Journal for Geometry and Graphics
Volume 5 (2001), No. 2, 133–144.

# Periodic and Aperiodic Figures on the Plane by Higher Dimensions

**Emil Molnár**[1], **Tilman Schulz**[2], **Jenő Szirmai**[1]

[1] *Department of Geometry, Budapest University of Technology and Economics*
*Egry József út 1, H.22., H-1111 Budapest XI, Hungary*
*emails: emolnar@mail.bme.hu, szirmai@math.bme.hu*

[2] *Department of Mathematics B, RWTH Aachen*
*D-52062 Aachen, Germany*
*email: tilman@momo.math.rwth-aachen.de*

*Dedicated to Prof. Gerhard* Geise *on the occasion of his* 70th *birthday*

**Abstract.** We extend de Bruijn's idea of constructing Penrose's non-periodic tilings of the plane to higher-dimensional analogons. On the base of $d$-dimensional space groups we can draw nice aperiodic coloured plane tilings with the aid of computers, especially interesting ones if $d+1$ is prime. Our proposed probabilistic method seems to produce attractive pictures, in particular.

*Key Words:* higher-dimensional space groups, two-dimensional projection, aperiodic tiling

*MSC 2000:* 52C22, 52C23

## 1. Introduction

Consider the "$(d+1)$-*rotation*" ($(d+1)$-cyclic transformation)

$$\mathbf{R}\colon\ \mathbf{e}_1 \ \to\ \mathbf{e}_2 \ \to\ \dots\ \to\ \mathbf{e}_d \ \to\ -\mathbf{e}_1-\mathbf{e}_2-\dots-\mathbf{e}_d\ (\ \to\ \mathbf{e}_1) \tag{1.1}$$

in the $d$-dimensional Euclidean space $\mathbb{E}^d$ for even $d$ in a coordinate system $(O,\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_d)$ to be fixed later more precisely.
That means in a matrix form $(\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_d)\to$

$$(\mathbf{R}\mathbf{e}_1,\mathbf{R}\mathbf{e}_2,\dots,\mathbf{R}\mathbf{e}_d)=(\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_{d-1},\mathbf{e}_d)\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1\\ 1 & 0 & 0 & \cdots & 0 & -1\\ 0 & 1 & 0 & \cdots & 0 & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \vdots & 1 & -1 \end{pmatrix}, \tag{1.2}$$

ISSN 1433-8157/\$ 2.50 

i.e., by use of the Einstein convention

$$\mathbf{e}_i \ \rightarrow \ \mathbf{e}_j R_i^j \ \text{ with } \det \mathbf{R} := \det(R_i^j) = 1.$$

The eigenvalues of $\mathbf{R}$ are the $(d+1)$-roots of one (except 1) of the characteristic equation $0 = \det(\lambda \mathbf{I}^d - \mathbf{R})$, that is the minimal polynomial for $\mathbf{R}$. By induction on $d$ we get

$$0 = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & 1 \\ -1 & \lambda & 0 & \cdots & 0 & 1 \\ 0 & -1 & \lambda & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & (\lambda+1) \end{vmatrix} = \lambda^d + \lambda^{d-1} + \ldots + \lambda + 1. \tag{1.3}$$

The complex conjugate eigenvalues

$$\lambda_1 = e^{i\frac{2\pi}{d+1}}, \ \lambda_d = e^{i\frac{2\pi d}{d+1}} =: \overline{\lambda}_1, \ \ldots$$

in pairs determine $\frac{d}{2}$ 2-dimensional $\mathbf{R}$-invariant real planes. Each of them can be chosen parallel to our computer screen to visualize the intersection of the $\mathbf{R}$-invariant collection of parallel hyperplane pencils, being defined in Section 4, with the screen.

So we get attractive $(d+1)$-periodic pictures, then also aperiodic ones by a unified algorithm, illustrated in our figures. The method is analogous to that of N.G. de Bruijn [4] which is very important in the theory of quasicrystals as well. In this direction we only refer to works of L. Danzer [6], A.W.M. Dress and his students [2] and of P. McMullen [7] (see also [1]). The general algorithm [10] for finding space groups in $\mathbb{E}^d$, gives the possibility to extend this method to further applications. This will also be indicated in Section 4.

## 2. The eigenvectors of R and its canonical form

Solving the eigenvector equation, according to (1.3), we get

$$(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d) \begin{pmatrix} \lambda^{d-1} + \cdots + \lambda + 1 \\ \vdots \\ \lambda + 1 \\ 1 \end{pmatrix} =: \mathbf{s}_\lambda \,, \tag{2.1}$$

a typical eigenvector to the eigenvalue $\lambda$ (up to a complex factor). Appropriate eigenvector pairs $\mathbf{s}_1, \mathbf{s}_d =: \overline{\mathbf{s}}_1, \ \ldots$ to the conjugate eigenvalues yield a new basis with

$$(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{d-1}, \mathbf{t}_d) := (\mathbf{s}_1, \overline{\mathbf{s}}_1, \ldots, \mathbf{s}_{d/2}, \overline{\mathbf{s}}_{d/2}) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2i} & \cdots & 0 & 0 \\ \frac{1}{2} & \frac{1}{2i} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2i} \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2i} \end{pmatrix} =$$

$$= (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1}, \mathbf{e}_d) \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \mathbf{M}, \tag{2.2}$$

where

$$\mathbf{M} = \begin{pmatrix} \lambda_1^{d-1} & \overline{\lambda}_1^{d-1} & \cdots & \lambda_{d/2}^{d-1} & \overline{\lambda}_{d/2}^{d-1} \\ \lambda_1^{d-2} & \overline{\lambda}_1^{d-2} & \cdots & \lambda_{d/2}^{d-2} & \overline{\lambda}_{d/2}^{d-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1 & \overline{\lambda}_1 & \cdots & \lambda_{d/2} & \overline{\lambda}_{d/2} \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2i} & \cdots & 0 & 0 \\ \frac{1}{2} & \frac{1}{2i} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2i} \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2i} \end{pmatrix}.$$

