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Geometry as Transformation

Robert A. Wiggs

University of Louisiana at Lafayette
128 Hugh Wallis Road, Lafayette, Louisiana 70508-2107, USA
email: wiggler@mindspring.com

Abstract. This paper utilizes geometry and graphics to demonstrate a process of three-dimensional spatial transformation. A transformation defines how structures behave in addition to how they are made. Here, the transformation process is applied to several different frames of reference. Material polyhedra transform into pairs of spatial lattices and the lattices transform into each other; the lattices also transform into twisted loops with exostructural and endostructural modes. The exo/endo interactions transform the hypercube into a pair of twisted loops and the twisted loops become a source for sculpture.

Key Words: polyhedra, lattice, twisted loops, hypercube, sculpture

1. Introduction

Classical polyhedra are composed of faces, vertices and edges. Leonard EULER (1707–1783) developed a formula for polyhedra which shows that the number of faces plus the number of vertices is always equal to the number of edges plus two ($F + V = E + 2$). In the cube, for example, ($6 + 8 = 12 + 2$). Subtracting two from the sum of the number of faces and vertices always gives the correct number of edges. This formula of EULER serves all polyhedra and many more geometrical configurations even when the angles are imprecise, faces do not plane and edge lengths are not conserved. It even applies to individual stones in a pile of limestone road gravel.

After EULER many different geometries were formulated (inverse, affine, projective and Riemannian) and their postulates and axioms became very complex. About 1882 the German mathematician, Felix KLEIN, proposed an idea that would help to integrate all of geometry. These statements about KLEIN reinforce his proposal; WEYL ([5], 1952) reports KLEIN believed that geometry is defined by a group of transformations and investigates everything that is invariant under the transformations of the given group. STEWART ([4], 1990) reports that according to KLEIN, geometry is not really about points, lines, angles and parallels; it is about transformations. It is transformations that truly characterize geometry. Likewise, in this paper the study of polyhedra is not just about faces, vertices, edges and angles, it is generally about transformations between polyhedral lattices.

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2. A material polyhedron

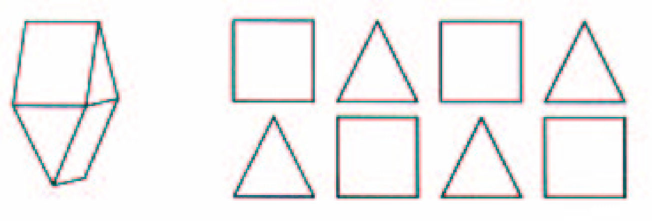

Figure 1: *Twist Octahedron*, a space filling polyhedron

Let's say that the polyhedron (left) in Fig. 1 is made of cardboard. It represents the three-dimensional self all-space filling prismatic *Twist Octahedron* discovered by this author. It is composed of eight faces, eight vertices and fourteen edges or $(8 + 8 = 14 + 2)$. The vertical axis of symmetry bisects the 2-fold polar edge line increments. It is also the 2-fold axis of rotation around which the polygon components are rigidly connected to each other. The form appears to be two trigonal prisms twisted together. On the right, the plane-faced polygon components are disassembled into individual classical polygons, making it easy to see that the four trigons and the four tetragons alternate in relation to each other. Hence, there is a 2-fold axial twist. It will be shown in the succeeding figures why the twist is fundamental to the transformation process.

3. A space lattice

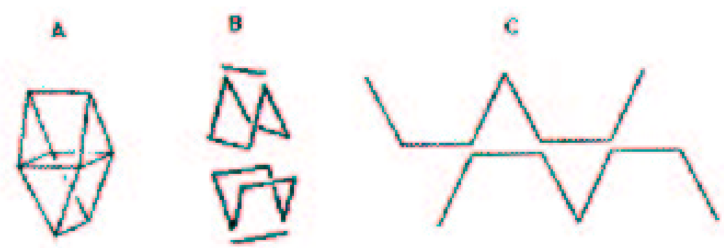

Figure 2: A prismatic octahedral space lattice

It is lines and not cardboard polygons that provide the structural components for the prismatic octahedral space lattice in Fig. 2. The lineal network of the lattice represents the space between the material polygons of Fig. 1. Unlike the cardboard version, the lattice hemispheres separate into angular equatorial rings (B), and the single polar line increments (top and bottom) lift off. The single line increments are called *"polar caps"*. The cap line increments are bisected by the 2-fold axis of symmetry. Looking from right to left (C to A), you can see that the structural components for the space lattice are a pair of point related lines in drawing C; the lines wrap into a pair of angular rings with the caps returning in drawing B; the caps are in place to complete the lattice in drawing A. The equatorial rings, called *"foundation sutures"*, are the primary structural components and the secondary components are the polar caps. Beyond the scope of this paper, higher numbered equatorial rings can be closed with circumferential, radial and compound caps to generate evolving families of polyhedral lattices. And EULER's formula still applies.

