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A Basic Evaluation Method of Subdivision Surfaces

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Abstract. A subdivision surface is a powerful tool to represent a smooth surface with arbitrary topology. It starts with a control polyhedron which roughly approximates the final surface. The polyhedron gradually approaches to the final surface by subdividing faces more closely to approximate the final surface. However, it is not so popular in engineering applications because of the procedural nature. Algorithms for parametric surfaces cannot be applicable to a procedural surface. This paper discusses a method for evaluating a subdivision surface with parametric values based on Stam's work. Our approach gained stability around an extraordinary point as well as the convergence of a normal vector at the point.

Key Words: Subdivision surface, Catmull-Clark surface, Loop surface

MSC 2000: 68U05, 53A05

1. Introduction

Subdivision surfaces have been considered as a powerful tool to represent smooth shapes with arbitrary topology. Although they are widely used in computer graphics applications, they are not so popular in engineering applications such as CAD/CAM/CAE. One of the reasons is their procedural nature. Most of algorithms for parametric surfaces are not applicable to the subdivision surfaces.

Recently a method for evaluating a surface point with certain parametric values was proposed by J. Stam [6]. This method allows us to treat subdivision surfaces as parametric surfaces. However, there still remain some problems on the robustness of the computation around extraordinary points. This paper discusses the issue on basic evaluations of subdivision surfaces and proposes an approach for the solution.

This paper proceeds with the overview of subdivision surfaces. Section 3 describes their mathematical formulation. Section 4 discusses the computation around extraordinary points. Section 5 explains a Catmull-Clark surface as an example of a subdivision surface. Finally, Section 6 concludes the paper.

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2. Subdivision Surface

A *subdivision surface* starts with a control polyhedron which roughly approximates the final surface. In each step of the subdivision, a new polyhedron is generated by subdividing faces into smaller faces which approximate the final surface more closely than the original faces. Therefore the polyhedron gradually approaches to the final surface.

There are several different types of subdivision surfaces, namely, a Catmull-Clark surface [1], a Doo-Sabin surface [2], a Loop surface [3], etc. Fig. 1 illustrates subdivision steps of a Catmull-Clark surface and a Loop surface. Every face is subdivided into small faces by inserting vertices. One can easily compute approximating polyhedra with any accuracy by recursively subdividing the polyhedron based on the subdivision rules. This is usually enough for the rendering purpose. However, they are merely approximations. A subdivision process do not calculate exact positions or normals of the surface.

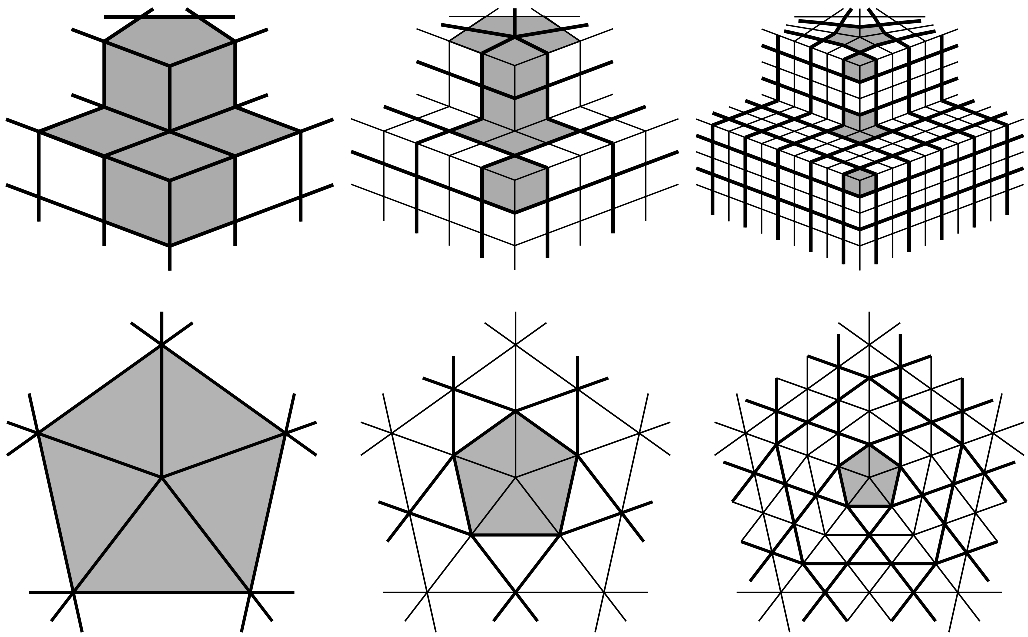


Figure 1: Subdivision steps of a Catmull-Clark surface (above) and Loop surface (bottom)

A *valence* of a vertex is the number of edges connected to the vertex. A vertex is said to be *regular* if its valence is four and six in case of rectangular and triangular patches, respectively. Otherwise, it is called an *extraordinary* vertex. A surface patch consisting only of regular vertices is *regular*. Many kinds of subdivision surfaces are designed such that regular patches converge to certain parametric surfaces, e.g., B-spline surfaces, box spline surfaces, and so on. During the subdivision process, the number of regular patches increases, while that of extraordinary vertices is constant. Most of the region of a subdivision surface become regular after several subdivision steps as shown in Fig. 1 where the patches containing extraordinary vertices are specified with shadows.