Thus, we can express the action of $\mathbf{R}$ on the basis $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{d-1}, \mathbf{t}_d)$:

$$\mathbf{R}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{d-1}, \mathbf{t}_d) = (\mathbf{R}\mathbf{s}_1, \mathbf{R}\overline{\mathbf{s}}_1, \dots, \mathbf{R}\mathbf{s}_{d/2}, \mathbf{R}\overline{\mathbf{s}}_{d/2}) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2i} & \cdots & 0 & 0 \\ \frac{1}{2} & \frac{1}{2i} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2i} \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2i} \end{pmatrix} =$$

$$= (\mathbf{s}_1, \overline{\mathbf{s}}_1, \dots, \mathbf{s}_{d/2}, \overline{\mathbf{s}}_{d/2})\mathbf{H} = (\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{d-1}, \mathbf{t}_d) \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ -i & i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & -i & i \end{pmatrix} \mathbf{H},$$

with

$$\mathbf{H} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \overline{\lambda}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{d/2} & 0 \\ 0 & 0 & \cdots & 0 & \overline{\lambda}_{d/2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2i} & \cdots & 0 & 0 \\ \frac{1}{2} & \frac{1}{2i} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2i} \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2i} \end{pmatrix}, \qquad (2.3)$$

i.e.,

$$\mathbf{R}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{d-1}, \mathbf{t}_d) = (\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{d-1}, \mathbf{t}_d) \begin{pmatrix} \cos\frac{2\pi}{d+1} & -\sin\frac{2\pi}{d+1} & \cdots & 0 & 0 \\ \sin\frac{2\pi}{d+1} & \cos\frac{2\pi}{d+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\frac{\pi d}{d+1} & -\sin\frac{\pi d}{d+1} \\ 0 & 0 & \cdots & \sin\frac{\pi d}{d+1} & \cos\frac{\pi d}{d+1} \end{pmatrix}$$

as usual canonical form. Moreover, (2.2) provides a real basis transform

$$(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{d-1}, \mathbf{t}_d) = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{d-1}, \mathbf{e}_d) \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \mathbf{W},$$

where

$$\mathbf{W} = \begin{pmatrix} \cos\frac{2\pi(d-1)}{d+1} & -\sin\frac{2\pi(d-1)}{d+1} & \cdots & \cos\frac{\pi d(d-1)}{d+1} & -\sin\frac{\pi d(d-1)}{d+1} \\ \cos\frac{2\pi(d-2)}{d+1} & -\sin\frac{2\pi(d-2)}{d+1} & \cdots & \cos\frac{\pi d(d-2)}{d+1} & -\sin\frac{\pi d(d-2)}{d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos\frac{2\pi}{d+1} & -\sin\frac{2\pi}{d+1} & \cdots & \cos\frac{\pi d}{d+1} & -\sin\frac{\pi d}{d+1} \\ 1 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{2.4}$$

not expanded further, because (2.2) is also comfortable through complex numbers. We use the Einstein convention by writing

$$\mathbf{t}_i = \mathbf{e}_j t_i^j \tag{2.5}$$

for the formula (2.4) or (2.2), respectively.

## 3. An R-invariant scalar product

For an $\mathbf{R}$-invariant symmetric positive definite scalar product

$$\langle\, ,\rangle:\ \mathbb{E}^d \times \mathbb{E}^d \ \rightarrow\ \mathbb{R}, \quad \langle \mathbf{e}_i, \mathbf{e}_j\rangle =: g_{ij} = g_{ji} \tag{3.1}$$

the Gramian $(g_{ij})$ has to be introduced by (1.2)

$$R_i^\alpha g_{\alpha\beta} R_j^\beta = g_{ij} \tag{3.2}$$

where $R_i^\alpha$ is the transposed of $R_\alpha^i$.
This is an equation system for the Gramian $g_{ij}$ of $\frac{d(d+1)}{2}$ parameters whose number can be reduced to $\frac{d}{2}$ as follows:

$$\begin{gathered} g_{11} = g_{22} = \cdots = g_{dd} =: g_0, \quad g_{12} = g_{23} = \cdots = g_{d-1,d} =: g_1, \\ g_{13} = g_{24} = \cdots = g_{d-2,d} =: g_2, \cdots, \ g_{1,d-1} = g_{2d} =: g_{d-2} =: g_{1d} =: g_{d-1}, \\ g_{dd} =: g_0 = dg_0 + 2(d-1)g_1 + 2(d-2)g_2 + \cdots + 2g_{d-1}, \\ g_{d-1,d} =: g_1 = -g_{d-1} - g_{d-2} \cdots - g_1 - g_0, \\ g_{d-2,d} =: g_2 = -g_{d-2} - g_{d-3} \cdots - g_1 - g_0 - g_1, \\ \vdots \\ g_{1,d} =: g_{d-1} = -g_1 - g_0 - g_1 \cdots - g_{d-2}. \end{gathered} \tag{3.3}$$

From this we get a solution (not uniquely) for any dimension $d \geq 2$

$$g_0 = 1,\ g_1 = \cdots = g_{d-1} = -\frac{1}{d},$$

i.e.

$$(g_{ij}) = \begin{pmatrix} 1 & -\frac{1}{d} & \cdots & -\frac{1}{d} & -\frac{1}{d} \\ -\frac{1}{d} & 1 & \cdots & -\frac{1}{d} & -\frac{1}{d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{d} & -\frac{1}{d} & \cdots & 1 & -\frac{1}{d} \\ -\frac{1}{d} & -\frac{1}{d} & \cdots & -\frac{1}{d} & 1 \end{pmatrix} =: (\langle \mathbf{e}_i, \mathbf{e}_j\rangle) \tag{3.4}$$

which provides a positive definite quadratic form $x^i g_{ij} x^j =: \langle \mathbf{x}, \mathbf{x}\rangle$ as it is well-known. Thus, the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1}, \mathbf{e}_d\}$ will be fixed in $\mathbb{E}^d$ up to isometry defined by the scalar product $\langle\, ,\rangle$ above.