4. A symmetrical glide/reflection transformation

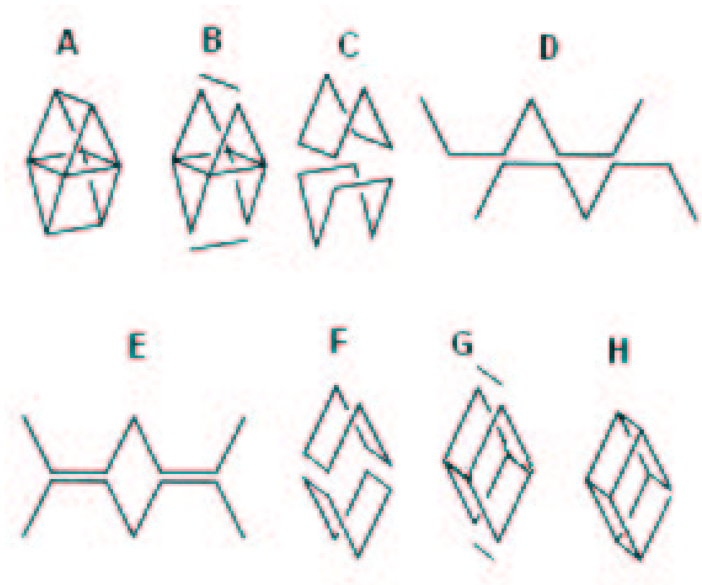

Figure 3: Transformation from *Twist Octahedron* to *Rhombic Cube*

The graphic sequence in Fig. 3 transforms a pair of polyhedral lattices into each other. The symmetry, called translational, generates an active glide/reflection transformation.

A. The transformation begins with the space lattice of the prismatic *Twist Octahedron*. The hemispheres are twisted in relation to each other on the 2-fold axis of symmetry parallel to the vertical plane of the page.

B. Two single line increments, polar caps, detach from the hemispheres and temporarily leave the scene.

C. The uncapped hemispheres separate at their point junction sties and become angular equatorial rings.

D. Remaining at the point junction positions, the rings unwrap into a pair of two-unit wavy lines. Each structural unit (__/) has three line increments and each unit also has both point and edge junction sites. There are two units in each line.

E. One of the lines translates (glides) either left or right from point to edge (mirror) junction sites in coincidence with the other line. Hermann Weyl calls it "slip reflection".

F. After the glide, the equatorial rings reform in the edge to edge position.

G. The rings are joined at their edge junction sites and the caps return.

H. The *Rhombic Cube* is formed when the caps are in place. The hemispheres are now untwisted in relation to each other on the 2-fold axis parallel to the vertical plane of the page.

The transformation can return from H to A, *Rhombic Cube* to prismatic *Twist Octahedron*, by the inverse route. This fulfills Felix Klein's invariance under transformation requirement.

The foundation suture space is manifest in a pair of equatorial rings, one for each hemisphere. The equatorial rings are not the great smooth geodesic circles of classical geometry but have peaks and troughs not unlike De Broglie's circular waves, as reported by Gribbin ([2], 1984). The suture line is the only lattice building component and consequently the polygon enclosures are the last to form. The lineal network, abstracted from between material polygons, functions in suture space and suture space is where all junction, separation and other interactions happen. All of this interaction and flexibility is possible, where none seemed to have existed before because of the action potential between the rings and caps. The interactions that drive the transformation are in the spatial lines between the polygons and not in the polygons themselves. EINSTEIN (1961) said: *"Physical objects are not in space but these objects are spatially extended. In this way the concept empty space loses it meaning".* Here the suture space contains the fundamentals for the transformation process. Here space becomes form. In the Buddhist heart sutra and subatomic interactions alike, emptiness becomes form and form becomes emptiness.

