computation (as for the definition see Karger/Novak [5], p. 88) yields $\lambda = 4$, $\kappa = -3$. It is, however, not positively semi-definite and will not be used here.

[3] According to Karger/Novak [5], p. 88, there is at least one invariant bilinear form, the so-called *'Killing form'*.

We choose some $\lambda, \kappa > 0$, getting a positively semi-definite bilinear form invariant with respect to the adjoint group in the tangent space T_7. According to O. Giering [3] we define a measure of the angle of two vectors $U = (u_0, \mathbf{u}^t, \hat{\mathbf{u}}^t), V = (v_0, \mathbf{v}^t, \hat{\mathbf{v}}^t)$ in the tangent space T_7:

Definition 1: *The angle $\angle(U, V)$ between two vectors U, V in T_7 is defined by*

$$\cos\angle(U,V) = \frac{F(U,V)}{\sqrt{F(U,U)}\ \sqrt{F(V,V)}} = \frac{\lambda u_0 v_0 + \kappa \hat{\mathbf{u}}^t \hat{\mathbf{v}}}{\sqrt{\lambda u_0^2 + \kappa \hat{\mathbf{u}}^t \hat{\mathbf{u}}}\ \sqrt{\lambda v_0^2 + \kappa \hat{\mathbf{v}}^t \hat{\mathbf{v}}}} \tag{20}$$

with $\lambda, \kappa > 0$.

Remarks:

1) This definition is invariant with respect to the adjoint group $Ad(G_7)$. It gives us the opportunity to compare two vectors out of T_7 and the corresponding infinitesimal transformations.
2) Evidently the angle $\angle(U, V)$ (20) only depends upon the ratio $\lambda : \kappa$. Putting $\tau := \lambda : \kappa$ we get
$$\cos\angle(U,V) = \frac{\tau u_0 v_0 + \hat{\mathbf{u}}^t \hat{\mathbf{v}}}{\sqrt{\tau u_0^2 + \hat{\mathbf{u}}^t \hat{\mathbf{u}}}\ \sqrt{\tau v_0^2 + \hat{\mathbf{v}}^t \hat{\mathbf{v}}}}, \tag{21}$$
where $\tau > 0$ can be chosen at will.

Now we are prepared to measure an angle between a vector $U \in T_7$ and the subspace T_6 (belonging to the group of Euclidean displacements):

7. The deviation of the tangent vector

We start with a vector[4] $U \in T_7$ given by $U := (u_0, \dot{\mathbf{u}}^t, \hat{\mathbf{u}}^t)$. The subspace T_6 is characterized by $u_0 = 0$. According to standard strategies of non-Euclidean geometry based on the positively semi-definite bilinear form (19) the vector space orthogonal to T_6 is given by $N = (1, \mathbf{n}^t, \mathbf{0}^t)$ with an arbitrary vector $\mathbf{n} \in \mathbb{R}^3$. We define the non-Euclidean orthogonal projection of our tangent vector U into the hyperplane T_6 by

$$V = U - u_0\ N = (0, \mathbf{u}^t - u_0 \mathbf{n}^t, \hat{\mathbf{u}}^t). \tag{22}$$

We now are ready to define:

Definition 2: *The angle $\varphi \in [0, \pi/2]$ between $U \neq O$ and T_6 is the angle between U and the orthogonal projection V (22). According to (20) we have*

$$\cos\varphi = \sqrt{\frac{\hat{\mathbf{u}}^t \hat{\mathbf{u}}}{\tau u_0^2 + \hat{\mathbf{u}}^t \hat{\mathbf{u}}}}, \tag{23}$$

where the value $\tau > 0$ can be chosen arbitrarily in $\mathbb{R}^+$.

Remarks:

[4] The computation of the infinitesimal transformation (vector U) to a given position of a Stewart-Gough platform will be presented in Section 8.

1) As the bilinear form F is only semi-definite, the perpendicular vector N is not uniquely defined. Equation (23) shows that this does not affect our definition.
2) The constant value $\tau \in \mathbb{R}^+$ can be regarded as some sort of non-linear scaling factor. In order to be able to compare different measurements it is necessary to choose the factor $\tau > 0$ once for all. We propose to take $\tau = 1$.
3) The angle (23) indicates the deviation of the infintitesimal transformation U assigned to the instantaneous equiform motion from the group of Euclidean displacements. This is exactly what we were looking for.
4) In case that $u_0 = 0$ the vector $U \neq O$ is tangent to a Euclidean motion. In general, we get $\varphi = 0$.
5) For $U = (0, \mathbf{u}^t, \mathbf{o}^t)$ Definition 2 does not work. U is part of the *'singular space'* of the quasi-elliptic structure of T_7. This subspace is characterized by $F(U, U) = 0$. The corresponding instantaneous motions are pure translations. They are contained in the subspace T_6 — we set $\varphi := 0$ by definition, iff $F(U, U) = 0$ and $U \neq O$.
6) For $U = O$ the equiform motion instantaneously is *'at rest'*. If the motion is viewed as a curve on the Lie group, the corresponding point will is a cusp. In these cases the definition of the tangent would have to use higher order derivatives. This does not occur in the case of Stewart-Gough platforms, so we do not go into detail here.