## 4. On R-invariant symmetric forms, in general

As $\mathbf{R}$ describes a representation of the cyclic group $C_{d+1}$ of order $d+1$, affording each nontrivial character exactly once, the space of all $\mathbf{R}$-invariant bilinear forms has dimension $d$, and the space of symmetric bilinear forms has dimension $\frac{d}{2}$. Determining the space of $G$-invariant forms $(g_{ij})$ for any matrix group $G$ requires solving linear equations of the form (3.2) for each generator of $G$ in the place of $R_i^j$. For a general algorithm determining all or some invariant (positive definite) quadratic forms, see [9]. In this case, however, the situation is more simple, as the group generated by $\mathbf{R}$ is cyclic, and hence Abelian.

First note, that $C_{d+1} \leq S_{d+1}$, where $S_{d+1}$ is the symmetric group on $d+1$ letters, e.g $\mathbf{e}_1, \dots, \mathbf{e}_{d+1}$, first. $S_{d+1}$ has an absolutely irreducible representation of degree $d$ over any field of characteristic 0, coming from its natural permutation module, subtracting the trivial constituent, i.e., describing the action on the quotient space modulo $\langle \mathbf{e}_1 + \dots + \mathbf{e}_{d+1} \rangle$. Note that this provides exactly the representation for $\mathbf{R}$ given in the introduction. Because this representation of $S_{d+1}$ is absolutely irreducible (and rational, hence equivalent to its dual), it is also uniform, and the $S_{d+1}$-invariant form (mod scalar factor) is given by (3.4). For a given quadratic form, e.g., for $(g_{ij})$ in (3.4) we can look for the integral matrix group leaving it invariant. In [3] for $d=4$ we find a maximal matrix group (of order $240 = 2 \cdot 5!$) 31/07/01 of Bravais lattice type XXII/$\mathbf{I}$ with generators

$$\mathbf{A}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}\begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{C}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \tag{4.1}$$

where $\mathbf{B}^2 \sim \mathbf{R}$ is integral ($\mathbb{Z}$) equivalent with our 5-cyclic transform. Surprisingly, a centred lattice basis with rational ($\mathbb{Q}$) matrix $\mathbf{W} = (W_i^j)$ as follows

$$(\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \hat{\mathbf{e}}_3\ \hat{\mathbf{e}}_4) = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3\ \mathbf{e}_4)\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}, \quad \hat{\mathbf{e}}_i = \mathbf{e}_j W_i^j, \tag{4.2}$$

provides us an isomorphic ($\mathbb{Q}$-equivalent) matrix group 31/07/02 of Bravais lattice type XXII/$\mathbf{II}$, with corresponding generators

$$\hat{\mathbf{A}}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbf{B}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbf{C}}\begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{4.3}$$

That means, e.g., $\mathbf{B} = \mathbf{W}\hat{\mathbf{B}}\mathbf{W}^{-1}$ holds with

$$\mathbf{W}^{-1} = \begin{pmatrix} -1 & -1 & -1 & 4 \\ -1 & -1 & 4 & -1 \\ -1 & 4 & -1 & -1 \\ 4 & -1 & -1 & -1 \end{pmatrix}, \text{ and } (\hat{g}_{ij}) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \tag{4.4}$$

will be the matrix of the corresponding invariant quadratic form (up to a scalar factor), as $W_i^\alpha g_{\alpha\beta} W_j^\beta = \frac{1}{10}\hat{g}_{ij}$ shows in case $d=4$. We say: $(g_{ij})$ characterizes the Bravais type XXII/$\mathbf{I}$, and $(\hat{g}_{ij})$ describes the $\mathbb{Q}$-equivalent Bravais type XXII/**II**. These maximal groups $\Gamma$ and $\hat{\Gamma}$ are $\mathbb{Q}$-equivalent but not $\mathbb{Z}$-equivalent ($\mathbb{Z}$ for integers), thus $\Gamma$ and $\hat{\Gamma}$ describe different space groups. Note, however, that the groups generated by $\mathbf{B}$ and $\hat{\mathbf{B}}$, respectively, are also $\mathbb{Z}$-equivalent, according to our starting arguments. A similar situation occurs in every even dimension $d$.