Fig. 2 shows two examples of subdivision steps. The original vertices are indicated with black dots and lines, while new vertices are shown as circles. The left figure shows a step of a Catmull-Clark surface with an extraordinary vertex of valence five, while the right one shows a Loop surface case with an extraordinary vertex of valence five as well. There are two kinds of new vertices, namely even vertices and odd vertices. An even vertex has

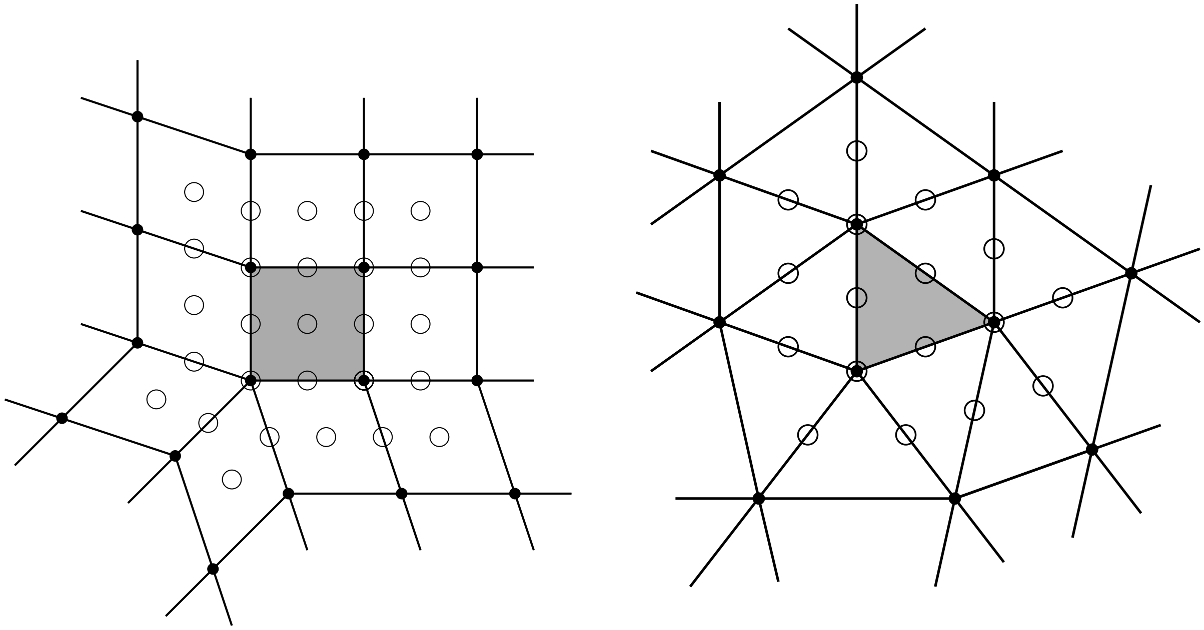


Figure 2: A subdivision step of a Catmull-Clark surface (left) and Loop surface (right).

an corresponding original vertex, while an odd vertex is a newly inserted vertex with the subdivision step. In case of a Catmull-Clark surface, odd vertices are further categorized into two types, i.e., edge vertices and face vertices, according to the locations where they are inserted.

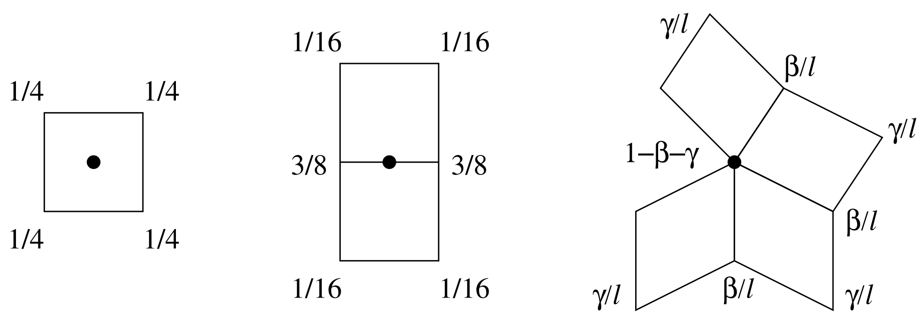


Figure 3: Catmull-Clark subdivision: masks for a face vertex (left), edge vertex (middle), and even vertex (right)