We have got

Theorem 2: *Any instantaneous equiform motion (viewed as an element U of the Lie algebra T_7) can be evaluated with respect to the subalgebra T_6 of the Euclidean displacement group. The semi-definite bilinear forms (19) which are invariant with respect to the adjoint group $Ad(G_7)$ provide the invariant definition of the angle φ between U and T_6 via (20). Any measurement of this type based on the semi-definite bilinear forms (19) can be gained from the one presented.*

In order to apply these techniques to a given position of a Stewart-Gough platform we have to compute the tangent vector U of the corresponding equiform motion:

8. The tangent vector assigned to a regarded position

In this section we will finally be able to present the *'rigidity rate'*.

We are given a Stewart-Gough platform at a position denoted by $(\mathbf{p}_i^*, \mathbf{q}_i^*)$ (see (1) and (3)) for $(i = 1, \ldots, 6)$. If we keep the rod lengths constant (see Section 3) we gain an equiform one-parameter motion of the upper platform with respect to the basis. We determine its infinitesimal transformation U at the given platform position.

Differentiation of the leg length conditions (4) results in

$$(\mathbf{p}_i^* - \mathbf{q}_i^*)^t \cdot \dot{\mathbf{q}}_i^* = 0 \quad (i = 1, \ldots, 6). \tag{24}$$

We substitute (8) and (9), use the notation (10) and get

$$(\mathbf{p}_i^* - \mathbf{q}_i^*)^t (u_0 \mathbf{q}_i^* + \mathbf{u} + \hat{\mathbf{u}} \times \mathbf{q}_i^*) = 0 \quad (i = 1, \ldots, 6). \tag{25}$$

Little calculus gives the shape

$$(u_0, \mathbf{u}^t, \hat{\mathbf{u}}^t) \begin{pmatrix} \mathbf{q}_i^{*t}(\mathbf{p}_i^* - \mathbf{q}_i^*) \\ \mathbf{p}_i^* - \mathbf{q}_i^* \\ \mathbf{q}_i^* \times \mathbf{p}_i^* \end{pmatrix} = 0 \quad (i = 1, \ldots, 6). \tag{26}$$

This is a homogeneous linear system of 6 equations for the 7 unknowns contained in U. Depending on the rank r of its system matrix we get at least a one-dimensional vector space of solutions $S \subset T_7$. Any $U \in S$ represents an infinitesimal equiform transformation of the upper platform which instantaneously keeps the rod lengths. $U \in T_6 \setminus \{O\}$ (i.e., $u_0 = 0$) tells us that the regarded position of the platform is[5] *'shaky'*. The angle of deviation of U from the Lie algebra T_6 of the Euclidean displacements is an indicator for the rigidity of the given platform position in the direction of U. We have to consider two cases:

Case A: (General case) $dim(S) = 1$ (i.e., the rank of the system equals 6).
We define

Definition 3: *The value of the angle φ between the vectors $U \in S \setminus \{O\}$ and the subspace T_6 (tangent to the Euclidean displacement group) is called the* '**rigidity rate of the regarded platform position**'. *According to Definition 2 (23) it can be evaluated by*

$$\varphi := \arccos \sqrt{\frac{\hat{\mathbf{u}}^t \hat{\mathbf{u}}}{\tau u_0^2 + \hat{\mathbf{u}}^t \hat{\mathbf{u}}}} \quad (\varphi \in [0, \pi/2]) \quad \textit{or} \tag{27}$$

$$\varphi := 0 \quad \textit{for} \quad F(U, U) = 0 \quad \textit{but} \quad U \neq O. \tag{28}$$

Remarks:

1) Proportional vectors $U \in S \setminus \{O\}$ give the same rigidity rate.
2) The definition of the rigidity rate is invariant with respect to the group of Euclidean similarities G_7: If we apply a similarity transformation to a given platform position we gain another platform and a corresponding position. The rigidity rate of that position will equal the one of the original platform position.
3) The second case (28) relates to Remark 5 (after Def. 2) and completes the definition of the rigidity rate for the case of an infinitesimal transformation corresponding to a pure translation.
4) The rigidity rate $\varphi = 0$ indicates a *'shaky position'*. The stiffest possible position has the rigidity rate $\varphi = \pi/2$.
5) In Definition 2 we discussed the influence of the constant τ which we suggest to set $\tau := 1$.

Case B: $dimS \geq 2$.
As S is contained in the 7-dimensional vector space T_7 it has at least a one-dimensional intersection with the 6-dimensional vector space T_6. There will always be vectors $U \neq 0$ in $S \cap T_6$. This is why we put $\varphi := 0$ for these cases.