## 5. The intersections

The computer screen will be chosen first as a 2-dimensional point set

$$\mathcal{C} := \left\{ \overrightarrow{OT} =: \mathbf{t} = \mathbf{t}_0 + \mathbf{t}_1 c^1 + \mathbf{t}_2 c^2 \mid c^1, c^2 \in \mathbb{R} \right\} \tag{5.1}$$

in our $d$-space spanned by $\{O, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1}, \mathbf{e}_d\}$, where

$$\mathbf{t}_0 = \mathbf{e}_i t_0^i \quad \text{and} \quad \mathbf{t}_1 = \mathbf{e}_j t_1^j, \ \mathbf{t}_2 = \mathbf{e}_k t_2^k. \tag{5.2}$$

Here $\mathbf{t}_0$ is given arbitrarily, $\mathbf{t}_1$ and $\mathbf{t}_2$ are due to (2.5), (2.4), (2.2). Now the 2-dimensional Gramian of $\{\mathbf{t}_1, \mathbf{t}_2\}$ can be chosen as

$$\langle \mathbf{t}_\alpha, \mathbf{t}_\beta \rangle =: \mathbf{t}_{\alpha\beta} = t_\alpha^i g_{ij} t_\beta^j = tt\delta_{\alpha\beta} \tag{5.3}$$

(the Kronecker symbol) $\alpha, \beta = 1, 2$ by (2.3) and the $\mathbf{R}$-invariance of the scalar product $\langle\, , \rangle$ above. That means $\{\mathbf{t}_1, \mathbf{t}_2\}$ will be fitted to the orthonormal basis of the screen by a similarity factor $t > 0$. Varying this factor later, we get larger or smaller picture on our screen. Now we define the important hyperplanes of the point lattice $L_O$ of the origin $O$, spanned by the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1}, \mathbf{e}_d\}$ in Sections 1 and 3

$$L_0 := \left\{ \ X \mid \overrightarrow{OX} = \mathbf{e}_1 x^1 + \ldots + \mathbf{e}_d x^d, \ x^1, \ldots, x^d \in \mathbb{Z} \right\} \tag{5.4}$$

with Gramian $g_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$ in (3.4).

**5.1.**

The dual vectors $\{\boldsymbol{\varepsilon}^j\}$ to $\{\mathbf{e}_i\}$ defined by

$$\boldsymbol{\varepsilon}^j \mathbf{e}_i = \delta_i^j \ \text{(Kronecker)} \tag{5.5}$$

in the dual space $\mathbb{E}_d$ of linear forms to $\mathbb{E}^d$ assign natural hyperplanes by equations

$$\boldsymbol{\varepsilon}^j \mathbf{x} = \boldsymbol{\varepsilon}^j \mathbf{e}_i x^i = x^j = \mathit{const.} \in \mathbb{Z}. \tag{5.6}$$

The transformation formula (1.2) provides the corresponding one to linear forms

$$\begin{pmatrix} \boldsymbol{\varepsilon}^1 \\ \boldsymbol{\varepsilon}^2 \\ \vdots \\ \boldsymbol{\varepsilon}^d \end{pmatrix} \to \begin{pmatrix} \boldsymbol{\varepsilon}^1 \mathbf{R}^{-1} \\ \boldsymbol{\varepsilon}^2 \mathbf{R}^{-1} \\ \vdots \\ \boldsymbol{\varepsilon}^d \mathbf{R}^{-1} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}^1 \\ \boldsymbol{\varepsilon}^2 \\ \vdots \\ \boldsymbol{\varepsilon}^d \end{pmatrix}, \tag{5.7}$$

indeed, by keeping the form values invariant

$$(\boldsymbol{\varepsilon}^j \mathbf{R}^{-1})(\mathbf{R}\mathbf{e}_i) = \boldsymbol{\varepsilon}^j \mathbf{e}_i = \delta_i^j. \tag{5.8}$$

That means

$$\mathbf{R}^{-1} : \boldsymbol{\varepsilon}^1 \;\to\; -\boldsymbol{\varepsilon}^1 + \boldsymbol{\varepsilon}^2 \;\to\; -\boldsymbol{\varepsilon}^2 + \boldsymbol{\varepsilon}^3 \;\to\; \dots \;\to\; -\boldsymbol{\varepsilon}^{d-1} + \boldsymbol{\varepsilon}^d \;\to\; \boldsymbol{\varepsilon}^d \; (\to \boldsymbol{\varepsilon}^1) \tag{5.9}$$

is a $(d+1)$-rotation according to (1.1)–(1.2).
The hyperplane pencil parallel to $\boldsymbol{\varepsilon}^1$ has equations

$$\boldsymbol{\varepsilon}^1 \mathbf{x} = x^1 = \mathit{const.} \in \mathbb{Z}.$$

This intersects our computer screen (4.1)–(4.2) in points $T(\mathbf{t})$ by

$$\boldsymbol{\varepsilon}^1 \mathbf{t} = t_0^1 + t_1^1 c^1 + t_2^1 c^2 = x^1 = \mathit{const.} \in \mathbb{Z} \tag{5.10}$$

as linear equation in the screen coordinates $c^1, c^2 \in \mathbb{R}$ for each fixed $x^1 \in \mathbb{Z}$. Here $t_0^1$ is determined by the origin $T_0(\mathbf{t}_0)$ of the screen in $\mathbb{E}^d$. The formula (2.2) or (2.4) gives $\mathbf{t}_1$ and $\mathbf{t}_2$, e.g.,

$$t_1^1 = 1 + \cos\frac{2\pi}{d+1} + \dots + \cos\frac{2\pi(d-1)}{d+1},\; t_2^1 = -\sin\frac{2\pi}{d+1} - \dots - \sin\frac{2\pi(d-2)}{d+1}. \tag{5.11}$$

Thus we get a parallel line pencil, coloured with the first colour (say red) in our screen. Then comes the second hyperplane pencil, parallel to $-\boldsymbol{\varepsilon}^1 + \boldsymbol{\varepsilon}^2$

$$(-\boldsymbol{\varepsilon}^1 + \boldsymbol{\varepsilon}^2)\mathbf{x} = -x^1 + x^2 = \mathit{const.} \in \mathbb{Z}.$$

This intersects our computer screen (5.1)–(5.2) as the equation

$$(-\boldsymbol{\varepsilon}^1 + \boldsymbol{\varepsilon}^2)\mathbf{t} = -t_0^1 + t_0^2 + (-t_1^1 + t_1^2)c^1 + (-t_2^1 + t_2^2)c^2 = \mathit{const.} \in \mathbb{Z} \tag{5.12}$$

prescribes for the screen coordinates $c^1, c^2 \in \mathbb{R}$. Thus we get the parallel line pencil, designed with the second colour (yellow) in the screen. Again (2.2) or (2.4) provides the coefficients. And so on up to $-\boldsymbol{\varepsilon}^d$, deriving the $(d+1)$-th line pencil in the screen. Until now the scalar product (3.4) did not play an explicit role.