Fig. 3 illustrates subdivision rules of a Catmull-Clark surface, called subdivision masks. The left figure depicts the subdivision mask for a face vertex. The middle and right ones show the the masks for an edge vertex and an even vertex, respectively. The new vertices are denoted with black dots. The numbers in the figure are the coefficients of the subdivision rule, i.e., the weights for calculating new vertices from the original vertices. For instance, a face vertex $\mathbf{v}$ is obtained by the following equation:

$$\mathbf{v} = \frac{1}{4}\left(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\right),$$

where $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ correspond to the original vertices of the face. In the mask for an even vertex, l stands for the valence of the vertex while β and γ are given by the following equations:

$$\beta = \frac{3}{2l}, \quad \gamma = \frac{1}{4l}.$$

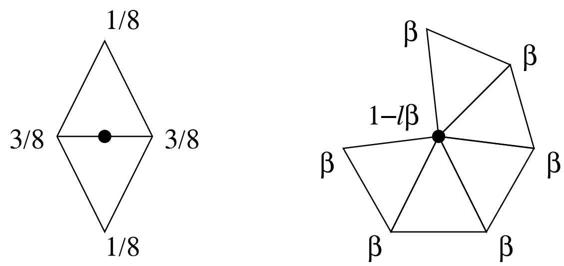


Figure 4: LOOP subdivision: masks for an edge vertex (left) and even vertex (right)

Fig. 4 indicates subdivision masks of a LOOP surface. The left figure shows the mask for an edge vertex, while the right one depicts that for an even vertex. Here, l is the valence of the vertex and β is computed from the following equation:

$$\beta = \frac{1}{l}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{l}\right)^2\right).$$

Every subdivision rule is given as a convex combination so that a new vertex is calculated with a weighted average of the original vertices. Thus the subdivided polyhedron consisting of the new vertices becomes more smooth and approximates the final surface more closely. However, it must be noted again that the polyhedron is only an approximation.

STAM proposed a way to compute an exact position of a surface point corresponding to certain parametric values by using the regular grids after some subdivision steps [6]. This allows us to see a subdivision surface as a parametric surface. Though, his method is not so stable around an extraordinary point. Especially the derivative computations suffer from instability. We have generalized the STAM's method to calculate differential properties.

3. Mathematical Formulation

This section explains a mathematical formulation of a subdivision surface. The discussion is applicable to the most of subdivision surfaces.

A subdivision surface patch is determined by K control points:

$$\mathbf{C} = (\mathbf{c}_1 \ \ldots \ \mathbf{c}_K)^T.$$

In order to simplify the problem, we assume that a patch contains at most one extraordinary point.

A subdivision surface has a parametrization even if it has an extraordinary point [5]. So that a surface can be defined with the following formula over a domain Ω:

$$\mathbf{S}(u,v)|_{(u,v)\in\Omega} = \sum_{i=1}^{K} b_i(u,v)\mathbf{c}_i = \mathbf{b}(u,v)^T\mathbf{C}. \qquad (1)$$

Here, $\mathbf{b}(u,v) = (b_1(u,v) \ \ldots \ b_K(u,v))^T$ is a set of basis functions. The basis functions must satisfy the partition of unity:

$$\sum_{i=1}^{K} b_i(u,v) = 1 \tag{2}$$

in order to guarantee the affine invariance so that a surface generated with affinely transformed control points coincides with the surface transformed by the same affine transformation. The domain is assumed to be a unit square in this paper: $\Omega = [0,1] \times [0,1]$. In case of a regular surface, it is determined by R control points with basis functions $\mathbf{b}_R(u,v) = (b_1(u,v) \ \ldots \ b_R(u,v))^T$ which are analytically defined, usually polynomials, such as B-spline functions and box spline functions. For instance, a regular Catmull-Clark patch consisting of $R = 16$ control points converges to a cubic B-spline surface. A regular Loop patch composed of $R = 12$ control points converges to a cubic box spline surface.

A subdivision step generates M control points, $\overline{\mathbf{C}}$, from the original control points, $\mathbf{C}$:

$$\overline{\mathbf{C}} = (\overline{\mathbf{c}}_1 \ \ldots \ \overline{\mathbf{c}}_M)^T = \bar{A}\mathbf{C},$$

where $\bar{A}$ is a $M \times K$ matrix representing the barycentric combination of the original control points. The surface is subdivided into $L+1$ subsurfaces defined over Ω with the new control points, $\overline{\mathbf{C}}$. In other words, the original patch consists of $L+1$ subpatches corresponding to the subdomains Ω_k:

$$\mathbf{S}(u,v)|_{(u,v)\in\Omega_k} = \mathbf{S}_k(u',v')|_{(u',v')\in\Omega},$$

where

$$\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_L \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \quad (i \neq j).$$

In the left case of Fig. 2, i.e., a Catmull-Clark surface, the numbers of the original control points and new points are $K = 18$ and $M = 27$, respectively. There are the $K = 11$ original control points and $M = 17$ new points in the right case, i.e., a Loop surface.