9. An example

In this section we want to compute the rigidity rate of some positions of a special Stewart-Gough platform. We start with the following anchor points (see Fig. 1)

$$\mathbf{p}_i^* := \begin{pmatrix} r\cos\alpha_i \\ r\sin\alpha_i \\ 0 \end{pmatrix} \tag{29}$$

[5]The dimension of the vector space $S \cap T_6$ could be used as a definition of the *'dimension of shakiness'*. This is a generalisation of the definition of shakiness of higher degree (see H. Stachel [12]).

for $i = 1, \dots, 6$ and a constant $r \in \mathbb{R}^+ \setminus \{0\}$ and

$$\begin{aligned} \alpha_1 &:= -\alpha, & \alpha_2 &:= \alpha \\ \alpha_3 &:= -\alpha + 2\pi/3, & \alpha_4 &:= \alpha + 2\pi/3 \\ \alpha_5 &:= -\alpha + 4\pi/3, & \alpha_6 &:= \alpha + 4\pi/3 \end{aligned} \tag{30}$$

where α denotes a constant shape parameter of the basis hexagon. The top hexagon of the platform is given by

$$\mathbf{q}_i := \begin{pmatrix} R\cos\beta_i \\ R\sin\beta_i \\ 0 \end{pmatrix} \tag{31}$$

for $i = 1, \dots, 6$ and $R \in \mathbb{R}^+ \setminus \{0\}$ and

$$\begin{aligned} \beta_1 &:= -\beta, & \beta_2 &:= \beta \\ \beta_3 &:= -\beta + 2\pi/3, & \beta_4 &:= \beta + 2\pi/3 \\ \beta_5 &:= -\beta + 4\pi/3, & \beta_6 &:= \beta + 4\pi/3 \end{aligned} \tag{32}$$

with another constant angle β. In order to avoid trivial cases (architectural singularity and possible self-motions!) we demand $\sin(\alpha - \beta) \neq 0$.

We want to consider positions with rotational symmetry with respect to the z^*-axis. They are given by

$$\mathbf{q}_i^* := \begin{pmatrix} R\cos(\beta_i + \psi) \\ R\sin(\beta_i + \psi) \\ k \end{pmatrix} \tag{33}$$

with $\psi \in [0, 2\pi)$ and $k \in \mathbb{R} \setminus \{0\}$. The constant value ψ is some *'twist angle'*, k is the height of the upper platform position with respect to the basis[6].

These input data are used to evaluate the system (26). It takes quite a few steps to get the following result:

$$U = \left(-\cos\psi,\ 0,\ 0,\ \frac{(R^2 - k^2)\cos\psi - Rr\cos(\alpha - \beta)}{k},\ 0,\ 0,\ \sin\psi \right). \tag{34}$$

According to our definition (27) the rigidity rate is

$$\varphi = \arccos\sqrt{\frac{\sin^2\psi}{\sin^2\psi + \tau\cos^2\psi}}, \tag{35}$$

especially for $\tau := 1$

$$\varphi = \arccos\sin\psi \quad (\varphi \in [0, \pi/2]). \tag{36}$$

So we get shaky positions ($\varphi = 0$) for $\psi = \pi/2$ and $\psi = 3\pi/2$, the stiffest configurations ($\varphi = \pi/2$) occur at $\psi = 0$ and $\psi = \pi$. Surprisingly, in this example the rigidity rate φ (36) measures the angular difference of ψ to the shaky positions $\psi = \pi/2$ and $\psi = 3\pi/2$. According to W. Wunderlich [14, 15, 16] *'snappy positions'* are to be expected for *'fairly small'* angles $\psi - \pi/2$ or $\psi - 3\pi/2$.

Figure 1 shows the platform of our example for $\psi = \pi/2 - \pi/12$. The rigidity value is $\varphi = \pi/12$ — a snappy configuration due to W. Wunderlich.

[6] Of course, this platform offers a great variety of other positions (without this rotational symmetry) which are out of consideration here.

10. Conclusions and final remarks

The *'rigidity rate'* (23) allows to compare different positions and different platforms in a geometrically invariant way (i.e., the definition is independant from any change of coordinates and scale). This is the crucial point of the whole matter.

To any position of a Stewart-Gough platform we assigned a tangent vector (in the Lie algebra). Basic notions of Lie group theory and differential geometry led to some measurement in the sense of the induced non-Euclidean Cayley-Klein geometries. This led to the desired rigidity rate defined as the deviation angle of the tangent vector with respect to a particular subspace.

The achieved result is comprehensive in the following sense: There is no other geometrically independent way of invariant measuring, as long as we stick to the induced positively semi-definite biliniear forms.

The basic idea of this paper can also be applied to other types of robots (e.g. 8-legged platforms): In order to define a tangent vector we then have to consider larger groups instead of the similarity group G_7 (e.g. the 12-dimensional affine group or the 15-dimensional projective group). Of course, the higher dimensions of these groups will result in higher dimensional subspaces in the tangent space of the group.

So the algorithm which we have worked out in detail is not at all confined to Stewart-Gough platforms. We do hope that our considerations will open a wide range of applications.

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Received February 2, 2001; final form September 24, 2001