## 5.2.

The hyperplanes orthogonal to $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{d-1}, \mathbf{e}_d, -\mathbf{e}_1 - \mathbf{e}_2 - \dots - \mathbf{e}_{d-1} - \mathbf{e}_d$, respectively, through the origin, have the equations for $\mathbf{x} = \mathbf{e}_i x^i$

$$\begin{gathered} 0 = \langle \mathbf{e}_1, \mathbf{x} \rangle = \langle \mathbf{e}_1, \mathbf{e}_i x^i \rangle = g_{1i} x^i, \text{ then } 0 = g_{2i} x^i, \dots, 0 = g_{di} x^i, \\ 0 = -(g_{1i} + g_{2i} + g_{3i} + \dots + g_{di}) x^i, \end{gathered} \tag{5.13}$$

respectively. The corresponding parallel hyperplane pencils, each having all points of $L_0$ by (5.4), will be

$$\begin{gathered} d g_{1i} x^i = \mathit{const.} \in \mathbb{Z}, \;\dots,\; d g_{di} x^i = \mathit{const.} \in \mathbb{Z}, \\ -x^1 - x^2 - \dots - x^d = \mathit{const.} \in \mathbb{Z}, \end{gathered} \tag{5.14}$$

respectively. These intersect our computer screen (4.1)–(4.2) in points $T(\mathbf{t})$ by

$$\begin{gathered} d g_{1i}(t_0^i + t_1^i c^1 + t_2^i c^2) = \mathit{const.} \in \mathbb{Z}, \;\dots,\; d g_{di}(t_0^i + t_1^i c^1 + t_2^i c^2) = \mathit{const.} \in \mathbb{Z}, \\ -(t_0^1 + \dots + t_0^d) - (t_1^1 + \dots + t_1^d)c^1 - (t_2^1 + \dots + t_2^d)c^2 = \mathit{const.} \in \mathbb{Z}, \end{gathered} \tag{5.15}$$

respectively, $d+1$ parallel line pencils in coordinates $c^1, c^2 \in \mathbb{R}$, designed by $d+1$ colours. Thus we get an $\mathbf{R}$-invariant picture on our computer screen $\mathcal{C}$ by (5.1)–(5.2), again.

## 6. Examples and closing remarks

For $d = 2$ our $\mathbb{E}^2$ is just the computer screen

$$(\mathbf{e}_1, \mathbf{e}_2) \rightarrow (\mathbf{R}\mathbf{e}_1, \mathbf{R}\mathbf{e}_2) = (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix};$$

$$(\mathbf{t}_1, \mathbf{t}_2) = (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{pmatrix},$$

$$g_{ij} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad t_{\alpha\beta} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$

Let $\mathbf{t}_0 = \mathbf{0}$. Then we get the 3 line pencils, as indicated in Subsection **5.1**,

$$\frac{1}{2}c^1 - \frac{\sqrt{3}}{2}c^2 = const. \in \mathbb{Z}, \quad \frac{1}{2}c^1 + \frac{\sqrt{3}}{2}c^2 = const. \in \mathbb{Z} \text{ and } 1c^1 = const. \in \mathbb{Z}. \tag{6.1}$$

For the construction of Subsection **5.2** we get in (5.15)

$$\begin{gathered} 2\left(-\tfrac{\sqrt{3}}{2}\right)c^2 = const. \in \mathbb{Z}, \quad 2\left(\tfrac{3}{4}c^1 + \tfrac{\sqrt{3}}{4}c^2\right) = const. \in \mathbb{Z}, \\ -\tfrac{3}{2}c^1 + \tfrac{\sqrt{3}}{2}c^2 = const. \in \mathbb{Z}. \end{gathered} \tag{6.2}$$

Both pictures serve the well-known regular triangle tiling up to a similarity constant (Fig. 1).