Only one of the subpatches has the same topology as the original patch and the rests are regular patches:

$$\mathbf{S}_k(u',v') = \begin{cases} \mathbf{b}(u',v')^T P_0 \overline{\mathbf{C}} & \text{if} \quad k = 0, \\ \mathbf{b}_R(u',v')^T P_k \overline{\mathbf{C}} & \text{if} \quad k = 1, \ldots, L \end{cases}$$

Here, P_k implies a picking matrix for selecting either K or R control points from the M control points, $\overline{\mathbf{C}}$. P_0 is a $K \times M$ matrix. Otherwise, P_k is a $R \times M$ matrix. Each row of P_k is filled with zeros except for a one in the column corresponding to the control points of each subpatch.

Each subpatch $\mathbf{S}_k$ is mathematically defined over Ω while it coincides with a part of the original surface $\mathbf{S}$ corresponding to the subdomain Ω_k. There are $L+1$ domain transformations between the original surface and the subsurfaces:

$$\phi_k\colon \ \Omega_k \to \Omega.$$

Since every subpatch but $\mathbf{S}_0$ is analytically defined, the patch of our interest is $\mathbf{S}_0$ defined by K control points. A subdivision matrix A generates K control points of the subpatch $\mathbf{S}_0$ from the original control points:

$$\mathbf{C}^{(1)} = P_0\overline{\mathbf{C}} = A\mathbf{C}^{(0)} = A\mathbf{C}.$$

Here we have introduced a new notation, $\mathbf{C}^{(i)}$, standing for control points of the i-th level subdivision. Because of the definition, A is a $K \times K$ matrix consisting of K rows of the matrix $\bar{A}$.

The corresponding subdomain of the original patch Ω_0 is also denoted as $\Omega^{(1)}$ to specify the level of subdivision. The domain transformation from $\Omega^{(1)}$ to Ω is denoted with ϕ instead of ϕ_0:

$$\phi\colon\ \Omega^{(1)} = \Omega_0^{(0)} \to \Omega$$

By applying the surface evaluation scheme given by Equation (1), the subsurface of the first subdivision level can be represented as below:

$$\begin{aligned} \mathbf{S}(u,v)|_{(u,v)\in\Omega^{(1)}} &= \mathbf{b}(u^{(1)},v^{(1)})^T \mathbf{C}^{(1)}\big|_{(u^{(1)},v^{(1)})\in\Omega} \\ &= \mathbf{b}(\phi(u,v))^T A\mathbf{C}\big|_{(u,v)\in\Omega^{(1)}} . \end{aligned}$$

Every subdivision step is processed with the same subdivision matrix A:

$$\mathbf{C}^{(i)} = A\mathbf{C}^{(i-1)}.$$

So the control points of the n-th subdivision level are given by the following equation:

$$\mathbf{C}^{(n)} = A^n \mathbf{C}. \tag{3}$$

The corresponding subdomain is as below:

$$(u^{(n)},v^{(n)})\big|_{(u^{(n)},v^{(n)})\in\Omega} = \phi^n(u,v)|_{(u,v)\in\Omega^{(n)}} .$$

Fig. 5 illustrates the domain subdivision of a Catmull-Clark surface. For each step, a domain $\Omega^{(i)}$ is subdivided into four subdomains, $\Omega_0^{(i)}$, $\Omega_1^{(i)}$, $\Omega_2^{(i)}$, and $\Omega_3^{(i)}$, where $\Omega_0^{(i)}$ corresponds to a domain of the $i+1$-th level subdivision $\Omega^{(i+1)}$.

Finally, the following equation for the n-th level subsurface is obtained:

$$\begin{aligned} \mathbf{S}(u,v)|_{(u,v)\in\Omega^{(n)}} &= \mathbf{b}(u^{(n)},v^{(n)})^T \mathbf{C}^{(n)}\big|_{(u^{(n)},v^{(n)})\in\Omega} \\ &= \mathbf{b}(\phi^n(u,v))^T A^n\mathbf{C}\big|_{(u,v)\in\Omega^{(n)}} . \end{aligned} \tag{4}$$

We are ready for evaluating a point of the subdivision surface with parameters (u,v).

1. Find n and k such that given (u,v) is in $\Omega_k^{(n)}$ where $k \neq 0$.
2. Evaluate the following equation:

$$\mathbf{S}(u,v)|_{(u,v)\in\Omega_k^{(n)}} = \mathbf{b}_R(\phi_k(\phi^n(u,v)))^T P_k \bar{A} A^n \mathbf{C}.$$

This evaluation process is rather expensive if n is large, because the multiplication of the subdivision matrix A is not so trivial. Furthermore, the equation cannot evaluate a point of parameters $(0,0)$, i.e., at the extraordinary point. The eigenspace analysis may be used for accelerating the evaluation but also for examining the convergence at the extraordinary point.