5. Twisted loops

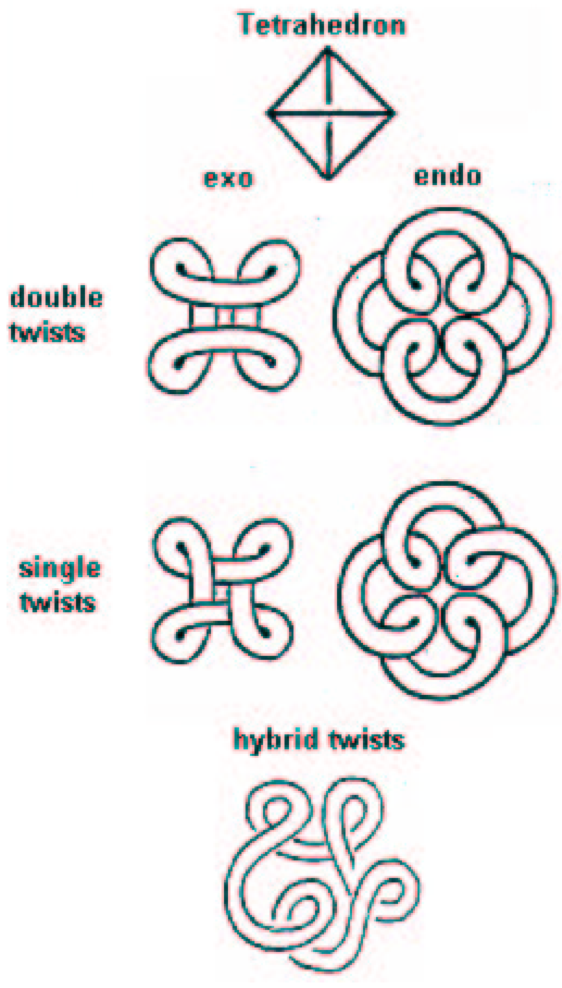

Figure 4: Twists

In addition to the glide/reflection transformation, polyhedral lattices also transform into twisted loops with exostructural and endostructural modes of interaction. Fig. 4 shows the exo and endo loops twisting around the four spatial coordinates or vertices of the 2-fold tetrahedron. The 2-fold axis of symmetry, bisecting polar edges, upon which the loop figures twist and untwist, is perpendicular to the plane of the page. The double twists turn right and left simultaneously and the single twists turn either right or left one twist at a time. The twists can also turn in hybrid (exo/endo – single/double) combinations. Space lattices and twisted loops have parallel interactions in common; they both generate pairs of spatial configurations that can transform into each other by twisting and untwisting. These two sources of behavioral information, ubiquitous in nature, offer extended potential for the study of twisted loops. Perhaps it could help in the understanding of something as large as the interaction between the tectonic plates of the earth and something as small as protein folding and the twisting and untwisting of the DNA molecule.