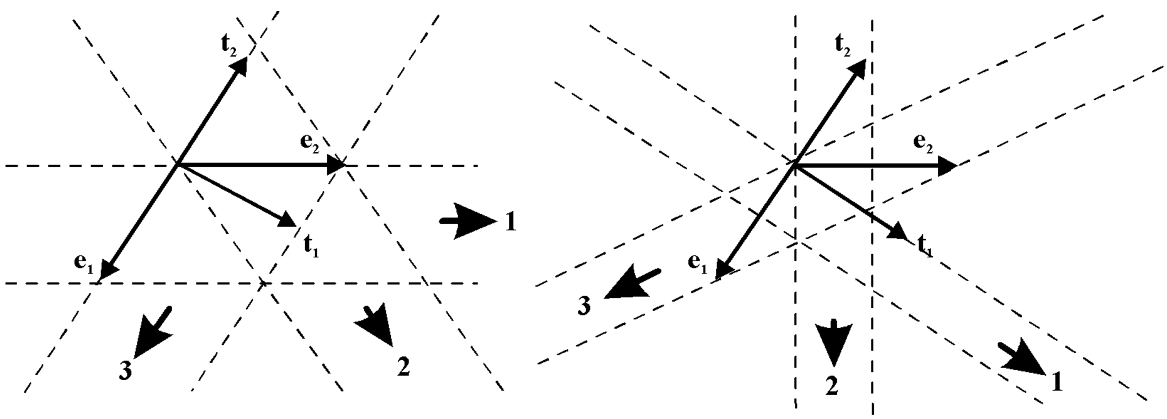


Figure 1: The construction to (6.1) and (6.2), respectively, $d = 2$

Varying the origin $T_0(\mathbf{t}_0)$ of the screen, the 3 pencils do not move, since they are defined to the coordinate system $(O, \mathbf{e}_1, \mathbf{e}_2)$. But if we allow inhomogeneous linear forms for a pencil, e.g.,

$$\left(\frac{1}{2}, \boldsymbol{\varepsilon}^1\right)\mathbf{x} := \frac{1}{2} + \boldsymbol{\varepsilon}^1\mathbf{x} = \frac{1}{2} + x^1 = const. \in \mathbb{Z},$$

then the origin $O$ may not lie on any pencil line (Fig. 2).

The most interesting dimensions are $d = 4, 6, 10, 12, \ldots$, when $d+1$ is prime. Moreover, the planes $\{\mathbf{t}_3, \mathbf{t}_4\}, \ldots, \{\mathbf{t}_{d-1}, \mathbf{t}_d\}$ by (2.2), (2.4) can play the former role of $\{\mathbf{t}_1, \mathbf{t}_2\}$, with permuted colours.

In Subsection **5.1** any form $\boldsymbol{\varepsilon}$ and its $\mathbf{R}^{-1}$ images by (5.7) can be chosen because of **5.2** such that we choose the $d+1$ linear forms

$$\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2 - \ldots - \boldsymbol{\varepsilon}^d, \quad \boldsymbol{\varepsilon}\mathbf{R}^{-1} = -\boldsymbol{\varepsilon}^1 + d\boldsymbol{\varepsilon}^2 - \ldots - \boldsymbol{\varepsilon}^d, \quad \ldots, \quad \boldsymbol{\varepsilon}\mathbf{R}^{-d} = -\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2 - \ldots - \boldsymbol{\varepsilon}^d \tag{6.3}$$

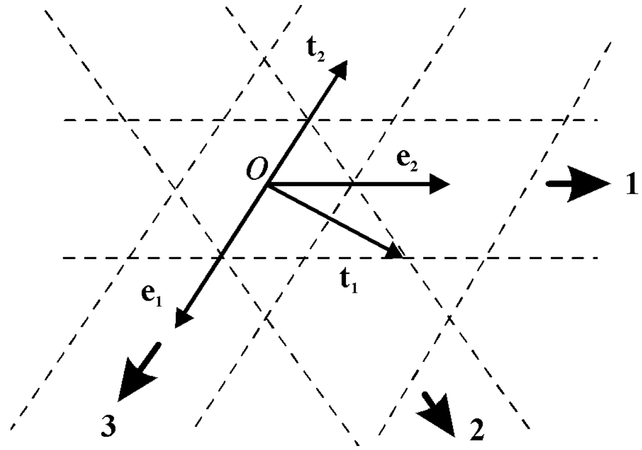


Figure 2: The construction with inhomogeneous linear form to (6.1)

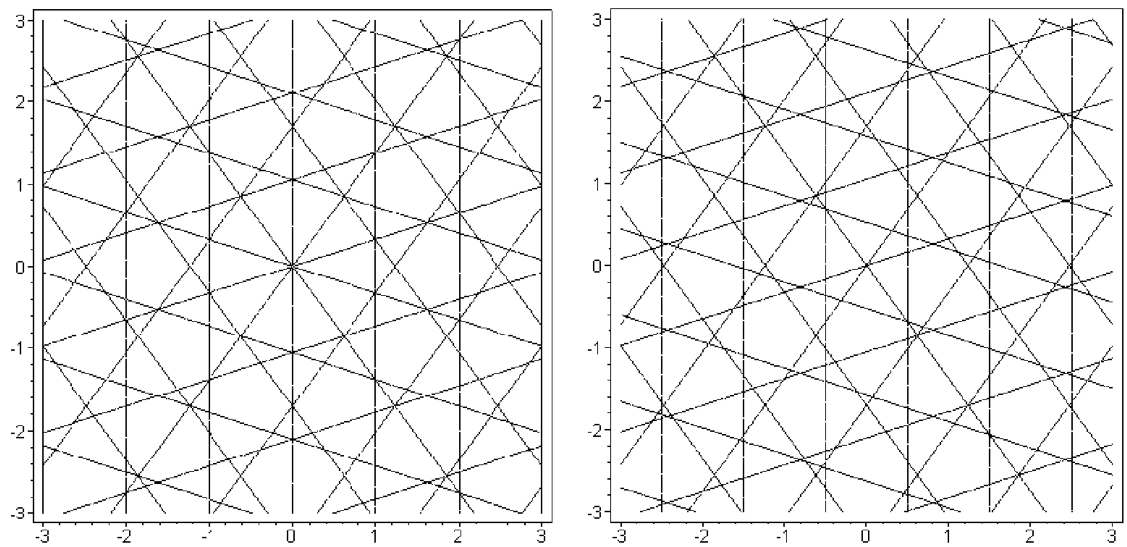

Figure 3: Dimension 4; the screen is parallel to $(\mathbf{t}_1, \mathbf{t}_2)$, intersected by coordinate 3-planes (a) through the origin, (b) translated origin

for the $d+1$ hyperplane pencils in Subsection **5.1**.