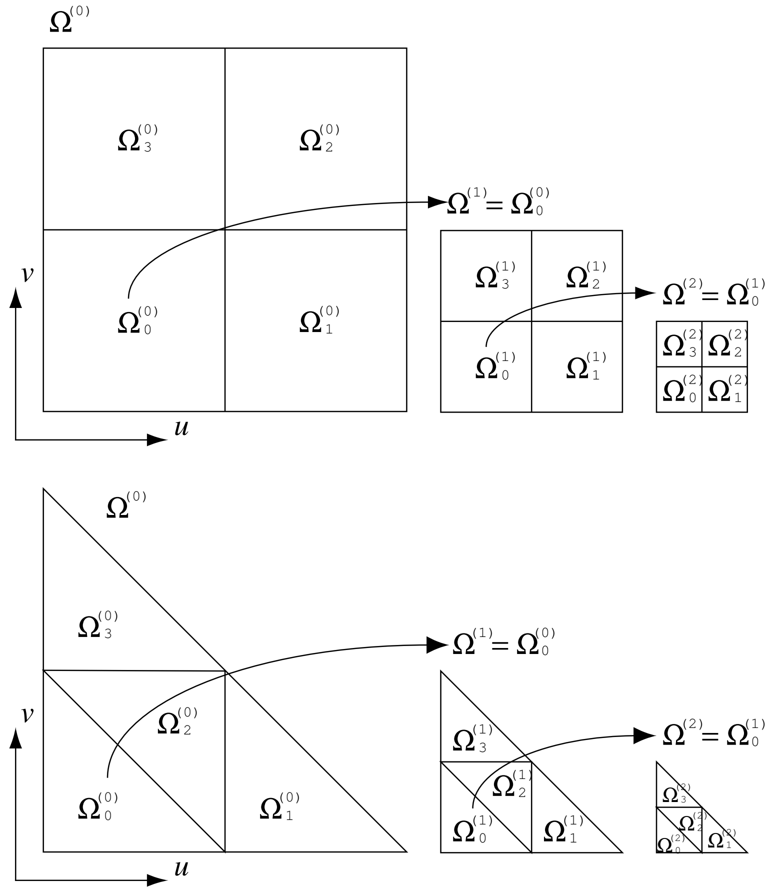


Figure 5: Domains of a Catmull-Clark surface (above) and Loop surface (below)

4. Eigenspace Analysis

An eigenvector $\mathbf{v}_i$ and an eigenvalue λ_i of a matrix A satisfies the following relation:

$$A\mathbf{v}_i = \mathbf{v}_i \lambda_i \, .$$

It is known that subdivision matrices of many types of subdivision surfaces have K linearly independent eigenvectors [4]. This indicates that the subdivision matrix A can be written with the diagonal matrix Λ and the invertible matrix V:

$$A = V \Lambda V^{-1}$$

where

$$V = (\mathbf{v}_1 \ldots \mathbf{v}_K) \, , \quad \Lambda = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_K \end{pmatrix} .$$

By substituting the matrix A with the diagonalized form, Equation (3) of n-th level control points yields the following equation:

$$\mathbf{C}^{(n)} = A^n \mathbf{C} = V \Lambda^n V^{-1} \mathbf{C}.$$

This indicates that a subdivision surface can be evaluated without multiplying the $K \times K$ matrix A in (4):

$$\begin{aligned} \mathbf{S}(u,v)|_{(u,v)\in\Omega^{(n)}} &= \mathbf{b}(\phi^n(u,v))^T V\Lambda^n V^{-1}\mathbf{C} \\ &= \sum_{i=1}^{K} \left(\mathbf{b}(\phi^n(u,v))^T\mathbf{v}_i\right)\lambda_i^n\hat{\mathbf{c}}_i \end{aligned} \tag{5}$$

where $\hat{\mathbf{C}} = (\hat{\mathbf{c}}_1 \ \ldots \ \hat{\mathbf{c}}_K)^T = V^{-1}\mathbf{C}$ represents the control points in the eigenspace.