6. The hypercube

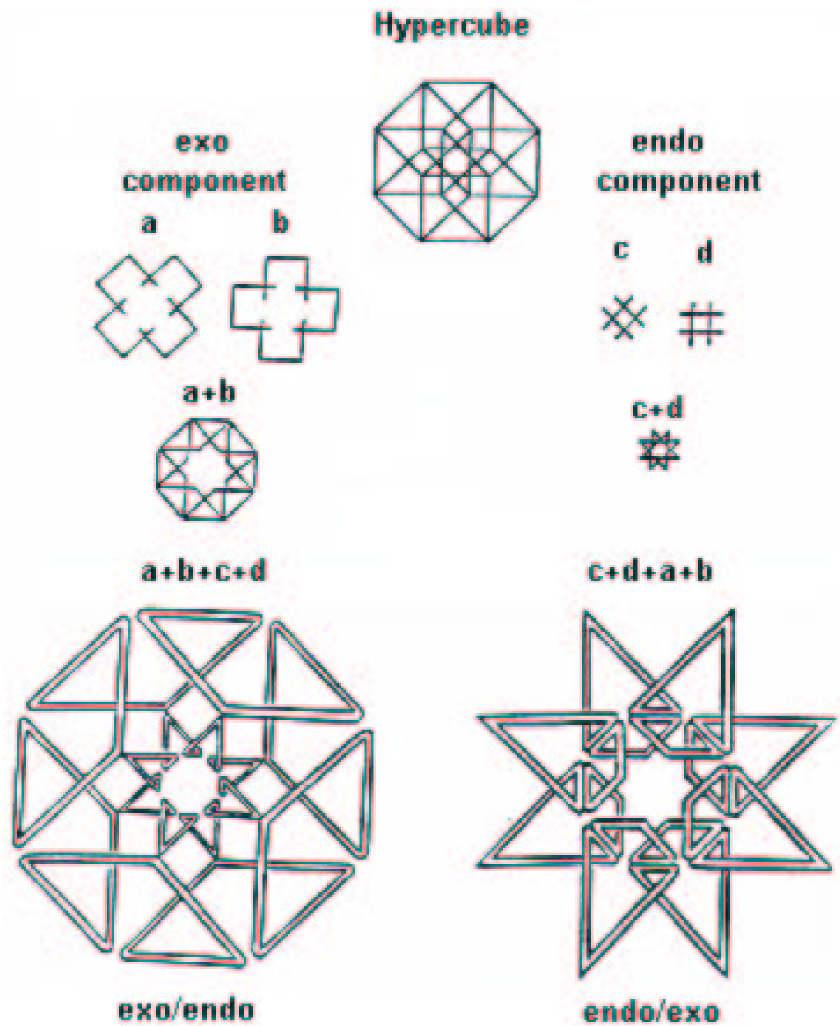

Figure 5: Hypercube

Since EINSTEIN, we can think in four space/time dimensions. This means that we have access to four subdivisions of thought projection; up/down, right/left, in/out and now/then (XYZT). The eureka moment of discovery comes when they all suddenly fit together at the same time. Timing is critical.

The hypercube introduces us to the possibility of even higher spatial dimensions, however, these ideas will not be discussed here. Instead, the hypercube will be transformed into a pair

Figure 6

Figure 7

of 3-D twisted loops. Incidentally, the hypercube is the logo for the International Society for Geometry and Graphics.

Like the space lattice in Fig. 2, the hypercube in this study is not composed of rigid polygons. It is composed of twisted loops. Many kinds of three-dimensional structures, both simple and complex, can be transformed into twisted loops. The hypercube loops are a case in point. In Fig. 5 the hypercube is transformed into a pair of twisted loops with reciprocal exostructural and endostructural twist modes.

The exo component (A+B), on the left, forms a lineal shell and the endo component (C+D), on the right, forms the nucleus inside the shell. The shell and the nucleus join on the left (A+B+C+D) at point junction sites. The transformation continues when the components (C+D) and (A+B) change places and join on the right (C+D+A+B) at edge junction sites. This flexible interchange is possible because there are no rigid polygon components to stop the action. There are only the twists upon the loops. The loops are free to return to a neutral untwisted condition, thereby conserving the action potential and again fulfilling Felix Klein's invariance under transformation requirement. With this kind of spatial freedom all of the parts of any given system could conceivably be connected by one single world loop.

7. Twisted loop sculpture

When twisted wire loops are filled in with a malleable material, the surfaces between the twists are what topologists call saddles. A saddle surface is curved both negatively and positively like a Pringle's potato chip. The twisted loop saddle sculptures shown here are

Figure 8

Figure 9

quite different from the mathematical Werner Boy's and Steiner's *Roman surfaces* and others. Those surfaces, as shown by Peterson ([3], 1990) are based on the interpenetrant crossings of knots. The saddles, on the other hand are generated by unknotted exostructural and endostructural twists with minimal surfaces akin to liquid soap membranes stretched across twisted wire frames. The simplest twisted loop is a loop with a single twist, turned either outside the loop (8) or inside the loop (◎) and then there is a continuation on up the scale to higher numbers of twists and to hybrid combinations. The four sculptures shown here have hybrid twists and they are asymmetrical. There is no restriction on the number or on the kinds of arrangements the twists can take. Theoretically, it appears that the parts of any one source object or any group of objects can be connected and defined as a twisted loop. After thirty years, this part of the research is just beginning. Upon discovery of the suture interactions between the parts of material systems, we can see that the material and the suture are not exclusive opposites. They coexist as two aspects of the same system.

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