In cases $d > 2$ the change of the origin $T_0(\mathbf{t}_0)$ of the screen $\mathcal{C}$ will cause changes in the $d+1$ line pencils.

In Figures 3–6 we illustrate $d = 4$, and $d = 6$ for 5 and 7 order rotational symmetry, respectively. For $d = 4$ we have the scalar product at (3.4)

$$g_0 = 1,\ g_1 = g,\ g_2 = g_3 = -\frac{1}{2} - g, \tag{6.4}$$

in general, for so called decagonal lattices with one free parameter $g$ ($g = -\frac{1}{4}$ yields $g_2 = g_3 = -\frac{1}{4}$ as well in (3.4) which describes the so-called icosahedral lattice in Section 5).

<u>*Remarks:*</u>

1. Although the planar pictures for given $d$ can be drawn up to similarity without any $d$-dimensional theory, the method might give some advantages from various aspects emphasized no more in details.

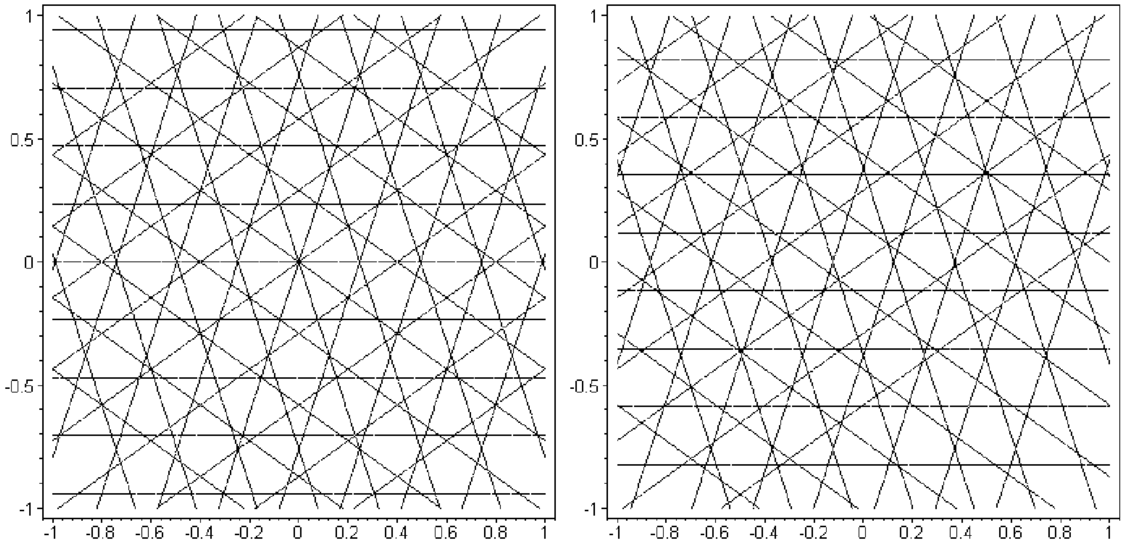


Figure 4: Dimension 4, the screen is parallel to $(\mathbf{t}_1, \mathbf{t}_2)$, intersected by coordinate 3-planes orthogonal to $\mathbf{e}_1, \mathbf{e}_2, \ldots, -\mathbf{e}_1 - \ldots - \mathbf{e}_d$, respectively
(a) through the origin, (b) translated origin

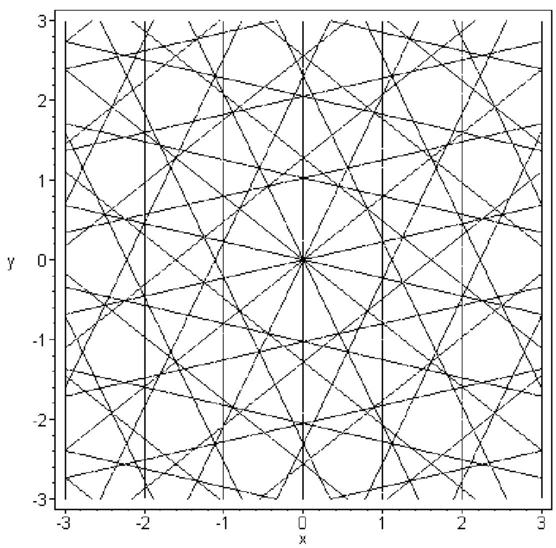


Figure 5: Dimension 6, the screen parallel to $(\mathbf{t}_1, \mathbf{t}_2)$, intersected by coordinate 5-planes through the origin

2. Observe that the line pencils may have only one points (the centre) when all the $d+1$ lines meet, since any 2-plane $\{\mathbf{t}_1, \mathbf{t}_2\}$ by (2.4) will have only one common point with the $d+1$ hyperplanes because of the irrationality of the $d+1$-roots of one ($\neq 1$).
3. The $d+1$ line pencils determine $\frac{d}{2}$ congruence types of parallelograms which can form a nonregular tiling, by cancelling superflous lines. We suggest the reader to construct a probabilistic algorithm for such a nonregular tiling (see Fig. 6,a-b).
4. All of our algorithms by coloured lines provide attractive pictures (nicer than our illustrations without colours) and some imaginations about higher-dimensional Euclidean