We introduce a convention for indexing the eigenvalues λ_i as below:

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_K|.$$

In order to converge the subdivision surface, the limits of control points must also converge. So any eigenvalue is less than or equal to one: $1 \geq |\lambda_i|$. Since a subdivision step generates new control points with barycentric combinations of the original control points, all rows of A sum up to 1. Therefore, the first eigenvalue and its corresponding eigenvector is obvious:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = (1 \ \ldots \ 1)^T.$$

The above condition together with the partition of unity of the basis function, i.e., Equation (2), makes the first term of (5) constant:

$$\begin{aligned} \mathbf{S}(u,v)|_{(u,v)\in\Omega_k^{(n)}} &= \hat{\mathbf{c}}_1 + \sum_{i=2}^{K} \left(\mathbf{b}(\phi^n(u,v))^T\mathbf{v}_i\right)\lambda_i^n\hat{\mathbf{c}}_i \\ &= \hat{\mathbf{c}}_1 + \sum_{i=2}^{K} \left(\mathbf{b}_R(\phi_k(\phi^n(u,v)))^T P_k\bar{A}\mathbf{v}_i\right)\lambda_i^n\hat{\mathbf{c}}_i. \end{aligned} \tag{6}$$

Derivatives of the subdivision surface are obtained just by taking partial derivative:

$$\begin{aligned} \mathbf{S}_u(u,v)|_{(u,v)\in\Omega_k^{(n)}} &= \sum_{i=2}^{K} \left(\frac{\partial\mathbf{b}(\phi^n(u,v))}{\partial u}^T\mathbf{v}_i\right)\lambda_i^n\hat{\mathbf{c}}_i \\ &= \sum_{i=2}^{K} \left(\frac{\partial\mathbf{b}_R(\phi_k(\phi^n(u,v)))}{\partial u}^T P_k\bar{A}\mathbf{v}_i\right)\lambda_i^n\hat{\mathbf{c}}_i. \end{aligned} \tag{7}$$

Equation (6) indicates the convergence of a surface point at the extraordinary point. The convergent position can be computed by taking the limit of (6):

$$\lim_{n\to\infty} \mathbf{S}(u,v)|_{(u,v)\in\Omega^{(n)}} = \hat{\mathbf{c}}_1,$$

if $1 > |\lambda_2|$. The convergence was firstly discussed by Doo and Sabin [2].

Equation (7) is related to the convergence condition of a normal at the extraordinary point specified by Reif [5]. The normal exists if and only if $\lambda_2 = \lambda_3$ is real with multiplicity 2, i.e., $1 > |\lambda_2| = |\lambda_3| > |\lambda_4|$. In this case, a normal vector of the n-th level subdivision is given in the following formula:

$$\begin{aligned} \mathbf{n}(u,v)|_{(u,v)\in\Omega^{(n)}} &= \mathbf{S}_u(u,v) \times \mathbf{S}_v(u,v)|_{(u,v)\in\Omega^{(n)}} \\ &= c\,(\hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3)\,\lambda_2^{2n} + o(\lambda_2^{2n}), \end{aligned} \tag{8}$$

where $o(\lambda_2^{2n})$ indicates that the limit of the rest terms converges to zero faster than λ_2^{2n}. Therefore, the normal at the extraordinary point is perpendicular to the plane spanned by the vectors, $\hat{\mathbf{c}}_2$ and $\hat{\mathbf{c}}_3$. However, Equation (8) does not specify the orientation of the normal vector, because it depends on the sign of the coefficient c.

5. Example: Catmull-Clark Surface

This section treats a Catmull-Clark surface as an example of subdivision surfaces especially for discussing differential properties. A Catmull-Clark patch with an extraordinary point of valence N consists of $K = 2N + 8$ control points. A subdivision step generates $M = K + 9$ control points representing four subsurfaces which correspond to the following subdomains of the original surface:

$$\Omega_0 = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \qquad \Omega_1 = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right],$$
$$\Omega_2 = \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right], \qquad \Omega_3 = \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right].$$

This means that the domain transformation of the subdivision step is a simple scaling transformation of factor two:

$$(u^{(n)}, v^{(n)})\big|_{(u^{(n)},v^{(n)})\in\Omega} = (2^n u, 2^n v)\big|_{(u,v)\in\Omega^{(n)}}.$$

Therefore, the Catmull-Clark surface of (u, v) is given by the following equation:

$$\mathbf{S}(u,v)\big|_{(u,v)\in\Omega_k^{(n)}} = \sum_{i=1}^{K} \left(\mathbf{N}(\phi_k(2^n u, 2^n v))^T P_k \bar{A}\mathbf{v}_i\right) \lambda_i^n \hat{\mathbf{c}}_i \tag{9}$$

$$= \hat{\mathbf{c}}_1 + \sum_{i=2}^{K} \left(\mathbf{N}(\phi_k(2^n u, 2^n v))^T P_k \bar{A}\mathbf{v}_i\right) \lambda_i^n \hat{\mathbf{c}}_i, \tag{10}$$

where $\mathbf{N}(u, v)$ denotes uniform cubic B-spline functions. The first order derivative of the Catmull-Clark surface is given as below:

$$\mathbf{S}_u(u,v)\big|_{(u,v)\in\Omega_k^{(n)}} = 2^{n+1} \sum_{i=1}^{K} \left(\mathbf{N}_u(\phi_k(2^n u, 2^n v))^T P_k \bar{A}\mathbf{v}_i\right) \lambda_i^n \hat{\mathbf{c}}_i \tag{11}$$

$$= 2^{n+1} \sum_{i=2}^{K} \left(\mathbf{N}_u(\phi_k(2^n u, 2^n v))^T P_k \bar{A}\mathbf{v}_i\right) \lambda_i^n \hat{\mathbf{c}}_i. \tag{12}$$