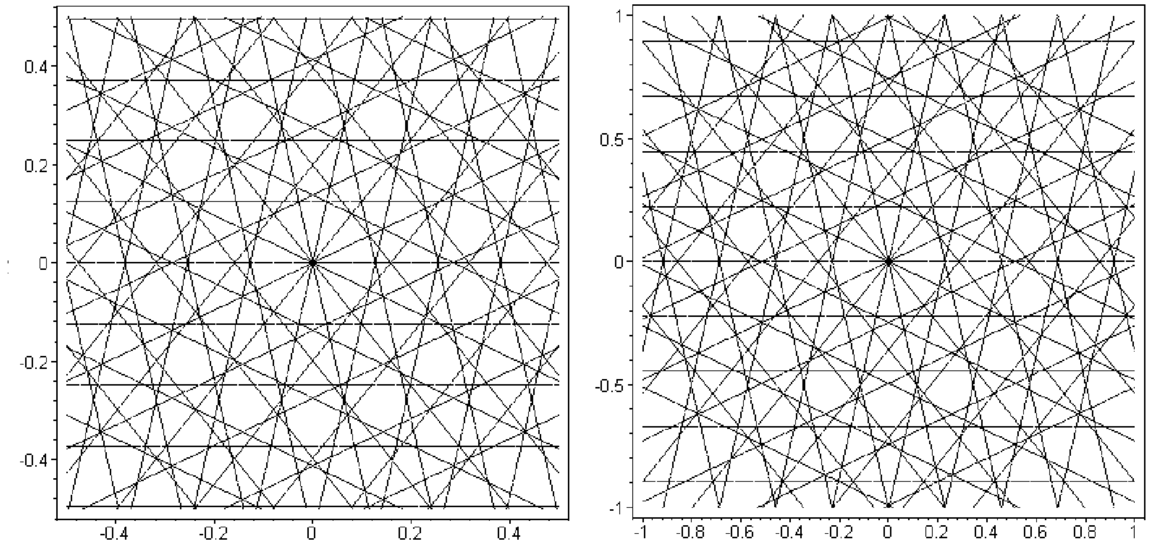


Figure 6: Dimension 6, the pencils are induced by 5-planes orthogonal to $\mathbf{e}_1$, $\mathbf{e}_2$, …, $-\mathbf{e}_1 - \ldots - \mathbf{e}_6$. The screen is parallel to (a) $(\mathbf{t}_1, \mathbf{t}_2)$, (b) $(\mathbf{t}_3, \mathbf{t}_4)$, respectively.

geometry.

5. We again assume that $d$ is even. If we imagine 3-dimensional screen, i.e., such an intersection, then $d+1$ order periodicity can easily be achived in dimension $d+1$ by taking a basis vector invariant under rotation $\mathbf{R}$ due to equations (1.1)–(1.2)

$$\mathbf{e}_{d+1} = \mathbf{R}\mathbf{e}_{d+1}. \tag{6.5}$$

Then $\mathbf{t}_3 = \mathbf{e}_{d+1}$ in addition to (5.2), perpendicular to $\mathbf{t}_1$ and $\mathbf{t}_2$ will be the third "screen basis vector".
A similar phenomenon can be obtained by

$$\mathbf{R}\colon\ \mathbf{e}_1 \ \rightarrow\ \mathbf{e}_2 \ \rightarrow\ \ldots\ \rightarrow\ \mathbf{e}_d \ \rightarrow\ \mathbf{e}_{d+1} \ \rightarrow\ \mathbf{e}_1 \tag{6.6}$$

with a $\mathbf{R}$-invariant vector $\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{d+1}$.

## Acknowledgments

The authors wish to thank the Hungarian National Scientific Fund (OTKA) T020498 (1996). This research was carried out in the framework of collaboration between the Department of Mathematics B, RWTH Aachen, and Department of Geometry, BME Budapest, supported by D-4/99 DAAD (Germany).

## References

[1] M. Baake, P. Kramer, M. Schlottmann, D. Zeideler: *Planar patterns with fivefold symmetry as sections of periodic structures in 4-space.* In I. Hargittai (ed.): Quasicrystals, Networks, and Molecules of Fivefold symmetry, VCH Publishers New York 1990, Ch. 9, 127–157.

[2] J. Bohne, A.W.M. Dress, S. Fischer: *A simple proof for de Bruijn's dualization principle*. Combinatorial mathematics and application (Calcutta 1988), Special Issue of Sankhy/=a, Ser. A **54**, 77–84 (1992).

[3] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, H. Zassenhaus: *Crystallographic groups of four-dimensional space*. A. Wiley-Interscience 1978.

[4] N.G. de Bruijn: *Algebraic theory of Penrose's non-periodic tilings of the plane*. Kon. Nederl. Akad. Wetensch. Proc., Ser. A **84** (Indagationes Math. **84**), 38–52 and 53–66 (1981).

[5] N.G. de Bruijn: *Dualization of multigrids*. J. Physique Colloque **C3**, suppl. 7,47, C3-9–C3-18 (1986).

[6] L. Danzer: *Three dimensional analogs of the planar Penrose tilings and quasicrystals*. Discrete Math. **76**, no. 1, 1–7 (1989).

[7] P. McMullen: *Nondiscrete regular honeycombs*. In I. Hargittai (ed.): Quasicrystals, Networks, and Molecules of Fivefold symmetry, VCH Publishers New York, Ch. 10, 159–179 (1990).

[8] J. Opgenorth, W. Plesken, T. Schulz: *Crystallographic algorithms and tables*. Acta Cryst. **A.54**, 517–531 (1998).

[9] W. Plesken, B. Souvignier: *Constructing rational representations of finite groups*. Exp. Math. **5**, no. 1, 39–48 (1996).

[10] W. Plesken, T. Schulz: *Counting crystallographic groups in low dimensions*. Exp. Math. **9**, no. 3, 407–411 (2000).

Received January 3, 2001; final form October 31, 2001