Equations (9) and (11) are the same as those specified by Stam [6]. Our main contributions are (10) and (12) which are obtained by pre-evaluating the first terms. The equations indicate the convergence of a surface point as well as a normal at the extraordinary point as described in Section 4. Furthermore, the equations, especially of derivatives, become much more numerically stable.

The first order derivative (12) near the extraordinary point is governed by the first two terms, $\lambda_2^n \hat{\mathbf{c}}_2$ and $\lambda_3^n \hat{\mathbf{c}}_3$. Thus, the normal vector at the extraordinary point is given by the following equation:

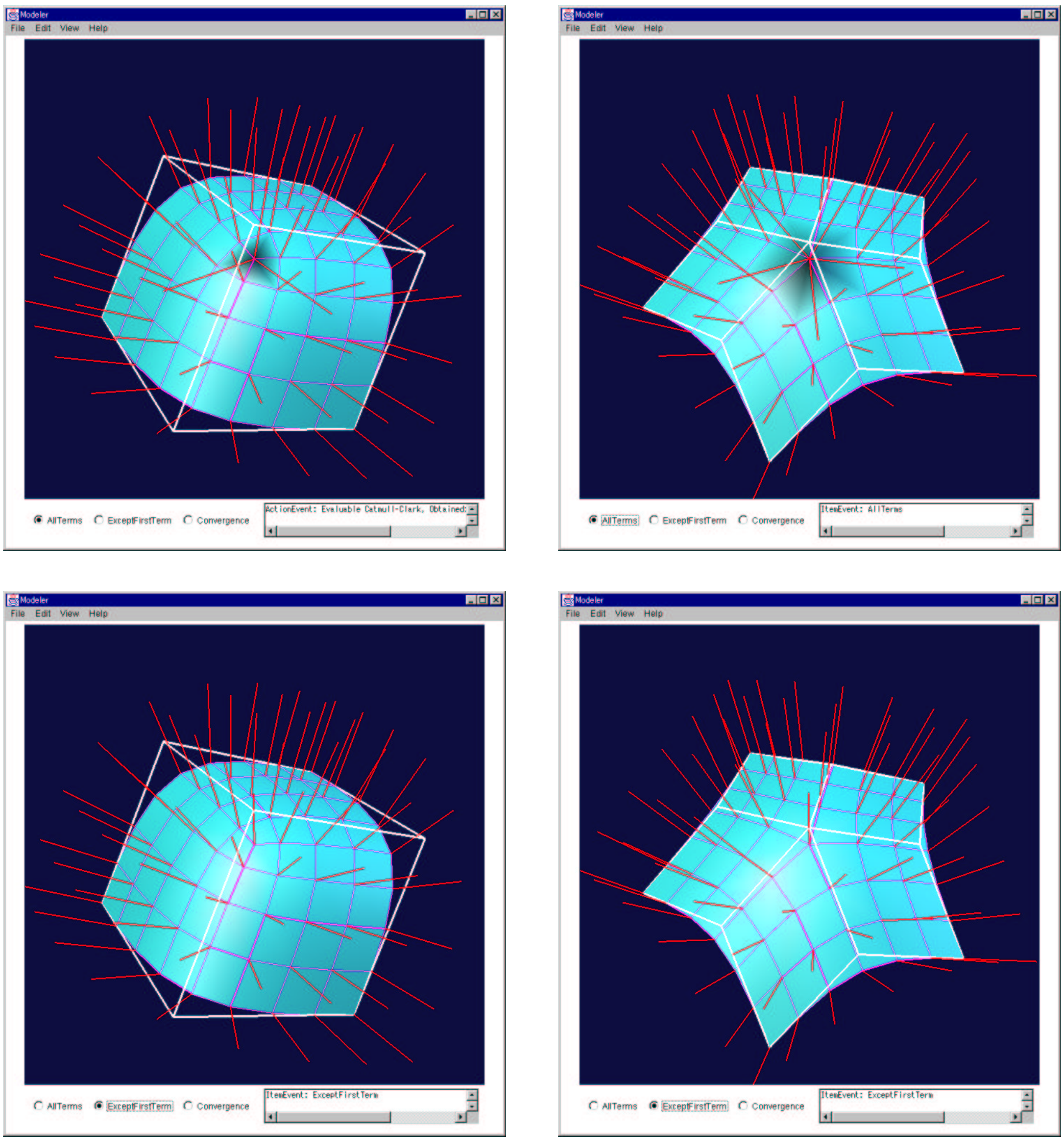
$$\begin{aligned}\mathbf{n}(0,0) &= \lim_{n\to\infty} \mathbf{S}_u(u,v) \times \mathbf{S}_v(u,v)\big|_{(u,v)\in\Omega_k^{(n)}} \\ &= \lim_{n\to\infty} \left(2 f_k (2\lambda_2)^{2n} \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3 + o\left((2\lambda_2)^{2n}\right)\right),\end{aligned} \tag{13}$$

where

$$f_k = \left(\mathbf{N}_u(0,0)^T P_k \bar{A}\mathbf{v}_2\right)\left(\mathbf{N}_v(0,0)^T P_k \bar{A}\mathbf{v}_3\right) - \left(\mathbf{N}_u(0,0)^T P_k \bar{A}\mathbf{v}_3\right)\left(\mathbf{N}_v(0,0)^T P_k \bar{A}\mathbf{v}_2\right).$$

Although f_k varies depending on k, the sign of f_k is consistent. Equation (13) gives a normal at the extraordinary point.

Fig. 6 shows some results of the normal vector calculation. The left three figures show extraordinary points of valence three while the right three figures show those of valence five. The top two figures on page 5 are calculated with (11), while the middle and bottom two figures are generated with (12) and (13), respectively. Due to the instability caused by the first term of (11), the left figures have wrong normal vectors around the extraordinary points which results dark spots of shading calculation. Other figures indicate that Equations (12) and (13) are capable of evaluating proper normal vectors around extraordinary points.



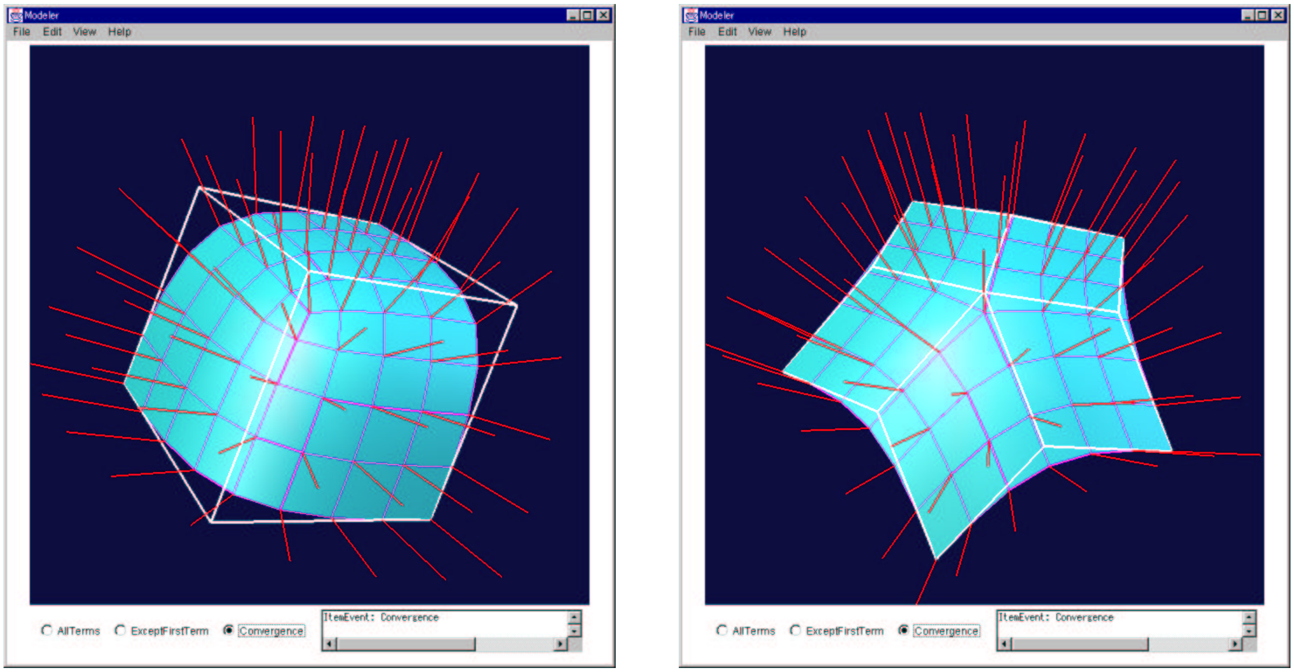


Figure 6: Normal vectors around extraordinary points of valence three and five

6. Conclusion

This paper discussed an approach to modify the Stam's method for evaluating a subdivision surface around an extraordinary point. A simplified version of the evaluation formula is obtained based on the investigation into the first eigenvalue and its eigenvector of a subdivision matrix. The achievements of the modification were verified with some experiments.

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